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Bifurcation of periodic solutions with large periods for a delay differential equation

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Abstract We consider a one-parameter family of delay differential equations which has been proposed as a model for a prize and prove that at a critical parameter where the linearization at equilibrium has a double zero eigenvalue periodic solutions bifurcate off with periods descending from infinity.

Keywords Delay differential equation · Takens-Bogdanov singularity · Bifurcation · Periodic solution

AMS Subject Classification. 34K18, 34K13, 37G15

1 Introduction

In [1] the delay differential equation

$$\dot{x}(t) = a(x(t) - x(t-1)) - b|x(t)|x(t)$$

with parameters $a > 0$, $b > 0$ was introduced as a model for a price under the sole influence of rational behaviour of trading agents, with fundamental economic quantities being constant. A detailed derivation of the model is given in [2]. Very roughly, the linear term on the right-hand side of the equation corresponds to the tendency to follow the trend, while the quadratic term represents the tendency to turn back when the price is too far from an equilibrium defined by the fundamental economic quantities. A similar model was proposed and discussed in [3].

Multiplication of solutions by $\frac{b}{a}$ reduces the model to the equation

$$a^{-1}\dot{x}(t) = x(t) - x(t-1) - |x(t)|x(t) \tag{1}$$

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with the single parameter $a > 0$, which is more convenient for mathematical analysis.

Incidentally, recall that solutions are either differentiable functions $x : \mathbb{R} \to \mathbb{R}$ which satisfy Eq. 1 everywhere, or continuous real functions on some interval of the form $[t_0 - 1, \infty)$, $t_0 \in \mathbb{R}$, which are differentiable and satisfy Eq. 1 for all $t > t_0$. Let C denote the Banach space of continuous real functions on the initial interval $[-1, 0]$, with the norm given by

$$\|\phi\| = \max_{-1 \le t \le 0} |\phi(t)|.$$

Data $\phi \in C$ uniquely determine solutions $x = x^{a,\phi}$ on $[-1, \infty)$ by the initial conditions $x|[-1, 0] = \phi$. The equations

$$F_a(t, \phi) = x_t, \quad x = x^{a,\phi} \quad \text{and} \quad x_t(s) = x(t + s)$$

define a semi-flow $F_a : [0, \infty) \times C \to C$. The only constant solution of Eq. 1 is zero; $0 \in C$ is the only stationary point of the semi-flow F_a. Let us also recall that slowly oscillating solutions are defined by the property that for any pair of zeros $z < z'$ we have $z + 1 < z'$.

Early numerical observations in [5] indicated that the equilibrium $0 \in C$ is asymptotically stable for $a < 1$, and that for $a > 1$ there are stable periodic orbits, with amplitudes becoming small and minimal periods growing to infinity as $a \searrow 1$. The main results in [1] establish asymptotic stability of the equilibrium for $a < 1$, instability for $a > 1$, and existence of a slowly oscillating periodic solution with orbit in the boundary of a global centre-unstable manifold for $a > 1$. In [12] the asymptotic shape of the periodic solutions found in [1], for $a \to \infty$, is determined. There are also other periodic orbits, see Sect. 1 of [12], and the result on asymptotic shape has implications on the number and location of continua in the global bifurcation diagram which are beyond the earlier numerical observations.

The present paper deals with bifurcation of periodic solutions from zero at $a = 1$. There is no Hopf bifurcation, since the only imaginary eigenvalue of the generator of the linearized semi-flow is $0 \in \mathbb{C}$. The multiplicity of this eigenvalue is 2 at $a = 1$ and 1 otherwise. So for a close to 1 the situation may be seen as part of a Takens-Bogdanov scenario with two parameters (see, e.g., [7]). Notice, however, that the non-linear part in Eq. 1 is given by the C^1-function

$$f : \mathbb{R} \ni \xi \mapsto (1 - |\xi|)\xi \in \mathbb{R}$$

which has no second derivative at 0. This means that the standard approach to identify the local dynamics, i.e., centre manifold reduction, expansion and truncation of the resulting vectorfields on the plane and transformation to a normal form, seems not amenable here, due to lack of smoothness.

The main result of the present paper is the following.

Theorem 1. *For every $\epsilon > 0$ there exist $a = a_\epsilon \in (1 - \epsilon, 1 + \epsilon)$ so that Eq. 1 has a slowly oscillating periodic solution $p_\epsilon : \mathbb{R} \to (-\epsilon, \epsilon)$ with minimal period τ_ϵ given by three consecutive zeros. We have*

$$\tau_\epsilon \to \infty \quad \text{as} \quad \epsilon \to 0.$$

The proof of the theorem begins in Sect. 2 with preliminary results on the semi-flows F_a, on linearization at the stationary point 0, and on the change of stability of the stationary point when the parameter a crosses the critical level 1. As in [1] we choose a certain complementary subspace $N \subset C$ of the two-dimensional centre space L_1 at $a = 1$ and a basis of L_1, to the effect that projection along N onto L_1 and taking coordinates coincides with the evaluation map

$$E : C \ni \phi \mapsto (\phi(0), \phi(-1)) \in \mathbb{R}^2.$$

This implies that for $a \geq 1$ not too large coordinates on centre-unstable manifolds are given by the map E.

The analysis of plane curves of the form $t \mapsto (x(t), x(t-1)) = Ex_t$, or equivalent ones, along special slowly oscillating solutions x has been used successfully in numerous papers on periodic solutions of delay differential equations, beginning with work of Kaplan and Yorke [8]. It is closely related to the discrete Lyapunov function from [10, 11]. For further references, see [8–11]. The identification of the evaluation map with coordinates on invariant manifolds in [1] is similar to an earlier construction in Chapter 6 of [9].

Section 3 deals with a centre manifold W_* for the augmented system

$$\begin{aligned} \dot{x}(t) &= \alpha(t)[x(t) - x(t-1) - |x(t)|x(t)] \\ \dot{\alpha}(t) &= 0 \end{aligned} \tag{2}$$

and with the foliation of W_* by locally positively invariant manifolds W_a of the semi-flows F_a; W_1 is a centre manifold and W_a for $a > 1$ are center-unstable manifolds.

In Sect. 4 it is shown that solutions $x : [-1, \infty) \to \mathbb{R}$ which start in smaller manifolds $W^a \subset W_a$ are slowly oscillating, continuously differentiable (on all of $[-1, \infty)$), and have slowly oscillating derivatives.

In the beginning of Sect. 5 one sees among others that the coordinates given by the evaluation E on each manifold W^a fill a fixed open square Q. The result of Sect. 4 is used in the proof of Proposition 5.2 about coordinate curves

$$[0, \infty) \ni t \mapsto (x(t), x(t-1)) \in \mathbb{R}^2$$

along solutions $x = x^{a,\phi}$ which start at initial data $\phi \in W^a \setminus \{0\}$ with coordinates $E\phi$ in the graph $f \subset \mathbb{R}^2$. For these curves the graph f and the vertical axis are global transversals. Propositions 5.3–5.6 and Corollary 5.1 deal with the case $a = 1$. We need that in this case the coordinate curves keep winding around 0. Moreover, the two maps which are given by successive intersections of the coordinate curves with the branches of f in the first and third quadrant need to have limit 0 at $0 \in |\mathrm{R}^2$. The inherent difficulty is, loosely spoken, that the angular speed tends to zero for curves in neighbourhoods shrinking to zero. The proof of the desired winding behaviour and of the limit relation is based on the exclusion of small solutions without zeros on unbounded intervals in Propositions 5.3 and 5.5. Proposition 5.3 generalizes Corollary 5.2 of [1]; its proof by a short comparison argument is new.

It is in Sect. 5 where our approach to local bifurcation diverges from the standard one: We study the coordinate flows on the square directly, without expansion and truncation of underlying vectorfields and recourse to normal forms. The choice of coordinates explained earlier makes this relatively easy. For an analysis of the dynamics on two-dimensional invariant manifolds which uses spectral projections instead of the projection along N see [13, 14] and the references given in [14].

Section 6 completes the proof of the theorem. The change of stability of the stationary point in the manifolds W^a when the parameter crosses the level 1, the information about coordinate curves from Sect. 5 and elementary continuity arguments yield bifurcation of slowly oscillating periodic solutions with small amplitude. Depending on the stability properties of the stationary point at $a = 1$ we obtain supercritical, subcritical, or critical bifurcation (Propositions 6.1–6.3). This is slightly better than stated in the theorem. The last assertion of the theorem is a consequence of the final Proposition 6.4, which makes precise what has been said earlier about angular speed: Small amplitudes of slowly oscillating solutions imply large distances between successive zeros.

The question whether the stationary point is asymptotically stable for $a = 1$ remains open. An affirmative answer would imply supercritical bifurcation.

Notation. For a Banach space B and a real number $r > 0$ the open ball with centre 0 and radius r is denoted by $B(r)$. S_B stands for the unit sphere in B. The closure and boundary of a subset $M \subset B$ are denoted by $\overline{M}$ and ∂M, respectively. If a curve $c : I \to B$, $I \subset \mathbb{R}$ an interval with inner points, is differentiable at $t \in I$ then we set

$$c'(t) = \dot{c}(t) = Dc(t)1 \in B.$$

Spectra of closed linear operators in a Banach space B over the field $\mathbb{R}$ are defined by complexification.

For a semi-flow $S : [0, \infty) \times M \to M$ and for $t \geq 0$, $\phi \in M$, $I \subset [0, \infty)$, $U \subset M$ we use the abbreviations $S(t, U) = S(\{t\} \times U)$, $S(I, \phi) = S(I \times \{\phi\})$, $S(I, U) = S(I \times U)$.

Solutions of non-autonomous delay differential equations

$$\dot{x}(t) = g(t, x(t), x(t-1)), \tag{3}$$

for $g : [t_0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$, $t_0 \in \mathbb{R}$, are continuous real-valued functions on $[t_0 - 1, \infty)$ which are differentiable and satisfy Eq. 3 for all $t > t_0$. In case g is continuous a solution has a right derivative at t_0, and the differential equation holds at t_0 as well. For a function $g : \mathbb{R}^3 \to \mathbb{R}$ we consider solutions on intervals of the form $[t_0 - 1, \infty)$ as before and solutions on $\mathbb{R}$ which are differentiable and satisfy Eq. 3 for all $t \in \mathbb{R}$.

In the sequel, equations are numbered separately in each section. Equation j from Sect. k is quoted in Sect. $m \neq k$ as Eq. $k.j$. Analogous conventions apply to propositions, corollaries, etc.

2 Preliminaries

In this section we recall basic facts about Eq. 1.1 with parameter $a > 0$. For proofs which are omitted, compare [1] and [4]. If $x : [-1, \infty) \to \mathbb{R}$ is a solution of Eq. (1.1) then the restriction $x|[0, \infty)$ is differentiable, and Eq. (1.1) is satisfied at $t = 0$ with the right derivative of x. The next result is a variant of Proposition 3.2 in [1].

Proposition 1 *Let $a > 0$ and let $x : [-1, \infty) \to \mathbb{R}$ be a solution of Eq. (1.1) without zeros on some interval (t, ∞) with $t \geq 0$.*

(i) If $sign(\dot{x}(s)) = -sign(x(s))$ on (t, ∞) then

$$\lim_{s\to\infty} x(s) = 0.$$

(ii) If $sign(\dot{x}(s)) = sign(x(s))$ for some $s > t$ then $\dot{x}$ has a zero $z > s$.

Proof 1. Proof of (i). If $c = \lim_{u\to\infty} x(u) \neq 0$ then

$$\dot{x}(u) = a(f(x(u)) - x(u-1)) \to a(f(c) - c) \neq 0 \text{ as } u \to \infty,$$

and we arrive at a contradiction.

2. Proof of (ii). Suppose $\dot{x}$ has no zero on (s, ∞). In case x is bounded it follows that $c = \lim_{u\to\infty} x(u) \neq 0$, which yields a contradiction as in part 1. In case x is unbounded and positive we infer that x is increasing on (s, ∞) with $x(u) \to \infty$ as $u \to \infty$. Using this and the inequality

$$\dot{x}(u) = a(f(x(u)) - x(u-1)) \leq a(f(x(u)) \text{ on } (t+1, \infty)$$

we get $\dot{x}(u) < 0$ for u sufficiently large, which yields a contradiction. The proof in the remaining case is analogous. □

The semi-flow

$$F_a : [0, \infty) \times C \ni (t, \phi) \mapsto x_t^{a,\phi} \in C$$

is continuous. All maps $F_a(t, \cdot)$, $t \geq 0$, and the restriction of F_a to $(1, \infty) \times C$ are continuously differentiable. We have

$$D_2 F_a(t, \phi)\chi = v_t^{a,\phi,\chi}$$

with the unique solution $v^{a,\phi,\chi} : [-1, \infty) \to \mathbb{R}$ of the variational equation

$$a^{-1}\dot{v}(t) = v(t) - v(t-1) - 2|x^{a,\phi}(t)|v(t),$$

with initial condition $v_0 = \chi \in C$. For $t > 1$ and $\phi \in C$,

$$D_1 F_a(t, \phi)1 = \dot{x}_t^{a,\phi}.$$

The curves $F_a(\cdot, \phi)$ have right derivatives at $t = 1$, given by the derivatives $\dot{x}_1^{a,\phi} \in C$ of the continuously differentiable functions $x_1^{a,\phi}$. For $a > 0$, $\phi \in C$, $x = x^{a,\phi}$, and $t \geq 1$,

$$D_2 F_a(t-1, x_1)\dot{x}_1 = \dot{x}_t.$$

Linearization at the unique stationary point $0 \in C$ yields a strongly continuous semi-group given by

$$T_a(t) = D_2 F_a(t, 0).$$

The variational equation along the zero solution is

$$a^{-1}\dot{v}(t) = v(t) - v(t-1). \tag{1}$$

The generator G_a of the semi-group is defined on

$$\mathrm{dom}_a = \{\chi \in C : \chi \text{ continuously differentiable}, a^{-1}\dot{\chi}(0) = \chi(0) - \chi(-1)\},$$

and $G_a\chi = \dot{\chi}$. Its spectrum spec_a is given by the zeros of the characteristic function

$$\mathbb{C} \ni z \mapsto z - a(1 - \mathrm{e}^{-z}) \in \mathbb{C}.$$

Obviously 0 is a zero for every $a > 0$; it is a simple eigenvalue of G_a for $0 < a < 1$ and for $a > 1$ and a double eigenvalue at $a = 1$. So there is always a non-trivial linear centre space; the stationary point $0 \in C$ is non-hyperbolic. For $0 < a \neq 1$ there is a unique real solution $u_a \neq 0$, $u_a < 0$ for $0 < a < 1$ and $u_a > 0$ for $a > 1$. u_a is a simple eigenvalue of G_a. The non-real $z \in \mathrm{spec}_a$ form complex conjugate pairs of simple eigenvalues, with $|\mathrm{Im}\, z| > 2\pi$ for each $a > 0$, $\mathrm{Re}\, z < u_a$ for $0 < a < 1$, and $\mathrm{Re}\, z < 0$ for $a \geq 1$. For any $a > 0$ there is a leading pair; more precisely, there is a continuous map

$$(0, \infty) \ni a \mapsto z_a \in \mathbb{C}$$

with $z_a \in \mathrm{spec}_a \setminus \mathbb{R}$ and $\mathrm{Re}\, z < \mathrm{Re}\, z_a$ for all $z \in \mathrm{spec}_a \setminus (\mathbb{R} \cup \{z_a, \overline{z_a}\})$.

