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On the structure of positive solutions of a class of fourth order nonlinear differential equations

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Abstract A detailed analysis is made of the structure of positive solutions of fourth-order differential equations of the form

$$(p(t)|x''|^{\alpha-1}x'')'' + q(t)|x|^{\beta-1}x = 0, \tag{A}$$

under the assumption that α, β are positive constants, $p(t)$, $q(t)$ are positive continuous functions on $[a, \infty)$, and $p(t)$ satisfies

$$\int_a^\infty t^{1+(1/\alpha)}(p(t))^{-1/\alpha}\,\mathrm{d}t < \infty.$$

Keywords Fourth-order nonlinear differential equation · Positive solution · Oscillation

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1 Introduction

This paper is concerned with the oscillatory and nonoscillatory behavior of fourth-order nonlinear differential equations of the form

$$(p(t)|x''|^{\alpha-1}x'')'' + q(t)|x|^{\beta-1}x = 0, \tag{A}$$

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where $\alpha > 0$, $\beta > 0$ are constants and $p, q : [a, \infty) \to (0, \infty)$ are continuous functions, $a \geq 0$. It is convenient to use the asterisk notation

$$\xi^{\gamma *} = |\xi|^{\gamma-1}\xi = |\xi|^{\gamma} \operatorname{sgn} \xi, \quad \xi \in \mathbb{R}, \quad \gamma > 0, \tag{1.1}$$

in terms of which Eq. (A) is expressed in the form

$$(p(t)(x'')^{\alpha *})'' + q(t)x^{\beta *} = 0.$$

By a solution of (A) we mean a function $x : [T_x, \infty) \to \mathbb{R}$ which has the property that $p(t)|x''|^{\alpha-1}x''$ is twice continuously differentiable and satisfies Eq. (A) at every point of $[T_x, \infty)$. Those solutions of (A) which vanish identically in some neighborhood of infinity will be excluded from our consideration. A solution of (A) is called oscillatory if it has a sequence of zeros clustering at $t = \infty$ and nonoscillatory otherwise.

Oscillation theory for equations of the type (A) was first developed by Wu [6], who established sharp oscillation criteria for the case where the function $p(t)$ in (A) satisfies the integral conditions

$$\int_a^\infty t(p(t))^{-1/\alpha}\, \mathrm{d}t = \infty, \quad \int_a^\infty t^{1/\alpha}(p(t))^{-1/\alpha}\, \mathrm{d}t = \infty. \tag{1.2}$$

His intention was to fully generalize the oscillation theorems of Kusano and Naito [3, 4] for semilinear equations of the form

$$(p(t)x'')'' + q(t)|x|^{\beta-1}x = 0 \tag{B}$$

to quasilinear equations of the type (A) involving nonlinear differential operators.

Wu's theory has been enriched with information about the asymptotic behavior of nonoscillatory solutions of (A) provided by Naito and Wu [5] and Kamo and Usami [1, 2] for the case where $p(t)$ satisfies (1.2), or one of the following

$$\int_a^\infty t(p(t))^{-1/\alpha}\, \mathrm{d}t = \infty, \quad \int_a^\infty t^{1/\alpha}(p(t))^{-1/\alpha}\, \mathrm{d}t < \infty, \tag{1.3}$$

$$\int_a^\infty t(p(t))^{-1/\alpha}\, \mathrm{d}t < \infty, \quad \int_a^\infty t^{1/\alpha}(p(t))^{-1/\alpha}\, \mathrm{d}t = \infty, \tag{1.4}$$

$$\int_a^\infty t(p(t))^{-1/\alpha}\, \mathrm{d}t < \infty, \quad \int_a^\infty t^{1/\alpha}(p(t))^{-1/\alpha}\, \mathrm{d}t < \infty. \tag{1.5}$$

The aim of this paper is to proceed further along the path paved by Wu to investigate the possibility of enlarging the class of equations of the type (A) for which oscillation theory of a similar nature can also be developed. We will impose the integral condition on $p(t)$

$$\int_a^\infty t^{1+(1/\alpha)}(p(t))^{-1/\alpha}\, \mathrm{d}t < \infty \tag{1.6}$$

and show that, for Eq. (A) with $p(t)$ satisfying this condition, it is possible to gain a detailed knowledge of the structure of nonoscillatory solutions, from which an effective criterion for oscillation of all solutions can be drawn.

The main body of the paper is divided into three sections. In Sect. 1 we classify the totality of nonoscillatory solutions of (A) into several types according to their asymptotic behavior at infinity. There a crucial role is played by the functions

$$\varphi_1(t) = \int_t^\infty (s-t)(p(s))^{-1/\alpha}\,\mathrm{d}s, \quad \varphi_2(t) = \int_t^\infty (s-t)s^{1/\alpha}(p(s))^{-1/\alpha}\,\mathrm{d}s,$$
$$\varphi_3(t) = 1, \quad \varphi_4(t) = t, \tag{1.7}$$

which are the particular solutions of the unperturbed differential equation

$$(p(t)|x''|^{\alpha-1}x'')'' = 0. \tag{A$_0$}$$

In Sect. 2 we establish the existence of nonoscillatory solutions of (A) belonging to each of the classes appearing on the classification list. Fixed point techniques are used for this purpose. Analysis of the structure of nonoscillatory solutions of (A) allows us to indicate explicit conditions under which all nonoscillatory solutions of (A) cease to exist. Such conditions are nothing else but oscillation criteria for (A). Our results, when specialized to the case $\alpha = 1$, will serve to supplement the oscillation theorems for (B) obtained in the papers [3, 4]. Some examples illustrating our main results will be presented in Sect. 3.

