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Regularity results for the gradient of solutions linear elliptic equations with $L^{1,\lambda}$ data

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Abstract We prove that if f belongs to the Morrey space $L^{1,\lambda}(\Omega)$, with $\lambda \in [0, n-2]$, and u is the solution of the problem

$$\begin{cases} -\text{div}(A(x)Du) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then Du belongs to the space $L^{q,\lambda}(\Omega)$, for any $q \in [1, \frac{n}{n-1}[$.

Keywords Elliptic equations · Measure data

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1 Introduction

In this paper we study the regularity of the solution of the following Dirichlet problem

$$\begin{cases} -\text{div}(A(x)Du) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is an open bounded subset of $I\!R^n$ $(n \geq 3)$, $A(x)$ is an elliptic symmetric matrix with L^∞-coefficients and f belongs to the Morrey space $L^{1,\lambda}(\Omega)$.

The study of linear elliptic equations with L^1-data has been started by G. Stampacchia (see [21, 22]) who introduced, by means of a duality method, the notion of *very weak solution*. The very weak solution is unique, belongs to the space $W_0^{1,q}(\Omega)$ for any $q \in [1, \frac{n}{n-1}[$ and satisfies (1) in the distributional sense.

It is well known that a solution in the sense of distributions of problem (1) is weaker than the very weak one and, by Serrin's counterexample [20], it is not

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unique. In paper [2], in the context of nonlinear equations, it has been proved that a distributional solution of (1), satisfying an additional condition, the so called *entropy solution* (see Sect. 2 for the definition), is unique.

We remark that a solution u of the problem (1) (very weak or entropy or distributional) belongs only to $\bigcap_{1\le q<\frac{n}{n-1}} W_0^{1,q}(\Omega)$; the best regularity for Du, that is $Du \in L^{\frac{n}{n-1}}(\Omega)$, can be achieved under the additional assumption $f \in L\log L(\Omega)$ or $f \in L^1_{\frac{n}{n-1}}(\Omega)$ (see [5, 6, 11, 17, 21, 22]).

Here, we will prove that if $f \in L^{1,\lambda}(\Omega)$, with $\lambda \in [0, n-2]$, and u is the solution of the problem (1) then Du belongs to the Morrey space $L^{q,\lambda}(\Omega)$, for any $q \in [1, \frac{n}{n-1}[$.

We point out that for $\lambda = 0$ we obtain as well the original result [21]; moreover, our upper bound on λ is not a restriction since $L^{1,\lambda}(\Omega) \subset H^{-1}(\Omega)$ for $\lambda > n-2$ and, in this case, we are out of the L^1-framework (see e.g. [12]).

Furthermore, our result is coherent with the $L^{2,\lambda}$ variational theory developed in the sixties by S. Campanato; it is also well known, in fact, that if $f \in L^{2,\lambda}(\Omega)$ and u is the variational solution of the problem (1) then $Du \in L^{2,\lambda}(\Omega)$ (see e.g. [21, 9]).

Finally, we remark that $L^{1,\lambda}(\Omega)$ is not contained nor contains $L\log L(\Omega)$ as well as $L^1_{\frac{n}{n-1}}(\Omega)$ (see Remark 2.3 and Appendix); consequently, our regularity result is independent of that studied in [5, 6, 11, 17].

Concerning the technique, we use the approximation method introduced by L. Boccardo and T. Gallouet in the L^1-setting for treating nonlinear equations (see e.g. [5, 6]) mixed with the technique created by S. Campanato for handling equations and systems of equations with $L^{2,\lambda}$-data (see e.g. [9]).

2 Main notations, functions spaces and statement of the results

In $\mathbb{R}^n$ $(n \ge 3)$, with generic point $x = (x_1, x_2, \ldots, x_n)$, we shall denote by Ω a bounded open nonempty set with diameter d_Ω.

For $\rho > 0$ and $x_o \in \mathbb{R}^n$ we define

$$B(x_o, \rho) = \{x \in \mathbb{R}^n : |x - x_o| < \rho\},$$
$$\Omega(x_o, \rho) = \Omega \cap B(x_o, \rho).$$

Moreover, if $u \in L^1(B)$ we denote by

$$u_B = \frac{1}{|B|}\int_B u(x)\,\mathrm{d}x$$

where $|B|$ is the n-dimensional Lebesgue measure of B.

Definition 2.1 **(Morrey's space)** Let $q \ge 1$ and $0 \le \lambda < n$. By $L^{q,\lambda}(\Omega)$ we denote the linear space formed by the functions $u \in L^q(\Omega)$ for which

$$\|u\|_{L^{q,\lambda}(\Omega)} = \sup_{x_o\in\Omega,\, 0<\rho\le d_\Omega} \left\{\rho^{-\lambda}\int_{\Omega(x_o,\rho)} |u(x)|^q\,\mathrm{d}x\right\}^{1/q} < +\infty.$$

$L^{q,\lambda}(\Omega)$ equipped with the aforementioned norm is a Banach space.

Let us introduce the truncation operator. For a given constant $k > 0$ we define the cut function $T_k : I\!R \to I\!R$ as

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k \\ k\,\mathrm{sign}(s) & \text{if } |s| > k. \end{cases}$$

For a function $f = f(x)$, $x \in \Omega$, we define the truncated function $f_k = T_k(f)$ pointwise: for every $x \in \Omega$ the value of f_k at x is just $T_k(f(x))$.

If $u : \Omega \to I\!R$, we set

$$D_i \equiv \frac{\partial}{\partial x_i}, \qquad Du = (D_i u)_{i=1,\dots,n}\,.$$

Let $A_{ij}(x)$, $i, j = 1, 2, \dots, n$, be functions for which the following conditions be satisfied:

$$A_{ij}(x) \in L^\infty(\Omega), \quad A_{ij}(x) = A_{ji}(x) \tag{2}$$

there exist two positive constants Λ_1 and Λ_2 such that[1]

$$\begin{gathered} \Lambda_1\, |\xi|^2 \le A_{ij}(x)\xi_i\xi_j \le \Lambda_2\, |\xi|^2 \\ \text{for a.a. } x \in \Omega, \forall \xi = (\xi_i) \in I\!R^n, \end{gathered} \tag{3}$$

We introduce here the notion of entropy solution.

Definition 2.2 Let $f \in L^1(\Omega)$. By an entropy solution of the problem (1) we mean a function $u \in L^1(\Omega)$ such that

$$\begin{cases} T_k(u) \in H_0^1(\Omega), \quad \forall k > 0 \\ \displaystyle\int_\Omega A_{ij}(x)\, D_j u\, D_i T_k(u-v)\,\mathrm{d}x \le \int_\Omega f\, T_k(u-v)\mathrm{d}x \;\; \forall v \in H_0^1(\Omega) \bigcap L^\infty(\Omega). \end{cases} \tag{4}$$

In [2] it has been proved that there exists an unique entropy solution u of the aforementioned problem.

