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Surfaces with high contact number and their characterization

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Abstract The notion of contact number $\mathfrak{c}_\#(M)$ of a Euclidean submanifold was introduced in an earlier article (Proc. Edinb. Math. Soc. 47:69–100, 2004) as the highest order of contact of geodesics and normal sections on the submanifold. It was proved in (Proc. Edinb. Math. Soc. 47:69–100, 2004) that the contact number relates closely with the notions of isotropic submanifolds and holomorphic curves. One important problem concerning contact number is to construct Euclidean submanifolds with high contact number. The purpose of this article is thus to construct Euclidean surfaces with high contact number and to provide simple geometric characterization of such surfaces.

Keywords Contact number · Pseudo-umbilical surface · Constant isotropic surface · Flat torus with high contact number · Normal section

Mathematics Subject Classification (2000) Primary 53C40, 53A10 Secondary 53B25, 53C42

1 Introduction

Let M be an n-dimensional submanifold in the Euclidean m-space $\mathbb{E}^m$. Denote by UM and $T^\perp M$ the unit tangent bundle and the normal bundle of M, respectively. For each $p \in M$ and $u \in U_pM$, there are two canonical unit speed curves on M associated with (p, u); one is the *geodesic* γ_u with $\gamma_u(0) = p$ and $\gamma_u'(0) = u$, the other is the *normal section* β_u defined as follows: Let $E(p, u)$ be the affine $(m - n + 1)$-subspace through p spanned by u and $T_p^\perp M$. The intersection of M and $E(p, u)$ gives rise to a unit speed curve $\beta_u(s)$ with $\beta_u(0) = p$ and $\beta_u'(0) = u$ (see [3] for details).

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It is well known that geodesics play very important role in differential geometry, calculus of variations, as well as in the theory of general relativity. On the other hand, normal sections have been studied by many geometers (see, for instance, [1, 3, 5–8, 10, 11]. In particular, C. U. Sánchez et al. showed that normal sections play some important roles in algebraic geometry as well as in differential geometry.

The curves γ_u and β_u are said to be in contact of order k if $\gamma_u^{(i)}(0) = \beta_u^{(i)}(0)$ for $i = 1, \ldots, k$, where $\gamma_u^{(i)}$ and $\beta_u^{(i)}$ denote the i-th derivatives of γ_u and β_u with respect to their arclength function. The submanifold is said to be *in contact of order* k if, for each (p, u), the geodesic γ_u and the normal section β_u at (p, u) are in contact of order k. If the submanifold M is in contact of order k for every natural number k, the *contact number* $\mathfrak{c}_\#(M)$ of M is defined to be ∞. Otherwise, the contact number $\mathfrak{c}_\#(M)$ of M is defined to be the largest natural number k such that M is in contact of order k and but not of order $k+1$ (see [4] for details).

From [4] we have the following.

Theorem A *For every submanifold M in a Euclidean space, we have:*

(1) The contact number $\mathfrak{c}_\#(M)$ of M is at least 2, i.e., $\mathfrak{c}_\#(M) \geq 2$.
(2) M is isotropic if and only if $\mathfrak{c}_\#(M) \geq 3$ holds.
(3) M is constant isotropic if and only if $\mathfrak{c}_\#(M) \geq 4$ holds.

Theorem B *A surface M in $\mathbb{E}^4$ is a non-planar holomorphic curve with respect to some orthogonal complex structure on $\mathbb{E}^4$ if and only if we have $\mathfrak{c}_\#(M) = 3$.*

Theorem C *A submanifold in a Euclidean space satisfies $\mathfrak{c}_\#(M) \geq k$ for some integer $k \geq 3$ if and only if each unit tangent vector u of M is an eigenvector of the shape operator $A_{(\bar{\nabla}^j h)(u^{j+2})}$ for $j = 0, \ldots, k-3$.*

Euclidean submanifolds with high contact number are not well understood. Very few such examples are known. Hence, one important problem concerning contact number is to construct nontrivial examples of Euclidean submanifolds with high contact number. The purpose of this article is thus to construct Euclidean surfaces with high contact number and to provide simple geometric characterization of such surfaces.

2 Preliminaries

Let M be an n-dimensional submanifold in $\mathbb{E}^m$. We choose a local field of orthonormal frames $\{e_1, \ldots, e_m\}$ in $\mathbb{E}^m$ such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M and hence $e_{n+1}, \ldots, e_m$ are normal to M. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M and $\mathbb{E}^m$, respectively. And let D denote the normal connection of M in $\mathbb{E}^m$.

We denote by $\omega^1, \ldots, \omega^m$ the field of dual frames. If we put $\tilde{\nabla} e_A = \Sigma_{B=1}^m \omega_A^B e_B$, $A, B = 1, \ldots, m$, then the structure equations of $\mathbb{E}^m$ are given by

$$d\omega^A = -\sum_{B=1}^m \omega_B^A \wedge \omega^B, \quad d\omega_B^A = -\sum_{C=1}^m \omega_C^A \wedge \omega_B^C, \quad \omega_A^B + \omega_B^A = 0. \tag{2.1}$$

Restricting these forms on M, we have $\omega^r = 0$ for $r = n+1, \ldots, m$. Thus, we get $0 = d\omega^r = -\Sigma_{i=1}^n \omega_i^r \wedge \omega^i$. Hence, by applying Cartan's lemma, we may write

$$\omega_i^r = \sum_{j=1}^{n} h_{ij}^r \omega^j, \quad h_{ij}^r = h_{ji}^r, \quad i, j = 1, \ldots n. \tag{2.2}$$

The second fundamental form h of M in $\mathbb{E}^m$ is given by $h = \Sigma h_{ij}^r \omega^i \omega^j e_r$.

For any two vectors x, y tangent to M and any vector ξ normal to M we have

$$\tilde{\nabla}_x y = \nabla_x y + h(x, y), \quad \tilde{\nabla}_x \xi = -A_\xi x + D_x \xi, \tag{2.3}$$

where A_ξ is the shape operator of M in $\mathbb{E}^m$ with respect to ξ. The second fundamental form and the shape operator are related by $\langle A_\xi x, y\rangle = \langle h(x, y), \xi\rangle$.

The covariant derivative $\bar{\nabla} h$ of h with respect to $TM \oplus T^\perp M$ is defined by

$$(\bar{\nabla}_x h)(y, z) = D_x h(y, z) - h(\nabla_x y, z) - h(y, \nabla_x z). \tag{2.4}$$

Sometimes we write $(\bar{\nabla}_x h)(y, z)$ as $(\bar{\nabla} h)(y, z, x)$ and we put $\bar{\nabla}^0 h = h$.

In general, the k-th $(k \geq 1)$ covariant derivative $\bar{\nabla}^k h$ of h is defined by

$$\begin{aligned}(\bar{\nabla}^k h)(x_1, x_2, \ldots, x_{k+2}) &= D_{x_{k+2}}((\bar{\nabla}^{k-1} h)(x_1, \ldots, x_{k+1})) \\ &- \sum_{i=1}^{k+1} (\bar{\nabla}^{k-1} h)(x_1, \ldots, \nabla_{x_{k+2}} x_i, \ldots, x_{k+1}).\end{aligned} \tag{2.5}$$

We simply denote $(\bar{\nabla}^q h)(\overbrace{x, \ldots, x}^{q+2 \text{ times}})$ by $(\bar{\nabla}^q h)(x^{q+2})$. It is well known that $\bar{\nabla}^k h$ is a normal-bundle-valued tensor field of type $(0, k+2)$.

The equations of Gauss, Codazzi and Ricci are given respectively by

$$R(x, y, z, w) = \langle h(x, w), h(y, z)\rangle - \langle h(x, z), h(y, w)\rangle, \tag{2.6}$$

$$(\bar{\nabla}_x h)(y, z) = (\bar{\nabla}_y h)(x, z), \tag{2.7}$$

$$R^D(x, y, \xi, \eta) = \langle [A_\xi, A_\eta](x), y\rangle, \tag{2.8}$$

where $R(x, y) = \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}$ and $R^D(x, y) = D_x D_y - D_y D_x - D_{[x,y]}$ are the curvature tensors of the tangent and normal bundles, respectively.

A submanifold M in a Riemannian manifold is said to be *isotropic* if, for each point $p \in M$, the length $\lambda = |h(u, u)|$ of the normal curvature vector $h(u, u)$ is independent of the choice of $u \in U_p M$. If $\lambda = |h(u, u)|$ is also independent of $p \in M$, then M is said to be *constant isotropic* (see, for instance, [9]).

We recall the following lemma due to O'Neill.

Lemma 1 ([9]) *A submanifold M is isotropic if and only if each $u \in UM$ is an eigenvector of $A_{h(u,u)}$.*

3 Flat surfaces with high contact number

The main result of this article is the following.

Theorem 1 *We have the following:*

(i) For any positive number a and integer $n \geq 2$, the map

$$\begin{aligned}\psi_{2n}^{a}(x,y)=\frac{1}{\sqrt{2n}\,a}\Bigg(&\cos\left(\sqrt{1+\cos\left(\frac{(2j-1)\pi}{2n}\right)}\,ax+\sqrt{1-\cos\left(\frac{(2j-1)\pi}{2n}\right)}\,ay\right),\\ &\sin\left(\sqrt{1+\cos\left(\frac{(2j-1)\pi}{2n}\right)}\,ax+\sqrt{1-\cos\left(\frac{(2j-1)\pi}{2n}\right)}\,ay\right),\\ &\cos\left(\sqrt{1+\cos\left(\frac{(2j-1)\pi}{2n}\right)}\,ax-\sqrt{1-\cos\left(\frac{(2j-1)\pi}{2n}\right)}\,ay\right),\\ &\sin\left(\sqrt{1+\cos\left(\frac{(2j-1)\pi}{2n}\right)}\,ax-\sqrt{1-\cos\left(\frac{(2j-1)\pi}{2n}\right)}\,ay\right)\Bigg)_{j=1,\ldots,n}\end{aligned} \tag{3.1}$$

defines an isometric immersion of a flat 2-torus $T_{2n,a}$ into $\mathbb{E}^{4n}$ whose contact number $\mathfrak{c}_{\#}(T_{2n,a})$ equals to $4n-2$. Moreover, the immersion has parallel mean curvature vector.