2 Classification of positive solutions

We begin by classifying the set of all possible nonoscillatory solutions of (A) according to their asymptotic behavior as $t \to \infty$. Clearly, it suffices to restrict our attention to the totality of eventually positive solutions of (A). Let $x(t)$ be an eventually positive solution of (A). Since, from (A), $(p(t)(x''(t))^{\alpha *})''$ is eventually negative, it follows that all the functions $x(t),\ x'(t),\ x''(t)$ and $(p(t)(x''(t))^{\alpha *})'$ are eventually monotone and one-signed. Using the fact that if a C^2-function $x(t)$ satisfies $x'(t) < 0$ and $x''(t) < 0$ for all large t, then $x(t) \to -\infty$ as $t \to \infty$, we see that the following four types of combination of the signs of $x'(t)$, $x''(t)$ and $(p(t)(x''(t))^{\alpha *})'$ are possible for an eventually positive solution $x(t)$ of (A):

$$(p(t)(x''(t))^{\alpha *})' > 0, x''(t) > 0, \quad x'(t) > 0; \quad \text{(I)}$$
$$(p(t)(x''(t))^{\alpha *})' > 0, x''(t) > 0, \quad x'(t) < 0; \quad \text{(II)}$$
$$(p(t)(x''(t))^{\alpha *})' > 0, x''(t) < 0, \quad x'(t) > 0; \quad \text{(III)}$$
$$(p(t)(x''(t))^{\alpha *})' < 0, x''(t) < 0, \quad x'(t) > 0. \quad \text{(IV)}$$

We first investigate the precise asymptotic behavior (order of growth or decay) as $t \to \infty$ of solutions belonging to these types I–IV.

(Solutions of type I). Let $x(t)$ be a positive solution of type I on $[t_0, \infty)$. The function $(p(t)(x''(t))^{\alpha *})'$ is positive and decreasing, so that it tends to a finite limit $\omega_3 \ge 0$ as $t \to \infty$. Integrating (A) from t to ∞, we have

$$(p(t)(x''(t))^{\alpha *})' = \omega_3 + \int_t^\infty q(s)(x(s))^\beta\,\mathrm{d}s, \quad t \ge t_0, \tag{2.1}$$

with the understanding that the integral on the right-hand side converges:

$$\int_t^\infty q(s)(x(s))^\beta\,\mathrm{d}s < \infty, \quad t \ge t_0. \tag{2.2}$$

We then integrate (2.1) on $[t_0, t]$ to obtain

$$x''(t) = (p(t))^{-1/\alpha} \left[\xi_2 + \int_{t_0}^{t} \left(\omega_3 + \int_s^{\infty} q(r)(x(r))^{\beta} \, dr \right) ds \right]^{1/\alpha}, \tag{2.3}$$

where $\xi_2 = p(t_0)(x''(t_0))^{\alpha} > 0$. Since $\int_{t_0}^{t} \left(\omega_3 + \int_s^{\infty} q(r)(x(r))^{\beta} \, dr \right) ds = O(t)$ as $t \to \infty$, condition (1.6) enables us to integrate (2.3) over $[t, \infty)$:

$$x'(t) = \omega_1 - \int_t^{\infty} (p(s))^{-1/\alpha} \left[\xi_2 + \int_{t_0}^{s} \left(\omega_3 + \int_r^{\infty} q(\rho)(x(\rho))^{\beta} d\rho \right) dr \right]^{1/\alpha} ds \tag{2.4}$$

for $t \geq t_0$, where $\omega_1 = \lim_{t \to \infty} x'(t) > 0$. Integration of (2.4) on $[t_0, t]$ yields the integral equation for a type I-solution $x(t)$ of (A):

$$x(t) = \xi_0 + \int_{t_0}^{t} \left\{ \omega_1 - \int_s^{\infty} (p(r))^{-1/\alpha} \left[\xi_2 + \int_{t_0}^{r} \left(\omega_3 + \int_{\sigma}^{\infty} q(\rho)(x(\rho))^{\beta} \, d\rho \right) d\sigma \right]^{1/\alpha} dr \right\} ds, \; t \geq t_0, \tag{2.5}$$

where $\xi_0 = x(t_0) > 0$, which implies that $x(t)$ is asymptotic to a constant multiple of $\varphi_4(t) = t$ as $t \to \infty$. This fact is expressed by $x(t) \sim c\varphi_4(t)$ as $t \to \infty$, where the symbol $\sim$ is used to mean the asymptotic equivalence

$$f(t) \sim g(t) \;\text{ as }\; t \to \infty \Leftrightarrow \lim_{t \to \infty} \frac{f(t)}{g(t)} = 1. \tag{2.6}$$

(Solutions of type II). Let $x(t)$ be a solution of type II on $[t_0, \infty)$. There exists a finite limit $\omega_1 = \lim_{t\to\infty} x'(t) \leq 0$. If $\omega_1 < 0$, then the inequality $x'(t) \leq \omega_1, \; t \geq t_0$, yields $\lim_{t\to\infty} x(t) = -\infty$, an impossibility, and so ω_1 must be zero. Proceeding as in the above case and taking this fact into account, we arrive at (1.4) with $\omega_1 = 0$. Since the limit $\omega_0 = \lim_{t\to\infty} x(t) \geq 0$ is clearly finite, integrating (1.4) from t to ∞ shows that

$$x(t) = \omega_0 + \int_t^{\infty} (s - t)(p(s))^{-1/\alpha} \times \left[\xi_2 + \int_{t_0}^{s} \left(\omega_3 + \int_r^{\infty} q(\rho)(x(\rho))^{\beta} \, d\rho \right) dr \right]^{1/\alpha} ds, \quad t \geq t_0. \tag{2.7}$$