Here we assume that

$$f \in L^{1,\lambda}(\Omega), \quad \lambda \in [0, n-2] \tag{5}$$

and will prove the following result.

Theorem 2.1 *Assume that $\partial\Omega \in C^1$ and that hypotheses (2), (3) and (5) be satisfied. Let $u \in W_o^{1,q}(\Omega)$, $q \in [1, \frac{n}{n-1}[$, be the entropy solution of the problem (1).*

Then

$$D\,u \in L^{q,^{n+q+\lambda q-nq}}(\Omega)$$

and there exists a positive constant c depending on $n, q, \lambda, \Lambda_1, \Lambda_2, d_\Omega, \partial\Omega$, $\|f\|_{L^{1,\lambda}(\Omega)}$ such that

$$\|Du\|_{L^{q,\lambda}(\Omega)} \le c. \tag{6}$$

[1] Einstein's convention will be used throughout the paper.

As a consequence of the theorem we obtain the following corollary whose proof is contained in Sect. 5.

Corollary 2.1 *Under the same assumptions of the theorem we have*

$$u \in L^{\beta,\mu}(\Omega)$$

for all $\beta \in [1, \frac{q(n-\lambda-1)}{n-\lambda-2}[$, $\mu \in [0, N-\beta(N-\lambda-2)[$

Remark 2.1 As already mentioned in Sect. 1, for $\lambda = 0$ Theorem 2.1 gives us the result due to G. Stampacchia in [21].

Moreover, if $\lambda > n-2$ we are out of the L^1-framework since $L^{1,\lambda}(\Omega) \subset H^{-1}(\Omega)$ and in this case the Hölder continuity of u, Du has been studied in [12, 15].

Remark 2.2 If $u : \Omega \to I\!R$ is a measurable function we set

$$\Omega(u,\sigma) = \{x \in \Omega : |u(x)| > \sigma\}, \quad \forall \sigma > 0$$

and we say that $u \in \mathcal{M}_p(\Omega)$, $p \geq 1$, if $\exists K > 0$ such that

$$|\Omega(u,\sigma)| \leq \left(\frac{K}{\sigma}\right)^p \quad \forall \sigma > 0.$$

In [2] it has been proved that there exists an unique entropy solution u of the problem (1) such that $u \in \mathcal{M}_{\frac{n}{n-2}}(\Omega)$ and $Du \in \mathcal{M}_{\frac{n}{n-1}}(\Omega)$.

As a consequence of the inclusion

$$\mathcal{M}_{\frac{n}{n-1}}(\Omega) \subset L^{\frac{n-\lambda}{n-1},\lambda}(\Omega)$$

(see e.g. [9, Osservazione 8.II, p. 90]) the entropy solution u of the problem (1) with $f \in L^1(\Omega)$ has a gradient which belongs only to $L^{\frac{n-\lambda}{n-1},\lambda}(\Omega)$ while the assumption $f \in L^{1,\lambda}(\Omega)$, as stated in Theorem 2.1, improves the regularity of Du since

$$L^{q,\lambda}(\Omega) \subset L^{\frac{n-\lambda}{n-1},\lambda}(\Omega), \quad \forall q \geq \frac{n-\lambda}{n-1}.$$

Remark 2.3 In [5–8] the authors have proved that $Du \in L^{\frac{n}{n-1}}(\Omega)$ under the additional assumption $f \in L \log L(\Omega)$ or $f \in L^1_{\frac{n}{n-1}}(\Omega)$.

We mention that the Morrey space $L^{1,\lambda}(\Omega)$, with $\lambda > 0$, is not contained nor contains the space $L\ \log\ L$ and analogously $L^{1,\lambda}(\Omega)$ is not contained nor contains the space $L^1_{\frac{n}{n-1}}(\Omega)$ (see Appendix for the definitions).

Remark 2.4 Under the same assumptions of Corollary 2.1, in [12, 13] it was proved that the very weak solution u of the Dirichlet problem (1) belongs to the Morrey space $L^{r,\lambda}(\Omega)$ for any $r \in [1, \frac{n-\lambda}{n-\lambda-2}[$.

Being

$$\frac{q(n-\lambda)}{n-\lambda-q} < \frac{n-\lambda}{n-\lambda-2},$$
$$\forall q \in [1, \tfrac{n}{n-1}[, \quad \forall \lambda \in [0, n-2[$$

our regularity of the solution u is weaker than the previous one.

On the other hand, existing $q \in [1, \frac{n}{n-1}[$ such that

$$\frac{n}{n-2} < \frac{q(n-\lambda)}{n-\lambda-q}, \quad \forall\lambda \in]0, n-2],$$

from the Corollary 2.1 immediately follows the improvement of the analogous results achieved in [5, 11] (see Remark 2.3).

3 Auxiliary results

In this section we assume that the structural conditions (2) and (3) hold and we consider a weak solution v of the linear equation

$$-D_i(A_{ij}(x)D_j v) = 0 \quad \text{in } \Omega \tag{7}$$

that is, a function $v \in H^1(\Omega)$ such that

$$\int_\Omega A_{ij}(x)D_j v D_i \varphi \,\mathrm{d}x = 0, \quad \forall\varphi \in H_0^1(\Omega).$$

The key step in our paper will be to prove the following theorem.

Theorem 3.1 **(Saint–Venant Principle)** *Let $v \in H^1(\Omega)$ be a weak solution of Eq. 7.*

Then there exist two constants $\mu = \mu(n, \Lambda_1, \Lambda_2) \in]0, 1[$ and $c = c(n, q, \Lambda_1, \Lambda_2) > 0$ such that

$$\|Dv\|^q_{L^q(B(x_o,\rho_1))} \le c\left(\frac{\rho_1}{\rho_2}\right)^{n-q+\mu q} \|Dv\|^q_{L^q(B(x_o,\rho_2))} \tag{8}$$

$\forall x_o \in \Omega$, $\forall 0 \le \rho_1 \le \rho_2 < \operatorname{dist}(x_o, \partial\Omega)$, $\forall q \in [1, 2[$.

Before proving the previous theorem let us premise two useful lemmata which are interesting by themselves.

Lemma 3.1 *Let $v \in H^1(\Omega)$ be a weak solution of Eq. (7).*

Then there exist two constants $\mu = \mu(n, \Lambda_1, \Lambda_2) \in]0, 1[$ and $c = c(n, q, \Lambda_1, \Lambda_2) > 0$ such that

$$\operatorname{osc}_{B(x_o,\rho_1)} v \le c\left(\frac{\rho_1}{\rho_2}\right)^{\mu} \rho_2^{-n/q} \|v\|_{L^q(B(x_o,2\rho_2))} \tag{9}$$

$\forall x_o \in \Omega$, $\forall 0 < \rho_1 \le \rho_2 < \frac{1}{2}\operatorname{dist}(x_o, \partial\Omega)$, $\forall q \in [1, 2]$.