(ii) For any positive number a and integer $n \geq 1$, the map

$$\begin{aligned}\psi_{2n+1}^{a}(x,y)=\frac{1}{\sqrt{2n+1}\,a}\Bigg(&\cos(\sqrt{2}ax),\sin(\sqrt{2}ax),\\ &\Bigg(\cos\left(\sqrt{1+\cos\left(\frac{2j\pi}{2n+1}\right)}\,ax+\sqrt{1-\cos\left(\frac{2j\pi}{2n+1}\right)}\,ay\right),\\ &\sin\left(\sqrt{1+\cos\left(\frac{2j\pi}{2n+1}\right)}\,ax+\sqrt{1-\cos\left(\frac{2j\pi}{2n+1}\right)}\,ay\right),\\ &\cos\left(\sqrt{1+\cos\left(\frac{2j\pi}{2n+1}\right)}\,ax-\sqrt{1-\cos\left(\frac{2j\pi}{2n+1}\right)}\,ay\right),\\ &\sin\left(\sqrt{1+\cos\left(\frac{2j\pi}{2n+1}\right)}\,ax-\sqrt{1-\cos\left(\frac{2j\pi}{2n+1}\right)}\,ay\right)\Bigg)_{j=1,\ldots,n}\Bigg)\end{aligned} \tag{3.2}$$

defines an isometric immersion of a flat 2-torus $T_{2n+1,a}$ into $\mathbb{E}^{4n+2}$ with $\mathfrak{c}_{\#}(T_{2n+1,a})=4n$ which has parallel mean curvature vector.

(iii) Conversely, if M is a flat surface in $\mathbb{E}^{2k}$ $(k \geq 3)$ with $\mathfrak{c}_{\#}(M) \geq 2k-2$ and with parallel mean curvature vector, then M is either an open portion of a totally geodesic 2-plane or an open portion of the flat torus $T_{k,a}$ for some $a>0$ defined above and the immersion is congruent to the ψ_k^a defined either by (3.1) or by (3.2), according to $k=2n \geq 4$ or $k=2n+1 \geq 3$.

Proof (1) Let $k=2n$ be an even number ≥ 4 and let a be a positive number. Denote by $\mathbb{E}^2$ the Euclidean 2-plane equipped with the standard Euclidean metric $g=dx^2+dy^2$. Consider the map $\psi=\psi_{2n}^{a}:\mathbb{E}^2\to\mathbb{E}^{4n}$ defined by (3.1). It is easy to verify that ψ is an isometric immersion and ψ is doubly periodic with respect to x and y. Thus, ψ is an isometric immersion of a 2-torus, say $T_{2n,a}$ into $\mathbb{E}^{2k}$.

If we put $e_1 = \partial\psi/\partial x, e_2 = \partial\psi/\partial y$, then e_1, e_2 are orthonormal parallel vector fields. Let us put

$$\begin{aligned}
e_3 &= -a\psi,\ e_4 = \sqrt{2}\left(e_3 - \frac{\psi_{xx}}{a}\right),\ e_5 = \frac{\sqrt{2}}{a}\psi_{xy} \\
e_6 &= \frac{\sqrt{2}}{a}\frac{\partial e_4}{\partial x} - \psi_x, \quad e_7 = -\frac{\sqrt{2}}{a}\frac{\partial e_5}{\partial x} - \psi_y, \\
e_{2j} &= \frac{(-1)^{j-1}\sqrt{2}}{a}\frac{\partial e_{2j-2}}{\partial x} - e_{2j-4}, \\
e_{2j+1} &= \frac{(-1)^{j}\sqrt{2}}{a}\frac{\partial e_{2j-1}}{\partial x} - e_{2j-3}, \qquad j = 4, \ldots, 2n-1, \\
e_{4n} &= \frac{1}{a}\frac{\partial e_{4n-1}}{\partial x} - \frac{1}{\sqrt{2}}e_{4n-3}.
\end{aligned} \tag{3.3}$$

Then, by a direct computation, we know that $e_3, \ldots, e_{4n}$ are orthonormal normal vector fields of $T_{2n,a}$ in $\mathbb{E}^{4n}$ which satisfy

$$\begin{aligned}
&\psi_{xx} = ae_3 - \frac{a}{\sqrt{2}}e_4, \quad \psi_{xy} = \frac{a}{\sqrt{2}}e_5, \quad \psi_{yy} = ae_3 + \frac{a}{\sqrt{2}}e_4, \\
&\frac{\partial e_3}{\partial x} = -a\psi_x, \quad \frac{\partial e_3}{\partial y} = -a\psi_y, \\
&\frac{\partial e_4}{\partial x} = \frac{a}{\sqrt{2}}(\psi_x + e_6), \quad \frac{\partial e_4}{\partial y} = \frac{a}{\sqrt{2}}(e_7 - \psi_y), \\
&\frac{\partial e_5}{\partial x} = -\frac{a}{\sqrt{2}}(\psi_y + e_7), \quad \frac{\partial e_5}{\partial y} = \frac{a}{\sqrt{2}}(e_6 - \psi_x), \\
&\frac{\partial e_{2j}}{\partial x} = \frac{(-1)^j a}{\sqrt{2}}(e_{2j-2} + e_{2j+2}), \quad \frac{\partial e_{2j}}{\partial y} = \frac{a}{\sqrt{2}}(e_{2j+3} - e_{2j-1}), \\
&\frac{\partial e_{2j+1}}{\partial x} = \frac{(-1)^{j+1} a}{\sqrt{2}}(e_{2j-1} + e_{2j+3}), \quad \frac{\partial e_{2j+1}}{\partial y} = \frac{a}{\sqrt{2}}(e_{2j+2} - e_{2j-2}), \\
&\frac{\partial e_{4n-2}}{\partial x} = -\frac{a}{\sqrt{2}}e_{4n-4}, \quad \frac{\partial e_{4n-2}}{\partial y} = -\frac{a}{\sqrt{2}}e_{4n-3} + ae_{4n}, \\
&\frac{\partial e_{4n-1}}{\partial x} = \frac{a}{\sqrt{2}}e_{4n-3} + ae_{4n}, \quad \frac{\partial e_{4n-1}}{\partial y} = -\frac{a}{\sqrt{2}}e_{4n-4}, \\
&\frac{\partial e_{4n}}{\partial x} = -ae_{4n-1}, \quad \frac{\partial e_{4n}}{\partial y} = -ae_{4n-2},
\end{aligned} \tag{3.4}$$

where $j = 3, \ldots, 2n-2$. In particular, (3.4) implies that

$$h(e_1, e_1) = ae_3 - \frac{a}{\sqrt{2}}e_4,\ h(e_1, e_2) = \frac{a}{\sqrt{2}}e_5,\ h(e_2, e_2) = ae_3 + \frac{a}{\sqrt{2}}e_4. \tag{3.5}$$

Hence, the surface is constant isotropic. Using (3.4) we also know that

$$
\begin{aligned}
&D_{e_1}e_3 = D_{e_2}e_3 = 0,\\
&D_{e_1}e_4 = \frac{a}{\sqrt{2}}e_6, \quad D_{e_2}e_4 = \frac{a}{\sqrt{2}}e_7,\\
&D_{e_1}e_5 = -\frac{a}{\sqrt{2}}e_7, \quad D_{e_2}e_5 = \frac{a}{\sqrt{2}}e_6,\\
&D_{e_1}e_{2j} = \frac{(-1)^j a}{\sqrt{2}}(e_{2j-2}+e_{2j+2}), \quad D_{e_2}e_{2j} = \frac{a}{\sqrt{2}}(e_{2j+3}-e_{2j-1}),\\
&D_{e_1}e_{2j+1} = \frac{(-1)^{j+1} a}{\sqrt{2}}(e_{2j-1}+e_{2j+3}), \ D_{e_2}e_{2j+1} = \frac{a}{\sqrt{2}}(e_{2j+2}-e_{2j-2}),\\
&D_{e_1}e_{4n-2} = -\frac{a}{\sqrt{2}}e_{4n-4}, \ D_{e_2}e_{4n-2} = -\frac{a}{\sqrt{2}}e_{4n-3}+ae_{4n},\\
&D_{e_1}e_{4n-1} = \frac{a}{\sqrt{2}}e_{4n-3}+ae_{4n}, \ D_{e_2}e_{4n-1} = -\frac{a}{\sqrt{2}}e_{4n-4},\\
&D_{e_1}e_{4n} = -ae_{4n-1}, \ D_{e_2}e_{4n} = -ae_{4n-2}, \quad j = 3,\dots,2n-2.
\end{aligned} \tag{3.6}
$$

For any real number θ, let us put $e_\theta = \cos\theta e_1 + \sin\theta e_2$. Then, we have $\nabla_{e_\theta}e_\theta = 0$, since e_1, e_2 are parallel. From (3.6) we find

$$
\begin{aligned}
&h(e_\theta, e_\theta) = ae_3 - \frac{\cos 2\theta}{\sqrt{2}}ae_4 + \frac{\sin 2\theta}{\sqrt{2}}ae_5,\\
&D_{e_\theta}e_3 = 0, \quad D_{e_\theta}e_4 = \frac{a}{\sqrt{2}}(\cos\theta e_6 + \sin\theta e_7),\\
&D_{e_\theta}e_5 = \frac{a}{\sqrt{2}}(\sin\theta e_6 - \cos\theta e_7),\\
&D_{e_\theta}e_{2j} = \frac{a}{\sqrt{2}}((-1)^j\cos\theta(e_{2j-2}+e_{2j+2}) + \sin\theta(e_{2j+3}-e_{2j-1})),\\
&D_{e_\theta}e_{2j+1} = \frac{a}{\sqrt{2}}(\sin\theta(e_{2j+2}-e_{2j-2}) - (-1)^j\cos\theta(e_{2j-1}+e_{2j+3})),\\
&\qquad j = 3,\dots,2n-2,\\
&D_{e_\theta}e_{4n-2} = -\frac{a}{\sqrt{2}}(\cos\theta e_{4n-4} + \sin\theta e_{4n-3}) + a\sin\theta e_{4n},\\
&D_{e_\theta}e_{4n-1} = \frac{a}{\sqrt{2}}(\cos\theta e_{4n-3} - \sin\theta e_{4n-4}) + a\cos\theta e_{4n},\\
&D_{e_\theta}e_{4n} = -a\sin\theta e_{4n-2} - a\cos\theta e_{4n-1}.
\end{aligned} \tag{3.7}
$$