There are two possible cases: either $\omega_0 > 0$ or $\omega_0 = 0$. If $\omega_0 > 0$, then it is clear that $x(t) \sim \omega_0 = \omega_0 \varphi_3(t)$ as $t \to \infty$. Suppose that $\omega_0 = 0$. Then, as easily verified, we have $\lim_{t\to\infty} x(t)/\varphi_2(t) = \omega_3^{1/\alpha}$, which implies that $x(t) \sim \omega_3^{1/\alpha} \varphi_2(t)$ or $x(t) = o(\varphi_2(t))$ as $t \to \infty$ according as $\omega_3 > 0$ or $\omega_3 = 0$. Furthermore, we find that

$$\lim_{t \to \infty} \frac{x(t)}{\varphi_1(t)} = \left[\xi_2 + \int_{t_0}^{\infty} \int_s^{\infty} q(r)(x(r))^{\beta} \, dr \, ds \right]^{1/\alpha}, \tag{2.8}$$

where the repeated integral on the right-hand side diverges or converges according to whether

$$\int^{\infty} tq(t)(x(t))^{\beta}\, \mathrm{d}t = \infty \quad \text{or} \quad \int^{\infty} tq(t)(x(t))^{\beta}\, \mathrm{d}t < \infty, \tag{2.9}$$

respectively. In the former case we have $\lim_{t\to\infty} x(t)/\varphi_1(t) = \infty$, and in the latter case we have $x(t) \sim c\varphi_1(t)$ as $t \to \infty$, where c denotes the right-hand side of (2.8). Summarizing the above arguments, we conclude that the four patterns of asymptotic behavior are possible for a type II-solution $x(t)$ of (A):

$$x(t) \sim c_0, \quad x(t) \sim c_1\varphi_1(t), \quad x(t) \sim c_2\varphi_2(t), \quad \varphi_1(t) \prec x(t) \prec \varphi_2(t),$$

as $t \to \infty$, where $c_i > 0,\ i = 0,\ 1,\ 2$, are constants and the symbol $f(t) \prec g(t)$ is used to mean

$$\lim_{t\to\infty} \frac{g(t)}{f(t)} = \infty. \tag{2.10}$$

(Solutions of type III). Let $x(t)$ be a positive solution of type III on $[t_0, \infty)$. In this case we have (2.1) with $\omega_3 = 0$. In fact, if $\omega_3 > 0$, then integrating $(p(t)(x''(t))^{\alpha *})' \ge \omega_3,\ t \ge t_0$, we have $x''(t) > 0, t \ge t_0$, which is impossible for a type-III solution of (A). We now integrate (2.1) ($\omega_3 = 0$) on $[t_0, t]$, obtaining

$$-x''(t) = (p(t))^{-1/\alpha} \left[\xi_2 - \int_{t_0}^{t} \int_{s}^{\infty} q(r)(x(r))^{\beta}\, \mathrm{d}r\, \mathrm{d}s\right]^{1/\alpha}, t \ge t_0, \tag{2.11}$$

where $\xi_2 = p(t_0)(-x''(t_0))^{\alpha}$. We note that the repeated integral on the right-hand side of (2.11) must remain bounded as $t \to \infty$, so that the second relation of (2.9) is satisfied. We now put $\omega_1 = \lim_{t\to\infty} x'(t)$. Since $\omega_1 \ge 0$ is finite, (2.11) can be integrated over $[t, \infty)$ to yield

$$x'(t) = \omega_1 + \int_{t}^{\infty} (p(s))^{-1/\alpha} \left[\xi_2 - \int_{t_0}^{s} \int_{r}^{\infty} q(\rho)(x(\rho))^{\beta}\, \mathrm{d}\rho\, \mathrm{d}r\right]^{1/\alpha} \mathrm{d}s, \quad t \ge t_0. \tag{2.12}$$

Integrating (2.12) on $[t_0, t]$, we have

$$\begin{aligned} x(t) = \xi_0 + \int_{t_0}^{t} \Bigg\{\omega_1 + \int_{s}^{\infty} (p(r))^{-1/\alpha} \\ \times \left[\xi_2 - \int_{t_0}^{r} \int_{\rho}^{\infty} q(\sigma)(x(\sigma))^{\beta}\, \mathrm{d}\sigma\, \mathrm{d}\rho\right]^{1/\alpha} \mathrm{d}r\Bigg\}\, \mathrm{d}s \end{aligned} \tag{2.13}$$

for $t \ge t_0$, where $\xi_0 = x(t_0) > 0$, whence we find that if $\omega_1 > 0$, then $x(t) \sim \omega_1 t = \omega_1\varphi_4(t)$ as $t \to \infty$, and if $\omega_1 = 0$, then $x(t) \sim \omega_0 = \omega_0\varphi_3(t)$ as $t \to \infty$ for some $\omega_0 > 0$. It follows that a solution of type III satisfies either $x(t) \sim c_3\varphi_3(t)$ or $x(t) \sim c_4\varphi_4(t)$ as $t \to \infty$ for some constant $c_3 > 0$ or $c_4 > 0$.