Proof Fixed $x_o \in \Omega$, $\rho_2 \in]0, \frac{1}{2}\operatorname{dist}(x_o, \partial\Omega)[$ and $q \in [1, 2]$, by Theorem 5.4 (part 2 p. 80) from [14] we have, $\forall q \in [1, 2]$,

$$\|v\|^2_{L^2(B(x_o,\rho_2))} \le c(n) \sup_{B(x_o,\rho_2)} |v|^2 \rho_2^n \le c(n,q)\, \rho_2^{n-2n/q} \|v\|^2_{L^q(B(x_o,2\rho_2))}. \tag{10}$$

On the other hand, by De Giorgi's regularity theorem (see [10] or [16, p. 75]) and previous inequality we deduce, $\forall 0 < \rho_1 \le \rho_2$,

$$\begin{aligned} \operatorname{osc}_{B(x_o,\rho_1)} v &\le c(n, q, \Lambda_1, \Lambda_2)\, (\rho_1/\rho_2)^{\mu}\, \rho_2^{-n/2} \|v\|_{L^2(B(x_o,2\rho_2))} \\ &\le c(n, q, \Lambda_1, \Lambda_2)\, (\rho_1/\rho_2)^{\mu}\, \rho_2^{-n/q} \|v\|_{L^q(B(x_o,2\rho_2))}. \end{aligned}$$

□

Lemma 3.2 *Let $v \in H^1(\Omega)$ be a weak solution of Eq. 7 .*
Then there exists a positive constant $c = c(n, q, \Lambda_1, \Lambda_2)$ such that

$$\|Dv\|^q_{L^q(B(x_o,\rho))} \le c\rho^{-q} \|v - h\|^q_{L^q(B(x_o,3\rho))} \tag{11}$$

$\forall x_o \in \Omega$, $\forall \rho \in]0, \frac{1}{3}\operatorname{dist}(x_o, \partial\Omega)[$, $\forall q \in [1, 2[$, $\forall h \in \mathbb{R}$.

Proof Fixed $x_o \in \Omega$, $\rho \in]0, \frac{1}{3}\operatorname{dist}(x_o, \partial\Omega)[$, $q \in [1, 2[$ and $h \in \mathbb{R}$, by Cacciop-poli's inequality (see e.g. [14, p. 24]) one has:

$$\|Dv\|^2_{L^2(B(x_o,\rho))} \le c(\Lambda_1, \Lambda_2)\rho^{-2} \|v - h\|^2_{L^2(B(x_o,2\rho))}.$$

On the other hand, Hölder's inequality and the previous inequality yield

$$\begin{aligned} \|Dv\|^q_{L^q(B(x_o,\rho))} &\le c(n)\rho^{n(1-q/2)} \|Dv\|^q_{L^2(B(x_o,\rho))} \\ &\le c(n, q, \Lambda_1, \Lambda_2)\rho^{n(1-q/2)-q} \|v - h\|^q_{L^2(B(x_o,2\rho))}. \end{aligned} \tag{12}$$

As the function $v - h$ is still a solution of problem (7), again by Theorem 5.4 from [14] (see (10)) and (12) we deduce

$$\|Dv\|^q_{L^q(B(x_o,\rho))} \le c(n, q, \Lambda_1, \Lambda_2)\rho^{n(1-q/2)-q+nq/2-n} \|v - h\|^q_{L^q(B(x_o,3\rho))}.$$

□

Proof of Theorem 3.1.
Let us fix $x_o \in \Omega$, $\rho_2 \in]0, \operatorname{dist}(x_o, \partial\Omega)[$, $0 < \rho_1 \le \rho_2$ and $q \in [1, 2[$. First, we will prove our inequality for $\rho_1 \in]0, \frac{1}{6}\rho_2[$.
Applying formula (9) in $B(x_o, 3\rho_1)$ with $h = v_{B(x_o,\rho_2)}$ we obtain

$$\begin{aligned} \left|v - v_{B(x_o,3\rho_1)}\right| &\le \operatorname{osc}_{B(x_o,3\rho_1)} v = \operatorname{osc}_{B(x_o,3\rho_1)} \left(v - v_{B(x_o,\rho_2)}\right) \\ &\le c \left(\frac{\rho_1}{\rho_2}\right)^{\mu} \rho_2^{-n/q} \left\|v - v_{B(x_o,\rho_2)}\right\|_{L^q(B(x_o,\rho_2))}. \end{aligned} \tag{13}$$

Joining together formulae (11) and (13), and using Poincaré's inequality we deduce

$$\begin{aligned} \|Dv\|^q_{L^q(B(x_o,\rho_1))} &\le c\, \rho_1^{-q} \left\|v - v_{B(x_o,3\rho_1)}\right\|^q_{L^q(B(x_o,3\rho_1))} \\ &\le c\, \rho_1^{n-q+\mu q} \rho_2^{-n-\mu q} \left\|v - v_{B(x_o,\rho_2)}\right\|^q_{L^q(B(x_o,\rho_2))} \\ &\le c\, (\rho_1/\rho_2)^{n-q+\mu q} \|Dv\|^q_{L^q(B(x_o,\rho_2))}. \end{aligned}$$

As far as it concerns the case $\rho_1 \in [\frac{1}{6}\rho_2, \rho_2]$ we immediately have

$$\begin{aligned} \|Dv\|^q_{L^q(B(x_o,\rho_1))} &= (\rho_1/\rho_2)^{n-q+\mu q} (\rho_2/\rho_1)^{n-q+\mu q} \|Dv\|^q_{L^q(B(x_o,\rho_1))} \\ &\le\ c\, (\rho_1/\rho_2)^{n-q+\mu q} \|Dv\|^q_{L^q(B(x_o,\rho_2))} \end{aligned}$$

and the previous two formulae imply the thesis.

4 Interior and boundary estimates: Global regularity

In this section we will connect the technique developed in [6] with the nowadays classical method of S. Campanato.

For a given function $f \in L^{1,\lambda}(\Omega)$ let us consider a sequence of functions $\{f_k\}_{k\in \mathbb{N}}$ such that

(i) $f_k \in H^{-1}(\Omega) \cap L^{1,\lambda}(\Omega)$, $\forall k \in \mathbb{N}$,
(ii) $f_k \to f$ in $L^1(\Omega)$ as $k \to +\infty$,
(iii) $\|f_k\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}, \quad \forall k \in \mathbb{N}$,
(iv) $\|f_k\|_{L^{1,\lambda}(\Omega)} \le \|f\|_{L^{1,\lambda}(\Omega)}, \quad \forall k \in \mathbb{N}$.

An example of sequence satisfying the above requirements is the sequence $\{T_k(f)\}_{k\in \mathbb{N}}$.

Fixed $k \in \mathbb{N}$, let $u_k \in H_0^1(\Omega)$ be a weak solution of the equation

$$-D_i(A_{ij}(x)D_j u_k) = f_k \quad \text{in } \Omega \tag{14}$$

that is,

$$\begin{cases} u_k \in H_0^1(\Omega) \\ \displaystyle\int_\Omega A_{ij}(x)D_j u_k\, D_i\varphi\, \mathrm{d}x = \int_\Omega f_k\, \varphi\, \mathrm{d}x, \quad \forall \varphi \in H_0^1(\Omega). \end{cases}$$

We will prove, first, the following theorem.