If we put

$$
\begin{aligned}
&E_{2j} = \cos(j\theta)e_{2j} + (-1)^{j-1}\sin(j\theta)e_{2j+1}, \quad j = 2,3,\dots,2n-1,\\
&E_{4n} = K_{4n} = \sqrt{2}\sin(2n\theta)e_{4n},\\
&K_{2j} = \cos((4n-j)\theta)e_{2j} + (-1)^j\sin((4n-j)\theta)e_{2j+1}, \ j = 2,\dots,2n-1,
\end{aligned} \tag{3.8}
$$

then we obtain from (3.7) and (3.8) that

$$\begin{aligned} D_{e_\theta} E_4 &= \frac{a}{\sqrt{2}} E_6, \\ D_{e_\theta} E_{2j} &= (-1)^j \frac{a}{\sqrt{2}} (E_{2j-2} + E_{2j+2}), \quad j = 3, 4, \ldots, 2n-2, \\ D_{e_\theta} E_{4n-2} &= -\frac{a}{\sqrt{2}} E_{4n-4} + \frac{a}{\sqrt{2}} E_{4n}, \\ D_{e_\theta} E_{4n} &= -\sqrt{2} a \sin(2n\theta) \left(\sin\theta e_{4n-2} + \cos\theta e_{4n-1}\right), \\ D_{e_\theta} K_{2j} &= (-1)^j \frac{a}{\sqrt{2}} (K_{2j-2} + K_{2j+2}), \quad j = 3, 4, \ldots, 2n-2, \\ D_{e_\theta} K_{4n-2} &= -\frac{a}{\sqrt{2}} (K_{4n-4} + E_{4n}). \end{aligned} \tag{3.9}$$

Moreover, from (3.4) and (3.8), we find

$$\begin{aligned} A_{E_4} e_\theta &= -\frac{a}{\sqrt{2}} e_\theta, \\ A_{K_4} e_\theta &= \frac{a}{\sqrt{2}} (\cos((4n-1)\theta) e_1 - \sin((4n-1)\theta) e_2). \\ A_{E_6} e_\theta &= \cdots = A_{E_{4n}} e_\theta = A_{K_6} e_\theta = \cdots = A_{K_{4n}} e_\theta = 0. \end{aligned} \tag{3.10}$$

Applying (2.5), (3.7), (3.8), (3.9) and $\nabla_{e_\theta} e_\theta = 0$, we obtain

$$\begin{aligned} (\nabla h)(e_\theta^3) &= -\frac{a^2}{2} E_6, \\ (\nabla^\ell h)\left(e_\theta^{\ell+2}\right) &= (-1)^{[\ell/2]-1} \left(\frac{a}{\sqrt{2}}\right)^{\ell+1} \left\{ \sum_{t=1}^{[\ell/2]} \left(\binom{\ell}{t} - \binom{\ell}{t-1} \right) E_{2\ell-4t+4} + E_{2\ell+4} \right\}, \\ (\nabla^s h)\left(e_\theta^{s+2}\right) &= (-1)^{[s/2]-1} \left(\frac{a}{\sqrt{2}}\right)^{s+1} \left\{ \sum_{t=[s/2]-n+2}^{[s/2]} \left(\binom{s}{t} - \binom{s}{t-1} \right) E_{2s-4t+4} \right. \\ &\quad \left. - \sum_{t=1}^{[s/2]-n+1} \left(\binom{s}{t} - \binom{s}{t-1} \right) K_{8n+4t-2s-4} - K_{8n-2s-4} \right\}, \end{aligned} \tag{3.11}$$

where $\ell = 2, \ldots, 2n-3$; $s = 2n-2, \ldots, 4n-4$, and $[\ell/2]$ denotes the largest integer $\leq \ell/2$. Using (3.10) and (3.11) we find

$$\begin{aligned} A_{h(e_\theta, e_\theta)} e_\theta &= \frac{a^2}{2} e_\theta, \\ A_{\bar\nabla^{2j-1} h\left(e_\theta^{2j+1}\right)} e_\theta &= 0, \quad j = 1, \ldots, 2n-2, \end{aligned}$$

$$A_{\bar{\nabla}^{2j}h\left(e_\theta^{2j+2}\right)}e_\theta = \left(\frac{-a^2}{2}\right)^{j+1}\left(\binom{2j}{j-1} - \binom{2j}{j}\right)e_\theta, \ j = 1, \ldots, 2n-3,$$
$$A_{\bar{\nabla}^{4n-4}h\left(e_\theta^{4n-2}\right)}e_\theta = \left(\frac{a^2}{2}\right)^{2n-1}\left\{\left(\binom{4n-4}{2n-2} - \binom{4n-4}{2n-3}\right)e_\theta - \cos(4n-1)\theta e_1 + \sin(4n-1)\theta e_2\right\} \tag{3.12}$$

for each $e_\theta = \cos\theta e_1 + \sin\theta e_2$. Therefore, e_θ is an eigenvector of $A_{\bar{\nabla}^t h(e_\theta^{t+2})}$ for $t = 1, \ldots, 4n-5$. Moreover, since $n \geq 2$, the last equation of (3.12) implies that not every e_θ is an eigenvector of $A_{\bar{\nabla}^{4n-4}h(e_\theta^{4n-2})}$. Consequently, by applying Theorem C, we conclude that the contact number of M is equal to $2k-2$.

Moreover, it follows from (3.5) and (3.6) that the immersion has parallel mean curvature vector. This proves statement (i).

(2) Let $k = 2n+1 \geq 3$. Consider the map $\psi^a_{2n+1} : \mathbb{E}^2 \to \mathbb{E}^{4n+2}$ defined by (3.2). Then, ψ^a_{2n+1} is an isometric immersion. As in the proof of statement (i), we simply denote ψ^a_{2n+1} by ψ and put $e_1 = \partial\psi/\partial x, e_2 = \partial\psi/\partial y$. Then, e_1, e_2 are orthonormal parallel vector fields. Moreover, ψ induces an isometric immersion of a 2-torus, say $T_{2n+1,a}$, into $\mathbb{E}^{4n+2}$. Let us put

$$\begin{aligned}
e_3 &= -a\psi, \ e_4 = \sqrt{2}\left(e_3 - \frac{\psi_{xx}}{a}\right), \ e_5 = \frac{\sqrt{2}}{a}\psi_{xy}, \\
e_6 &= \frac{\sqrt{2}}{a}\frac{\partial e_4}{\partial x} - \psi_x, \quad e_7 = -\frac{\sqrt{2}}{a}\frac{\partial e_5}{\partial x} - \psi_y, \\
e_{2j} &= \frac{(-1)^{j-1}\sqrt{2}}{a}\frac{\partial e_{2j-2}}{\partial x} - e_{2j-4}, \\
e_{2j+1} &= \frac{(-1)^{j}\sqrt{2}}{a}\frac{\partial e_{2j-1}}{\partial x} - e_{2j-3}, \qquad j = 4, \ldots, 2n, \\
e_{4n+2} &= \frac{1}{a}\frac{\partial e_{4n}}{\partial x} - \frac{1}{\sqrt{2}}e_{4n-2}.
\end{aligned} \tag{3.13}$$

Then, $e_3, \ldots, e_{4n+2}$ are orthonormal normal vector fields satisfying

$$\begin{aligned}
&\psi_{xx} = ae_3 - \frac{a}{\sqrt{2}}e_4, \quad \psi_{xy} = \frac{a}{\sqrt{2}}e_5, \quad \psi_{yy} = ae_3 + \frac{a}{\sqrt{2}}e_4, \\
&\frac{\partial e_3}{\partial x} = -a\psi_x, \quad \frac{\partial e_3}{\partial y} = -a\psi_y, \\
&\frac{\partial e_4}{\partial x} = \frac{a}{\sqrt{2}}(\psi_x + e_6), \quad \frac{\partial e_4}{\partial y} = \frac{a}{\sqrt{2}}(e_7 - \psi_y), \\
&\frac{\partial e_5}{\partial x} = -\frac{a}{\sqrt{2}}(\psi_y + e_7), \quad \frac{\partial e_5}{\partial y} = \frac{a}{\sqrt{2}}(e_6 - \psi_x),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial e_{2j}}{\partial x} &= \frac{(-1)^j a}{\sqrt{2}}(e_{2j-2}+e_{2j+2}), \quad \frac{\partial e_{2j}}{\partial y} = \frac{a}{\sqrt{2}}(e_{2j+3}-e_{2j-1}),\\
\frac{\partial e_{2j+1}}{\partial x} &= \frac{(-1)^{j+1} a}{\sqrt{2}}(e_{2j-1}+e_{2j+3}), \quad \frac{\partial e_{2j+1}}{\partial y} = \frac{a}{\sqrt{2}}(e_{2j+2}-e_{2j-2}),\\
&\qquad j = 3, \dots, 2n-1, \\
\frac{\partial e_{4n}}{\partial x} &= \frac{a}{\sqrt{2}}e_{4n-2}+ae_{4n+2}, \quad \frac{\partial e_{4n}}{\partial y} = -\frac{a}{\sqrt{2}}e_{4n-1},\\
\frac{\partial e_{4n+1}}{\partial x} &= -\frac{a}{\sqrt{2}}e_{4n-1}, \quad \frac{\partial e_{4n+1}}{\partial y} = -\frac{a}{\sqrt{2}}e_{4n-2}+ae_{4n+2},\\
\frac{\partial e_{4n+2}}{\partial x} &= -ae_{4n}, \quad \frac{\partial e_{4n+2}}{\partial y} = -ae_{4n+1}.
\end{aligned} \tag{3.14}$$

From (3.14) we may obtain

$$\begin{aligned}
&D_{e_1}e_3 = D_{e_2}e_3 = 0, \quad D_{e_1}e_4 = \frac{a}{\sqrt{2}}e_6, \quad D_{e_2}e_4 = \frac{a}{\sqrt{2}}e_7,\\
&D_{e_1}e_5 = -\frac{a}{\sqrt{2}}e_7, \quad D_{e_2}e_5 = \frac{a}{\sqrt{2}}e_6,\\
&D_{e_1}e_{2j} = \frac{(-1)^j a}{\sqrt{2}}(e_{2j-2}+e_{2j+2}), \quad D_{e_2}e_{2j} = \frac{a}{\sqrt{2}}(e_{2j+3}-e_{2j-1}),\\
&D_{e_1}e_{2j+1} = \frac{(-1)^{j+1} a}{\sqrt{2}}(e_{2j-1}+e_{2j+3}), \quad D_{e_2}e_{2j+1} = \frac{a}{\sqrt{2}}(e_{2j+2}-e_{2j-2}),\\
&D_{e_1}e_{4n} = \frac{a}{\sqrt{2}}e_{4n-2} = ae_{4n+2}, \quad D_{e_2}e_{4n} = -\frac{a}{\sqrt{2}}e_{4n-1},\\
&D_{e_1}e_{4n+1} = \frac{a}{\sqrt{2}}e_{4n-1}, \quad D_{e_2}e_{4n+1} = -\frac{a}{\sqrt{2}}e_{4n-2}+ae_{4n+2},\\
&D_{e_1}e_{4n} = -ae_{4n}, \quad D_{e_2}e_{4n+2} = -ae_{4n+1}, \quad j = 3, \dots, 2n-1.
\end{aligned} \tag{3.15}$$