(Solutions of type IV). Let $x(t)$ be a positive solution of type IV on $[t_0, \infty)$. Our first task is to integrate (A) twice over $[t_0, t]$. As a result, we have

$$-x''(t) = (p(t))^{-1/\alpha} \left[\xi_2 + \xi_3(t - t_0) + \int_{t_0}^{t} (t - s)q(s)(x(s))^{\beta}\, \mathrm{d}s\right]^{1/\alpha}, \ t \ge t_0, \tag{2.14}$$

where $\xi_2 = p(t_0)(-x''(t_0))^\alpha$ and $\xi_3 = (p(t)(-x''(t))^\alpha)'$ at $t = t_0$. The existence of the finite limit $\omega_1 = \lim_{t\to\infty} x'(t) \geq 0$ implies the integrability of the right-hand side of (2.14) on $[t_0, \infty)$, in particular, the convergence of the integral

$$\int_{t_0}^{\infty} (p(t))^{-1/\alpha} \left[\int_{t_0}^{t} (t-s)q(s)(x(s))^\beta \, \mathrm{d}s\right]^{1/\alpha} \mathrm{d}t < \infty. \tag{2.15}$$

Integrating (2.14) on $[t, \infty)$ and then integrate the resulting equation from t_0 to t, we obtain

$$x(t) = \xi_0 + \omega_1(t-t_0) + \int_{t_0}^{t}\int_{s}^{\infty} (p(r))^{-1/\alpha} \Bigg[\xi_2 + \xi_3(r-t_0) + \int_{t_0}^{r} (r-\rho)q(\rho)(x(\rho))^\beta \, \mathrm{d}\rho\Bigg]^{1/\alpha} \mathrm{d}r \, \mathrm{d}s, \quad t \geq t_0, \tag{2.16}$$

where $\xi_0 = x(t_0) > 0$. From (2.16) we see that $\omega_1 > 0$ implies $x(t) \sim \omega_1 t = \omega_1\varphi_4(t)$ as $t \to \infty$, while $\omega_1 = 0$ implies $\lim_{t\to\infty} x(t)/t = 0$, i.e. $x(t) \prec \varphi_4(t)$ as $t \to \infty$. Furthermore, if $\omega_1 = 0$ in (2.16), then it can be shown that $\lim_{t\to\infty} x(t) = \omega_0$ (i.e. $x(t) \sim \omega_0\varphi_3(t)$) for some $\omega_0 > 0$ or $\lim_{t\to\infty} x(t) = \infty$ (i.e. $\varphi_3(t) \prec x(t)$ as $t \to \infty$) according to whether

$$\int_{t_0}^{\infty} t(p(t))^{-1/\alpha} \left[\int_{t_0}^{t} (t-s)q(s)(x(s))^\beta \, \mathrm{d}s\right]^{1/\alpha} \mathrm{d}t < \infty \tag{2.17}$$

or

$$\int_{t_0}^{\infty} t(p(t))^{-1/\alpha} \left[\int_{t_0}^{t} (t-s)q(s)(x(s))^\beta \, \mathrm{d}s\right]^{1/\alpha} \mathrm{d}t = \infty. \tag{2.18}$$

Thus, a solution $x(t)$ of type IV has the property that

$$x(t) \sim c_3\varphi_3(t) \quad \text{or} \quad x(t) \sim c_4\varphi_4(t) \quad \text{or} \quad \varphi_3(t) \prec x(t) \prec \varphi_4(t) \quad \text{as } t \to \infty$$

for some constant $c_3 > 0$ or $c_4 > 0$.

3 Existence of positive solutions

It has been shown above that a positive solution $x(t)$ of Eq. (A) enjoys one of the asymptotic properties:

$$x(t) \sim c_i\varphi_i(t), \;\; i \in \{1,\ 2,\ 3,\ 4\}, \;\; \varphi_1(t) \prec x(t) \prec \varphi_2(t), \;\; \varphi_3(t) \prec x(t) \prec \varphi_4(t),$$

where $c_i,\ i = 1,\ 2,\ 3,\ 4$, are positive constants.

Our purpose here is to prove that all of the above cases may occur.

(A) We first discuss the existence of a solution $x(t)$ which is asymptotic to a constant multiple of $\varphi_4(t) = t$ as $t \to \infty$.

Theorem 3.1 *Equation* (A) *has a positive solution* $x(t)$ *of type I such that* $x(t) \sim ct$ *as* $t \to \infty$ *for some* $c > 0$ *if and only if*

$$\int_a^\infty t^\beta q(t)\,\mathrm{d}t < \infty. \tag{3.1}$$

Theorem 3.2 *Equation* (A) *has a positive solution* $x(t)$ *of type* III *such that* $x(t) \sim ct$ *as* $t \to \infty$ *for some* $c > 0$ *if and only if*

$$\int_a^\infty t^{\beta+1} q(t)\,\mathrm{d}t < \infty. \tag{3.2}$$

Theorem 3.3 *Equation* (A) *has a positive solution* $x(t)$ *of type* IV *such that* $x(t) \sim ct$ *as* $t \to \infty$ *for some* $c > 0$ *if and only if*

$$\int_a^\infty (p(t))^{-1/\alpha} \left[\int_a^t (t-s)s^\beta q(s)\,\mathrm{d}s\right]^{1/\alpha} \mathrm{d}t < \infty. \tag{3.3}$$

Proof of Theorem 3.1. Let a constant $c > 0$ be fixed arbitrarily. Choose $t_0 \geq a$ so that

$$2Q^{1/\alpha} \int_{t_0}^\infty t^{1/\alpha}(p(t))^{-1/\alpha}\,\mathrm{d}t \leq c^{1-(\beta/\alpha)}, \quad \text{where } Q = \int_0^\infty t^\beta q(t)\,\mathrm{d}t. \tag{3.4}$$