Notice that there are no non-trivial imaginary eigenvalues, and Hopf bifurcation of periodic solutions does not occur.

The spectra of the maps $T_a(t)$, $t > 0$, are given by

$$\{\mathrm{e}^{zt} : z \in \mathrm{spec}_a\} \cup \{0\},$$

and for $z \in \mathrm{spec}_a$ the generalized eigenspace of (the complexification of) G_a coincides with the generalized eigenspace of the eigenvalue e^{zt} of the (complexification of the) operator $T_a(t)$.

Set $u_0 = 0$. For $a > 0$ let $L_a \subset C$ denote the two-dimensional leading realified generalized eigenspace, given by the spectral set $\{u_a, 0\}$, and let $Q_a \subset C$ denote the complementary realified generalized eigenspace given by $\mathrm{spec}_a \setminus \{u_a, 0\}$.

The centre space L_1 consists of the segments v_t of the solutions $v : \mathbb{R} \to \mathbb{R}$ of Eq. 1 with $a = 1$ which have the form

$$v(t) = c + dt, \quad t \in \mathbb{R},$$

with real constants c, d. Notice that all non-trivial solutions of this form are slowly oscillating with at most one zero. We have

$$L_1 = \mathbb{R}\mathbf{1} + \mathbb{R}\, id|[-1, 0],$$

with $\mathbf{1}(t) = 1$ for $-1 \leq t \leq 0$.

For $0 < a \neq 1$ the space L_a consists of the segments of the solutions $v : \mathbb{R} \to \mathbb{R}$ of Eq. 1 of the form

$$v(t) = c + d\,\eta_a(t), \quad t \in \mathbb{R},$$

with real constants c, d and

$$\eta_a(t) = e^{u_a t}.$$

Again, all non-trivial solutions v of this form and their segments v_t have at most one zero.

For the non-linear equation (1.1) there is a change of stability at $a = 1$. The zero solution is asymptotically stable for $0 < a < 1$ and unstable for $a > 1$. Moreover, the instability properties of the stationary point in centre-unstable manifolds which are stated at the beginning of Sect. 6 in [1] imply the following: If $a > 1$ and if W is any two-dimensional C^1-submanifold of C which is locally positively invariant with respect to F_a and satisfies

$$T_0 W = L_a$$

then no flowline $F_a(\cdot, \phi)$ in $W \setminus \{0\}$ tends to 0 as $t \to \infty$.

Next we introduce a complementary space of L_1 in C which later will facilitate the description of projections of flowlines $F_a(\cdot, \phi)$ in case the solution $x^{a,\phi}$ is slowly oscillating.

Notice that a solution of Eq. (1.1) is slowly oscillating if and only if all segments belong to the set

$$Z = \{\phi \in C : \phi \text{ has at most 1 zero}\}.$$

Obviously, for every $a > 0$,

$$L_a \subset Z \cup \{0\}.$$

For the closed subspace

$$N = \{\phi \in C : \phi(-1) = 0 = \phi(0)\}$$

of codimension 2 we have $N \cap Z = \emptyset$ and

$$C = L_1 \oplus N.$$

The functions

$$\lambda_- = \mathbf{1} + id|[-1, 0] \quad \text{and} \quad \lambda_0 = -id|[-1, 0]$$

form a basis of L_1 with

$$\lambda_-(-1) = 0, \qquad \lambda_-(0) = 1, \qquad \lambda_0(-1) = 1, \qquad \lambda_0(0) = 0,$$

which implies that the projection $P : C \to C$ along N onto L_1 is given by

$$P\phi = \phi(0)\lambda_- + \phi(-1)\lambda_0.$$

The isomorphism

$$K : L_1 \ni y_1\lambda_- + y_2\lambda_0 \mapsto (y_1, y_2) \in \mathbb{R}^2$$

and the evaluation

$$E : C \ni \phi \mapsto (\phi(0), \phi(-1)) \in \mathbb{R}^2$$

satisfy

$$E = K \circ P.$$

For any solution $x : [-1, \infty) \to \mathbb{R}$ of Eq. (1.1) and $t > 0$,

$$\dot{x}(t) = 0 \text{ if and only if } x(t-1) = f(x(t)),$$

i.e., if and only if

$$Ex_t \in f \subset \mathbb{R}^2.$$

Incidentally, notice that for a slowly oscillating solution $x : [-1, \infty) \to \mathbb{R}$ of Eq. (1.1) all zeros in $(0, \infty)$ are simple.

It is convenient to introduce the closed hyperplane

$$H_0 = \{\phi \in C : \phi(0) = 0\} \subset N$$

and the projections

$$p_1 : C^2 \ni (\phi, \psi) \mapsto \phi \in C, \quad p_2 : C^2 \ni (\phi, \psi) \mapsto \psi \in C.$$

On the space C^2 we use the norm given by

$$\|(\phi, \psi)\| = \|\phi\| + \|\psi\|.$$

3 The augmented system

The segments (x_t, α_t) of the solutions $(x, \alpha) : [-1, \infty) \to \mathbb{R}^2$ to the augmented system (1.2) define a semi-flow F_* on the space C^2. A look at the second equation of system (1.2) yields the following result.

Proposition 1 *For ϕ, ψ in C and $t \geq 0$,*

$$p_1 F_*(t, (\phi, \psi)) = F_{\psi(0)}(t, \phi) \text{ and } p_2 F_*(t, (\phi, \psi))(0) = \psi(0).$$

The linear variational equation of system (1.2) along the constant solution $\mathbb{R} \ni t \mapsto (0, 1) \in \mathbb{R}^2$ is the system

$$\begin{aligned} \dot{v}(t) &= v(t) - v(t-1) \\ \dot{w}(t) &= 0. \end{aligned} \tag{1}$$

For the associated strongly continuous semi-group of the operators

$$T_*(t) = D_2 F_*(t, 0), \quad t \geq 0,$$

on C^2 we have

$$T_*(t)(\phi, \chi) = (T_1(t)\phi, S(t)\chi),$$

with the semi-group $(S(t))_{t \geq 0}$ on C given by the segments w_t of the solutions $w : [-1, \infty) \to \mathbb{R}$ to the equation

$$\dot{w}(t) = 0.$$

Let $G_* : \mathrm{dom}_* \to C$ and $G_S : \mathrm{dom}_S \to C$ denote the generators of the semigroups $(T_*(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively. Then

$$\mathrm{dom}_* = \mathrm{dom}_1 \times \mathrm{dom}_S \text{ and } G_*(\phi, \chi) = (G_1(\phi), G_S(\chi)).$$

The characteristic functions for the spectra of G_S and G_* are

$$\mathrm{char}_S : \mathbb{C} \ni z \mapsto z \in \mathbb{C}$$

and

$$\mathrm{char}_* : \mathbb{C} \ni z \mapsto z(z - (1 - \mathrm{e}^{-z})) \in \mathbb{C},$$

respectively. $z = 0$ is a simple zero of $char_S$ and a triple zero of $char_*$. The associated one- and three-dimensional realified generalized eigenspaces of G_S and G_* are $L_S = \mathbb{R}\,\mathbf{1}$ and

$$L_* = L_1 \times L_S,$$

respectively.

Proposition 2 $C = L_S \oplus H_0$

Proof The equation

$$\phi = \phi(0)\,\mathbf{1} + (\phi - \phi(0)\,\mathbf{1})$$

yields $C = L_S + H_0$. For $\phi \in L_S \cap H_0$, $\phi(0) = 0$ and $\phi = \phi(0)\,\mathbf{1}$, hence $\phi = 0$. Consequently, $L_S \cap H_0 = \{0\}$. □

It is not hard to show that the complementary realified generalized eigenspaces of L_S and L_* are the hyperplane H_0 and the space

$$Q_* = Q_1 \times H_0,$$

of codimension 3, respectively. We shall not make use of this, but need that the closed subspace

$$N_* = N \times H_0$$

of codimension 3 is a complement of L_*, i.e.,

$$C^2 = L_* \oplus N_*,$$

which follows from the equation $L_* = L_1 \times L_S$ in combination with Proposition 2 and the decomposition $C = L_1 \oplus N$.

Let $P_* : C^2 \to C^2$ denote the projection along N_* onto L_*. Next, we state a version of the centre manifold theorem for the stationary point $(0, \mathbf{1})$ of the semiflow F_*.

Proposition 3 *There exist $r > 0$, $\rho \in (0, 1)$, a C^1-map $w_* : (0, \mathbf{1}) + L_*(r) \to N_*$ with $w_*(0, \mathbf{1}) = 0$, $Dw_*(0, \mathbf{1}) = 0$, and an open neighbourhood U_* of $(0, \mathbf{1})$ in C^2 so that for all (ϕ, ψ) in*

$$W_* = \{(\lambda, \sigma) + w_*(\lambda, \sigma) : (\lambda, \sigma) \in (0, \mathbf{1}) + L_*(r)\}$$

and for all $t \geq 0$ with $F_*([0, t], (\phi, \psi)) \subset U_*$ we have

$$F_*(t, (\phi, \psi)) \in W_*.$$

The map $F_*(1,\cdot)$ *defines a* C^1*-diffeomorphism from* $W_*\cap U_*$ *onto an open neighbourhood of* $(0,\mathbf{1})$ *in* W_*. *If* $(x,\alpha):\mathbb{R}\to\mathbb{R}^2$ *is a solution to system (1.2) with* $(x_t,\alpha_t)\in(0,\mathbf{1})+C^2(\rho)$ *for all* $t\in\mathbb{R}$ *then*

$$(x_t,\alpha_t)\in W_* \text{ for all } t\in\mathbb{R}.$$

Corollary 1 *For all* $a>0$ *with* $|a-1|<\rho$,

$$w_*(0,a\,\mathbf{1})=0.$$

Proof For $a>0$ with $|a-1|<\rho$ the constant solution $\mathbb{R}\ni t\mapsto(0,a)\in\mathbb{R}^2$ to system (1.2) has all segments in $(0,\mathbf{1})+C^2(\rho)$. It follows that $(0,a\,\mathbf{1})\in W_*\cap L_*$, which implies $w_*(0,a\,\mathbf{1})=0$. □

The next result improves the invariance property of the centre manifold slightly. Choose $\delta\in(0,\rho)$ and $\bar\delta>0$ so that

$$L_1(\delta)\times(1-\delta,1+\delta)\,\mathbf{1}\subset(0,\mathbf{1})+L_*(r),$$
$$L_1(\delta)\times(1-\delta,1+\delta)\,\mathbf{1}+N(\bar\delta)\times H_0(\bar\delta)\subset U_*,$$

and

$$w_*(\chi)\in N(\bar\delta)\times H_0(\bar\delta)\text{ for all }\chi\in L_1(\delta)\times(1-\delta,1+\delta)\,\mathbf{1}.$$

Corollary 2 *For every* $\chi\in L_1(\delta)\times(1-\delta,1+\delta)\,\mathbf{1}$ *and* $t>0$ *so that* $\phi=\chi+w_*(\chi)$ *satisfies*

$$P_*F_*([0,t],\phi)\subset L_1(\delta)\times(1-\delta,1+\delta)\,\mathbf{1}$$

we have

$$F_*(t,\phi)\in W_*.$$

Proof Suppose $F_*(t,\phi)\notin W_*$. The set

$$M=\{s\in[0,t]:F_*(s,\phi)\notin W_*\}$$

is given by

$$(id-P_*)F_*(s,\phi)\neq w_*(P_*F_*(s,\phi))$$

and consequently open in $[0,t]$. By the assumption, $t\in M$. It follows that $t_0=\inf M$ satisfies $0\le t_0<t$. We have $0\notin M$, and by the choice of δ and $\bar\delta$, $\phi\in U_*$. Continuity yields

$$F_*([0,t_1],\phi)\in U_*$$

for some $t_1\in(0,t)$. By Proposition 3,

$$F_*([0,t_1],\phi)\subset W_*.$$

Hence, $[0,t_1]\cap M=\emptyset$, and thereby $0<t_1\le t_0$. As M is open in $[0,1]$, we obtain $t_0\notin M$. Arguing as before, with t_0 in place of 0 and $F_*(t_0,\phi)$ in place of ϕ, we find $t_1^*>t_0$ with

$$F_*([t_0,t_1^*],\phi)\subset W_*.$$

Recall $F_*(s,\phi)\in W_*$ for $0\le s<t_0$. It follows that

$$F_*(s,\phi)\in W_*\text{ for all }s\in[0,t_1^*],$$

in contradiction to $t_0=\inf M$. □

The centre manifold of the augmented system yields two-dimensional locally positively invariant C^1-submanifolds $W_a \subset C$ for the semi-flows F_a: For $a \in (1-\delta, 1+\delta)$ consider the C^1-map

$$w_a : L_1(\delta) \ni \lambda \mapsto p_1 w_*(\lambda, a\,\mathbf{1}) \in N$$

and define

$$W_a = \{\lambda + w_a(\lambda) : \lambda \in L_1(\delta)\}.$$

By Corollary 1, $w_a(0) = 0$.