where $j = 3, \dots, 2n-2$. Thus, we also have

$$\begin{aligned}
&h(e_\theta, e_\theta) = ae_3 - \frac{\cos 2\theta}{\sqrt{2}}ae_4 + \frac{\sin 2\theta}{\sqrt{2}}ae_5,\\
&D_{e_\theta}e_3 = 0, \quad D_{e_\theta}e_4 = \frac{a}{\sqrt{2}}(\cos\theta e_6 + \sin\theta e_7),\\
&D_{e_\theta}e_5 = \frac{a}{\sqrt{2}}(\sin\theta e_6 - \cos\theta e_7),\\
&D_{e_\theta}e_{2j} = \frac{a}{\sqrt{2}}((-1)^j\cos\theta(e_{2j-2}+e_{2j+2}) + \sin\theta(e_{2j+3}-e_{2j-1})),\\
&D_{e_\theta}e_{2j+1} = \frac{a}{\sqrt{2}}(\sin\theta(e_{2j+2}-e_{2j-2}) - (-1)^j\cos\theta(e_{2j-1}+e_{2j+3})),\\
&\qquad j = 3, \dots, 2n-1,
\end{aligned} \tag{3.16}$$

$$
\begin{aligned}
D_{e_\theta} e_{4n} &= \frac{a}{\sqrt{2}}(\cos\theta e_{4n-2} - \sin\theta e_{4n-1}) + a\cos\theta e_{4n+2},\\
D_{e_\theta} e_{4n+1} &= -\frac{a}{\sqrt{2}}(\cos\theta e_{4n-1} + \sin\theta e_{4n-2}) + a\sin\theta e_{4n+2},\\
D_{e_\theta} e_{4n+2} &= -a\cos\theta e_{4n} - a\sin\theta e_{4n+1}.
\end{aligned}
$$

If we put

$$
\begin{aligned}
E_{2j} &= \cos(j\theta)e_{2j} + (-1)^{j-1}\sin(j\theta)e_{2j+1},\\
E_{4n+2} = K_{4n+2} &= \sqrt{2}\cos((2n+1)\theta)e_{4n+2},\\
K_{2t} &= \cos((4n-t+2)\theta)e_{2t} + (-1)^t\sin((4n-t+2)\theta)e_{2t+1}
\end{aligned}
\tag{3.17}
$$

for $j = 2, \ldots, 2n;\ t = 2, \ldots, 2n$, then (3.16) and (3.17) imply that

$$
\begin{aligned}
D_{e_\theta} E_4 &= \frac{a}{\sqrt{2}} E_6,\\
D_{e_\theta} E_{2j} &= (-1)^j \frac{a}{\sqrt{2}}(E_{2j-2} + E_{2j+2}), \quad j = 3, 4, \ldots, 2n,\\
D_{e_\theta} E_{4n+2} &= -\sqrt{2}a\cos((2n+1)\theta)\left(\cos\theta e_{4n} + \sin\theta e_{4n+1}\right),\\
D_{e_\theta} K_{2j} &= (-1)^j \frac{a}{\sqrt{2}}(K_{2j-2} + K_{2j+2}), \quad j = 3, 4, \ldots, 2n.
\end{aligned}
\tag{3.18}
$$

Moreover, we obtain from (3.14) and (3.17) that

$$
\begin{aligned}
A_{E_4} e_\theta &= -\frac{a}{\sqrt{2}} e_\theta,\\
A_{K_4} e_\theta &= \frac{a}{\sqrt{2}}(-\cos((4n+1)\theta)e_1 + \sin((4n+1)\theta)e_2),\\
A_{E_6} e_\theta &= \cdots = A_{E_{4n+2}} e_\theta = A_{K_6} e_\theta = \cdots = A_{K_{4n+2}} e_\theta = 0.
\end{aligned}
\tag{3.19}
$$

After long computation we obtain from (2.5), (3.14), (3.18), (3.19) and $\nabla_{e_\theta} e_\theta = 0$ that

$$
\begin{aligned}
(\nabla h)\big(e_\theta^3\big) &= -\left(\frac{a}{\sqrt{2}}\right)^2 E_6,\\
(\nabla^j h)\big(e_\theta^{j+2}\big) &= (-1)^{[j/2]-1}\left(\frac{a}{\sqrt{2}}\right)^{j+1} \left\{ \sum_{t=1}^{[j/2]} \left(\binom{j}{t} - \binom{j}{t-1} \right) E_{2j-4t+4} \right.\\
&\qquad \left. + E_{2j+4} \right\}, \quad j = 2, \ldots, 2n-1,\\
(\nabla^r h)\big(e_\theta^{r+2}\big) &= (-1)^{[r/2]-1}\left(\frac{a}{\sqrt{2}}\right)^{r+1} \left\{ \sum_{t=[r/2]-n+1}^{[r/2]} \left(\binom{r}{t} - \binom{r}{t-1} \right) E_{2r-4t+4} \right.\\
&\qquad \left. + \sum_{t=1}^{[r/2]-n} \left(\binom{r}{t} - \binom{r}{t-1} \right) K_{8n+4t-2r} + K_{8n-2r} \right\},\\
r &= 2n, \ldots, 4n-2.
\end{aligned}
\tag{3.20}
$$

By applying (3.19) and (3.20) we find

$$
\begin{aligned}
A_{h(e_\theta,e_\theta)}e_\theta &= \frac{a^2}{2}e_\theta,\\
A_{\bar\nabla^{2j-1}h\left(e_\theta^{2j+1}\right)}e_\theta &= 0, \quad j=1,\dots,2n-2,\\
A_{\bar\nabla^{2j}h\left(e_\theta^{2j+2}\right)}e_\theta &= \left(\frac{-a^2}{2}\right)^{j+1}\left(\binom{2j}{j-1}-\binom{2j}{j}\right)e_\theta, \quad j=1,\dots,2n-2,\\
A_{\bar\nabla^{4n-2}h\left(e_\theta^{4n}\right)}e_\theta &= -\left(\frac{a^2}{2}\right)^{2n}\left\{\left(\binom{4n-2}{2n-1}-\binom{4n-2}{2n-2}\right)e_\theta\right.\\
&\qquad\left. +\cos((4n+1)\theta)e_1-\sin((4n+1)\theta)e_2\right\},\\
e_\theta &= \cos\theta e_1+\sin\theta e_2.
\end{aligned}
\tag{3.21}
$$

So e_θ is an eigenvector of $A_{\bar\nabla^t h(e_\theta^{t+2})}$ for $t=1,\dots,4n-3$. The last equation in (3.21) shows that e_θ is not an eigenvector of $A_{\bar\nabla^{4n-2}h(e_\theta^{4n})}$ in general. Thus, Theorem C implies that the contact number is equal to $4n=2k-2$. It follows from (3.15) that the immersion has parallel mean curvature vector. This proves statement (ii).

(3) If $k=3$, statement (iii) follows from Theorem 4 of [4]. So, we only need to prove statement (iii) for $k\geq 4$. In order to do so, we need the following lemma.

Lemma 2 *Let $\phi: M\to\mathbb{E}^m$ $(k\geq 4)$ be an isometric immersion of a flat surface M into Euclidean m-space with parallel mean curvature vector. If the contact number of M satisfies $\mathfrak{c}_\#(M)\geq 2k-2$ with $k\geq 4$, then there exists a coordinate system $\{x,y\}$ and orthonormal normal vector fields $e_3,\dots,e_m$ on M satisfying*

$$
\begin{aligned}
&\phi_{xx}=ae_3-\frac{a}{\sqrt2}e_4,\quad \phi_{xy}=\frac{a}{\sqrt2}e_5,\quad \phi_{yy}=ae_3+\frac{a}{\sqrt2}e_4,\quad H=ae_3\\
&D_{e_1}e_3=D_{e_2}e_3=0,\quad D_{e_1}e_4=\frac{a}{\sqrt2}e_6,\quad D_{e_2}e_4=\frac{a}{\sqrt2}e_7,\\
&D_{e_1}e_5=-\frac{a}{\sqrt2}e_7,\quad D_{e_2}e_5=\frac{a}{\sqrt2}e_6,\\
&D_{e_1}e_{2j}=\frac{(-1)^j a}{\sqrt2}(e_{2j-2}+e_{2j+2}),\quad D_{e_2}e_{2j}=-\frac{a}{\sqrt2}(e_{2j-1}-e_{2j+3}),\\
&D_{e_1}e_{2j+1}=\frac{(-1)^{j+1}a}{\sqrt2}(e_{2j-1}+e_{2j+3}),\ D_{e_2}e_{2j+1}=-\frac{a}{\sqrt2}(e_{2j-2}-e_{2j+2}),\\
&D_{e_1}e_{2k-2}=(-1)^{k-1}\left(\frac{a}{\sqrt2}e_{2k-4}+a\delta e_{2k}\right),\\
&D_{e_2}e_{2k-2}=-\frac{a}{\sqrt2}e_{2k-3}+a\eta e_{2k}+a\varphi e_{2k+1},\\
&D_{e_1}e_{2k-1}=(-1)^k a\left(\frac{1}{\sqrt2}e_{2k-3}+\eta e_{2k}+\varphi e_{2k+1}\right),\\
&D_{e_2}e_{2k-1}=-\frac{a}{\sqrt2}e_{2k-4}+a\delta e_{2k}
\end{aligned}
\tag{3.22}
$$

for $j=3,\dots,k-2$, where δ,η,φ are functions, $e_1=\partial/\partial x$ and $e_2=\partial/\partial y$.

Proof of Lemma 2. Let $\phi : M \to \mathbb{E}^m$ be an isometric immersion of a flat surface with parallel mean curvature vector and $\mathfrak{c}_\#(M) \geq 2k-2$. Then, Lemma 2 and Lemma 3 of [5] and Theorem A imply that M is a pseudo-umbilical constant isotropic surface. Also, from Theorem C we know that each unit vector e_θ is an eigenvector of $A_{(\bar{\nabla}^j h)(e_\theta^{j+2})}$ for $j = 0, 1, 2, \ldots, 2k-5$.

When M is minimal in $\mathbb{E}^m$, M is totally geodesic according to the equation of Gauss. In this case, M is an open part of a totally geodesic 2-plane with contact number ∞.

Now, let us assume that M is a non-minimal surface in $\mathbb{E}^m$. Since M a pseudo-umbilical surface with nonzero parallel mean curvature vector H, we may choose e_3 so that (a) the mean curvature vector H is equal to ae_3 with $a = |H| > 0$ and (b) $De_3 = 0$. Also, we may choose e_4 satisfying $h(e_1, e_1) = ae_3 - be_4$.