Theorem 4.1 *Assume that hypotheses (2), (3), and (5) hold, and let u_k be the solution of problem (14).*

Then

$$D\, u_k \in L^{q,\lambda}_{\mathrm{loc}}(\Omega), \quad \forall q \in \left[1, \frac{n}{n-1}\right[, \quad \forall k \in \mathbb{N},$$

and for all $H \subset\subset \Omega$ there exists a positive constant c depending on n, q, λ, Λ_1, Λ_2, d_Ω, $\mathrm{dist}(\bar{H}, \partial\Omega)$ *such that*

$$\|Du_k\|_{L^{q,\lambda}(H)} \le c\left[\|Du_k\|_{L^q(\Omega)} + \|f\|_{L^{1,\lambda}(\Omega)}\right], \quad \forall k \in \mathbb{N}. \tag{15}$$

Proof Fix $k \in \mathbb{N}$, $x_o \in \Omega$ and $\rho \in]0, \mathrm{dist}(x_o, \partial\Omega)]$

In $B(x_o, \rho)$ we can write $u_k = v_k + w_k$ where $v_k \in H^1(B(x_o, \rho))$ is a weak solution of the problem

$$-D_i(A_{ij}(x)D_j v_k) = 0 \quad \text{in } B(x_o, \rho)$$

and $w_k \in H_0^1(B(x_o, \rho))$ is the weak solution of the Dirichlet problem

$$\begin{cases} -D_i(A_{ij}(x)D_j w_k) = f_k & \text{in } B(x_o, \rho) \\ w_k = 0 & \text{on } \partial B(x_o, \rho) \end{cases} \tag{16}$$

Since any weak solution of the problem (16) is also a very weak solution of the same problem then, by formula (8.6) page 220 from [22] (see also [22, p. 108], [16, Theorem B.2, p. 63], and [21, Theorem 9.1]) and item (iv), it follows that, for any $q \in [1, \frac{n}{n-1}[$,

$$\begin{aligned} \|w_k\|^q_{W^{1,q}(B(x_o,\rho))} &\leq c(n,q,\Lambda_1)\,\rho^{n+q-nq}\,\|f_k\|^q_{L^1(B(x_o,\rho))} \\ &\leq c(n,q,\Lambda_1)\,\rho^{n+q+\lambda q-nq}\,\|f\|^q_{L^{1,\lambda}(\Omega)}. \end{aligned} \tag{17}$$

Gathering together (8) and (17) we deduce, for any $\sigma < \rho$,

$$\begin{aligned} &\|D\,u_k\|^q_{L^q(B(x_o,\sigma))} \\ &\leq c(n,q,\Lambda_1,\Lambda_2)\left[\left(\frac{\sigma}{\rho}\right)^{n-q+\mu q}\|Dv_k\|^q_{L^q(B(x_o,\rho))} \right.\\ &\qquad \left. +\,\rho^{n+q+\lambda q-nq}\,\|f\|^q_{L^{1,\lambda}(\Omega)}\right] \\ &\leq c(n,q,\Lambda_1,\Lambda_2)\left[\left(\frac{\sigma}{\rho}\right)^{n-q+\mu q}\|Du_k\|^q_{L^q(B(x_o,\rho))} \right.\\ &\qquad \left. +\,\rho^{n+q+\lambda q-nq}\,\|f\|^q_{L^{1,\lambda}(\Omega)}\right]. \end{aligned}$$

An application of Lemma 1.1, p. 7, from [9][2] to the aforementioned inequality gives

$$\begin{aligned} &\|D\,u_k\|^q_{L^q(B(x_o,\sigma))} \\ &\leq c(n,q,\Lambda_1,\Lambda_2)\left[\left(\frac{\sigma}{\rho}\right)^{n+q+\lambda q-nq}\|Du_k\|^q_{L^q(B(x_o,\rho))} \right. \\ &\left. +\,\sigma^{n+q+\lambda q-nq}\,\|f\|^q_{L^{1,\lambda}(\Omega)}\right]. \end{aligned} \tag{18}$$

Let now $H \subset\subset \Omega$, $x_o \in H$ and set $d_o = \operatorname{dist}(\overline{H}, \partial\Omega)$.
Since $d_o \leq \operatorname{dist}(x_o, \partial\Omega)$, inequality (18) rewritten for $\rho = d_o$ gives

$$\sigma^{-\lambda}\,\|D\,u_k\|^q_{L^q(B(x_o,\sigma)\cap H)} \leq c(n,q,\lambda,d_o)\big[\|Du_k\|^q_{L^q(\Omega)} + \|f\|^q_{L^{1,\lambda}(\Omega)}\big] \quad \forall \sigma \in]0, d_0[.$$

On the other hand, if $d_0 < d_H$, for any $\sigma \in [d_o, d_H]$ we obtain

$$\begin{aligned} &\sigma^{-\lambda}\,\|D\,u_k\|^q_{L^q(B(x_o,\sigma)\cap H)} \\ &= (d_0)^{-\lambda}\left(\frac{\sigma}{d_0}\right)^{-\lambda}\|D\,u_k\|^q_{L^q(B(x_o,\sigma)\cap H)} \\ &\leq (d_0)^{-\lambda}\,\|D\,u_k\|^q_{L^q(\Omega)}. \end{aligned}$$

The previous two inequalities readily yield (15). □

Now we will deduce boundary and then global regularity from the previous interior result through an extension technique and the successive standard "flattening and covering" arguments.

[2] Set $\varphi(\sigma) = \|D\,u_k\|^q_{L^q(B(x_o,\sigma))}$, $A = c(n,q,\Lambda_1,\Lambda_2)$, $\alpha = n-q+\mu q$, $\beta = n+q+\lambda q-nq$, $\Phi(\sigma) = c(n,q,\Lambda_1,\Lambda_2)\,\|f\|^q_{L^{1,\lambda}(\Omega)}$, $\varepsilon = \mu q + q(n-\lambda-2)$.

This technique is illustrated in Lemma 2.18 and in Theorem 2.19 of [24]. We will reproduce here the main steps for the reader's convenience.

If $y = (y_1, \dots, y_{n-1}, 0)$ we define

$$B^+(y, \rho) = \{x \in B(y, \rho) : x_n > 0\},$$
$$\Gamma(y, \rho) = \{x \in B(y, \rho) : x_n = 0\}.$$

Fixed $R_1 > 0$ and $k \in I\!N$, we begin investigating a solution of the problem

$$\begin{cases} u_k \in H^1(B^+(y, R_1)) \\ u_{|\Gamma(y,R_1)} = 0 \\ \int_{B^+(y,R_1)} A_{ij}(x) D_j u_k \, D_i \varphi \, \mathrm{d}x = \int_{B^+(y,R_1)} f_k \, \varphi \, \mathrm{d}x \\ \forall \varphi \in H_0^1(B^+(y, R_1)) \end{cases} \tag{19}$$

assuming that hypotheses (2), (3), and (5) hold with $\Omega = B^+(y, R_1)$.