Proposition 4 *(i) (Local positive invariance) For every $a \in (1-\delta, 1+\delta)$, $\phi \in W_a$ and $t \geq 0$ with*

$$P F_a([0, t], \phi) \subset L_1(\delta)$$

we have $F_a(t, \phi) \subset W_a$.

(ii) (Injectivity) For every $a \in (1-\delta, 1+\delta)$, ϕ and $\hat{\phi}$ in W_a and $t > 0$ with $F_a([0, t], \phi) \cup F_a([0, t], \hat{\phi}) \subset W_a$ and $F_a(t, \phi) = F_a(t, \hat{\phi})$,

$$\phi = \hat{\phi}.$$

(iii) For every integer $j > 0$ and every $\epsilon \in (0, \delta)$ there exists $\delta_{j\epsilon} \in (0, \delta)$ so that for all $a \in (1-\delta_{j\epsilon}, 1+\delta_{j\epsilon})$,

$$W_a \cap (L_1(\delta_{j\epsilon}) + N) \subset F_a(j, W_a \cap (L_1(\epsilon) + N)).$$

Proof 1. Proof of (i). We have $\phi = \lambda + w_a(\lambda)$ with $\lambda \in L_1(\delta)$. Set $\psi = (\lambda, a\,\mathbf{1}) + w_*(\lambda, a\,\mathbf{1}) \in W_*$. Notice that $p_2\psi(0) = a + 0 = a$.

1.1. Proof of $P_* F_*([0, t], \psi) \subset L_1(\delta) \times (1-\delta, 1+\delta)\,\mathbf{1}$. Let $s \in [0, t]$ be given. We have $p_1\psi = \lambda + p_1 w_*(\lambda, a\,\mathbf{1}) = \lambda + w_a(\lambda) = \phi$. Using Proposition 1 we infer $p_1 F_*(s, \psi) = F_a(s, p_1\psi) = F_a(s, \phi) \in L_1(\delta) + N$. Again by Proposition 1, $p_2 F_*(u, \psi)(0) = p_2\psi(0) = a$ for all $u \geq 0$. Hence, $p_2 F_*(s, \psi) \in a\,\mathbf{1} + H_0$, and thereby

$$F_*(s, \psi) = (p_1 F_*(s, \psi), p_2 F_*(s, \psi)) \in L_1(\delta) \times (1-\delta, 1+\delta)\,\mathbf{1} + N \times H_0,$$

which yields the assertion.

1.2. Using Corollary 2 we infer $F_*(t, \psi) \in W_*$, and

$$F_*(t, \psi) = (\tilde{\lambda}, \tilde{a}\,\mathbf{1}) + w_*(\tilde{\lambda}, \tilde{a}\,\mathbf{1})$$

for some $\tilde{\lambda} \in L_1(\delta)$ and $\tilde{a} \in (1-\delta, 1+\delta)$. It follows that

$$a = p_2 F_*(t, \psi)(0) = \tilde{a} + p_2 w_*(\ldots)(0) = \tilde{a} + 0 = \tilde{a},$$

and thereby

$$F_a(t, \phi) = p_1 F_*(t, \psi) = \tilde{\lambda} + p_1 w_*(\tilde{\lambda}, a\,\mathbf{1}) = \tilde{\lambda} + w_a(\tilde{\lambda}) \in W_a.$$

2. Proof of (ii). Set $\lambda = P\phi$, $\hat{\lambda} = P\hat{\phi}$, $\psi = (\lambda, a\,\mathbf{1}) + w_*(\lambda, a\,\mathbf{1}) \in W_*$, $\hat{\psi} = (\hat{\lambda}, a\,\mathbf{1}) + w_*(\hat{\lambda}, a\,\mathbf{1}) \in W_*$. Then

$$\phi = \lambda + w_a(\lambda), \qquad \hat{\phi} = \hat{\lambda} + w_a(\hat{\lambda}), \qquad \phi = p_1\psi, \qquad \hat{\phi} = p_1\hat{\psi}.$$

Using Proposition 1 we obtain for the largest integer j in $[0, t]$ the equations

$$p_1F_*(j+1, \psi) = F_a(j+1, \phi) = F_a(j+1, \hat{\phi}) = p_1F_*(j+1, \hat{\psi})$$

and

$$p_2F_*(j+1, \psi) = a\,\mathbf{1} = p_2F_*(j+1, \hat{\psi}).$$

Hence,

$$F_*(j+1, \psi) = F_*(j+1, \hat{\psi}). \tag{2}$$

2.1. Proof of $F_*([0, t], \psi) \subset U_*$. Let $s \in [0, t]$. Then

$$p_1F_*(s, \psi) = F_a(s, \phi) \in W_a \subset L_1(\delta) + N(\bar{\delta}),$$

by the choice of δ and $\bar{\delta}$. For $\theta \in [-1, 0]$ we have

$$(p_2F_*(s, \psi))(\theta) = a \text{ in case } 0 \le s + \theta,$$

and

$$(p_2F_*(s, \psi))(\theta) = a + (p_2w_*(\lambda, a\,\mathbf{1}))(s + \theta) \text{ in case } s + \theta < 0.$$

Observe that $p_2w_*(\lambda, a\,\mathbf{1}) \in H_0(\bar{\delta})$. It follows that $p_2F_*(s, \psi) \in a\,\mathbf{1} + H_0(\bar{\delta})$. Finally,

$$F_*(s, \psi) \in (L_1(\delta) + N(\bar{\delta})) \times ((1 - \delta, 1 + \delta)\mathbf{1} + H_0(\bar{\delta})) \subset U_*.$$

2.2 Analogously, we get $F_*([0, t], \hat{\psi}) \subset U_*$. An application of Proposition 3 yields

$$F_*(k, \psi) \in W_* \cap U_* \ni F_*(k, \hat{\psi}) \text{ for all integers } k \in [0, j].$$

Using the preceding equations, the injectivity of $F_*(1, \cdot)$ on $W_* \cap U_*$ (see Proposition 3), Eq. 2 and induction we infer $\psi = \hat{\psi}$, hence $\phi = p_1\psi = p_1\hat{\psi} = \hat{\phi}$.

3. Proof of (iii). Let $j \in \mathbb{N}$ and $\epsilon \in (0, \delta)$ be given. Proposition 3 implies that $F_*(j, \cdot)$ defines a diffeomorphism between neighbourhoods of $(0, \mathbf{1})$ in W_*. In particular, there exists $\hat{\delta} \in (0, \delta)$ so that

$$V = F_*(j, W_* \cap (C(\hat{\delta}) \times (\mathbf{1} + C(\hat{\delta})))$$

is an open neighbourhood of $(0, \mathbf{1})$ in W_*, and we may assume

$$Pp_1(W_* \cap (C(\hat{\delta}) \times (\mathbf{1} + C(\hat{\delta})))) \subset L_1(\epsilon).$$

Choose $\delta_{j\epsilon} \in (0, \delta)$ so that for every $a \in (1 - \delta_{j\epsilon}, 1 + \delta_{j\epsilon})$ and for each $\lambda \in L_1(\delta_{j\epsilon})$ we have

$$(\lambda, a\,\mathbf{1}) + w_*(\lambda, a\,\mathbf{1}) \in V.$$

Let $a \in (1-\delta_{j\epsilon}, 1+\delta_{j\epsilon})$, $\lambda \in L_1(\delta_{j\epsilon})$, and consider $\phi = \lambda + w_a(\lambda) \in W_a \cap (L_1(\delta_{j\epsilon}) + N)$. Set $\psi = (\lambda, a\,\mathbf{1}) + w_*(\lambda, a\,\mathbf{1})$. Then $p_1\psi = \phi$ and $(p_2\psi)(0) = a$, and by the preceding construction,

$$V \ni \psi = F_*(j, \hat{\psi}) \text{ for some } \hat{\psi} \in W_* \cap (C(\hat{\delta}) \times (\mathbf{1} + C(\hat{\delta}))).$$

By Proposition 1, $p_2\psi$ is constant with value $(p_2\hat{\psi})(0) = (p_2\psi)(0) = a$. Also,

$$\hat{\psi} = (\hat{\lambda}, \hat{a}\,\mathbf{1}) + w_*(\hat{\lambda}, \hat{a}\,\mathbf{1})$$

for some $(\hat{\lambda}, \hat{a}\,\mathbf{1}) \in L_*(r)$, and

$$\hat{a} = (p_2\hat{\psi})(0) = a \in (1-\delta_{j\epsilon}, 1+\delta_{j\epsilon}),$$

$$\hat{\lambda} = Pp_1\hat{\psi} \in Pp_1(W_* \cap (C(\hat{\delta}) \times (\mathbf{1} + C(\hat{\delta})))) \subset L_1(\epsilon).$$

Hence,

$$p_1\hat{\psi} = \hat{\lambda} + p_1 w_*(\hat{\lambda}, \hat{a}\,\mathbf{1}) = \hat{\lambda} + p_1 w_*(\hat{\lambda}, a\,\mathbf{1}) = \hat{\lambda} + w_a(\hat{\lambda}) \in W_a.$$

It follows that

$$\phi = p_1\psi = p_1 F_*(j, \hat{\psi}) = F_a(j, p_1\hat{\psi}) \in F_a(j, W_a \cap (L_1(\epsilon) + N)).$$

□

Parts (ii) and (iii) of the preceding proposition are not optimal, of course. The maps $F_a(t, \cdot)$ define diffeomorphisms between neighbourhoods of 0 in W_a, but we shall not need this in the sequel.

Proposition 5 *There exists $\delta_1 \in (0, \delta)$ so that for every $a \in (1-\delta_1, 1+\delta_1)$,*

$$T_0 W_a = L_a.$$

Proof

1. Let $a \in (1-\delta, 1+\delta)$ be given. Using Proposition 4 (i), continuity, and a compactness argument we find that each $F_a(t, \cdot)$, $t \geq 0$, maps a neighbourhood of the stationary point 0 in W_a into W_a. It follows that $T_a(t)T_0W_a = D_2F_a(t, 0)T_0W_a \subset T_0W_a$ for all $t \geq 0$. The eigenvalues of the endomorphism E_a induced by $T_a(1)$ on T_0W_a are also eigenvalues of $T_a(1)$, and T_0W_a is contained in the realified generalized eigenspace of $T_a(1)$ given by these eigenvalues.
2. Using $Dw_*(0, \mathbf{1}) = 0$ we find $Dw_1(0) = 0$, and thereby $T_0W_1 = \{\lambda + Dw_1(0)\lambda : \lambda \in L_1\} = L_1$. It follows that the endomorphism E_1 has a double eigenvalue $e^0 = 1$. By continuity, there exists $\delta_1 \in (0, \delta)$ so that for every $a \in (1-\delta_1, 1+\delta_1)$ the leading complex conjugate pair $\{z_a, \overline{z_a}\} \subset \mathrm{spec}_a$ and the eigenvalues z of E_a satisfy

$$|e^{z_a}| < |z|.$$

For such a we conclude that the eigenvalues of E_a are $e^0 = 1$ and e^{u_a}, and thereby $T_0W_a \subset L_a$. □

According to a remark in the preceding section we obtain from Propositions 4 (i) and 5 the following result on instability.

Corollary 3 *For $a \in (1, 1+\delta_1)$ no curve $F_a(\cdot, \phi)$ with trace in $W_a \setminus \{0\}$ tends to 0 as $t \to \infty$.*

4 Slowly oscillating solutions

In the present section it is shown that for a close to 1 solutions on $[-1,\infty)$ which start from data in restrictions of the manifolds W_a are slowly oscillating, continuously differentiable, and have slowly oscillating derivatives. We begin with a variant of Proposition 3.1 in [1]; for the proof, see [1] and the remark following Proposition 2.3 in [12].

Proposition 1 *Suppose $h:[0,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous and locally Lipschitz continuous with respect to the second variable, and $h(t,0)=0$ for all $t\geq 0$. Suppose also that for any $t_0\geq 0$, $t_1\geq t_0$, $w_0\in\mathbb{R}$, and for any continuous function $g:[t_0,t_1]\to\mathbb{R}$ the unique maximal solution to the initial value problem*

$$\dot{w}(t)=h(t,w(t))+g(t),\;\; w(t_0)=w_0,$$

is defined on $[t_0,t_1]$. Let $a>0$ and let $v:[-1,\infty)\to\mathbb{R}$ be a solution of the equation

$$\dot{v}(t)=h(t,v(t))-a\,v(t-1).$$

(i) In case $0\neq v_0\in\overline{Z}$,

$$v_t\in\overline{Z}\text{ for all } t\geq 0,$$

and for some $t\in[0,3]$ the segment v_t has no zero. For all $s\geq 3$,

$$v_s\in Z.$$

(ii) In case $z\geq 0$ is a zero of v and v has no zero on $[z-1,z)$ we have

$$0\neq sign\,v(t)=-sign\,v(z-1)\text{ for all } t\in(z,z+1].$$

Before we can make use of Proposition 1 we need another auxiliary result.