Since $h(e_2, e_2) = 2H - h(e_1, e_1)$ and $|h(e_1, e_1)| = |h(e_2, e_2)|$ with $H \neq 0$, we have $h(e_2, e_2) = ae_3 + be_4$ for some function b. Because M is isotropic, we obtain from Lemma 1 that

$$\langle h(e_1, e_2), h(e_1, e_1)\rangle = \langle h(e_1, e_2), h(e_2, e_2)\rangle = 0. \tag{3.23}$$

So we may choose e_5 so that $h(e_1, e_2) = \delta e_5$ for some function δ. Hence, we have

$$h(e_1, e_1) = ae_3 + be_4, \quad h(e_2, e_2) = ae_3 - be_4, \quad h(e_1, e_2) = \delta e_5. \tag{3.24}$$

If we put $e_\theta = \cos\theta e_1 + \sin\theta e_2$, then (3.23) and (3.24) imply that

$$|h(e_\theta, e_\theta)|^2 = a^2 + b^2 + 4\cos^2\theta \sin^2\theta(\delta^2 - b^2). \tag{3.25}$$

Because M is isotropic, (3.25) yields $\delta^2 = b^2$. Moreover, it follows from the flatness and Gauss' equation that $a^2 = 2b^2$. Therefore, we may choose $e_1, \ldots, e_{2k}$ which satisfy

$$h(e_1, e_1) = ae_3 - \frac{a}{\sqrt{2}}e_4, \quad h(e_2, e_2) = ae_3 + \frac{a}{\sqrt{2}}e_4, \quad h(e_1, e_2) = \frac{a}{\sqrt{2}}e_5. \tag{3.26}$$

In particular, we know from (3.26) that each first normal space is a 3-dimensional space spanned by e_3, e_4, e_5. Because M is flat, there is a local coordinate system $\{x, y\}$ so that $g = dx^2 + dy^2$. Hence, if we put $e_1 = \partial/\partial x$ and $e_2 = \partial/\partial y$, we get $\omega_1^2 = 0$. Combining this with (2.5) and (3.26) gives

$$\begin{aligned} &h(e_\theta, e_\theta) = ae_3 - \frac{a}{\sqrt{2}}(\cos 2\theta e_4 - \sin 2\theta e_5), \\ &(\bar{\nabla}^\ell h)\big(e_\theta^{\ell+2}\big) = D_{e_\theta}\big((\bar{\nabla}^{\ell-1}h)\big(e_\theta^{\ell+1}\big)\big), \quad \ell = 1, 2, \ldots. \end{aligned} \tag{3.27}$$

From (2.4), (3.26), (3.27) and $De_3 = 0$, we find

$$\begin{aligned} (\bar{\nabla}_{e_j}h)(e_1, e_1) &= -\frac{a}{\sqrt{2}}\sum_{r=5}^m \omega_4^r(e_j)e_r, \\ (\bar{\nabla}_{e_j}h)(e_1, e_2) &= \frac{a}{\sqrt{2}}\sum_{r=4}^m \omega_5^r(e_j)e_r, \\ (\bar{\nabla}_{e_j}h)(e_2, e_2) &= \frac{a}{\sqrt{2}}\sum_{r=3}^m \omega_4^r(e_j)e_r, \quad j = 1, 2. \end{aligned} \tag{3.28}$$

Applying (3.28) and the equation of Codazzi, we get

$$\omega_4^5 = 0, \quad \omega_4^r(e_2) = -\omega_5^r(e_1), \quad \omega_4^r(e_1) = \omega_5^r(e_2), \quad r = 6, \dots, m. \tag{3.29}$$

It follows from (3.28) and (3.29) that

$$(\bar{\nabla} h)\big(e_\theta^3\big) = a \sum_{r=6}^{m} F_\theta^r e_r, \quad F_\theta^r = -\frac{\cos 3\theta}{\sqrt{2}} \omega_4^r(e_1) - \frac{\sin 3\theta}{\sqrt{2}} \omega_4^r(e_2). \tag{3.30}$$

So, by (2.5), (3.23), (3.26), (3.27), and (3.30), we have

$$(\bar{\nabla}^2 h)\big(e_\theta^4\big) = a \sum_{t=4}^{5} \sum_{r=6}^{m} F_\theta^r \left(\cos\theta\, \omega_r^t(e_1) + \sin\theta\, \omega_r^t(e_2)\right) e_t + \sum_{s=6}^{m} f_s e_s, \tag{3.31}$$

for some functions $f_6, \dots, f_{2k}$. Therefore, we have

$$\begin{aligned} A_{(\bar{\nabla}^2 h)\left(e_\theta^4\right)} e_\theta = \frac{a^2}{\sqrt{2}} \sum_{r=6}^{m} F_\theta^r \big\{ & \big(\cos 2\theta\, \omega_4^r(e_1) + \sin 2\theta\, \omega_4^r(e_2)\big) e_1 \\ & + \big(\cos 2\theta\, \omega_4^r(e_2) - \sin 2\theta\, \omega_4^r(e_1)\big) e_2 \big\} \end{aligned} \tag{3.32}$$

Since e_θ is an eigenvector of $A_{(\bar{\nabla}^2 h)(e_\theta^4)}$ by assumption, we have

$$\big\langle A_{(\bar{\nabla}^2 h)(e_\theta^4)} e_\theta, \sin\theta\, e_1 - \cos\theta\, e_2 \big\rangle = 0.$$

Thus, by applying (3.30) and (3.32), we obtain

$$\sum_{r=6}^{m} \big\{ 2\omega_4^r(e_1)\omega_4^r(e_2) \cos 6\theta + \big(\omega_4^r(e_2)^2 - \omega_4^r(e_1)^2\big) \sin 6\theta \big\} = 0. \tag{3.33}$$

Since this equation holds for every θ, we obtain

$$\sum_{r=6}^{2k} \omega_4^r(e_1)\omega_4^r(e_2) = 0, \quad \sum_{r=6}^{2k} \omega_4^r(e_1)^2 = \sum_{r=6}^{m} \omega_4^r(e_2)^2. \tag{3.34}$$

By taking the derivative of $\omega_4^5 = 0$ and applying (3.29), we find

$$\sum_{r=6}^{m} \omega_4^r(e_1)^2 + \sum_{r=6}^{m} \omega_4^r(e_2)^2 = a^2. \tag{3.35}$$

Thus, by (3.34), and (3.35), we discover that

$$\langle D_{e_1} e_4, D_{e_2} e_4 \rangle = 0, \quad |D_{e_1} e_4| = |D_{e_2} e_4| = \frac{a}{\sqrt{2}}. \tag{3.36}$$

Also, from (3.29), we have $D_{e_2} e_4 = -D_{e_1} e_5$, $D_{e_1} e_4 = D_{e_2} e_5$. Hence, we may choose $e_6, \dots, e_{2k}$ so that

$$D_{e_1} e_4 = D_{e_2} e_5 = \frac{a}{\sqrt{2}} e_6, \quad D_{e_2} e_4 = -D_{e_1} e_5 = \frac{a}{\sqrt{2}} e_7. \tag{3.37}$$

From the equation of Ricci, (3.23) and (3.37) we find

$$0 = R^D(e_1, e_2, e_4, e_r) = \frac{a}{\sqrt{2}}\big(\omega_7^r(e_1) - \omega_6^r(e_2)\big), \quad r = 6, \dots, m, \tag{3.38}$$

which ensures that $\omega_6^r(e_2) = \omega_7^r(e_1)$ for $r = 6, \dots, m$. Similarly, it follows from $0 = R^D(e_1, e_2, e_5, e_r)$, $r = 6, \dots, m$, that $\omega_6^r(e_1) = -\omega_7^r(e_2)$. Combining these yields $\omega_6^7 = 0$. Therefore, we may choose $e_6, \dots, e_m$ such that

$$\begin{aligned}
&De_3 = 0, \quad D_{e_1}e_4 = \frac{a}{\sqrt{2}}e_6, \quad D_{e_2}e_4 = \frac{a}{\sqrt{2}}e_7\\
&D_{e_1}e_5 = -\frac{a}{\sqrt{2}}e_7, \quad D_{e_2}e_5 = \frac{a}{\sqrt{2}}e_6,\\
&D_{e_1}e_6 = -\frac{a}{\sqrt{2}}e_4 - a\alpha e_8, \quad D_{e_2}e_6 = -\frac{a}{\sqrt{2}}e_5 + a\beta e_8 + a\gamma e_9,\\
&D_{e_1}e_7 = \frac{a}{\sqrt{2}}e_5 + a\beta e_8 + a\gamma e_9, \quad D_{e_2}e_7 = -\frac{a}{\sqrt{2}}e_4 + a\alpha e_8.
\end{aligned} \tag{3.39}$$

If $k = 4$, this is (3.22) with $\delta = a\alpha$, $\eta = a\beta$, $\varphi = a\gamma$. Next, let us assume that $k \geq 5$. Then, we have $c_\#(M) \geq 8$ according to the hypothesis. Thus, Theorem C implies that e_θ is an eigenvector of $A_{(\bar\nabla^j h)(e_\theta^{j+2})}$, $j = 0, 1, \dots, 5$.

From (3.30), (3.31), and (3.39) we get

$$(\bar\nabla h)\big(e_\theta^3\big) = -\frac{a^2}{2}(\cos 3\theta e_6 + \sin 3\theta e_7), \tag{3.40}$$

Hence, by (3.27), (3.39), and (3.40), we find

$$\begin{aligned}
(\bar\nabla^2 h)\big(e_\theta^4\big) = {} & \frac{a^3}{2\sqrt{2}}(\cos 2\theta e_4 - \sin 2\theta e_5)\\
& + \frac{a^3}{2}(\alpha \cos 4\theta - \beta \sin 4\theta)e_8 - \frac{a^3}{2}\gamma \sin 4\theta e_9,
\end{aligned} \tag{3.41}$$

which implies that $A_{(\bar\nabla^2 h)(e_\theta^4)}e_\theta = -(a^4/4)e_\theta$. In views of (3.39) we may put

$$\begin{aligned}
D_{e_1}e_8 &= a\alpha e_6 - a\beta e_7 + \omega_8^9(e_1)e_9 + \sum_{r=10}^m \omega_8^r(e_1)e_r,\\
D_{e_2}e_8 &= -a\beta e_6 - a\alpha e_7 + \omega_8^9(e_2)e_9 + \sum_{r=10}^m \omega_8^r(e_2)e_r,\\
D_{e_1}e_9 &= -a\gamma e_7 - \omega_8^9(e_1)e_8 + \sum_{r=10}^m \omega_9^r(e_1)e_r,\\
D_{e_2}e_9 &= -a\gamma e_6 - \omega_8^9(e_2)e_8 + \sum_{r=10}^m \omega_9^r(e_2)e_r.
\end{aligned} \tag{3.42}$$