Define the set $X_1 \subset C[t_0, \infty)$ and the operator $\mathcal{F}_1 : X_1 \to C[t_0, \infty)$ by

$$X_1 = \left\{x \in C[t_0, \infty) : \ \frac{1}{2}c(t-t_0) \leq x(t) \leq c(t-t_0), \ \ t \geq t_0\right\} \tag{3.5}$$

and

$$\mathcal{F}_1 x(t) = \int_{t_0}^t \left\{c - \int_s^\infty (p(r))^{-1/\alpha} \times \left[\int_{t_0}^r \int_\rho^\infty q(\sigma)(x(\sigma))^\beta\,\mathrm{d}\sigma\,\mathrm{d}\rho\right]^{1/\alpha} \mathrm{d}r\right\} \mathrm{d}s, \quad t \geq t_0. \tag{3.6}$$

Because of (3.4) $\mathcal{F}_1$ maps X_1 into itself. It can be shown that $\mathcal{F}_1$ is a continuous mapping by means of the Lebesgue dominated convergence theorem, and that the set $\mathcal{F}_1(X_1)$ is precompact in $C[t_0, \infty)$ with the help of the Ascoli–Arzelà theorem. Therefore, by the Schauder–Tychonoff fixed point theorem, there exists an element $x_1 \in X_1$ such that $x_1 = \mathcal{F}_1 x_1$ which is a special case of the integral Eq. (2.5). It is easy to verify that the function $x_1 = x_1(t)$ is a positive type I-solution of (A) on $[t_0, \infty)$ satisfying $x_1(t) \sim ct$ as $t \to \infty$. □

Proof of Theorems 3.2 and 3.3 (Theorem 3.2) Let $c > 0$ be any constant, choose $t_0 \geq a$ so that

$$\int_{t_0}^\infty (p(t))^{-1/\alpha}\,\mathrm{d}t \leq 1 \quad \text{and} \quad 2^\beta \int_{t_0}^\infty t^{\beta+1} q(t)\,\mathrm{d}t \leq c^{\alpha-\beta} \tag{3.7}$$

and consider the set of functions

$$X_2 = \{x \in C[t_0, \infty)\colon \ c(t-t_0) \leq x(t) \leq 2c(t-t_0), \ \ t \geq t_0\}. \tag{3.8}$$

The desired solution of (A) of type III is obtained, via the Schauder–Tychonoff theorem, as a fixed point $x_2 \in X_2$ of the operator $\mathcal{F}_2$ defined by

$$\mathcal{F}_2 x(t) = \int_{t_0}^{t} \left\{ c + \int_{s}^{\infty} (p(r))^{-1/\alpha} \times \left[c^\alpha - \int_{t_0}^{r} \int_{\rho}^{\infty} q(\sigma)(x(\sigma))^\beta \, \mathrm{d}\sigma \, \mathrm{d}\rho \right]^{1/\alpha} \mathrm{d}r \right\} \mathrm{d}s, \quad t \geq t_0. \tag{3.9}$$

Note that $x_2 = \mathcal{F}_2 x_2$ is an integral equation of the type (2.13).

(Theorem 3.3) Given a $c > 0$, choose $t_0 \geq a$ so that

$$2^{\beta/\alpha} \int_{t_0}^{\infty} (p(t))^{-1/\alpha} \left[\int_{t_0}^{t} (t-s) s^\beta q(s) \, \mathrm{d}s \right]^{1/\alpha} \mathrm{d}t \leq c^{1-(\beta/\alpha)}, \tag{3.10}$$

and define the integral operator $\mathcal{F}_3$ by

$$\mathcal{F}_3 x(t) = c(t - t_0) + \int_{t_0}^{t} \int_{s}^{\infty} (p(r))^{-1/\alpha} \times \left[\int_{t_0}^{r} (r-\rho) q(\rho)(x(\rho))^\beta \, \mathrm{d}\rho \right]^{1/\alpha} \mathrm{d}r \, \mathrm{d}s, \quad t \geq t_0. \tag{3.11}$$

Then, by the Schauder–Tychonoff theorem $\mathcal{F}_3$ has a fixed element x_3 in the set X_2 defined by (3.8), and since this fixed element $x_3 = x_3(t)$ satisfies an integral equation of the type (2.16) it provides a solution of (A) of type IV with the desired asymptotic property.

Noting that (3.2) $\Rightarrow$ (3.1) $\Rightarrow$ (3.3), we have the following corollary.

Corollary 3.1 *Equation* (A) *has a positive solution* $x(t)$ *such that* $x(t) \sim ct$ *as* $t \to \infty$ *for some* $c > 0$ *if and only if* (3.3) *is satisfied.*

(B) Let us now provide criteria for the existence of solutions for (A) which are asymptotic to constant multiples of $\varphi_3(t) = 1$ as $t \to \infty$.

Theorem 3.4 *Equation* (A) *has a positive solution* $x(t)$ *of type* II *such that* $x(t) \sim c$ *as* $t \to \infty$ *for some* $c > 0$ *if and only if*

$$\int_{a}^{\infty} q(t) \, \mathrm{d}t < \infty. \tag{3.12}$$

Theorem 3.5 *Equation* (A) *has a positive solution* $x(t)$ *of type* III *such that* $x(t) \sim c$ *as* $t \to \infty$ *for some* $c > 0$ *if and only if*

$$\int_{a}^{\infty} t q(t) \, \mathrm{d}t < \infty. \tag{3.13}$$

Theorem 3.6 *Equation* (A) *has a positive solution* $x(t)$ *of type* IV *such that* $x(t) \sim c$ *as* $t \to \infty$ *for some* $c > 0$ *if and only if*

$$\int_{a}^{\infty} t (p(t))^{-1/\alpha} \left[\int_{a}^{t} (t-s) q(s) \, \mathrm{d}s \right]^{1/\alpha} \mathrm{d}t < \infty. \tag{3.14}$$

All the above theorems can also be proved by means of the Schauder–Tychonoff fixed point theorem.