Proposition 2 *(i) For every $a\in(1-\delta,1+\delta)$ and $\phi\in W_a\setminus\{0\}$, $P\phi\neq 0$ and*

$$\|\|\phi\|^{-1}\phi-\|P\phi\|^{-1}P\phi\|\leq 2\sup_{\lambda\in L_1(\|P\phi\|)}\|Dw_a(\lambda)\|.$$

(ii) For every $a\in(1-\delta,1+\delta)$ and $\phi\in W_a\setminus\{0\}$ with $F_a([0,2],\phi)\subset W_a$, $\psi=\dot{x}_1^{a,\phi}$ satisfies $\psi\neq 0\neq P\psi$ and

$$\|\|\psi\|^{-1}\psi-\|P\psi\|^{-1}P\psi\|\leq 2\|Dw_a(PF_a(1,\phi))\|.$$

(iii) There exists $d>0$ so that for every $\lambda\in L_1\cap S_C$ there is $t=t(\lambda)\in[0,2]$ with

$$|[T_1(t)\lambda](\theta)|\geq d>0\text{ for all }\theta\in[-1,0].$$

Proof 1. Proof of (i). From $0 \neq \phi = P\phi + w_a(P\phi)$ and $w_a(0) = 0$ we get $P\phi \neq 0$. It follows that

$$\begin{aligned}
&\| \|\phi\|^{-1}\phi - \|P\phi\|^{-1}P\phi\| \le \|P\phi\|^{-1}\|\phi - P\phi\| + |\|\phi\|^{-1} - \|P\phi\|^{-1}|\|\phi\| \\
&= \|P\phi\|^{-1}\|w_a(P\phi)\| + \frac{|\|P\phi\| - \|\phi\||}{\|\phi\|\|P\phi\|}\|\phi\| \le \|P\phi\|^{-1}\|w_a(P\phi)\| \\
&\quad + \frac{\|P\phi - \phi\|}{\|P\phi\|} \\
&= 2\|P\phi\|^{-1}\|w_a(P\phi)\| \le 2 \sup_{\lambda \in L_1(\|P\phi\|)} \|Dw_a(\lambda)\|.
\end{aligned}$$

2. Proof of (ii). Let $x = x^{a,\phi}$.
   2.1. Proof of $\psi \neq 0$. Assume $\psi = 0$. Then x_1 is constant. Equation 1.1 yields that also $x_0 = f \circ x_1 - a^{-1}\dot{x}_1$ is constant. By continuity, x is constant on $[-1, 1]$. Then $0 = a^{-1}\dot{x}(t) = x(t) - x(t-1) - |x(t)|x(t) = -|x(t)|x(t)$ on $(0, 1]$ yields $x(t) = 0$ on $[-1, 1]$, in contradiction to $x_0 = \phi \neq 0$.
   2.3. Proof of $P\psi \neq 0$. Differentiating the curve $[1, 2] \ni t \mapsto x_t = Px_t + w_a(Px_t) \in C$ at $t = 1$ we obtain

$$0 \neq \psi = P\psi + Dw_a(Px_1)P\psi,$$

   which implies $P\psi \neq 0$.
   2.4. It follows that

$$\begin{aligned}
&\| \|\psi\|^{-1}\psi - \|P\psi\|^{-1}P\psi\| \le \|P\psi\|^{-1}\|\psi - P\psi\| \\
&\quad + |\|\psi\|^{-1} - \|P\psi\|^{-1}|\|\psi\| \\
&\le \|P\psi\|^{-1}\|Dw_a(Px_1)P\psi\| + \frac{\|P\psi - \psi\|}{\|\psi\|\|P\psi\|}\|\psi\| \\
&\quad = 2\|P\psi\|^{-1}\|Dw_a(Px_1)P\psi\|.
\end{aligned}$$

3. Proof of (iii). From the fact that $L_1 \setminus \{0\} \subset Z$ is positively invariant under the operators $T_1(t)$, $t \ge 0$, we deduce that for each $\lambda \in L_1 \cap S_C$ there exists $t(\lambda) \in [0, 2]$ so that $T_1(t(\lambda))\lambda$ has no zero. Moreover, for some $d(\lambda) > 0$,

$$|[T_1(t(\lambda))\lambda](\theta)| > d(\lambda) > 0 \text{ on } [-1, 0].$$

Continuity implies that for each $\lambda \in L_1 \cap S_C$ there exists $\epsilon(\lambda) > 0$ so that for all $\hat{\lambda} \in \lambda + C(\epsilon(\lambda))$ we have

$$|[T_1(t(\lambda))\hat{\lambda}](\theta)| > d(\lambda) > 0 \text{ on } [-1, 0].$$

The compact circle $L_1 \cap S_C$ is covered by a finite number of such neighbourhoods $\lambda + C(\epsilon(\lambda))$. Define d by the minimum of the associated constants $d(\lambda)$. □

Proposition 3 *There exists $\delta_2 \in (0, \delta)$ so that*

$$F_a(t, \phi) \in Z$$

for all $a \in (1 - \delta_2, 1 + \delta_2)$, all $\phi \in (W_a \setminus \{0\}) \cap C(\delta_2)$ and all $t \ge 2$.

Proof 1. Let $a \in (1-\delta, 1+\delta)$, $\phi \in W_a \setminus \{0\}$, $t \in [0,2]$. Using Proposition 2 (i) we infer $P\phi \neq 0$ and

$$
\begin{aligned}
&\| \|\phi\|^{-1} F_a(t,\phi) - \|P\phi\|^{-1} T_1(t) P\phi \| \\
&\le \|\phi\|^{-1} \|F_a(t,\phi) - T_a(t)\phi\| + \| \|\phi\|^{-1} T_a(t)\phi - \|P\phi\|^{-1} T_1(t) P\phi\| \\
&= \|\phi\|^{-1} \| \int_0^1 [D_2F_a(t, s\phi) - D_2F_a(t,0)]\phi ds \| + \| \|\phi\|^{-1} T_a(t)\phi \\
&\quad - \|P\phi\|^{-1} T_1(t) P\phi \| \\
&\le \|\phi\|^{-1} \sup_{\psi \in C(\|\phi\|)} \|D_2F_a(t,\psi) - D_2F_a(t,0)\| \|\phi\| + \| \|\phi\|^{-1} T_a(t)\phi \\
&\quad - \|P\phi\|^{-1} T_1(t) P\phi \|
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{\psi \in C(\|\phi\|)} \|D_2F_a(t,\psi) - D_2F_a(t,0)\| + \| \|\phi\|^{-1} T_a(t)\phi - \|P\phi\|^{-1} T_1(t) P\phi \| \\
&\le \sup_{\psi \in C(\|\phi\|)} \|D_2F_a(t,\psi) - D_2F_a(t,0)\| + \|T_a(t) - T_1(t)\| \\
&\quad + \|T_1(t)\| \| \|\phi\|^{-1}\phi - \|P\phi\|^{-1} P\phi \| \\
&\le \sup_{\psi \in C(\|\phi\|)} \|D_2F_a(t,\psi) - D_2F_a(t,0)\| + \|T_a(t) - T_1(t)\| \\
&\quad + \|T_1(t)\| \, 2 \sup_{\lambda \in L_1(\|P\phi\|)} \|Dw_a(\lambda)\|.
\end{aligned}
$$

2. There exists $c > 0$ with

$$\|T_1(t)\| \le c \text{ on } [0,2].$$

Using $Dw_*(0, \mathbf{1}) = 0$ and the definition of the maps w_a we find $\delta_{21} \in (0, \delta)$ so that for all $a \in (1-\delta_{21}, 1+\delta_{21})$ and all $\lambda \in L_1(\delta_{21})$,

$$\|Dw_a(\lambda)\| \le \frac{d}{6c}.$$

Continuous dependence on parameters for solutions to variational equations shows that in addition we may assume

$$\|D_2F_a(t,\psi) - D_2F_a(t,0)\| < \frac{d}{3}$$

for $a \in (1-\delta_{21}, 1+\delta_{21})$, $\psi \in C(\delta_{21})$ and $t \in [0,2]$, and

$$\|T_a(t) - T_1(t)\| < \frac{d}{3}$$

for such a and t.

3. Choose $\delta_2 \in (0, \delta_{21})$ so that P maps $C(\delta_2)$ into $L_1(\delta_{21})$. Let $a \in (1-\delta_2, 1+\delta_2)$, $\phi \in (W_a \setminus \{0\}) \cap C(\delta_2)$. Choose $t = t(\|P\phi\|^{-1}P\phi) \in [0,2]$ according to Proposition 2 (iii). The estimates in parts 1 and 2 combined with Proposition 2 (iii) imply

$$|[\|\phi\|^{-1} F_a(t,\phi)](\theta)| > 0 \text{ on } [-1,0].$$

So $F_a(t,\phi)$ has no zero. Now one can apply Proposition 1 (ii) with $h(t,\xi) = a\,f(\xi)$ and deduce $F_a(s,\phi) \in Z$ for all $s \ge t$. □

Proposition 4 *There exists* $\delta_3 \in (0, \delta)$ *so that*

$$D_1 F_a(t, \phi) 1 = \dot{x}_t^{a,\phi} \in Z$$

for all $a \in (1 - \delta_3, 1 + \delta_3)$*, all* $\phi \in (W_a \setminus \{0\}) \cap C(\delta_3)$ *and all* $t \geq 3$.

Proof 1. Proof that there exists $\delta_{31} \in (0, \delta)$ so that for all $a \in (1 - \delta_{31}, 1 + \delta_{31})$ and $\phi \in (W_a \setminus \{0\}) \cap C(\delta_{31})$ we have

$$F_a([0, 3], \phi) \subset W_a.$$

Continuity and compactness arguments and the fact that $(0, \mathbf{1})$ is a stationary point for the semi-flow F_* altogether imply that there exists $\delta_{31} \in (0, \delta)$ so that

$$P p_1 F_*([0, 3], C(\delta_{31}) \times (\mathbf{1} + C(\delta_{31}))) \subset L_1(\delta).$$

For $a \in (1 - \delta_{31}, 1 + \delta_{31})$, $0 \neq \phi \in W_a \cap C(\delta_{31})$ and $t \in [0, 3]$ we get

$$\begin{aligned} P F_a(t, \phi) &= P p_1 F_*(t, (\phi, a\, \mathbf{1})) \text{ (Proposition 3.1)} \\ &\in L_1(\delta), \end{aligned}$$

and Proposition 3.4 (i) yields the assertion.

2. For $a \in (1 - \delta_{31}, 1 + \delta_{31})$ and $\phi \in (W_a \setminus \{0\}) \cap C(\delta_{31})$ consider the solution $x = x^{a,\phi}$. Recall

$$D_2 F_a(t - 1, x_1) \dot{x}_1 = \dot{x}_t \text{ for all } t \geq 1.$$

By Proposition 2 (ii), $\dot{x}_1 \neq 0 \neq P\dot{x}_1$, and for all $t \in [1, 3]$,

$$\begin{aligned} &\| \|\dot{x}_1\|^{-1} \dot{x}_t - T_1(t-1) \|P\dot{x}_1\|^{-1} P\dot{x}_1 \| \\ &\leq \| \|\dot{x}_1\|^{-1} D_2 F_a(t-1, x_1)\dot{x}_1 - T_a(t-1)\|\dot{x}_1\|^{-1}\dot{x}_1 \| + \|T_a(t-1) \\ &\quad - T_1(t-1)\| + \|T_1(t-1)\| \| \|\dot{x}_1\|^{-1}\dot{x}_1 - \|P\dot{x}_1\|^{-1} P\dot{x}_1 \| \\ &\leq \|D_2 F_a(t-1, x_1) - D_2 F_a(t-1, 0)\| + \|T_a(t-1) - T_1(t-1)\| \\ &\quad + \|T_1(t-1)\|\, 2\, \|D w_a(P x_1)\| \text{ (see Proposition 2 (ii)).} \end{aligned}$$

3. As in the proof of Proposition 3 there exists $\delta_3 \in (0, \delta_{31})$ so that for all $a \in (1 - \delta_3, 1 + \delta_3)$, all $\phi \in (W_a \setminus \{0\}) \cap C(\delta_3)$ and all $s \in [0, 2]$ we have

$$\begin{aligned} \|D_2 F_a(s, F_a(1, \phi)) - D_2 F_a(s, 0)\| &< \frac{d}{3}, \\ \|T_a(s) - T_1(s)\| &< \frac{d}{3}, \\ 2\, \|T_1(s)\| \|D w_a(P F_a(1, \phi))\| &< \frac{d}{3}. \end{aligned}$$

Let $a \in (1 - \delta_3, 1 + \delta_3)$, $\phi \in (W_a \setminus \{0\}) \cap C(\delta_3)$, and set $x = x^{a,\phi}$. By Proposition 2 (ii), $\dot{x}_1 \neq 0 \neq P\dot{x}_1$. Using the estimates in part 2 and the previous inequalities we obtain for $s = t(\|P\dot{x}_1\|^{-1} P\dot{x}_1) \in [0, 2]$ (see Proposition 2 (iii)) the estimate

$$\| \|\dot{x}_1\|^{-1} \dot{x}_{s+1} - T_1(s) \|P\dot{x}_1\|^{-1} P\dot{x}_1 \| < d$$

which implies that $\dot{x}_{s+1}$ has no zero. Now one can apply Proposition 1 (ii) to the solution $\dot{x}(\cdot + 1) : [-1, \infty) \to \mathbb{R}$ of the equation

$$\dot{v}(t) = a(1 - 2|x(t+1)|)v(t) - a\, v(t-1)$$

and deduce that $D_1 F_a(t, \phi)1 = \dot{x}_t \in Z$ for all $t \geq s + 1$. □

Corollary 1 *There exists* $\delta_4 \in (0, \delta)$ *so that for every* $a \in (1 - \delta_4, 1 + \delta_4)$ *and for every* $\phi \in W_a \setminus \{0\}$ *with* $P\phi \in L_1(\delta_4)$ *the solution* $x = x^{a,\phi}$ *is continuously differentiable everywhere and slowly oscillating, with slowly oscillating derivative* $\dot{x} : [-1, \infty) \to \mathbb{R}$.

Proof Choose $\epsilon \in (0, \delta)$ so that for $a \in (1 - \epsilon, 1 + \epsilon)$,

$$W_a \cap (L_1(\epsilon) + N) \subset C(\min\{\delta_2, \delta_3\}).$$

Proposition 3.4 (iii) guarantees the existence of $\delta_4 \in (0, \min\{\epsilon, \delta_2, \delta_3\})$ so that for $a \in (1-\delta_4, 1+\delta_4)$ and for each $\phi \in W_a$ with $P\phi \in L_1(\delta_4)$ we have $\phi = F_a(3, \hat{\phi})$ for some

$$\hat{\phi} \in W_a \cap (L_1(\epsilon) + N) \subset W_a \cap C(\min\{\delta_2, \delta_3\}).$$

Use Propositions 3 and 4 to complete the proof. □

We choose

$$\delta_0 \in (0, \min\{\delta_1, \delta_4\}),$$

set

$$W^a = W_a \cap (L_1(\delta_0) + N)$$

for $a \in (1 - \delta_0, 1 + \delta_0)$, and

$$Q = (-\delta_0, \delta_0)^2 \subset \mathbb{R}^2.$$

5 Behaviour of coordinate curves

In this section we describe the behaviour of the curves

$$X : [0, \infty) \ni t \mapsto Ex_t \in \mathbb{R}^2,$$

or equivalently,

$$[0, \infty) \ni t \mapsto (x(t), x(t-1)) \in \mathbb{R}^2$$

along continuously differentiable slowly oscillating solutions $x : [-1, \infty) \to \mathbb{R}$ of Eq. (1.1). Corollary 4.1 guarantees that solutions starting from initial data in W^a, $1 - \delta_0 < a < 1 + \delta_0$, have these regularity properties. For $\phi \in W^a$ and $y = E\phi$ we denote the curve X associated with $x^{a,\phi}$ by

$$X^{a,\phi} = X^{a,y}.$$

Recall that the restrictions $P|W^a$ and $E|W^a = K \circ P|W^a$ are injective.