Using $\omega_1^2 = 0$, (3.39), (3.41), and (3.42) we discover that

$$(\bar{\nabla}^3 h)\big(e_\theta^5\big) = \frac{a^4}{4}\big\{(1+\alpha^2+\beta^2+\gamma^2)\cos 3\theta + (\alpha^2-\beta^2-\gamma^2)\cos 5\theta$$
$$- 2\alpha\beta\sin 5\theta\big\}e_6 + \frac{a^4}{4}\big\{(1+\alpha^2+\beta^2+\gamma^2)\sin 3\theta$$
$$- (\alpha^2-\beta^2-\gamma^2)\sin 5\theta - 2\alpha\beta\cos 5\theta\big\}e_7$$
$$+ \frac{a^3}{2}\big\{\gamma(\omega_8^9(e_1)\cos\theta + \omega_8^9(e_2)\sin\theta)\sin 4\theta$$
$$+ (\alpha_u\cos\theta + \alpha_v\sin\theta)\cos 4\theta - (\beta_u\cos\theta + \beta_v\sin\theta)\sin 4\theta)\big\}e_8$$
$$+ \frac{a^3}{2}\big\{(\alpha\cos 4\theta - \beta\sin 4\theta)\big(\omega_8^9(e_1)\cos\theta + \omega_8^9(e_2)\sin\theta\big)$$
$$- (\gamma_u\cos\theta + \gamma_v\sin\theta)\sin 4\theta\big\}e_9$$
$$+ \frac{a^3}{2}\sum_{r=10}^{m}\big\{(\alpha\cos 4\theta - \beta\sin 4\theta)\big(\omega_8^r(e_1)\cos\theta + \omega_8^r(e_2)\sin\theta\big)$$
$$- \gamma\big(\omega_9^r(e_1)\cos\theta + \omega_9^r(e_2)\sin\theta\big)\sin 4\theta\big\}e_r, \tag{3.43}$$

which yields $A_{(\bar{\nabla}^3 h)(e_\theta^5)}e_\theta = 0$. Also, using (2.5), (3.39), (3.42), and (3.43), we find

$$(\bar{\nabla}^4 h)\big(e_\theta^6\big) = \frac{a^5}{4\sqrt{2}}\{(2\alpha\beta\sin 6\theta - (1+\alpha^2+\beta^2+\gamma^2)\cos 2\theta$$
$$- (\alpha^2-\beta^2-\gamma^2)\cos 6\theta)e_4 - (2\alpha\beta\cos 6\theta$$
$$- (1+\alpha^2+\beta^2+\gamma^2)\sin 2\theta + (\alpha^2-\beta^2-\gamma^2)\sin 6\theta)e_5\} + \sum_{t=6}^{m} p_t e_t \tag{3.44}$$

for some functions $p_6, \ldots, p_m$. Therefore, we have

$$A_{(\bar{\nabla}^4 h)(e_\theta^6)}e_\theta = \frac{a^6}{8}\{(1+\alpha^2+\beta^2+\gamma^2)e_\theta + [(\alpha^2-\beta^2-\gamma^2)\cos 7\theta$$
$$- 2\alpha\beta\sin 7\theta]e_1 - [(\alpha^2-\beta^2-\gamma^2)\sin 7\theta + 2\alpha\beta\cos 7\theta]e_2\}. \tag{3.45}$$

Since e_θ is an eigenvector of $(\bar{\nabla}^4 h)(e_\theta^6)$ for any θ, we obtain from (3.45) that

$$\alpha^2 = \beta^2 + \gamma^2, \quad \alpha\beta = 0. \tag{3.46}$$

The second condition in (3.46) implies that we have either $\alpha = 0$ or $\beta = 0$.

If $\alpha = 0$ holds, then (3.46) gives $\beta = \gamma = 0$. In this case, (3.39) reduces to

$$De_3 = 0, \quad D_{e_1}e_4 = D_{e_2}e_5 = \frac{a}{\sqrt{2}}e_6, \quad D_{e_2}e_4 = -D_{e_1}e_5 = \frac{a}{\sqrt{2}}e_7$$
$$D_{e_1}e_6 = D_{e_2}e_7 = -\frac{a}{\sqrt{2}}e_4, \quad D_{e_1}e_7 = -D_{e_2}e_6 = \frac{a}{\sqrt{2}}e_5. \tag{3.47}$$

Thus, by $R^D(e_1, e_2, e_6, e_7) = 0$ and (3.47), we get $a^2 = 0$ which is a contradiction. Therefore, we have $\beta = 0$ and $\alpha^2 = \gamma^2$. Without loss of generality, we may assume that $\alpha = \gamma$. Hence, (3.47) and $R^D(e_1, e_2, e_6, e_7) = 0$ yield $\alpha^2 = 1/2$. Without loss of generality, we may assume that $\alpha = 1/\sqrt{2}$. Thus, (3.39) and (3.42) give

$$\begin{aligned}
&De_3 = 0, \quad D_{e_1}e_4 = \frac{a}{\sqrt{2}}e_6, \quad D_{e_2}e_4 = \frac{a}{\sqrt{2}}e_7\\
&D_{e_1}e_5 = -\frac{a}{\sqrt{2}}e_7, \quad D_{e_2}e_5 = \frac{a}{\sqrt{2}}e_6,\\
&D_{e_1}e_6 = -\frac{a}{\sqrt{2}}e_4 - \frac{a}{\sqrt{2}}e_8, \quad D_{e_2}e_6 = \frac{a}{\sqrt{2}}e_9 - \frac{a}{\sqrt{2}}e_5,\\
&D_{e_1}e_7 = \frac{a}{\sqrt{2}}e_5 + \frac{a}{\sqrt{2}}e_9, \quad D_{e_2}e_7 = \frac{a}{\sqrt{2}}e_8 - \frac{a}{\sqrt{2}}e_4,\\
&D_{e_1}e_8 = \frac{a}{\sqrt{2}}e_6 + \omega_8^9(e_1)e_9 + \sum_{r=10}^m \omega_8^r(e_1)e_r,\\
&D_{e_2}e_8 = -\frac{a}{\sqrt{2}}e_7 + \omega_8^9(e_2)e_9 + \sum_{r=10}^m \omega_8^r(e_2)e_r,\\
&D_{e_1}e_9 = -\frac{a}{\sqrt{2}}e_7 - \omega_8^9(e_1)e_8 + \sum_{r=10}^m \omega_9^r(e_1)e_r,\\
&D_{e_2}e_9 = -\frac{a}{\sqrt{2}}e_6 - \omega_8^9(e_2)e_8 + \sum_{r=10}^m \omega_9^r(e_2)e_r.
\end{aligned} \tag{3.48}$$

From (3.48) and $R^D(e_1, e_2, e_6, e_r) = 0$, $r \geq 8$, we obtain $\omega_9^r(e_1) = -\omega_8^r(e_2)$. Similarly, from (3.48) and $R^D(e_1, e_2, e_7, e_r) = 0$, $r \geq 8$, we find $\omega_9^r(e_2) = \omega_8^r(e_1)$. By combining these we obtain $\omega_8^9 = 0$. Therefore, we may choose e_{10} which satisfies $D_{e_1}e_8 = (a/\sqrt{2})e_6 + a\delta e_{10}$. Thus, we get $D_{e_2}e_9 = -(a/\sqrt{2})e_6 + a\delta e_{10}$. Similarly, if we choose e_{11} satisfying the condition $D_{e_2}e_8 = -(a/\sqrt{2})e_7 + a\eta e_{10} + a\varphi e_{11}$, we have $D_{e_1}e_9 = -(a/\sqrt{2})e_7 - a\eta e_{10} - a\varphi e_{11}$. Consequently, we obtain

$$\begin{aligned}
&De_3 = 0, \quad D_{e_1}e_4 = D_{e_2}e_5 = \frac{a}{\sqrt{2}}e_6,\\
&D_{e_2}e_4 = -D_{e_1}e_5 = \frac{a}{\sqrt{2}}e_7\\
&D_{e_1}e_6 = -\frac{a}{\sqrt{2}}e_4 - \frac{a}{\sqrt{2}}e_8, \quad D_{e_2}e_6 = -\frac{a}{\sqrt{2}}e_5 + \frac{a}{\sqrt{2}}e_9,\\
&D_{e_1}e_7 = \frac{a}{\sqrt{2}}e_5 + \frac{a}{\sqrt{2}}e_9, \quad D_{e_2}e_7 = -\frac{a}{\sqrt{2}}e_4 + \frac{a}{\sqrt{2}}e_8,\\
&D_{e_1}e_8 = \frac{a}{\sqrt{2}}e_6 + a\delta e_{10}, \quad D_{e_2}e_8 = -\frac{a}{\sqrt{2}}e_7 + a\eta e_{10} + a\varphi e_{11},\\
&D_{e_1}e_9 = -\frac{a}{\sqrt{2}}e_7 - a\eta e_{10} - a\varphi e_{11}, \quad D_{e_2}e_9 = -\frac{a}{\sqrt{2}}e_6 + a\delta e_{10}.
\end{aligned} \tag{3.49}$$

This proves Lemma 2 for $k = 5$.

Next, we shall prove Lemma 2 by induction. In order to do so, let us assume that Lemma 2 is true for a given integer $k = n \geq 5$. Now, suppose that the surface M has contact number $\mathfrak{c}_\#(M) \geq 2k$ and it has parallel mean curvature vector. We will prove that Lemma 2 also holds for $k = n + 1$.