We state the following lemma. □

Lemma 4.1 *Let u_k be a solution of problem (19). Then, for every $R \in]0, R_1[$, we have*

$$D\, u_k \in L^{q,\lambda}(B^+(y, R)), \quad \forall q \in \left[1, \frac{n}{n-1}\right[, \quad \forall k \in I\!N,$$

and there exists a positive constant c depending on $n, q, \lambda, \Lambda_1, \Lambda_2, R$ such that

$$\|Du_k\|_{L^{q,\lambda}(B^+(y,R))} \le c \left[\|Du_k\|_{L^q(B^+(y,R_1)} + \|f\|_{L^{1,\lambda}(B^+(y,R_1))}\right], \quad \forall k \in I\!N. \tag{20}$$

Proof.

Fixed $y = (y_1, \dots, y_{n-1}, 0)$, for a function $w(x)$, $x \in B(y, R_1)$, we define

$$\tilde{w}(x_1, \dots, x_{n-1}, x_n) = w(x_1, \dots, x_{n-1}, -x_n).$$

We now extend the functions A_{ij}, f_k, u_k a.e. to $B(y, R_1)$ by setting

$$\overline{A_{in}}(x_1, \dots, x_{n-1}, x_n) = \begin{cases} A_{in}(x_1, \dots, x_{n-1}, x_n) & \text{if } x_n > 0 \\ -A_{in}(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$
$$\text{for } i = 1, \dots, n-1,$$
$$\overline{A_{ij}}(x_1, \dots, x_{n-1}, x_n) = \begin{cases} A_{ij}(x_1, \dots, x_{n-1}, x_n) & \text{if } x_n > 0 \\ A_{ij}(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$
$$\text{for all the remaining values of } i, j,$$

$$\overline{f_k}(x_1, \dots, x_{n-1}, x_n) = \begin{cases} f_k(x_1, \dots, x_{n-1}, x_n) & \text{if } x_n > 0 \\ -f_k(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0, \end{cases}$$

$$\overline{u_k}(x_1, \dots, x_{n-1}, x_n) = \begin{cases} u_k(x_1, \dots, x_{n-1}, x_n) & \text{if } x_n \ge 0 \\ -u_k(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0. \end{cases}$$

Observe that $\overline{A_{ij}}$ satisfies assumption (2) and (3) with the same constants Λ_1, Λ_2.

Moreover, $\overline{f_k} \in L^{1,\lambda}(B(y, R_1))$ and $\overline{u_k} \in H^1(B(y, R_1))$.

Now, fixed a function $\varphi \in C^1_o(B(y, R_1))$, simple calculations show that

$$\begin{aligned}
&\int_{B(y,R_1)} \overline{A_{ij}}(x) D_j \overline{u_k}\, D_i\varphi \,\mathrm{d}x \\
&\quad = \int_{B^+(y,R_1)} A_{ij}(x) D_j u_k\, D_i(\varphi - \tilde{\varphi})\,\mathrm{d}x = \int_{B^+(y,R_1)} f_k\,(\varphi - \tilde{\varphi})\mathrm{d}x \\
&\quad = \int_{B(y,R_1)} \overline{f_k}\,\varphi\,\mathrm{d}x
\end{aligned} \tag{21}$$

and by a density argument the function $\overline{u_k}$ satisfies Eq. 14.

Thus, $\overline{u_k}$ satisfies inequality (18) and the thesis of the theorem follows by a change of variables. □

Since Ω is of class C^1 (see [19, p. 314]), for each $\overline{y} \in \partial\Omega$ there is a ball $\mathcal{B}(\overline{y}, R_o)$ and a C^1-function ζ defined on a domain $D \subset I\!R^{n-1}$ such that with respect to a suitable system of coordinates $\{y_1, \ldots, y_n\}$, with the origin at $\overline{y}$:

(a) the set $\partial\Omega \cap \mathcal{B}(\overline{y}, R_o)$ can be represented by an equation of the type:

$$y_n = \zeta(y_1, \ldots, y_{n-1}),$$

(b) each $y \in \Omega \cap \mathcal{B}(\overline{y}, R_o)$ satisfies

$$y_n < \zeta(y_1, \ldots, y_{n-1}).$$

For such domains the boundary can be locally straightened by means of the smooth transformation:

$$\begin{cases} \psi_i(y) = y_i - \bar{y}_i & \text{for } i = 1, 2, \ldots, n-1 \\ \psi_n(y) = y_n - \zeta(y_1, \ldots, y_{n-1}). \end{cases} \tag{22}$$

It turns out that $\psi(y) = (\psi_1(y), \ldots, \psi_n(y))$ is a $C^1(\mathcal{B}(\overline{y}, R_o))$-diffeomorphism verifying the following properties (see e.g. [18, p. 305] or Theorem V at page 375 from [8]):

(i) $\psi(\bar{y}) = 0$

(ii) $\psi(\mathcal{B}(\overline{y}, R_o) \cap \partial\Omega) = \{x \in I\!R^n : x_n = 0,\ |x_i| < R_o, \text{ for } i = 1, \ldots, n-1\}$,

(iii) there exist two positive constants α_1 and α_2, with $\alpha_1 \le \alpha_2$, such that

$$\alpha_1\,|y - \bar{y}| \le |\psi(y)| \le \alpha_2\,|y - \bar{y}|, \quad \forall y \in \mathcal{B}(y, R_o) \cap \Omega,$$

$$B^+(0, \alpha_1 R_o) \subset \psi(\mathcal{B}(\bar{y}, R_o) \cap \Omega) \subset B^+(0, \alpha_2 R_o),$$

$$\mathcal{B}\left(\overline{y}, \frac{\alpha_1}{\alpha_2} R_o\right) \cap \Omega \subset \psi^{-1}(B^+(0, \alpha_1 R_o)) \subset \mathcal{B}(\overline{y}, R_o) \cap \Omega. \tag{23}$$

Global regularity is now a consequence of Theorem 4.1 and Lemma 4.1 as in Theorem 2.19 from [24].

Fixed $k \in I\!N$, let $u_k \in H^1_0(\Omega)$ be the weak solution of the homogeneous Dirichlet problem associated to the equation

$$-D_i(A_{ij}(x) D_j u_k) = f_k \quad \text{in } \Omega \tag{24}$$

that is,

$$\begin{cases} u_k \in H_0^1(\Omega) \\ \int_\Omega A_{ij}(x) D_j u_k \, D_i \varphi \, \mathrm{d}x = \int_\Omega f_k \, \varphi \, \mathrm{d}x, \quad \forall \varphi \in H_0^1(\Omega). \end{cases}$$

Theorem 4.2 *Assume that hypotheses (2), (3), (5) hold and let u_k be the solution of problem (24).*

Then

$$D\, u_k \in L^{q,\lambda}(\Omega), \quad \forall q \in \left[1, \frac{n}{n-1}\right[, \quad \forall k \in I\!N,$$

and there exists a positive constant c depending on n, q, λ, Λ_1, Λ_2, d_Ω, $\partial\Omega$, $\|f\|_{L^{1,\lambda}(\Omega)}$ such that

$$\|Du_k\|_{L^{q,\lambda}(\Omega)} \le c, \quad \forall k \in I\!N. \tag{25}$$

Proof.