Proposition 1 *(i) Let $a > 0$, let $x : [-1, \infty) \to \mathbb{R}$ be a continuously differentiable and slowly oscillating solution to Eq. (1.1). The curve $X = (X_1, X_2)$ given by $X(s) = (x(s), x(s-1))$ for all $s \geq 0$ intersects the axis $\{0\} \times \mathbb{R}$ transversally, with*

$$\dot{X}_1(t) = \dot{x}(t) \left\{ \begin{matrix} < \\ > \end{matrix} \right\} 0 \text{ for } t \geq 0 \text{ with } X(t) \in \left\{ \begin{matrix} \{0\} \times (0, \infty) \\ \{0\} \times (-\infty, 0) \end{matrix} \right\}.$$

(ii) For all $a \in (1 - \delta_0, 1 + \delta_0)$,

$$KL_1(\delta_0) = Q = EW^a.$$

(iii) The map

$$(1 - \delta_0, 1 + \delta_0) \times Q \times [0, \infty) \ni (a, y, t) \mapsto X^{a,y}(t) \in \mathbb{R}^2$$

is continuous.

(iv) For all $a \in (1 - \delta_0, 1 + \delta_0)$, $\phi \in W^a$ and all $t \geq 0$ with $X^{a,\phi}([0, t]) \subset Q$,

$$F_a(t, \phi) \in W^a.$$

(v) For all $a \in (1 - \delta_0, 1 + \delta_0)$, $y \in Q$, $s \geq 0$ with $X^{a,y}([0, s]) \subset Q$, $t \geq 0$ and for $\hat{y} = X^{a,y}(s)$,

$$X^{a,\hat{y}}(t) = X^{a,y}(s + t).$$

(vi) For all $a \in (1 - \delta_0, 1 + \delta_0)$, all $y, \hat{y}$ in Q and all $t > 0$ with $X^{a,y}([0, t]) \cup X^{a,\hat{y}}([0, t]) \subset Q$ and $X^{a,y}(t) = X^{a,\hat{y}}(t)$,

$$y = \hat{y}.$$

Proof

1. Proof of (i). In case $X(t) \in \{0\} \times (0, \infty)$, or $x(t) = 0 < x(t-1)$, Eq. (1.1) yields $a^{-1}\dot{x}(t) = 0 - x(t-1) < 0$, with the right derivative in case $t = 0$. The proof for $X(t) \in \{0\} \times (-\infty, 0)$ is analogous.
2. Proof of (ii). For each $\lambda \in L_1(\delta_0)$ we have $\lambda = P\lambda = \lambda(0)\lambda_- + \lambda(-1)\lambda_0$, and therefore $K\lambda = (\lambda(0), \lambda(-1))$. Both components of the preceding pair are bounded by δ_0. Hence,

$$KL_1(\delta_0) \subset Q.$$

Conversely, let $y = (y_1, y_2) \in Q$ be given. Then

$$\lambda = y_1\lambda_- + y_2\lambda_0 \in L_1$$

satisfies $K\lambda = y$ and

$$\lambda(t) = y_1(1 + t) - y_2 t = y_1 + (-t)(y_2 - y_1) \in (-\delta_0, \delta_0)$$

for all $t \in [-1, 0]$. Consequently, $\lambda \in L_1(\delta_0)$, and we obtain

$$Q \subset KL_1(\delta_0).$$

The remaining part of (ii) follows from $EW^a = KPW^a = KL_1(\delta_0)$.

3. Assertion (iii) follows from

$$X^{a,y}(t) = EF_a(t, K^{-1}y + w_a(K^{-1}y)) = Ep_1F_*(t, (K^{-1}y, a\,\mathbf{1}) + w_* \times (K^{-1}y, a\,\mathbf{1})).$$

4. Proof of (iv). Suppose $X^{a,\phi}([0,t]) = EF_a([0,t],\phi) \subset Q$. As $E = K \circ P$ and K is an isomorphism, we obtain from part (ii) that $PF_a([0,t],\phi) \subset L_1(\delta_0)$. Next, Proposition 3.4 (i) yields $F_a(t,\phi) \in W_a$, and finally $PF_a(t,\phi) \in L_1(\delta_0)$ gives $F_a(t,\phi) \in W^a$.
5. Proof of (v). For $\phi \in W^a$ with $E\phi = y$ we obtain from (iv) that $F_a(s,\phi) \in W^a$. Set $\hat{\phi} = F_a(s,\phi)$. Then $E\hat{\phi} = EF_a(s,\phi) = X^{a,y}(s) = \hat{y}$, and consequently

$$X^{a,\hat{y}}(t) = EF_a(t,\hat{\phi}) = EF_a(t, F_a(s,\phi)) = EF_a(s+t,\phi) = X^{a,y}(t).$$

6. Proof of (vi). For ϕ and $\hat{\phi}$ in W^a with $E\phi = y$ and $E\hat{\phi} = \hat{y}$, part (iv) yields

$$F_a([0,t],\phi) \cup F_a([0,t],\hat{\phi}) \subset W^a.$$

Using this and $EF_a(t,\phi) = X^{a,y}(t) = X^{a,\hat{y}}(t) = EF_a(t,\hat{\phi})$ we get $F_a(t,\phi) = F_a(t,\hat{\phi})$. Proposition 3.4 (ii) gives $\phi = \hat{\phi}$, hence $y = E\phi = E\hat{\phi} = \hat{y}$. □

Recall that for any solution $x : [1,\infty) \to \mathbb{R}$ of Eq. (1.1) and for any $t > 0$ we have

$$\dot{x}(t) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \text{ if and only if } x(t-1) \quad \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} f(x(t)),$$

i.e., if and only if Ex_t is strictly below, on, or strictly above the graph $f \subset \mathbb{R}^2$. Let

$$f_< = \{y \in \mathbb{R}^2 : y_2 < f(y_1)\} \text{ and } f_> = \{y \in \mathbb{R}^2 : y_2 > f(y_1)\}.$$

Recall also that all zeros $z > 0$ of slowly oscillating solutions of Eq. (1.1) on $[-1,\infty)$ are simple.

Proposition 2 *Let $a > 0$, and let $x : [-1,\infty) \to \mathbb{R}$ be a continuously differentiable solution of Eq. (1.1) so that x and $\dot{x}$ are slowly oscillating. Let $t \geq 0$, $\dot{x}(t) = 0$, and $x(t) > 0$. Then*

$$\dot{x}(s) > 0 \textit{ on } [t-1,t),$$

and in case $0 < x(t) < 1$,

$$0 < x(t-1).$$

The curve $X : [0,\infty) \ni s \mapsto (x(s), x(s-1)) \in \mathbb{R}^2$ intersects f at t transversally, i.e.,

$$\dot{X}(t) = DX(t)1 = (\dot{x}(t), \dot{x}(t-1)) \notin T_{X(t)}f = \mathbb{R}(1, f'(x(t))),$$

and for some $\tau > 0$,

$$X(s) \in f_< \text{ on } (t-\tau, t) \cap [0,\infty) \text{ and } X(s) \in f_> \text{ on } (t, t+\tau).$$

In case $x(s) > 0$ *on* $[t,\infty)$ *we have* $\dot{x}(s) < 0$ *on* (t,∞) *and* $\lim_{s\to\infty} x(s) = 0$. *Otherwise there are zeros* $z \in (t,\infty)$ *of* x *and* $\zeta \in (z,\infty)$ *of* $\dot{x}$ *with*

$$\dot{x}(s) < 0 \text{ on } (t,\zeta),$$

and if in addition $0 < x(t) < 1$ *then*

$$X((t,\zeta)) \subset f_> \cap (x(\zeta), x(t)]^2.$$

Proof 1. By $x(t) > 0$, $x(t) > f(x(t))$. By $\dot{x}(t) = 0$, $f(x(t)) = x(t-1)$. Hence, $x(t) > x(t-1)$, and consequently $\dot{x}(s) > 0$ for some $s \in [t-1,t)$. As t is a zero of the slowly oscillating function $\dot{x}$ we infer $\dot{x}(s) > 0$ for all $s \in [t-1,t)$. In case $0 < x(t) < 1$ we have $x(t-1) = f(x(t)) > 0$. The assertion about transversality is obvious from $\dot{x}(t-1) > 0$ which gives $(\dot{x}(t), \dot{x}(t-1)) \in \{0\} \times (0,\infty)$.

2. Next we observe that in case $t > 0$ the derivative $\ddot{x}(t)$ of the slowly oscillating solution $\dot{x}$ to the variational equation

$$a^{-1}\dot{v}(t) = (1 - 2|x(t)|)v(t) - v(t-1)$$

equals $-a\dot{x}(t-1) < 0$; in case $t = 0$ this holds true for the right derivative of $\dot{x}$. Using the preceding statements and the fact that $\dot{x}$ is slowly oscillating we get $\dot{x}(s) < 0$ on $(t, t+1]$.

3. Consider the case $x(s) > 0$ on $[t,\infty)$. Suppose $\dot{x}(s) \geq 0$ for some $s > t+1$. Then there is a smallest zero ζ of $\dot{x}$ in (t,∞), and $t+1 < \zeta$. We have $\dot{x}(u) < 0$ on (t,ζ), and thereby $x(\zeta) < x(\zeta-1)$, in contradiction to

$$x(\zeta-1) = -a^{-1}\dot{x}(\zeta) + x(\zeta) - |x(\zeta)|x(\zeta) = x(\zeta) - |x(\zeta)|x(\zeta) < x(\zeta).$$

It follows that $\dot{x}(u) < 0$ on (t,∞). Proposition 2.1 (i) gives

$$\lim_{u\to\infty} x(u) = 0.$$

4. The case $x(s) \leq 0$ for some $s > t$. Then there is a smallest zero z of x in (t,∞). By part 2, $\dot{x}(u) < 0$ on $(t, t+1]$. If $z \leq t+1$ then obviously $\dot{x}(u) < 0$ on $(t, z]$. If $z > t+1$ then arguments as in part 3 yield $\dot{x}(u) < 0$ on $(t, z]$ as well.
In the subcase $x(u) \geq 0$ for some $u > z$ there is a smallest zero ζ of $\dot{x}$ in (z, u), and clearly $\dot{x}(u') < 0$ on (t,ζ).
In the other subcase $x(u) < 0$ on (z,∞) Proposition 2.1 (ii) shows that $\dot{x}$ has a zero in (z,∞). For the smallest zero ζ of $\dot{x}$ in (z,∞) we have $\dot{x}(u) < 0$ on $[z,\zeta)$.

5. The remaining assertions are now obvious. □

Remark Of course, the analogue of Proposition 2 for the case $\dot{x}(t) = 0$ with $x(t) < 0$ holds as well.

The next results concern the case $a = 1$. At first we show that curves $X^{1,y}$ starting at $y \in f$ with $0 < y_1 < \delta_0$ intersect the y_2-axis, or equivalently, that solutions $x : [-1, \infty) \to \mathbb{R}$ of Eq. (1.1) which start in $W^1 \setminus \{0\}$ and have a maximum at $t = 0$ must have a zero $z > 0$. This is similar to the result of Corollary 5.2 from [1]. Notice that the following proof is by a simple comparison argument, whereas in [1] a centre manifold reduction was used.

Proposition 3 *Every solution $x = x^{1,\phi}$ with $\phi \in W^1$ and $\dot{x}(0) = 0 < x(0)$ has a zero $z > 0$.*

Proof Suppose x has no positive zero. Then Proposition 2 yields $\dot{x}(t) < 0 < x(t)$ on $(0, \infty)$ and $x(t) \to 0$ as $t \to \infty$. On the interval $[1, 2]$ the function $\dot{x}$ has a negative maximum $c < 0$. There exist $b \in \mathbb{R}$ and $s \geq 2$ so that the solution $v : \mathbb{R} \ni t \mapsto b + ct \in \mathbb{R}$ of the linear equation

$$\dot{v}(t) = v(t) - v(t-1)$$

satisfies $v(t) \leq x(t)$ on $[1, \infty)$ and $v(s) = x(s)$. It follows that $\dot{v}(s) = \dot{x}(s)$. On the other hand,

$$\begin{aligned} \dot{x}(s) &= x(s) - x(s-1) - |x(s)|x(s) = v(s) - x(s-1) - |x(s)|x(s) \\ &\leq v(s) - v(s-1) - |x(s)|x(s) < v(s) - v(s-1) = \dot{v}(s), \end{aligned}$$

in contradiction to the previous equation. □

Set

$$\begin{aligned} &f_+ = \{y \in f : 0 < y_1\} \text{ and } f_{+0} = \{y \in f : 0 < y_1 < \delta_0\}, \\ &f_- = \{y \in f : y_1 < 0\} \text{ and } f_{-0} = \{y \in f : -\delta_0 < y_1 < 0\}. \end{aligned}$$

Recall $\delta_0 < 1$. By Proposition 3 the set

$$\Delta_- = \{(a, y) \in (1 - \delta_0, 1 + \delta_0) \times f_{+0} : X^{a,y}(z) \in \{0\} \times \mathbb{R} \text{ for some } z > 0\}$$

is non-empty, and

$$\{1\} \times f_{+0} \subset \Delta_-.$$

Proposition 2 implies that for each $(a, y) \in \Delta_-$ the smallest zero $z_+(a, y)$ of $X_1^{a,y}$ in $(0, \infty)$ satisfies $X_2^{a,y}(z_+(a, y)) > 0$, and that there is a smallest $t_-(a, y) > z_+(a, y)$ so that $X^{a,y}(t_-(a, y)) \in f$; we have $X^{a,y}(t_-(a, y)) \in f_-$ and

$$X^{a,y}(t) \in f_> \cap (X_1^{a,y}(t_-(a, y)), y_1]^2 \text{ on } (0, t_-(a, y)). \tag{1}$$

Using Proposition 1 (i) and (iii) and Proposition 2 we see that Δ_- is open in $(1 - \delta_0, 1 + \delta_0) \times f$, and that the maps

$$z_+ : \Delta_- \ni (a, y) \mapsto z_+(a, y) \in (0, \infty) \text{ and } t_- : \Delta_- \ni (a, y) \mapsto t_-(a, y) \in (0, \infty)$$

are continuous. It follows that the map

$$J_- : \Delta_- \ni (a, y) \mapsto X^{a,y}(t_-(a, y)) \in f_-$$

is continuous. Our next aim is to show that the map $J_-(1,\cdot) : f_{+0} \to f_-$ satisfies

$$\lim_{f_{+0}\ni y\to 0} J_-(1,y) = (0,0).$$

This requires some preparations. Observe that for $y \in f_{+0}$,

$$X^{1,y}(z_+(1,y)+1) \in (-\infty,0)\times\{0\}.$$

Proposition 4 *The map* $I : f_{+0} \ni y \mapsto X_1^{1,y}(z_+(1,y)+1) \in (-\infty,0)$ *satisfies*

$$\lim_{f_{+0}\ni y\to 0} I(y) = 0.$$

Proof For $y \in f_{+0}$, $0 < y_1 < 1$. Proposition 2 therefore yields

$$X^{1,y}([0,z_+(1,y)]) \subset [0,y_1]^2.$$

Consequently,

$$\lim_{f_{+0}\ni y\to 0} X^{1,y}(z_+(1,y)) = (0,0).$$

Moreover, Proposition 1 (v) shows that for all $y \in f_{+0}$,

$$X^{1,y}(z_+(1,y)+1) = X^{1,\hat{y}}(1) \text{ with } \hat{y} = X^{1,y}(z_+(1,y)).$$

Apply Proposition 1 (iii). □

Incidentally, notice that there is no bound for $z_+(1,y)$ with $y \in f_{+0}$ small, and that one can not invoke continuity of the semi-flow F_1 and the fact that $0 \in C$ is stationary in order to obtain the previous result.