It follows from the assumption that there exist local coordinate system $\{x, y\}$ and orthonormal normal vector fields $e_3, \ldots, e_m$ such that

$$\begin{aligned}
&\phi_{xx} = ae_3 - \frac{a}{\sqrt{2}}e_4, \quad \phi_{xy} = \frac{a}{\sqrt{2}}e_5, \quad \phi_{yy} = ae_3 + \frac{a}{\sqrt{2}}e_4, \quad H = ae_3 \\
&D_{e_1}e_3 = D_{e_2}e_3 = 0, \quad D_{e_1}e_4 = \frac{a}{\sqrt{2}}e_6, \quad D_{e_2}e_4 = \frac{a}{\sqrt{2}}e_7, \\
&D_{e_1}e_5 = -\frac{a}{\sqrt{2}}e_7, \quad D_{e_2}e_5 = \frac{a}{\sqrt{2}}e_6, \\
&D_{e_1}e_{2j} = \frac{(-1)^j a}{\sqrt{2}}(e_{2j-2} + e_{2j+2}), \quad D_{e_2}e_{2j} = \frac{a}{\sqrt{2}}(e_{2j+3} - e_{2j-1}), \\
&D_{e_1}e_{2j+1} = \frac{(-1)^{j+1} a}{\sqrt{2}}(e_{2j-1} + e_{2j+3}), \quad D_{e_2}e_{2j+1} = \frac{a}{\sqrt{2}}(e_{2j+2} - e_{2j-2}), \\
&\qquad\qquad j = 3, \ldots, n-2, \\
&D_{e_1}e_{2n-2} = (-1)^{n-1}\Big(\frac{a}{\sqrt{2}}e_{2n-4} + a\delta e_{2n}\Big), \\
&D_{e_2}e_{2n-2} = -\frac{a}{\sqrt{2}}e_{2n-3} + a\eta e_{2n} + a\varphi e_{2n+1}, \\
&D_{e_1}e_{2n-1} = (-1)^n\Big(\frac{a}{\sqrt{2}}e_{2n-3} + a\eta e_{2n} + a\varphi e_{2n+1}\Big), \\
&D_{e_2}e_{2n-1} = -\frac{a}{\sqrt{2}}e_{2n-4} + a\delta e_{2n}.
\end{aligned} \tag{3.50}$$

In views of (3.50) we may put

$$\begin{aligned}
&D_{e_1}e_{2n} = (-1)^n a\{\delta e_{2n-2} - \eta e_{2n-1}\} + \omega_{2n}^{2n+1}(e_1)e_{2n+1} + \sum_{r=2n+2}^{2k} \omega_{2n}^r(e_1)e_r, \\
&D_{e_2}e_{2n} = -a\eta e_{2n-2} + a\delta e_{2n-1} + \omega_{2n}^{2n+1}(e_2)e_{2n+1} + \sum_{r=2n+2}^{m} \omega_{2n}^r(e_2)e_r, \\
&D_{e_1}e_{2n+1} = (-1)^{n+1}a\varphi e_{2n-1} - \omega_{2n}^{2n+1}(e_1)e_{2n} + \sum_{r=2n+2}^{m} \omega_{2n+1}^r(e_1)e_r, \\
&D_{e_2}e_{2n+1} = -a\varphi e_{2n-2} - \omega_{2n}^{2n+1}(e_2)e_{2n} + \sum_{r=2n+2}^{m} \omega_{2n+1}^r(e_2)e_r.
\end{aligned} \tag{3.51}$$

Let us put $e_\theta = \cos\theta e_1 + \sin\theta e_2$ as before. Then, (3.50) implies that

$$\begin{aligned}
&D_{e_\theta}e_3 = 0, \quad D_{e_\theta}e_4 = \frac{a}{\sqrt{2}}(\cos\theta e_6 + \sin\theta e_7), \\
&D_{e_\theta}e_5 = \frac{a}{\sqrt{2}}(\sin\theta e_6 - \cos\theta e_7),
\end{aligned}$$

$$
\begin{aligned}
D_{e_\theta} e_{2j} &= \frac{a}{\sqrt{2}}\{(-1)^j \cos\theta(e_{2j-2}+e_{2j+2}) - \sin\theta(e_{2j-1}-e_{2j+3})\},\\
D_{e_\theta} e_{2j+1} &= \frac{a}{\sqrt{2}}\{\sin\theta(e_{2j+2}-e_{2j-2}) - (-1)^j \cos\theta(e_{2j-1}+e_{2j+3})\},\\
&\qquad\qquad\qquad\qquad j = 3,\dots,n-2,\\
D_{e_\theta} e_{2n-2} &= \frac{a}{\sqrt{2}}((-1)^{n+1}\cos\theta e_{2n-4} - \sin\theta e_{2n-3})\\
&\quad + a((-1)^{n+1}\delta\cos\theta + \eta\sin\theta)e_{2n} + a\varphi\sin\theta e_{2n+1},\\
D_{e_\theta} e_{2n-1} &= \frac{a}{\sqrt{2}}((-1)^{n}\cos\theta e_{2n-3} - \sin\theta e_{2n-4})\\
&\quad + a((-1)^{n}\eta\cos\theta + \delta\sin\theta)e_{2n} + (-1)^n a\varphi\cos\theta e_{2n+1},\\
D_{e_\theta} e_{2n} &= a((-1)^n\delta\cos\theta - \eta\sin\theta)e_{2n-2} - a((-1)^n\eta\cos\theta - \delta\sin\theta)e_{2n-1}\\
&\quad + \omega_{2n}^{2n+1}(e_\theta)e_{2n+1} + \sum_{r=2n+2}^{m}\omega_{2n}^{r}(e_\theta)e_r,\\
D_{e_\theta} e_{2n+1} &= -a\varphi\sin\theta e_{2n-2} - (-1)^n a\varphi\cos\theta e_{2n-1} - \omega_{2n}^{2n+1}(e_\theta)e_{2n}\\
&\quad + \sum_{r=2n+2}^{m}\omega_{2n+1}^{r}(e_\theta)e_r.
\end{aligned}
\tag{3.52}
$$

If we put

$$
\begin{aligned}
E_{2\ell} &= \cos(\ell\theta)e_{2\ell} + (-1)^{\ell-1}\sin(\ell\theta)e_{2\ell+1},\\
P_{2(n-j)} &= \cos((n+j)\theta)e_{2(n-j)} + (-1)^{n-j}\sin((n+j)\theta)e_{2(n-j)+1},\\
Q_{2(n-j)} &= (-1)^{n-j}\sin((n+j)\theta)e_{2(n-j)} - \cos((n+j)\theta)e_{2(n-j)+1}
\end{aligned}
\tag{3.53}
$$

for $\ell = 2,\dots,n-1$ and $j = 1,\dots,n-2$, then we obtain from (3.52) and (3.53) that

$$
\begin{aligned}
D_{e_\theta} E_4 &= \frac{a}{\sqrt{2}}E_6,\\
D_{e_\theta} E_{2j} &= (-1)^j\frac{a}{\sqrt{2}}(E_{2j-2}+E_{2j+2}), \quad j = 3,\dots,n-2,\\
D_{e_\theta} E_{2n-2} &= \frac{(-1)^{n-1}a}{\sqrt{2}}E_{2n-4} - a((-1)^n\delta\cos(n\theta) - \eta\sin(n\theta))e_{2n}\\
&\quad + a\varphi\sin(n\theta)e_{2n+1},\\
D_{e_\theta} P_{2(n-1)} &= (-1)^{n-1}\frac{a}{\sqrt{2}}K_{2(n-2)} + a((-1)^{n-1}\delta\cos n\theta - \eta\sin n\theta)e_{2n}\\
&\quad - a\varphi\sin n\theta e_{2n+1},\\
D_{e_\theta} P_{2(n-j)} &= (-1)^{n-j}\frac{a}{\sqrt{2}}\big(P_{2(n-j-1)} + P_{2(n-j+1)}\big), \quad j = 2,\dots,n-2,
\end{aligned}
\tag{3.54}
$$

$$\begin{aligned} D_{e_\theta} Q_{2(n-1)} &= (-1)^n \frac{a}{\sqrt{2}} Q_{2(n-2)} + a((-1)^{n-1}\eta\cos n\theta + \delta\sin n\theta)e_{2n} \\ &\quad + (-1)^{n-1} a\varphi\cos n\theta e_{2n+1}, \\ D_{e_\theta} Q_{2(n-j)} &= (-1)^{n-j+1}\frac{a}{\sqrt{2}}\big(Q_{2(n-j-1)} + Q_{2(n-j+1)}\big), \quad j = 2,\ldots,n-2. \end{aligned}$$

Moreover, from (3.4) and (3.8), we find

$$\begin{aligned} A_{E_4} e_\theta &= -\frac{a}{\sqrt{2}} e_\theta, \\ A_{P_4} e_\theta &= -\frac{a}{\sqrt{2}}\big(\cos((2n-1)\theta)e_1 - \sin((2n-1)\theta)e_2\big), \\ A_{Q_4} e_\theta &= -\frac{a}{\sqrt{2}}\big(\sin((2n-1)\theta)e_1 + \cos((2n-1)\theta)e_2\big), \\ A_{E_{2j}} e_\theta &= A_{P_{2j}} e_\theta = A_{Q_{2j}} e_\theta = A_{e_6} = \cdots = A_{e_m} = 0, \quad j = 3,\ldots,n-1. \end{aligned} \tag{3.55}$$

By applying (3.40) and (3.52) we find

$$\begin{aligned} (\nabla h)\big(e_\theta^3\big) &= -\frac{a^2}{2} E_6, \quad (\nabla^2 h)\big(e_\theta^4\big) = \frac{a^3}{2\sqrt{2}}\{E_4 + E_8\}, \\ (\nabla^\ell h)\big(e_\theta^{\ell+2}\big) &= \frac{(-1)^{[(\ell+2)/2]} a^{\ell+1}}{2^{(\ell+1)/2}} \left\{ \sum_{t=1}^{[\ell/2]} \left(\binom{\ell}{t} - \binom{\ell}{t-1} \right) E_{2\ell-4t+4} + E_{2\ell+4} \right\}, \\ &\qquad \ell = 3,\ldots,n-3, \\ (\nabla^{n-3} h)\big(e_\theta^{n-1}\big) &= \frac{(-1)^{[(n-1)/2]} a^{n-2}}{2^{n/2-1}} \left\{ \sum_{t=1}^{[(n-3)/2]} \left(\binom{n-3}{t} \right.\right. \\ &\quad \left.\left. - \binom{n-3}{t-1} \right) E_{2n-2-4t} + E_{2n-2} \right\}, \\ (\nabla^{n-2} h)\big(e_\theta^{n}\big) &= \frac{(-1)^{[n/2]} a^{n-1}}{2^{(n-1)/2}} \left\{ \sum_{t=1}^{[n/2-1]} \left(\binom{n-2}{t} - \binom{n-2}{t-1} \right) E_{2n-4t} \right\} \\ &\quad + \frac{(-1)^{[(n+1)/2]} a^{n-1}}{2^{n/2-1}} \{((-1)^n \delta\cos(n\theta) \\ &\quad - \eta\sin(n\theta))e_{2n} - \varphi\sin(n\theta)e_{2n+1}\}. \end{aligned} \tag{3.56}$$