(i) Suppose that (3.12) holds. For any $c > 0$ choose $t_0 \geq a$ so that

$$2^{\beta/\alpha} Q^{1/\alpha} \int_{t_0}^{\infty} t^{1+(1/\alpha)} (p(t))^{-1/\alpha}\, dt \leq c^{1-(\beta/\alpha)}, \quad \text{where } Q = \int_0^{\infty} q(t)\, dt, \tag{3.15}$$

and define

$$\mathcal{G}_1 x(t) = c + \int_t^{\infty} (s-t)(p(s))^{-1/\alpha} \times \left[\int_{t_0}^{s} \int_r^{\infty} q(\rho)(x(\rho))^{\beta}\, d\rho\, dr \right]^{1/\alpha} ds, \quad t \geq t_0. \tag{3.16}$$

Then, $\mathcal{G}_1$ has a fixed point x_1 in the set

$$X = \{x \in C[t_0, \infty) \colon\ c \leq x(t) \leq 2c,\ \ t \geq t_0\}. \tag{3.17}$$

Since the integral equation $x_1 = \mathcal{G}_1 x_1$ is a special case of (2.7) it follows that this $x_1(t)$ is of type II and satisfies $x_1(t) \sim c$ as $t \to \infty$.

(ii) Suppose that (3.13) holds. For any $c > 0$ choose $t_0 \geq a$ so that

$$\int_{t_0}^{\infty} t(p(t))^{-1/\alpha}\, dt \leq 1 \quad \text{and} \quad 2^{\beta} \int_{t_0}^{\infty} t q(t)\, dt \leq c^{\alpha-\beta}. \tag{3.18}$$

Then, the operator $\mathcal{G}_2$ defined by

$$\mathcal{G}_2 x(t) = c + \int_{t_0}^{t} \int_s^{\infty} (p(r))^{-1/\alpha} \times \left[c^{\alpha} - \int_{t_0}^{r} \int_{\rho}^{\infty} q(\sigma)(x(\sigma))^{\beta}\, d\sigma\, d\rho \right]^{1/\alpha} dr\, ds, \quad t \geq t_0, \tag{3.19}$$

has a fixed point x in the set X given by (3.17). It is easy to check that the function $x_2 = x_2(t)$ is a solution of type III satisfying $x_2(t) \sim c$ as $t \to \infty$. The use of $\mathcal{G}_2$ was motivated by the derivation of (2.13).

(iii) In case (3.14) holds, one can construct a desired solution of type IV as a solution of an integral equation of the type (2.16), more precisely, as a fixed point of the operator $\mathcal{G}_3$ defined by

$$\mathcal{G}_3 x(t) = c + \int_{t_0}^{t} \int_s^{\infty} (p(r))^{-1/\alpha} \times \left[\int_{t_0}^{r} (r-\rho) q(\rho)(x(\rho))^{\beta}\, d\rho \right]^{1/\alpha} dr\, ds, \quad t \geq t_0, \tag{3.20}$$

where $c > 0$ is an arbitrarily given constant and $t_0 > 0$ is chosen so large that

$$2^{\beta/\alpha} \int_{t_0}^{\infty} t(p(t))^{-1/\alpha} \left[\int_{t_0}^{t} (t-s) q(s)\, ds \right]^{1/\alpha} dt \leq c^{1-(\beta/\alpha)}. \tag{3.21}$$

The existence of such an x_3 in the set X defined by (3.17) follows from the Schauder–Tychonoff fixed point theorem applied to $\mathcal{G}_3$.

From Theorems 3.4–3.6 combined with the observation that (3.13) $\Rightarrow$ (3.12) $\Rightarrow$ (3.14), we have the following result.

Corollary 3.2 *Equation* (A) *has a positive solution $x(t)$ such that $x(t) \sim c$ as $t \to \infty$ if and only if* (3.14) *is satisfied.*

(C) We now turn to the existence of decaying positive solutions (of type II) which behave like the functions

$$\varphi_1(t) = \int_t^\infty (s-t)(p(s))^{-1/\alpha}\,\mathrm{d}s, \quad \varphi_2(t) = \int_t^\infty (s-t)s^{1/\alpha}(p(s))^{-1/\alpha}\,\mathrm{d}s.$$

Theorem 3.7 *Equation* (A) *has a positive solution $x(t)$ such that $x(t) \sim c\varphi_1(t)$ as $t \to \infty$ for some $c > 0$ if and only if*

$$\int_a^\infty t(\varphi_1(t))^\beta q(t)\,\mathrm{d}t < \infty. \tag{3.22}$$

Theorem 3.8 *Equation* (A) *has a positive solution $x(t)$ such that $x(t) \sim c\varphi_2(t)$ as $t \to \infty$ for some $c > 0$ if and only if*

$$\int_a^\infty (\varphi_2(t))^\beta q(t)\,\mathrm{d}t < \infty. \tag{3.23}$$

Proof of Theorems 3.7 and 3.8. Our aim is to solve two integral equations of the type (2.7) which generate positive solutions of (A) with the desired asymptotic properties. Let $c > 0$ be any fixed constant and choose $t_i \geq 0,\ i = 1,\ 2$, large enough so that

$$2^\beta \int_{t_1}^\infty t(\varphi_1(t))^\beta q(t)\,\mathrm{d}t \leq (2^\alpha - 1)c^{\alpha-\beta}, \tag{3.24}$$

and

$$2^\beta \int_{t_2}^\infty (\varphi_2(t;t_2))^\beta q(t)\,\mathrm{d}t \leq (2^\alpha - 1)c^{\alpha-\beta}, \tag{3.25}$$

where

$$\varphi_2(t;t_2) = \int_t^\infty (s-t)(s-t_2)^{1/\alpha}(p(s))^{-1/\alpha}\,\mathrm{d}s, \quad t \geq t_2.$$