We need to control the behaviour of curves $X^{1,y}$ with $-1 < I(y)$ on the interval $[z_+(1,y)+1, t_-(1,y)]$, where they travel from $(-1,0)\times\{0\}$ to the branch f_- of f. The subsequent result can be viewed as an analogue of Proposition 3 for backward solutions. In part 2.2 of the proof we use a comparison argument as in the proof of Proposition 3.

Proposition 5 *There exists* $\delta_- \in (0,\delta_0)$ *so that for each* $y \in f_{-0}$ *with* $-\delta_- < y_1 < 0$ *there are* $\eta \in (y_1,0)$ *and* $t > 0$ *with*

$$y = X^{1,(\eta,0)}(t)$$

and

$$X^{1,(\eta,0)}((0,t)) \subset f_> \cap ((y_1,0)\times(y_2,0)).$$

Proof 1. Proposition 3.4 shows that there are open neighbourhoods U and V of 0 in W^1 such that $F_1([0,1],U) \subset W^1$ and $V \subset F_1(1,U)$. It follows that there is $\delta_- \in (0,\delta_0)$ so that for each $y \in (-\delta_-,\delta_-)^2$ there exists $\hat{y} \in Q$ with $y = X^{1,\hat{y}}(1)$ and $X^{1,\hat{y}}([0,1]) \subset Q$. We may assume $\delta_- < \frac{1}{2}$. Then f is strictly increasing on $(-\delta_-,0)$. Consider $y \in f_{-0}$ with $-\delta_- < y_1 < 0$. Then $y_2 = f(y_1) \in (y_1,0) \subset (-\delta_-,0)$.

2. Suppose
$$X^{1,(\eta,0)}(t) \neq y \text{ for all } \eta \in (y_1, 0) \text{ and } t \geq 0. \tag{2}$$
Let
$$Y = f_> \cap ((y_1, 0) \times (y_2, 0)).$$

2.1. Proof that there is a solution $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) with $a = 1$ so that $x_0 \in W^1$, $Ex_0 = y$, and
$$\dot{x}(t) < 0 \text{ and } y_1 < x(t) < 0 \text{ on } (-\infty, 0). \tag{3}$$

2.1.1. We have $y = X^{1,\hat{y}}(1)$ with
$$X^{1,\hat{y}}([0,1]) \subset Q \setminus \{(0,0)\}.$$
By transversality as in Proposition 2, $X^{1,\hat{y}}(t) \in f_>$ on some interval $(1-\tau, 1)$ with $0 < \tau \leq 1$. Hence, $\dot{X}_1^{1,\hat{y}}(t) < 0$ on this interval. So we may assume that on $(1-\tau, 1)$,
$$\begin{aligned} X^{1,\hat{y}}(t) &\in f_> \cap ((y_1, 0) \times (-\infty, 0)) = f_> \cap ((y_1, 0) \times (y_2, 0)) \\ &= Y \subset f_> \cap (-\delta_-, 0)^2. \end{aligned}$$

2.1.2. Proof of $X^{1,\hat{y}}([0,1)) \subset Y$. Suppose the assertion is false. Then there exists a largest $t \in [0, 1-\tau]$ with
$$X^{1,\hat{y}}(t) \notin Y.$$
It follows that $X^{1,\hat{y}}(t)$ is in ∂Y, and $X^{1,\hat{y}}((t,1)) \subset Y$. Using the assumption (2) and Proposition 1 (v) we obtain
$$X^{1,\hat{y}}(t) \notin (y_1, 0) \times \{0\}.$$
Hence, $(0,0) \neq X^{1,\hat{y}}(t) \in f_-$ or $X^{1,\hat{y}}(t) \in (\{y_1\} \times \mathbb{R}) \cap f_>$. In the first case transversality yields $X^{1,\hat{y}}(s) \in f_< \subset \mathbb{R}^2 \setminus Y$ for some $s \in (t,1)$, and we obtain a contradiction. In the second case we have $\dot{X}_1^{1,\hat{y}}(t) < 0$, and thereby $X_1^{1,\hat{y}}(s) < y_1$ for some $s \in (t,1)$, hence $X^{1,\hat{y}}(s) \in \mathbb{R}^2 \setminus Y$, and we get the same contradiction as before.

2.1.3. In particular, $\hat{y} \in Y$. Using arguments as in parts 2.1.1 and 2.1.2 and induction one extends the solution $x^{1,\phi}$ with $\phi \in W^1$ and $E\phi = y$ backward to a solution on $\mathbb{R}$ with property (3).

2.2. Let $c = \min \dot{x}|[-2,-1] < 0$. Choose $s_c \in [-2,-1]$ with $\dot{x}(s_c) = c$. There exist $b \in \mathbb{R}$ and $s \leq s_c$ so that the solution
$$v : \mathbb{R} \ni t \mapsto b + ct \in \mathbb{R}$$
of the linear variational equation
$$\dot{v}(t) = v(t) - v(t-1)$$
satisfies $v(s) = x(s)$ and $v(t) \geq x(t)$ on $(-\infty, s]$. In both cases $s < s_c$ and $s = s_c$ we have $\dot{v}(s) = \dot{x}(s)$. On the other hand,
$$\begin{aligned} \dot{x}(s) &= x(s) - x(s-1) - |x(s)|x(s) > x(s) - x(s-1) = v(s) - x(s-1) \\ &\geq v(s) - v(s-1) = \dot{v}(s), \end{aligned}$$
which yields a contradiction.

2.3. It follows that there exist $\hat{\eta} \in (y_1, 0)$ and $\hat{t} > 0$ with $y = X^{1,(\hat{\eta},0)}(\hat{t})$.

3. As in part 2.1.1 we have $X^{1,(\hat{\eta},0)}((\hat{t} - \tau, \hat{t})) \subset Y$ for some $\tau \in (0, \hat{t})$. Set

$$s = \max\{t \in [0, \hat{t} - \tau] : X^{1,(\hat{\eta},0)}(t) \in \mathbb{R}^2 \setminus Y\}.$$

Then $(0, 0) \neq X^{1,(\hat{\eta},0)}(s) \in \partial Y$. As in part 2.1.2 we exclude the cases $X^{1,(\hat{\eta},0)}(s) \in f_-$ and $X^{1,(\hat{\eta},0)}(s) \in \{y_1\} \times \mathbb{R}$. It follows that $X^{1,(\hat{\eta},0)}(s) \in (y_1, 0) \times \{0\}$, and Proposition 1 (v) shows that the assertion holds for $\eta = X_1^{1,(\hat{\eta},0)}(s)$ and $t = \hat{t} - s$. □

Part 2.2 of the preceding proof shows that in case $a = 1$ there are no solutions $x : \mathbb{R} \to \mathbb{R}$ of Eq. (1.1) so that x and $\dot{x}$ are negative on $(-\infty, 0)$. For any $a > 1$ this is false: The branches of the one-dimensional local unstable manifold at 0 consist of segments of solutions x so that x and $\dot{x}$ have no zero and are of the same sign on $(-\infty, 0)$. For a proof, see [1].

Proposition 6 $\lim_{f_{+0} \ni y \to 0} J_-(1, y) = (0, 0)$.

Proof 1. Let $\epsilon \in (0, \delta_-)$ be given. Proposition 5 guarantees the existence of $\eta \in (-\epsilon, 0)$ and $t > 0$ so that

$$(-\epsilon, f(-\epsilon)) = X^{1,(\eta,0)}(t) \text{ and } X^{1,(\eta,0)}((0, t)) \subset f_> \cap ((-\epsilon, 0) \times (f(-\epsilon), 0)).$$

In particular, $\dot{X}_1^{1,(\eta,0)}(s) < 0$ on $(0, t)$. It follows that the set $X^{1,(\eta,0)}([0, t]) \cup ((\eta, 0] \times \{0\})$ is given by a continuous function $g : [-\epsilon, 0] \to \mathbb{R}$. The open set

$$Y = \{\hat{y} \in \mathbb{R}^2 : -\epsilon < \hat{y}_1 < 0, f(\hat{y}_1) < \hat{y}_2 < g(\hat{y}_1)\}$$

is contained in the square $(-\epsilon, 0)^2$ and has the boundary

$$\partial Y = X^{1,(\eta,0)}([0, t]) \cup ((\eta, 0] \times \{0\}) \cup (f \cap ((-\epsilon, 0) \times \mathbb{R})).$$

By Proposition 4 there is $d > 0$ with

$$I(f_{+0} \cap ((0, d) \times \mathbb{R})) \subset (\eta, 0).$$

In the sequel we show that for $y \in f_{+0} \cap ((0, d) \times \mathbb{R})$ we have

$$J_-(1, y) = X^{1,y}(t_-(1, y)) \in \overline{Y} \subset [-\epsilon, 0]^2.$$

2. Let $y \in f_{+0} \cap ((0, d) \times \mathbb{R})$ be given. Then

$$X^{1,y}(z_+(1, y)+1) \in (\eta, 0)\times\{0\} \text{ and } X_2^{1,y}(s) < 0 \text{ on } (z_+(1, y)+1, t_-(1, y)),$$

by Proposition 2. It follows that for some $\tau > 0$,

$$X^{1,y}(s) \in Y \text{ on } (z_+(1, y) + 1, z_+(1, y) + 1 + \tau).$$

Suppose

$$X^{1,y}(t_-(1, y)) \in \mathbb{R}^2 \setminus \bar{Y}.$$

Then there exists $s \in [z_+(1, y) + 1 + \tau, t_-(1, y))$ with $X^{1,y}(s) \in \partial Y$. Proposition 2 shows that $X^{1,y}(s)$ is neither in $\mathbb{R} \times \{0\}$ nor in f. Consequently,

$$X^{1,y}(s) = X^{1,(\eta,0)}(u) \text{ for some } u \in (0, t).$$

Notice that for $0 < v < s$ we have

$$-\epsilon < X_1^{1,(\eta,0)}(u) = X_1^{1,y}(s) < X_1^{1,y}(v) < y_1.$$

due to Proposition 2. Again by Proposition 2,

$$X^{1,y}([0, s]) \subset (-\epsilon, y_1]^2 \subset Q.$$

Recall

$$X^{1,(\eta,0)}([0, u]) \subset Q.$$

2.1. In case $s = z_+(1, y) + 1 + u$ Proposition 1 (v) and (vi) yield

$$X^{1,y}(z_+(1, y) + 1) = X^{1,y}(s - u) = X^{1,(\eta,0)}(u - u) = (\eta, 0),$$

in contradiction to the choice of y.

2.2. In case $s < z_+(1, y) + 1 + u$ Proposition 1 (v) and (vi) yield

$$\begin{aligned} X^{1,(\eta,0)}(u - (s - (z_+(1, y) + 1))) &= X^{1,y}(s - (s - (z_+(1, y) + 1))) \\ &= X^{1,y}(z_+(1, y) + 1) \in (-\infty, 0) \times \{0\}, \end{aligned}$$

in contradiction to $X_2^{1,(\eta,0)}(v) < 0$ for $0 < v < t$.

2.3. In case $s > z_+(1, y) + 1 + u$ Proposition 1 (v) and (vi) yield

$$X^{1,y}(s - u) = X^{1,(\eta,0)}(u - u) = (\eta, 0),$$

in contradiction to $X_2^{1,y}(v) < 0$ on $(z_+(1, y) + 1, t_-(1, y))$.

2.4. It follows that

$$J_-(1, y) = X^{1,y}(t_-(1, y)) \in \overline{Y} \subset [-\epsilon, 0]^2.$$

□

By arguments as before, the set

$$\Delta_+ = \{(a, y) \in (1 - \delta_0, 1 + \delta_0) \times f_{-0} : X^{a,y}(z) \in \{0\} \times \mathbb{R} \text{ for some } z > 0\}$$

is non-empty with

$$\{1\} \times f_{-0} \subset \Delta_+$$

and open in $(1 - \delta_0, 1 + \delta_0) \times f$, and for each $(a, y) \in \Delta_+$ there exists a smallest local extremum $t_+(a, y) > 0$ of $X_1^{a,y} = x^{a,\phi}$, where $\phi \in W^a$ and $E\phi = y$. For each $(a, y) \in \Delta_+$, $X^{a,y}(t_+(a, y)) \in f_+$ and

$$X^{a,y}(t) \in f_< \cap [y_1, X_1^{a,y}(t_+(a, y)))^2 \text{ on } (0, t_+(a, y)). \tag{4}$$

The maps

$$t_+ : \Delta_+ \ni (a, y) \mapsto t_+(a, y) \in (0, \infty)$$

and

$$J_+ : \Delta_+ \ni (a, y) \mapsto X^{a,y}(t_+(a, y)) \in f_+$$

are continuous, and $J_+(1, \cdot) : f_{-0} \to f_+$ has limit $(0, 0)$ at $(0, 0)$.