Put $R_1 = \alpha_1 R_o$, if $z \in B^+(0, R_1)$ we set

$$\begin{aligned} A'_{ij}(z) &= A_{rs}(\psi^{-1}(z)) \frac{\partial \psi_i}{\partial y_r}(\psi^{-1}(z)) \frac{\partial \psi_j}{\partial y_s}(\psi^{-1}(z)) \, J(z), \\ f'_k(z) &= f_k(\psi^{-1}(z)) \, J(z), \\ u'_k(z) &= u_k(\psi^{-1}(z)), \end{aligned} \tag{26}$$

where $J(z)$ is the absolute value of the Jacobian determinant of $\psi^{-1}(z)$.

Let us observe that $A'_{ij}(z)$ still satisfy hypothesis (2).

Moreover, from the definition (26) it follows that

$$\Lambda_1 \, J \sum_{h=1}^n \left(\frac{\partial \psi_i}{\partial y_h} \eta_i \right)^2 \le A'_{ij} \eta_i \eta_j \le \Lambda_2 \, J \sum_{h=1}^n \left(\frac{\partial \psi_i}{\partial y_h} \eta_i \right)^2, \quad \forall \eta \in I\!R^n$$

so that, if $\eta \in I\!R^n$ and $|\eta| = 1$ we have

$$\begin{aligned} &\Lambda_1 \, \min_{\overline{B^+(0,R_1)}} J \, \min_{\overline{B^+(0,R_1)} \times \{|\eta|=1\}} \sum_{h=1}^n \left(\frac{\partial \psi_i}{\partial y_h} \eta_i \right)^2 \\ &\quad \le A'_{ij} \eta_i \eta_j \le \Lambda_2 \, \max_{\overline{B^+(0,R_1)}} (J \, |D\psi|^2). \end{aligned} \tag{27}$$

Thus, a change of variables in the equation yields

$$\begin{cases} u'_k \in H^1(B^+(0, R_1)) \\ u'_k = 0 \quad \text{on } \Gamma(0, R_1) \\ \int_{B^+(0,R_1)} A'_{ij} D_j u'_k \, D_i \varphi' \, \mathrm{d}z = \int_{B^+(0,R_1)} f'_k \, \varphi' \mathrm{d}z, \quad \forall \varphi' \in H_0^1(B^+(0, R_1)). \end{cases} \tag{28}$$

To (28) we apply Lemma 4.1 and so we conclude that Du'_k lies in $L^{q,\lambda}(B^+(0,R))$, $R \in]0, R_1[$, with norm estimate (20).

As a consequence, the vector-function $Du'_k(\psi(y))$, $y \in \mathcal{B}(\overline{y}, r) \cap \Omega$, $r \in]0, \frac{R_1}{\alpha_2}[$, belongs to $L^{q,\lambda}(\mathcal{B}(\overline{y}, r) \cap \Omega)$ (see (23)) that is, by the chain rule, $Du_k \in L^{q,\lambda}(\mathcal{B}(\overline{y}, r) \cap \Omega)$ and, by virtue of (20),

$$\|Du_k\|_{L^{q,\lambda}(\mathcal{B}(\overline{y},r)\cap\Omega)} \le c\left[\|Du_k\|_{L^q(\Omega)} + \|f\|_{L^{1,\lambda}(\Omega)}\right]. \tag{29}$$

Because $\partial\Omega$ is compact, there is a finite number of balls such as $\mathcal{B}(\overline{y}, r)$, say $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_\nu$, which cover $\partial\Omega$.

Moreover, there exists an open set $H_0 \subset\subset \Omega$ such that $H_0, \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_\nu$ cover $\overline{\Omega}$.

If $\{g_m\}_{m=0,1,\dots,\nu}$ is a partition of the unity relative to the above covering then it turns out

$$\|Du_k\|_{L^{q,\lambda}(\Omega)} \le c(\nu, q)\left[\|Du_k\|_{L^{q,\lambda}(H_0)} + \sum_{m=1}^{\nu} \|Du_k\|_{L^{q,\lambda}(\mathcal{B}_m(\overline{y},r)\cap\Omega)}\right]. \tag{30}$$

On the other hand, by (8.6) page 220 from [22] and item (iv), it follows that

$$\|Du_k\|_{L^q(\Omega)} \le c(n, q, \Lambda_1, d_\Omega)\|f\|_{L^{1,\lambda}(\Omega)}, \quad \forall k \in \mathbb{N} \tag{31}$$

so that the uniform norm estimate (25) now follows from (30) by joining together (15), (29) and (31). □

5 Proof of the Theorem 2.1.

We have already remarked (see (31)) that

$$\|Du_k\|_{L^q(\Omega)} \le c(n, q, \Lambda_1, d_\Omega)\|f\|_{L^{1,\lambda}(\Omega)}, \quad \forall k \in \mathbb{N}, \quad \forall q \in \left[1, \frac{n}{n-1}\right[.$$

This information allows us to deduce the following consequences

(a) $u_k \rightharpoonup u$ in $W^{1,q}(\Omega)$ as $k \to +\infty$,
(b) $u_k \to u$ in $L^q(\Omega)$ and a.e. in Ω as $k \to +\infty$,
(c) $Du_k \to Du$ a.e. in Ω as $k \to +\infty$ (see [6]),
(d) the function u is the entropy solution of the Dirichlet problem (1) (see [2]).

To conclude the proof we need only to show that $Du \in L^{q,\lambda}(\Omega)$.
To this purpose, let us fix $x_o \in \Omega$ and $\rho \in]0, d_\Omega]$.
Since, by *(c)*, we have

$$Du_k \to Du \quad \text{a.e. in } \Omega(x_o, \rho),$$

by virtue of Fatou's Lemma and (6) we obtain

$$\begin{aligned}\|D\,u\|^q_{L^q(\Omega(x_o,\rho))} &\le \liminf_{k\to+\infty} \|D\,u_k\|^q_{L^q(\Omega(x_o,\rho))}\\ &\le \rho^\lambda \liminf_{k\to+\infty} \|D\,u_k\|^q_{L^{q,\lambda}(\Omega)}\\ &\le c(n, q, \lambda, \Lambda_1, \Lambda_2, d_\Omega, \partial\Omega, \|f\|_{L^{1,\lambda}(\Omega)})\,\rho^\lambda.\end{aligned}$$

The previous inequality and *(d)* prove the thesis.