Let the sets $X_i,\ i = 1,\ 2$, of continuous functions be defined by

$$X_1 = \{x \in C[t_1,\infty):\ c\varphi_1(t) \leq x(t) \leq 2c\varphi_1(t),\ \ t \geq t_1\}, \tag{3.26}$$

$$X_2 = \{x \in C[t_2,\infty):\ c\varphi_2(t;t_2) \leq x(t) \leq 2c\varphi_2(t;t_2),\ \ t \geq t_2\}. \tag{3.27}$$

Application of the Schauder–Tychonoff theorem to the operators $\mathcal{H}_i$, $i = 1, 2$, defined by

$$\mathcal{H}_1 x(t) = \int_t^\infty (s-t)(p(s))^{-1/\alpha} \times \left[c^\alpha + \int_{t_1}^s \int_r^\infty q(\rho)(x(\rho))^\beta \,\mathrm{d}\rho \,\mathrm{d}r \right]^{1/\alpha} \mathrm{d}s, \quad t \geq t_1, \tag{3.28}$$

$$\mathcal{H}_2 x(t) = \int_t^\infty (s-t)(p(s))^{-1/\alpha} \times \left[\int_{t_1}^s \left(c^\alpha + \int_r^\infty q(\rho)(x(\rho))^\beta \,\mathrm{d}\rho \right) \mathrm{d}r \right]^{1/\alpha} \mathrm{d}s, \quad t \geq t_2, \tag{3.29}$$

shows that $\mathcal{H}_i$ has a fixed point x_i in the set X_i, which gives a positive solution $x_i = x_i(t)$ of (A) on $[t_i, \infty)$, $i = 1, 2$. It is a matter of easy calculation to verify that $x_i(t) \sim c_i \varphi_i(t)$ for some constant $c_i > 0$ as $t \to \infty$, $i = 1, 2$. This completes the proof of Theorems 3.7 and 3.8.

(D) It remains to examine positive solutions $x(t)$ of (A) satisfying $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$ or $\varphi_3(t) \prec x(t) \prec \varphi_4(t)$, which may well be called "intermediate"solutions of type II or of type IV, respectively.

From the discussions developed in Sect. 1 we see that an intermediate solution $x(t)$ of type II [respectively of type IV] satisfies both (2.2) and the first relation of (2.9) [respectively both (2.15) and (2.18)]. Therefore, (A) admits no intermediate solution of type II if either

$$\int_a^\infty q(t)(x(t))^\beta \,\mathrm{d}t = \infty \tag{3.30}$$

or

$$\int_a^\infty t q(t)(x(t))^\beta \,\mathrm{d}t < \infty \tag{3.31}$$

for all continuous $x(t)$ such that $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$, and (A) admits no intermediate solution of type IV if either

$$\int_a^\infty (p(t))^{-1/\alpha} \left[\int_a^t (t-s) q(s)(x(s))^\beta \,\mathrm{d}s \right]^{1/\alpha} \mathrm{d}t = \infty \tag{3.32}$$

or

$$\int^\infty t(p(t))^{-1/\alpha} \left[\int_{t_0}^t (t-s) q(s)(x(s))^\beta \,\mathrm{d}s \right]^{1/\alpha} \mathrm{d}t < \infty \tag{3.33}$$

for all continuous $x(t)$ such that $\varphi_3(t) \prec x(t) \prec \varphi_4(t)$. We remark in addition that condition (3.30) for all $x(t)$ with $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$ [respectively (3.32) for all $x(t)$ with $\varphi_3(t) \prec x(t) \prec \varphi_4(t)$] precludes the existence of any positive solution of type II [respectively of type IV] for Eq. (A).

Unlike those solutions of (A) which behave like $\varphi_i(t)$, $i = 1, 2, 3, 4$, it seems difficult to establish criteria characterizing the existence of intermediate solutions for (A), and we have to be content with presenting sufficient conditions which are applicable to some special cases of (A). □

Theorem 3.9 *If* (3.23) *holds and if*

$$\int_a^\infty t(\varphi_1(t))^\beta q(t)\,\mathrm{d}t = \infty, \tag{3.34}$$

then (A) *has a positive solution* $x(t)$ *such that* $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$.

Theorem 3.10 *If* (3.3) *holds and if*

$$\int_a^\infty t(p(t))^{-1/\alpha}\left[\int_a^t (t-s)q(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}t = \infty, \tag{3.35}$$

then (A) *has a positive solution* $x(t)$ *such that* $\varphi_3(t) \prec x(t) \prec \varphi_4(t)$.

Proof of Theorem 3.9. Choose $t_0 \geq \max\{a, 1\}$ so that

$$2^{\beta/\alpha}\int_{t_0}^t q(t)(\varphi_2(t))^\beta\,\mathrm{d}t \leq 1, \tag{3.36}$$

and define

$$X_1 = \{x \in C[t_0,\infty):\ \varphi_1(t) \leq x(t) \leq 2^{1/\alpha}\varphi_2(t), \quad t \geq t_0\}, \tag{3.37}$$

$$\mathcal{I}x(t) = \int_t^\infty (s-t)(p(s))^{-1/\alpha} \times \left(1 + \int_{t_0}^s \int_r^\infty q(\rho)(x(\rho))^\beta\,\mathrm{d}\rho\,\mathrm{d}r\right)^{1/\alpha}\mathrm{d}s, \quad t \geq t_0. \tag{3.38}$$