Corollary 1 *For every $\epsilon \in (0, \delta_0)$ there exist $y = y^\epsilon \in f_+$ with $(1, y) \in \Delta_-$ and $(1, J_-(1, y)) \in \Delta_+$ so that with $\hat{y} = J_-(1, y)$ we have*

$$X^{1,y}(t) \in (-\epsilon, \epsilon)^2 \text{ on } [0, t_-(1, y) + t_+(1, \hat{y}],$$
$$X^{1,y}(t) \in f_> \cap (\hat{y}_1, y_1]^2 \text{ on } (0, t_-(1, y)),$$

and

$$X^{1,y}(t) \in f_< \cap [\hat{y}_1, X_1^{a,\hat{y}}(t_+(1, \hat{y})))^2 \text{ on } (t_-(1, y), t_-(1, y) + t_+(1, \hat{y})).$$

Proof Use Proposition 6 and $\lim_{f_{-0} \ni \hat{y} \to 0} J_+(1, \hat{y}) = (0, 0)$ in order to find $y \in f_+$ with $0 < y_1 < \epsilon$, $-\epsilon < X_1^{1,y}(t_-(1, y)) = J_-(1, y)_1 < 0$, and $0 < X_1^{1,\hat{y}}(t_+(1, \hat{y})) < \epsilon$, where $\hat{y} = X^{1,y}(t_-(1, y)) = J_-(1, y)$. Properties (1) and (4) yield

$$X^{1,y}((0, t_-(1, y))) \subset f_> \cap (\hat{y}_1, y_1]^2 \subset (-\epsilon, \epsilon)^2$$

and

$$X^{1,\hat{y}}((0, t_+(1, \hat{y}))) \subset f_< \cap [\hat{y}_1, X_1^{1,\hat{y}}(t_+(1, \hat{y})))^2 \subset (-\epsilon, \epsilon)^2.$$

Use Proposition 1 (v) to complete the proof. □

Proposition 7 *Let (a, y) and (a, y^*) in Δ_- be given with $y_1^* < y_1$ and $X^{a,y}([0, t_-(a, y)]) \subset Q$. Then*

$$X_1^{a,y}(t_-(a, y)) < X_1^{a,y^*}(t_-(a, y^*)).$$

Proof Consider ϕ and ϕ^* in W^a with $E\phi = y$ and $E\phi^* = y^*$, and set $x = x^{a,\phi}$, $x^* = x^{a,\phi^*}$, $X = X^{a,y}$, $X^* = X^{a,y^*}$. Set $s = t_-(a, y)$ and $s^* = t_-(a, y^*)$. From $\dot{x} < 0$ on $(0, s)$ and $\dot{x}^* < 0$ on $(0, s^*)$ we infer that the arcs $X([0, s])$ and $X^*([0, s^*])$ can be written as continuous maps

$$A : [x(s), x(0)] \to \mathbb{R} \text{ and } B : [x^*(s^*), x^*(0)] \to \mathbb{R},$$

respectively, with

$$(x(s), A(x(s))) = X(s) \in f_-, \ (x^*(s^*), B(x^*(s^*))) = X^*(s^*) \in f_-,$$
$$(x(0), A(x(0))) = X(0) \in f_+, \ (x^*(0), B(x^*(0))) = X^*(0) \in f_+,$$

and

$$(\xi, A(\xi)) \in f_> \text{ on } (x(s), x(0)), \ (\xi, B(\xi)) \in f_> \text{ on } (x^*(s^*), x^*(0)).$$

Assume $x^*(s^*) \le x(s)$. Then the graphs A and B intersect. Notice that $X^*(0) \in f_+$ is different from $X(0)$ and does not belong to $X((0, s))$, and that $X(0) \in (x^*(0), \infty) \times \mathbb{R}$ does not belong to $X^*([0, s^*])$. It follows that there exist $t \in (0, s]$ and $t^* \in (0, s^*]$ with $X(t) = X^*(t^*)$. Proposition 2 shows that the arc $X^*([0, t^*])$ is contained in

$$\begin{aligned}[x^*(t^*), x^*(0)] \times [\min\{0, x^*(t^*)\}, x^*(0)] &= [x(t), x^*(0)] \\ &\quad \times [\min\{0, x(t)\}, x^*(0)] \\ &\subset [x(s), x(0)]^2 \subset Q.\end{aligned}$$

Now the inclusion $X([0, s]) \subset Q$ and arguments as in part 2 of the proof of Proposition 6 yield a contradiction. □

Remark The analogue of Proposition 7 for Δ_+ instead of Δ_- holds as well.

6 Bifurcation

We complete the proof of the theorem from Sect. 1. It is convenient to consider for $\epsilon > 0$ and $a > 0$ the following property.

($P_{\epsilon a}$) Eq. (1.1) has a slowly oscillating periodic solution $p : \mathbb{R} \to (-\epsilon, \epsilon)$ whose minimal period is given by three consecutive zeros. There is a first zero $z > -1$ and $0 < x(t) \le x(0)$ on $(-1, z)$.

Proposition 1 (Supercritical bifurcation) *Let $\epsilon \in (0, \delta_0)$ and assume $y = y^{(\epsilon)}$ satisfies*

$$X_1^{1,y}(t_-(1, y) + t_+(1, J_-(1, y))) < y_1.$$

Then there exists $a_\epsilon \in (1, 1+\epsilon)$ so that ($P_{\epsilon a}$) holds for every $a \in (1, a_\epsilon)$.

Proof 1. Let $\epsilon \in (0, \delta_0)$ be given and consider $y = y^\epsilon$ as in Corollary 5.1. Then $(1, y) \in \Delta_-$, and $(1, \hat{y})$ with $\hat{y} = J_-(1, y) = X^{1,y}(t_-(1, y))$ belongs to Δ_+, and

$$J_{+,1}(1, J_-(1, y)) = X_1^{1,\hat{y}}(t_+(1, \hat{y})) = X_1^{1,y}(t_-(1, y) + t_+(1, \hat{y})) < y_1,$$

by Proposition 5.1 (v). Choose $t_0 > t_-(1, y) + t_+(1, \hat{y})$ with $X^{1,y}([0, t_0]) \subset (-\epsilon, \epsilon)^2$. Using Proposition 5.1 (iii), the openness of Δ_- and Δ_+, and the continuity of t_-, J_-, t_+ and J_+ we find $a_\epsilon \in (1, 1+\epsilon)$ so that for every $a \in (1, a_\epsilon)$ the following holds: $(a, y) \in \Delta_-$, (a, y^*) with $y^* = J_-(a, y)$ belongs to Δ_+,

$$t_-(a, y) + t_+(a, y^*) < t_0,$$

$$J_{+,1}(a, J_-(a, y)) < y_1,$$

and

$$X^{a,y}(t) \in (-\epsilon, \epsilon)^2 \text{ on } [0, t_0].$$

Let $a \in (1, a_\epsilon)$ be given and set $X = X^{a,y}$, $s_1 = t_-(a, y)$, $y^* = J_-(a, y) = X^{a,y}(s_1)$ and $t_1 = s_1 + t_+(a, y^*)$. Using Proposition 5.1 (v) we get

$$X(s_1 + t) = X^{a,y*}(t) \text{ on } [0, t_+(a, y^*)].$$

From (5.1) and (5.4) in combination with the preceding statement we infer

$$X((0, s_1)) \subset f_> \cap (X_1(s_1), X_1(0)]^2 = f_> \cap (X_1(s_1), y_1]^2$$

and

$$X((s_1, t_1)) \subset f_< \cap [X_1(s_1), X_1(t_1))^2;$$

we have $X(s_1) \in f_-$, $X(t_1) \in f_+$, and

$$X_1(t_1) < X_1(0).$$

In particular,

$$X([0, t_1]) \subset (-\epsilon, \epsilon)^2 \subset Q,$$

and

$$X^{a,X(t_1)}(t) = X(t_1 + t) \text{ for all } t \geq 0, \tag{1}$$

due to Proposition 5.1 (v).

2. Proof of $(a, X(t_1)) \in \Delta_-$. Proposition 5.1 (iv) yields $F_a(t_1, \phi) \in W^a$ for $\phi \in W^a$ with $E\phi = y$. Let $x = x^{a,\phi}$. Suppose x has no zero in (t_1, ∞). Then Proposition 5.2 gives $\dot{x}(t) < 0$ on (t_1, ∞), $\lim_{t\to\infty} x(t) = 0$, and $0 < x(t) < \delta_0 < 1$ on $[t_1 - 1, \infty)$. Proposition 5.1 (iv) yields $x_t \in W^a$ on $[t_1, \infty)$, and we obtain a contradiction to Corollary 3.3. It follows that there is a smallest zero of x in (t_1, ∞). From this and from $x_{t_1} = F_a(t_1, \phi) \in W^a$ we obtain $(a, X(t_1)) \in \Delta_-$.
3. Set $s_2 = t_1 + t_-(a, X(t_1))$. An application of Proposition 5.7 to (a, y) and $(a, X(t_1))$ in combination with (1) yields

$$x(s_1) < x(s_2).$$

We have

$$X(s_2) = X^{a,X(t_1)}(t_-(a, X(t_1))) \in f_-,$$
$$X(t) \in f_> \text{ on } (t_1, s_2),$$

and

$$X([t_1, s_2]) \subset [x(s_2), x(t_1)]^2 \subset [x(s_1), x(0)]^2 \subset (-\epsilon, \epsilon)^2.$$

4. As in part 2 we get $(a, X(s_2)) \in \Delta_+$. Using the remark following Proposition 5.7 and arguments as in part 3 we find $t_2 > s_2$ with

$$X([s_2, t_2]) \subset (-\epsilon, \epsilon)^2, \ \ X((s_2, t_2)) \subset f_<, \ \ X(t_2) \in f_+,$$

and

$$x(t_2) < x(t_1).$$

Let $t_0 = 0$. Induction shows that the local maxima and local minima of x on $(-1, \infty)$ form strictly increasing sequences $(t_n)_0^\infty$ and $(s_n)_1^\infty$, respectively, with $t_0 = 0$, so that for all integers $n \geq 0$, $t_n < s_{n+1} < t_{n+1}$ and

$$-\epsilon < x(s_{n+1}) < x(s_{n+2}) < 0 < x(t_{n+1}) < x(t_n) < \epsilon.$$

Also, $x_t \in W^a$ for all $t \geq 0$.

5. Using the preceding statement and Corollary 3.3 we infer that

$$\text{(I)} \quad \lim_{n\to\infty} x(t_n) > 0 \quad \text{or} \quad \text{(II)} \quad \lim_{n\to\infty} x(s_n) < 0.$$

5.1 Case (I). Set $p_1^* = \lim_{n\to\infty} x(t_n) > 0$. Then $p^* = (p_1^*, f(p_1^*))$ belongs to $f_{+0} \cap (-\epsilon, \epsilon)^2$. Set $X^* = X^{a,p^*}$, and for $\phi^* \in W^a$ with $E\phi^* = p^*$, $x^* = x^{a,\phi^*}$. The inequality $0 < p_1^* < y_1 = x(0)$ and arguments as in parts 2–4 show that there are a first local minimum $s_1^* > 0$ and a first local maximum $t_1^* > s_1^*$ of x^*, with

$$-\epsilon < x(s_1) < x^*(s_1^*) < 0 < x^*(t_1^*) < x(t_1) < \epsilon.$$

It follows that

$$|x^*(t)| < \epsilon \text{ on } [-1, t_1^*],$$

and

$$x_t^* \in W^a \text{ on } [0, t_1^*].$$

Continuity of the maps J_- and J_+ implies that for $n \to \infty$ the points

$$\begin{aligned}(x(t_{n+1}), f(x(t_{n+1}))) &= X(t_{n+1}) = J_+(a, J_-(a, (X(t_n))))\\ &= J_+(a, J_-(a, (x(t_n), f(x(t_n)))))\end{aligned}$$

converge to

$$X^*(0) = p^* = J_+(a, J_-(a, p^*)) = X^*(t_1^*).$$

Hence, $\phi^* = F_a(t_1^*, \phi^*)$. It follows that x^* extends to the desired periodic solution p_ϵ of Eq. (1.1), with minimal period t_1^*.

5.2 The proof in case (II) is analogous. □

The next result establishes subcritical bifurcation in case the curve $X^{1,y^{(\epsilon)}}$ of Corollary 5.1 spirals outward. The following proof differs from the preceding one in that the periodic orbit is not obtained as a limit cycle.

Proposition 2 *Let $\epsilon \in (0, \delta_0)$ and assume $y = y^{(\epsilon)}$ satisfies*

$$X_1^{1,y}(t_-(1, y) + t_+(1, J_-(1, y))) > y_1.$$

Then there exists $a_\epsilon \in (1-\epsilon, 1)$ so that $(P_{\epsilon a})$ holds for every $a \in (a_\epsilon, 1)$.

Proof 1. As in the proof of Proposition 1 we obtain $a_\epsilon \in (1-\epsilon, 1)$ so that for each $a \in (a_\epsilon, 1)$ the following holds: $(a, y) \in \Delta_-$, $y^* = J_-(a, y)$ belongs to Δ_+, and $X = X^{a,y}$, $s_1 = t_-(a, y)$, and $t_1 = s_1 + t_+(a, y^*)$ satisfy

$$\begin{aligned}
&X([0, t_1]) \subset (-\epsilon, \epsilon)^2,\\
&X(t) \in f_> \cap (X_1(s_1), y_1]^2 = f_> \cap (X_1(s_1), X_1(0)]^2 \text{ on } (0, s_1),\\
&X(t) \in f_< \cap [X_1(s_1), X_1(t_1))^2 \text{ on } (s_1, t_1),\\
&X(s_1) \in f_- \text{ and } X(t_1) \in f_+,\\
&X_1(0) = y_1 < X_1(t_1).
\end{aligned}$$

Let $a \in (a_\epsilon, 1)$ be given. For $\phi \in W^a$ with $E\phi = y$ set $x = x^{a,\phi}$ and $s_1 = t_-(a, y)$. The set M of all $\hat{y}_1 \in (0, y_1]$ so that for $\hat{y} = (\hat{y}_1, f(\hat{y}_1))$ we have $(a, \hat{y}) \in \Delta_-$, $J_-(a, \hat{y}) \in \Delta_+$, and

$$\hat{y}_1 < J_{+,1}(a, J_-(a, \hat{y}))$$

contains y_1 and is open in $(0, y_1]$.