By applying (3.52), (3.53), (3.54), and (3.56) we find

$$\begin{aligned} (\nabla^{n-1} h)\big(e_\theta^{n+1}\big) &= \frac{(-1)^{[(n+1)/2]} a^n}{2^{n/2}} \left\{ \sum_{t=1}^{[(n-1)/2]} \left(\binom{n-1}{t} - \binom{n-1}{t-1} \right) E_{2n+2-4t} \right. \\ &\quad + (\delta^2 + \eta^2 + \varphi^2 - 1)E_{2n-2} + (\delta^2 - \eta^2 - \varphi^2)P_{2n-2} \\ &\quad \left. + 2\delta\eta Q_{2n-2} \right\} + \sum_{r=2n}^{m} p_{n-1,r} e_r \end{aligned} \tag{3.57}$$

for some functions $p_{n-1,r}$. Hence, by using (3.52), (3.54), and (3.57), we have

$$
\begin{aligned}
(\nabla^n h)(e_\theta^{n+2}) &= \frac{(-1)^{[(n+2)/2]}a^{n+1}}{2^{(n+1)/2}}\left\{\sum_{t=1}^{[n/2]}\left(\binom{n}{t}-\binom{n}{t-1}\right)E_{2n+4-4t}\right. \\
&\quad + (\delta^2+\eta^2+\varphi^2-1)E_{2n-4}+(\delta^2-\eta^2-\varphi^2)P_{2n-4} \\
&\quad \left. - 2\,\delta\eta Q_{2n-4}\right\} + \sum_{r=2n-2}^{m} p_{n,r}e_r, \\
&\cdots\cdots \\
(\nabla^{2n-4}h)(e_\theta^{2n-2}) &= (-1)^{n-1}\frac{a^{2n-3}}{2^{n-3/2}}\left\{\left(\frac{\binom{2n-4}{n-3}}{n-2}+\delta^2+\eta^2+\varphi^2-1\right)E_4\right. \\
&\quad \left. + (\delta^2-\eta^2-\varphi^2)P_4+(-1)^{n-3}2\delta\eta Q_4\right\}+\sum_{r=6}^{m} p_{2n-4,r}e_r
\end{aligned}
\tag{3.58}
$$

for some functions $p_{\alpha,r}$. Hence, (3.55) and (3.58) imply that

$$
\begin{aligned}
A_{(\nabla^{2n-4}h)(e_\theta^{2n-2})}e_\theta &= (-1)^n\frac{a^{2n-2}}{2^{n-1}}\left\{\left(\frac{\binom{2n-4}{n-3}}{n-2}+\delta^2+\eta^2+\varphi^2-1\right)e_\theta\right. \\
&\quad +(\delta^2-\eta^2-\varphi^2)(\cos((2n-1)\theta)e_1-\sin((2n-1)\theta)e_2) \\
&\quad \left. +(-1)^{n-3}2\delta\eta\,(\sin((2n-1)\theta)e_1+\cos((2n-1)\theta)e_2)\right\}.
\end{aligned}
\tag{3.59}
$$

Since we have $\mathfrak{c}_\#(M)\geq 2n$, Theorem C implies that each $e_\theta=\cos\theta e_1+\sin\theta e_2$ is an eigenvector of $A_{(\nabla^{2n-4}h)(e_\theta^{2n-2})}$. Therefore, (3.59) implies that

$$
\delta^2=\eta^2+\varphi^2,\quad \delta\eta=0. \tag{3.60}
$$

If $\delta=0$, then $\eta=\varphi=0$. Thus, by using $R^D(e_1,e_2,e_{2n-2},e_{2n-1})=0$ and (3.50), we get $a=0$ which is a contradiction. Hence, we obtain $\eta=0$. So we get $\delta^2=\varphi^2$.

Without loss of generality we may assume that $\delta=\varphi$. Hence, from (3.50) and $R^D(e_1,e_2,e_{2n-2},e_{2n-1})=0$ we find that $\delta^2=1/2$. Without loss of generality, we may put $\delta=1/\sqrt{2}$. Hence, by applying (3.50) and $R^D(e_1,e_2,e_{2n-2},e_r)=0$ for $r\geq 2n$, we may obtain $\omega_{2n+1}^r(e_1)=(-1)^{n-1}\omega_{2n}^r(e_2)$.

Similarly, from (3.50) and $R^D(e_1,e_2,e_{2n-1},e_r)=0$, $r\geq 2n$, we find $\omega_{2n+1}^r(e_2)=(-1)^n\omega_{2n}^r(e_1)$. By combining these we get $\omega_{2n}^{2n+1}=0$. Therefore, we may choose e_{2n+2}, e_{2n+3} such that

$$\begin{aligned} D_{e_1}e_{2n} &= (-1)^n\left(\frac{a}{\sqrt{2}}e_{2n-2} + a\tilde{\delta}e_{2n+2}\right), \\ D_{e_2}e_{2n+1} &= -\frac{a}{\sqrt{2}}e_{2n-2} + a\tilde{\delta}e_{2n+2}, \\ D_{e_2}e_{2n} &= -\frac{a}{\sqrt{2}}e_{2n-1} + a\tilde{\eta}e_{2n+2} + a\tilde{\varphi}e_{2n+3}, \\ D_{e_1}e_{2n+1} &= (-1)^{n+1}\left(\frac{a}{\sqrt{2}}e_{2n-1} + a\tilde{\eta}e_{2n+2} + a\tilde{\varphi}e_{2n+3}\right) \end{aligned} \tag{3.61}$$

for some functions $\tilde{\delta}, \tilde{\eta}, \tilde{\varphi}$. Consequently, we obtain (3.22) for $k = n+1$, This proves Lemma 2.

Now, let us assume that $\phi : M \to \mathbb{E}^{2k}$ ($k \geq 4$) is an isometric immersion of a flat surface into $\mathbb{E}^{2k}$ with contact number $\mathfrak{c}_\#(M) \geq 2k-2$ which has parallel mean curvature vector. Then, Lemma 2 implies that there exist a local coordinate $\{x, y\}$ and orthonormal vector fields $e_1 = \partial/\partial x$, $e_2 = \partial/\partial y$, $e_3, \ldots, e_{2k}$ on M such that

$$\begin{aligned} &\phi_{xx} = ae_3 - \frac{a}{\sqrt{2}}e_4, \quad \phi_{xy} = \frac{a}{\sqrt{2}}e_5, \quad \phi_{yy} = ae_3 + \frac{a}{\sqrt{2}}e_4, \quad H = ae_3 \\ &D_{e_1}e_3 = D_{e_2}e_3 = 0, \quad D_{e_1}e_4 = \frac{a}{\sqrt{2}}e_6, \quad D_{e_2}e_4 = \frac{a}{\sqrt{2}}e_7, \\ &D_{e_1}e_5 = -\frac{a}{\sqrt{2}}e_7, \quad D_{e_2}e_5 = \frac{a}{\sqrt{2}}e_6, \\ &D_{e_1}e_{2j} = \frac{(-1)^j a}{\sqrt{2}}(e_{2j-2} + e_{2j+2}), \quad D_{e_2}e_{2j} = \frac{a}{\sqrt{2}}(e_{2j+3} - e_{2j-1}), \\ &D_{e_1}e_{2j+1} = \frac{(-1)^{j+1} a}{\sqrt{2}}(e_{2j-1} + e_{2j+3}), \quad D_{e_2}e_{2j+1} = \frac{a}{\sqrt{2}}(e_{2j+2} - e_{2j-2}), \\ &\qquad j = 3, \ldots, k-2, \\ &D_{e_1}e_{2k-2} = (-1)^{k-1}\left(\frac{a}{\sqrt{2}}e_{2k-4} + a\delta e_{2k}\right), \quad D_{e_2}e_{2k-2} = a\eta e_{2k} - \frac{a}{\sqrt{2}}e_{2k-3}, \\ &D_{e_1}e_{2k-1} = (-1)^k\left(\frac{a}{\sqrt{2}}e_{2k-3} + a\eta e_{2k}\right), \quad D_{e_2}e_{2k-1} = a\delta e_{2k} - \frac{a}{\sqrt{2}}e_{2k-4}, \\ &D_{e_1}e_{2k} = (-1)^k\left(a\delta e_{2k-2} - a\eta e_{2k-1}\right), \quad D_{e_2}e_{2k} = -a\eta e_{2k-2} - a\delta e_{2k-1} \end{aligned} \tag{3.62}$$

for some functions δ, η, φ. From $R^D(e_1, e_2, e_{2k-2}, e_{2k-1}) = 0$ and (3.62), we find

$$\eta^2 + \delta^2 = 1. \tag{3.63}$$

From $R^D(e_1, e_2, e_{2k-2}, e_{2k}) = R^D(e_1, e_2, e_{2k-1}, e_{2k}) = 0$ and (3.62), we have

$$\frac{\partial \eta}{\partial x} = -\frac{\partial \delta}{\partial y}, \quad \frac{\partial \eta}{\partial y} = \frac{\partial \delta}{\partial x}. \tag{3.64}$$

Solving (3.63) and (3.64) yields

$$\delta = \sin t, \quad \eta = \cos t \tag{3.65}$$

for some constant t.

CASE (α): $k = 2n \geq 4$. In this case, if we put

$$\begin{aligned}
&\bar{e}_3 = e_3,\ \bar{e}_{4j} = \cos t e_{4j} - \sin t e_{4j+1},\\
&\bar{e}_{4j+1} = \sin t e_{4j} + \cos t e_{4j+1},\quad \bar{e}_{4j+2} = \cos t e_{4j+2} + \sin t e_{4j+3},\\
&\bar{e}_{4j+3} = -\sin t e_{4j+2} + \cos t e_{4j+3},\ \bar{e}_{4n} = e_{4n}
\end{aligned}$$

for $j = 1, \ldots, n-1$, then it follows from (3.62) and (3.65) that the orthonormal normal frame $\{\bar{e}_3, \ldots, \bar{e}_{4n}\}$ satisfies (3.6). Therefore, the immersion $\phi : M \to \mathbb{E}^{4n}$ satisfies (3.4). Consequently, by applying the fundamental rigid theorem of submanifolds (see [2]), we conclude that the immersion is congruent to the one defined by (3.1).

CASE (β): $k = 2n + 1 \geq 5$. If we put

$$\begin{aligned}
&\bar{e}_3 = e_3,\ \bar{e}_{4j} = \sin t e_{4j} - \cos t e_{4j+1},\\
&\bar{e}_{4j+1} = \cos t e_{4j} + \sin t e_{4j+1},\quad \bar{e}_{4j+2} = \sin t e_{4j+2} + \cos t e_{4j+3},\\
&\bar{e}_{4j+3} = -\cos t e_{4j+2} + \sin t e_{4j+3},\quad \bar{e}_{4n} = \sin t e_{4n} - \cos t e_{4n+1},\\
&\bar{e}_{4n+1} = \cos t e_{4n} + \sin t e_{4n+1},\quad \bar{e}_{4n+2} = e_{4n+2}
\end{aligned}$$

for $j = 1, \ldots, n-1$, then, (3.62) and (3.65) imply that $\{\bar{e}_3, \ldots, \bar{e}_{4n+2}\}$ satisfies (3.15). Hence, the immersion $\phi : M \to \mathbb{E}^{4n+2}$ satisfies the partial differential system (3.14) which shows that the immersion is congruent to the one defined by (3.2). This completes the proof of statement (iii). Consequently, we complete the proof of the theorem. □

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