Proof of the Corollary

Since $Du \in L^{q,\lambda}(\Omega)$ and $\partial\Omega \in C^1$ implies that Ω has the cone property (see e.g. [1, p. 67]), by virtue of Theorem 1.2, page 72 from [7], it will be enough to prove that $u \in L^{q,\lambda}(\Omega)$.

We start by observing that by Sobolev embedding theorem and the standard properties of Morrey spaces it turns out that

$$u \in L^{q^*,0}(\Omega) \subset L^{q,\mu_o}(\Omega), \quad \forall \mu_o \in]0, q].$$

(i) If $\lambda \in]0, q]$ then $u \in L^{q,\lambda}(\Omega)$.

(ii) If $\lambda > q$ then $u, Du \in L^{q,\mu_o}(\Omega)$ and, by virtue of Theorem 1.2, page 72 from [7], we have

$$u \in L^{q,\mu}(\Omega), \quad \forall \mu < q + \mu_o.$$

If now $\lambda < q + \mu_0$ then $u, Du \in L^{q,\lambda}(\Omega)$ and we stop.

If instead $\lambda \geq q + \mu_o$ then $u, Du \in L^{q,\mu_1}(\Omega)$, with $\mu_1 = \frac{q}{2} + \mu_o$, and a new application of the quoted theorem yields

$$u,\ Du \in L^{q,\mu}(\Omega), \quad \forall \mu < q + \mu_1.$$

Iterating the previous procedure we can prove that

$$u, Du \in L^{q,\mu_m}(\Omega), \quad \forall m \in I\!N$$

with

$$\mu_m = \begin{cases} \lambda & \text{if } \lambda < q + \mu_{m-1} \\ \dfrac{q}{2} + \mu_{m-1} & \text{if } \lambda \geq q + \mu_{m-1}. \end{cases}$$

Since, for all $m \in I\!N$,

$$\mu_m = \begin{cases} \lambda & \text{if } \lambda < q + \mu_{m-1} \\ m\frac{q}{2} + \mu_0 & \text{if } \lambda \geq q + \mu_{m-1}, \end{cases}$$

after a finite number of steps we obtain

$$u, Du \in L^{q,\lambda}(\Omega)$$

and this concludes the proof of the corollary.

Remark 5.1 The result of Theorem 2.1 can be readily extended to nonlinear operators with special structure.[3]

Let us consider the solution u (see [4]) of the following Dirichlet problem

$$\begin{cases} -\text{div}(A(x, Du)) = f \in L^{1,\lambda}(\Omega) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{32}$$

where $A(x, \xi)$ is a vector-function satisfying the following assumptions:

(i) $A : \Omega \times I\!R^n \to I\!R^n$ is a C^1-function in ξ for a.e. x and measurable in x for every ξ;

[3] The general nonlinear problem will be the topic of a forthcoming article.

(ii) there exists $\alpha > 0$ such that

$$\langle A(x,\xi),\xi\rangle \geq \alpha|\xi|^2$$

holds for every ξ and a.e. x, where $\langle\cdot,\cdot\rangle$ means the scalar product in $I\!R^n$;

(iii) for every $\xi, \eta \in I\!R^n$, with $\xi \neq \eta$, and a.e. $x \in \Omega$ there holds

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle > 0;$$

(iv) there exists $\gamma > 0$ such that

$$|A(x,\xi)| \leq \gamma(g(x) + |\xi|)$$

holds for every $\xi \in I\!R^n$ with $g \in L^2(\Omega)$;

(v) $A(x,0) = 0$ for a.e. $x \in \Omega$.

Indeed, fixed $k \in I\!N$, let $u_k \in H^1(\Omega)$ be the weak solution of the following Dirichlet problem

$$\begin{cases} -\mathrm{div}(A(x, Du_k)) = f_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Proceeding as in [4, p. 457], we can see that u_k satisfies the linear Dirichlet problem

$$\begin{cases} -\mathrm{div}(M_k(x)Du_k) = f_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{33}$$

where

$$M_k(x) = \int_0^1 D_\xi A(x, t\, Du_k(x))\, \mathrm{d}t$$

are equi-elliptic and equi-bounded matrices (see the hypotheses mentioned earlier).

To problem (33) we can thus apply the Theorem 4.2 to obtain that $D\,u_k$ lies in $L^{q,\lambda}(\Omega)$ with norm estimate, and conclude the proof with a passage to the limit as in the proof of Theorem 2.1.

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A Appendix

In this section we are going to construct some examples which point out the relationships among the spaces $L^{1,\lambda}$, $L\log L(\Omega)$ and $L^1_{\frac{n}{n-1}}$.

Let us start with the definitions of the last two spaces.

Definition A.1 We denote by $L\ \log\ L(\Omega)$ the space of measurable functions $f : \Omega \to I\!R$ such that

$$\int_\Omega |f|\ \log(1 + |f|)\,\mathrm{d}x < +\infty.$$

Definition A.2 We denote by $L^1_{\frac{n}{n-1}}(\Omega)$ the space of the functions $f \in L^1(\Omega)$ such that

$$\int_0^{|\Omega|} [\sigma\, f^{**}(\sigma)]^{\frac{n}{n-1}} \frac{\mathrm{d}\sigma}{\sigma} < +\infty$$

where

$$f^{**}(\sigma) = \frac{1}{\sigma} \sup_{|F|=\sigma} \int_F |f|\, \mathrm{d}x.$$

We reproduce now (see Esempio 1, page 3 from [13]) a helpful tool for constructing $L^{1,1}$-functions.

Remark .1 Let $f(x) \in L^1(B(0,\epsilon))$ then the function $g : \Omega = B(0,\epsilon)\times]0,1[\to I\!R$ defined by the law

$$g(x,t) = f(x)$$

belongs to the space $L^{1,1}(\Omega)$.

Indeed, fixed $y_o = (x_o, t_o) \in \Omega$ and $\rho \in]0, d_\Omega]$, we have

$$\begin{aligned}\int_{\Omega(y_o,\rho)} |g(x,t)|\, \mathrm{d}x\, \mathrm{d}t &= \int_{\Omega(y_o,\rho)} |f(x)|\, \mathrm{d}x\, \mathrm{d}t \\ &\le \int_{(B(0,\epsilon)\cap B(x_o,\rho))\times(]0,1[\cap]t_0-\rho,t_0+\rho[)} |f(x)|\, \mathrm{d}x\, \mathrm{d}t \\ &\le \rho \int_{B(0,\epsilon)} |f(x)|\, \mathrm{d}x\end{aligned}$$

Finally, we are ready to review the relationships among the aforementioned spaces.

(a) $L \log L \not\subset L^{1,\lambda}$.

It is well known that the function $f(x) = |x|^{-\alpha}$ belongs to $L\ \log\ L(B(0,R))$ for $\alpha \in]0,n[$. On the other hand, for any $\rho \in]0,R]$, it turns out

$$\rho^{-\lambda} \int_{B(0,\rho)\cap B(0,R)} |x|^{-\alpha}\, \mathrm{d}x = \rho^{-\lambda} \int_{B(0,\rho)} |x|^{-\alpha}\, \mathrm{d}x$$

and the right-hand side blows up as $\rho \to 0^+$ if $\alpha \in]n-\lambda, n[$.