2. Proof of $(0, y_1]\backslash M \neq \emptyset$. As 0 is asymptotically stable there is a neighbourhood U of 0 in C so that for all $\phi \in U$,

$$F_a([0, \infty), \phi) \subset C(y_1) \tag{2}$$

and

$$\lim_{t\to\infty} F_a(t, \phi) = 0. \tag{3}$$

Choose $\phi^* \in W^a \cap U$ with $E\phi^* \in f_+$. The inclusion (2) in combination with Proposition 5.1 (iv) shows that the solution $x^* = x^{a,\phi^*}$ satisfies $x^*_t \in W^a$ for all $t \geq 0$. Using (3) and the fact that the local maxima of x^* form a non-empty set of isolated points we find a local maximum $t_m \geq 0$ with

$$x^*(t) < x^*(t_m) \text{ on } (t_m, \infty).$$

Set $\hat{\phi} = x^*_{t_m}$, $\hat{y} = E\hat{\phi}$, and $\hat{x} = x^{a,\hat{\phi}} = x^*(t_m + \cdot)$. Then

$$\hat{y}_1 = \hat{x}(0) = x^*(t_m) \in (0, y_1),$$

by (2). Assume $\hat{y}_1 \in M$. Set $\hat{s} = t_-(a, \hat{y})$, $\check{y} = J_-(a, \hat{y})$, and $\hat{t} = \hat{s}+t_+(a, \check{y})$. Using (2) and Proposition 5.2 we see that on $[-1, \hat{s}]$ the inequalities

$$-\epsilon < -y_1 < x^*(t_m + \hat{s}) = \hat{x}(\hat{s}) \leq \hat{x}(t) \leq \hat{x}(0) = \hat{y}_1 < y_1 < \epsilon$$

hold. From this we infer

$$X^{a,\check{y}}(t) = X^{a,\hat{y}}(\hat{s} + t) \text{ for all } t \geq 0,$$

by Proposition 5.1 (v). It follows that

$$J_+(a, J_-(a, \hat{y})) = J_+(a, \check{y}) = X^{a,\check{y}}(t_+(a, \check{y})) = X^{a,\hat{y}}(\hat{s} + t_+(a, \check{y})) = X^{a,\hat{y}}(\hat{t}),$$

hence

$$J_{+,1}(a, J_-(a, \hat{y})) = \hat{x}(\hat{t}) = x^*(t_m + \hat{t}) < x^*(t_m) = \hat{y}_1,$$

in contradiction to the definition of M.

3. Let $\bar{y}_1 = \max((0, y_1] \setminus M)$. Then $0 < \bar{y}_1 < y_1$. Set $\bar{y} = (\bar{y}_1, f(\bar{y}_1)) \in (-\epsilon, \epsilon)^2$. Set $\bar{X} = X^{a,\bar{y}}$, and for $\bar{\phi} \in W^a$ with $E\bar{\phi} = \bar{y}$, $\bar{x} = x^{a,\bar{\phi}}$. In the sequel we show that $\bar{x}$ extends to the desired periodic solution.
4. Proof of $(a, \bar{y}) \in \Delta_-$.
   4.1. Assume the contrary. Then Proposition 5.2 yields $0 < \bar{x}(t) < \bar{y}_1$ on $(0, \infty)$ and $\lim_{t\to\infty} \bar{x}(t) = 0$. Choose a neighbourhood V of 0 in C with

$$F_a([0, \infty), V) \subset C(\bar{y}_1).$$

Choose $u \geq 0$ so that $\bar{x}_u \in V$. By continuity there exists $\hat{y}_1 \in (\bar{y}_1, y_1)$ so that the solution $\hat{x} = x^{a,\hat{\phi}}$ with $\hat{\phi} \in W^a$, $E\hat{\phi} = \hat{y}$, $\hat{y} = (\hat{y}_1, f(\hat{y}_1))$, satisfies

$$0 < \hat{x}(t) \text{ on } [0, u] \text{ and } \hat{x}_u \in V.$$

Set $\hat{X} = X^{a,\hat{y}}$.

4.2. Suppose $(a,\hat{y}) \in \Delta_-$. Then $u < t_-(a,\hat{y})$, by Proposition 5.2, and

$$\hat{x}(t_-(a,\hat{y})) \leq \hat{x}(t) \leq \hat{y}_1 < y_1 \text{ on } [-1, t_-(a,\hat{y})],$$

and by the choice of V,

$$|\hat{x}(t)| < \bar{y}_1 < \hat{y}_1 < \epsilon \text{ on } [t_-(a,\hat{y}), \infty).$$

In particular, $\hat{X}([0,\infty)) \subset Q$, and for $\check{y} = J_-(a,\hat{y}) = \hat{X}(t_-(a,\hat{y}))$ Proposition 5.1 (v) yields

$$X^{a,\check{y}}(s) = \hat{X}(t_-(a,\hat{y}) + s) \text{ for all } s \geq 0.$$

4.3. Suppose $(a,\hat{y}) \in \Delta_-$ and in addition $(a,\check{y}) \in \Delta_+$. Then we obtain

$$\begin{aligned} J_{+,1}(a, J_-(a,\hat{y})) &= J_{+,1}(a,\check{y}) = X_1^{a,\check{y}}(t_+(a,\check{y})) = \hat{X}_1(t_-(a,\hat{y}) + t_+(a,\check{y})) \\ &= \hat{x}(t_-(a,\hat{y}) + t_+(a,\check{y})) < \hat{y}_1. \end{aligned}$$

4.4. It follows that $\hat{y} \in (\bar{y}_1, y_1) \setminus M$, in contradiction to the definition of $\bar{y}_1$.

5. Let $\bar{s} = t_-(a,\bar{y})$. Proposition 5.7 yields

$$x(s_1) < \bar{x}(\bar{s}).$$

It follows that

$$-\epsilon < x(s_1) < \bar{x}(\bar{s}) \leq \bar{x}(t) \leq \bar{x}(0) = \bar{y}_1 < y_1 < \epsilon \tag{4}$$

on $[-1,\bar{s}]$, and $\bar{X}([0,\bar{s}]) \subset [\bar{x}(\bar{s}), \bar{x}(0)]^2 = [\bar{x}(\bar{s}), \bar{y}_1]^2 \subset (-\epsilon,\epsilon)^2$. Let $\tilde{y} = \bar{X}(\bar{s}) = J_-(a,\bar{y})$. By Proposition 5.1 (v),

$$X^{a,\tilde{y}}(t) = \bar{X}(\bar{s} + t) \text{ for all } t \geq 0. \tag{5}$$

6. Proof of $(a, J_-(a,\bar{y})) \in \Delta_+$.

6.1. Assume the contrary. Then Proposition 5.2 yields $\bar{x}(\bar{s}) < \bar{x}(t) < 0$ on $(\bar{s},\infty)$ and $\lim_{t\to\infty} \bar{x}(t) = 0$. Choose a neighbourhood V of 0 in C with

$$F_a([0,\infty), V) \subset C(\bar{y}_1).$$

Choose $u > \bar{s}$ so that $\bar{x}_u \in V$. Recall that on $[\bar{s}, u]$, $\bar{x}(t) < 0$. By continuity there exists $\hat{y}_1 \in (\bar{y}_1, y_1)$ so that $\hat{y} = (\hat{y}_1, f(\hat{y}_1))$ satisfies $(a,\hat{y}) \in \Delta_-$, and for $\hat{s} = t_-(a,\hat{y})$,

$$\hat{s} < u \text{ and } x(s_1) < \hat{x}(\hat{s});$$

moreover the solution $\hat{x} = x^{a,\hat{\phi}}$ with $\hat{\phi} \in W^a$ and $E\hat{\phi} = \hat{y}$ satisfies

$$\hat{x}(t) < 0 \text{ on } [\hat{s}, u] \text{ and } \hat{x}_u \in V.$$

6.2. On $[-1, \hat{s})$ we obtain

$$-\epsilon < x(s_1) < \hat{x}(\hat{s}) < \hat{x}(t) \le \hat{x}(0) = \hat{y}_1 < y_1 < \epsilon,$$

by Proposition 5.2. In particular, $\hat{X} = X^{a,\hat{y}}$ satisfies $\hat{X}([0, \hat{s}]) \subset Q$. Using this and Proposition 5.1 (v) we infer that for $y^\circ = \hat{X}(\hat{s}) = J_-(a, \hat{y})$ and for all $t \ge 0$,

$$X^{a,y^\circ}(t) = \hat{X}(\hat{s} + t).$$

6.3. Suppose $(a, y^\circ) \in \Delta_+$. Then

$$J_+(a, J_-(a, \hat{y})) = J_+(a, y^\circ) = X^{a,y^\circ}(t_+(a, y^\circ)) = \hat{X}(\hat{s} + t_+(a, y^\circ)),$$

hence

$$J_{+,1}(a, J_-(a, \hat{y})) = \hat{x}(\hat{s} + t_+(a, y^\circ)) < \bar{y}_1$$

(by the choice of V)

$$< \hat{y}_1.$$

6.4. It follows that $\hat{y}_1 \notin M$, in contradiction to the definition of $\bar{y}_1$.

7. From $\bar{y}_1 \in (0, y_1) \setminus M$ we have

$$J_{+,1}(a, J_-(a, \bar{y})) \le \bar{y}_1.$$

Continuity and the definition of M combined yield

$$J_{+,1}(a, J_-(a, \bar{y})) = \bar{y}_1.$$

As $\bar{y} \in f$ and J_+ maps into f we obtain

$$J_+(a, J_-(a, \bar{y})) = \bar{y}.$$

On $[-1, t_+(a, \tilde{y})]$ we have

$$\begin{aligned} -\epsilon < x(s_1) < \bar{x}(\bar{s}) = \tilde{y}_1 = X_1^{a,\tilde{y}}(0) \le X_1^{a,\tilde{y}}(t) \le X_1^{a,\tilde{y}}(t_+(a, \tilde{y})) \\ = J_{+,1}(a, \tilde{y}) = \bar{y}_1 < y_1 < \epsilon, \end{aligned}$$

by the remark following Proposition 5.2. Set $\bar{t} = \bar{s} + t_+(a, \tilde{y})$. Using (5) we infer that on $[\bar{s}, \bar{t}]$,

$$\bar{x}(t) = X_1^{a,\tilde{y}}(t - \bar{s}) \in (-\epsilon, \epsilon).$$

Combining this with (4) we get $\bar{x}(t) \in (-\epsilon, \epsilon)$ on $[-1, \bar{t}]$. Proposition 5.1 (iv) gives

$$\bar{x}_{\bar{t}} \in W^a.$$

We have

$$\begin{aligned} E\bar{\phi} = \bar{y} = J_+(a, J_-(a, \bar{y})) = J_+(a, \tilde{y}) = X^{a,\tilde{y}}(t_+(a, \tilde{y})) \\ = \bar{X}(\bar{s} + t_+(a, \tilde{y})) = \bar{X}(\bar{t}) = E\bar{x}_{\bar{t}}, \end{aligned}$$

and the injectivity of E on W^a gives

$$\bar{x}_0 = \bar{\phi} = \bar{x}_{\bar{t}};$$

$\bar{x}$ extends to a periodic solution with the properties stated in $(\mathrm{P}_{\epsilon a})$. □

Proposition 3 *Let $\epsilon \in (0, \delta_0)$ and assume $y = y^{(\epsilon)}$ satisfies*

$$X_1^{1,y}(t_-(1, y) + t_+(1, J_-(1, y))) = y_1.$$

Then $(P_{\epsilon 1})$ holds.

Proof Recall Corollary 5.1. Consider $\phi \in W^1$ with $E\phi = y$. Set $\hat{y} = J_-(1, y)$ and $\tau = t_-(1, y) + t_+(1, \hat{y})$. Using Proposition 5.1 (iv) we infer $F_1(t, \phi) \in W^1$ on $[0, \tau]$. The previous hypothesis and the relations $y \in f$, $X^{1,y}(\tau) \in f$ yield

$$EF_1(\tau, \phi) = X^{1,y}(\tau) = y = E\phi.$$

By the injectivity of E on W^1, $F_1(\tau, \phi) = \phi$, which implies the assertion. □

It remains to show that the minimal periods of the periodic solutions found in Propositions 1–3 tend to infinity as $\epsilon \to 0$. This is a consequence of the next result.

Proposition 4 *Let $a > 0$. Suppose the solution $x : [-1, \infty) \to \mathbb{R}$ of Eq. (1.1) and $z > 0$ satisfy $x(0) \geq x(t) > 0$ on $(-1, z)$ and $x(z) = 0$. Then*

$$-\frac{\log x(0)}{a(1 - x(0))} \leq z.$$

Proof Let $m = x(0)$. On $(0, z)$,

$$\begin{aligned} a^{-1}\dot{x}(t) &= x(t) - x(t-1) - |x(t)|x(t) \geq x(t) - m - m\,x(t) \\ &= (1 - m)\left(x(t) - \frac{m}{1 - m}\right). \end{aligned}$$

Hence,

$$x(t) - \frac{m}{1 - m} \geq \left(x(0) - \frac{m}{1 - m}\right) \mathrm{e}^{a(1-m)t},$$

or

$$x(t) \geq \frac{m}{1 - m} + \frac{m}{1 - m}((1 - m) - 1)\mathrm{e}^{a(1-m)t} = \frac{m}{1 - m}\big(1 - m\,\mathrm{e}^{a(1-m)t}\big).$$

It follows that z is not smaller than the zero ζ of the decreasing function

$$t \mapsto \frac{m}{1 - m}\big(1 - m\,\mathrm{e}^{a(1-m)t}\big),$$

$$\zeta = -\frac{\log m}{a(1 - m)}.$$

□

Remark The analogue of Proposition 4 for solutions which have a zero $z > 0$ with $x(0) \leq x(t) < 0$ on $(-1, z)$ holds as well – apply Proposition 4 to the solution $-x$ of Eq. (1.1).

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