(b) $L^{1,\lambda} \not\subset L \log L$.

Let $0 < \epsilon < 1$ and

$$f(x) = \frac{1}{|x|^n \log^2 |x|^n}, \quad x \in B(0,\epsilon).$$

By Remark .1 the function $g : \Omega = B(0,\epsilon)\times]0,1[\to I\!R$ defined by the law

$$g(x,t) = f(x)$$

belongs to the space $L^{1,1}(\Omega)$.

On the other hand, the maximal function of g does not belong to $L^1(\Omega)$ (see [3, p. 118]) and thus, by virtue of Theorem 6.7, page 250 from [3], g does not belong to the space $L\ \log\ L$.

(c) $L^1_{\frac{n}{n-1}} \not\subset L^{1,\lambda}$.

The function

$$f(x) = \frac{1}{|x|^n \log^2 |x|^n}, \quad x \in B(0,\epsilon), \quad 0 < \epsilon < 1,$$

belongs to the space $L^1_{\frac{n}{n-1}}(B(0,\epsilon))$ (see [11]) but direct calculations show that f does not belong to $L^{1,\lambda}(B(0,\epsilon))$ for any $\lambda \in]0,n[$.

(d) $L^{1,\lambda} \not\subset L^1_{\frac{n}{n-1}}$.

Let $n > 2$, $\Omega \subset I\!R^{n-1}$ and choose a function $f \in L^1(\Omega)$ such that $f \notin L^1_{\frac{n-1}{n-2}}(\Omega)$.

By virtue of Remark .1, the function $g : \Omega \times]0, 1[\to I\!R$ defined by the law

$$g(x, t) = f(x)$$

belongs to the space $L^{1,1}(\Omega \times]0, 1[)$.

We will prove now that $g \notin L^1_{\frac{n}{n-1}}(\Omega \times]0, 1[)$.

In fact, if g belongs to $L^1_{\frac{n}{n-1}}(\Omega \times]0, 1[)$ then, by definition (see [11]), we would have

$$\int_0^{|\Omega\times]0,1[|} [\sigma\, g^{**}(\sigma)]^{\frac{n}{n-1}} \frac{d\sigma}{\sigma} < +\infty$$

where

$$g^{**}(\sigma) = \frac{1}{\sigma} \sup_{|F|=\sigma} \int_F |g(x, t)|\, dx\, dt.$$

Observing now that

$$f^{**}(\sigma) \le g^{**}(\sigma), \quad \forall \sigma \in]0, |\Omega|[$$

and that $\sigma\, f^{**}(\sigma)$ is an increasing function (see page 176 from [23]), we deduce

$$\int_0^{|\Omega|} [\sigma\, f^{**}(\sigma)]^{\frac{n-1}{n-2}} \frac{d\sigma}{\sigma} = \int_0^{|\Omega|} [\sigma\, f^{**}(\sigma)]^{\frac{n}{n-1}} [\sigma f^{**}(\sigma)]^{\frac{n-1}{n-2}-\frac{n}{n-1}} \frac{d\sigma}{\sigma}$$
$$\le [|\Omega| f^{**}(|\Omega|)]^{\frac{n-1}{n-2}-\frac{n}{n-1}} \int_0^{|\Omega|} [\sigma\, g^{**}(\sigma)]^{\frac{n}{n-1}} \frac{d\sigma}{\sigma} < +\infty$$

whence $f \in L^1_{\frac{n-1}{n-2}}(\Omega)$ which contradicts the assumption.

References

1. Adams, R.A.: Sobolev Spaces. Academic Press, New York (1975)
2. Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M., Vazquez, J.L.: An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Sc. Normale Sup. Pisa IV **XXII**(2) (1995)
3. Bennett, C., Sharpley, R.: Interpolation of operators. In: Pure and Applied Mathematics. Academic Press, New York (1988)
4. Boccardo, L.: Problemi differenziali ellittici e parabolici con dati misure. Boll. U.M.I. **11-A**(7) (1997)
5. Boccardo, L., Gallouët, T.: Non linear elliptic and parabolic equations involving measure data. J. Func. Anal. **87**(1), 149–169 (1989)
6. Boccardo, L., Gallouët, T.: Nonlinear elliptic equations with right-hand side measures. Comm. Partial Diff. Eqs. **17**, 641–655 (1992)
7. Campanato, S.: Proprietá di inclusione per spazi di Morrey. Ricerche Mat. **12** (1963)
8. Campanato, S.: Equazioni ellittiche del II^o ordine e spazi $\mathcal{L}^{2,\lambda}$. Ann. Mat. Pura Appl. **69** (1965)
9. Campanato, S.: Sistemi Ellittici in forma divergenza. Regolarità all'interno. Quaderni Scuola Normale Superiore Pisa, Pisa (1980)
10. De Giorgi, E.: Sulla differenziabilitá e l'analiticità delle estremali degli integrali multipli regolari. Memorie dell'Accademia delle Scienze di Torino, Serie 3 **3**(I) (1957)
11. Del Vecchio, T.: Nonlinear elliptic equations with measure data. Potential Anal. **4**(2) (1995)
12. Di Fazio, G.: Poisson equations and Morrey spaces. J. Math. Anal. Appl. **163**(1) (1992)
13. Di Fazio, G.: Regolarità delle soluzioni generalizzate del problema di Dirichlet in ipotesi minime sui coefficienti. Ph.D. Dissertation, Catania (1993)
14. Giaquinta, M.: Introduction to regularity theory for nonlinear elliptic systems. Lecture in Mathematics. Birkhäuser Verlag (1993)

15. Kilpeläinen, T.: Hölder continuity of solutions to quasilinear elliptic equations involving measures, Potential Anal. **3** (1994)
16. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)
17. Kovalesvskii, A.A.: Integrability of solutions of nonlinear elliptic equations with right-hand sides from classes close to L^1. Math. Notes **70**(3), 337–346 (2001)
18. Kufner, A., John, O., Fučik, S.: Function Spaces. Noordhoff Int. Pub. Leyden. Academia Prague (1977)
19. Miranda, C.: Istituzioni di Analisi Funzionale Lineare. U.M.I. (1978)
20. Serrin, J.: Pathological solutions of elliptic differential equations. Ann. Sc. Norm. Sup. Pisa (1964)
21. Stampacchia, G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Ann. Ist. Fourier, Grenoble **15**(1) (1965)
22. Stampacchia, G.: Équations elliptiques du second ordre à coefficients discontinus. Université de Montréal (1966)
23. Talenti, G.: Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces. Ann. Mat. Pura e Appl. **120**(4) (1979)
24. Troianiello, G.M.: Elliptic Differential Equations and Obstacle Problems. The University Series in Mathematics. Plenum Press, New York (1987)