Because of (3.36) $\mathcal{I}$ maps X_1 into itself. This fact combined with the continuity of $\mathcal{I}$ and the relative compactness of $\mathcal{I}(X_1)$ guarantees the existence of a function $x_1 \in X_1$ satisfying the integral equation $x_1 = \mathcal{I}x_1$. It follows that $x_1(t)$ is a solution of (A) on $[t_0, \infty)$ and satisfies

$$\lim_{t\to\infty}\frac{x_1(t)}{\varphi_2(t)} = \lim_{t\to\infty}\left[\frac{1 + \int_{t_0}^t\int_s^\infty q(r)(x(r))^\beta\,\mathrm{d}r\,\mathrm{d}s}{t}\right]^{1/\alpha} = 0, \tag{3.39}$$

and

$$\lim_{t\to\infty}\frac{x_1(t)}{\varphi_1(t)} \geq \lim_{t\to\infty}\left[\int_{t_0}^t (s-t_0)q(s)(\varphi_1(s))^\beta\,\mathrm{d}s\right]^{1/\alpha} = \infty. \tag{3.40}$$

Consequently, $x_1(t)$ is an intermediate solution satisfying $\varphi_1(t) \prec x_1(t) \prec \varphi_2(t)$. □

Proof of Theorem 3.10. Let $t_0 \geq \max\{a, 1\}$ be so large that

$$2^{\alpha/\beta}\int_{t_0}^\infty (p(t))^{-1/\alpha}\left[\int_{t_0}^t (t-s)s^\beta q(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}t \leq 1. \tag{3.41}$$

Let $\mathcal{J}$ denote the integral operator

$$\mathcal{J}x(t) = 1+\int_{t_0}^{t}\int_{s}^{\infty}(p(r))^{-1/\alpha}\left[\int_{t_0}^{r}(r-\rho)q(\rho)(x(\rho))^{\beta}\,\mathrm{d}\sigma\right]^{1/\alpha}\mathrm{d}r\,\mathrm{d}s,\quad t\geq t_0. \tag{3.42}$$

Then $\mathcal{J}$ is shown to have a fixed point in the set

$$X_2 = \{x\in C[t_0,\infty):\ 1\leq x(t)\leq 2t,\ t\geq t_0\}. \tag{3.43}$$

From the integral equation $x_2 = \mathcal{J}x_2$ one sees that $x_2 = x_2(t)$ is a solution of (A) on $[t_0,\infty)$, and moreover, with the help of (3.34), that $x_2(t)$ satisfies

$$\lim_{t\to\infty}\frac{x_2(t)}{t} = \lim_{t\to\infty}\int_{t}^{\infty}(p(s))^{-1/\alpha}\left[\int_{t_0}^{s}(s-r)q(r)(x_2(r))^{\beta}\mathrm{d}r\right]^{1/\alpha}\mathrm{d}s = 0, \tag{3.44}$$

and

$$x_2(t)\geq\int_{t_0}^{t}(s-t_0)(p(s))^{-1/\alpha}\left[\int_{t_0}^{s}(s-r)q(r)\,\mathrm{d}r\right]^{1/\alpha}\mathrm{d}s\to\infty\quad\text{as } t\to\infty. \tag{3.45}$$

Thus $x_2(t)$ is an intermediate solution of (A) such that $\varphi_3(t)\prec x_2(t)\prec x_4(t)$. □

Remark 3.1 Theorem 3.9 applies only to the special case of (A) with $\alpha>\beta$, since otherwise the inequality $t(\varphi_1(t))^{\beta}\leq(\varphi_2(t))^{\beta},\ t\geq\max\{a,1\}$, would imply that conditions (3.23) and (3.34) are inconsistent.

Remark 3.2 Let us consider the conditions

$$\int_{a}^{\infty}q(t)(\varphi_1(t))^{\beta}\,\mathrm{d}t=\infty, \tag{3.46}$$

$$\int_{a}^{\infty}(p(t))^{-1/\alpha}\left[\int_{a}^{t}(t-s)q(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}t=\infty. \tag{3.47}$$

It (3.46) [respectively (3.47)] holds, then (A) possesses no positive solutions of type II [respectively of type IV], since it implies (3.30) for all $x(t)$ such that $\varphi_1(t)\prec x(t)\prec\varphi_2(t)$ [respectively (3.32) for all $x(t)$ such that $\varphi_3(t)\prec x(t)\prec\varphi_4(t)$]. Furthermore, since (3.46) and (3.47) clearly imply

$$\int_{a}^{\infty}tq(t)(\varphi_1(t))^{\beta}\,\mathrm{d}t=\infty \tag{3.48}$$

and

$$\int_{a}^{\infty}(p(t))^{-1/\alpha}\left[\int_{a}^{t}(t-s)s^{\beta}q(s)\,\mathrm{d}s\right]^{1/\alpha}\mathrm{d}t=\infty, \tag{3.49}$$

respectively, from Corollaries 3.1 and 3.2 it follows that (A) has none of the positive solutions that are asymptotic to constant multiples of $\varphi_3(t)=1$ or $\varphi_4(t)=t$ as $t\to\infty$. Summarizing the above observations, we have the following "oscillation theorem" for Eq. (A).

Theorem 3.11 *If* (3.46) *and* (3.47) *hold, then* (A) *has no nonoscillatory solution, that is, all proper solutions of* (A) *are oscillatory.*

4 Examples

Our main results developed in the preceding section will be illustrated by the following examples.

Example 4.1 Consider the equation

$$(e^{\lambda t}|x''|^{\alpha-1}x'')'' + ke^{\mu t}|x|^{\beta-1}x = 0, \quad t \geq 0, \tag{4.1}$$

where $\lambda > 0$, μ and $k > 0$ are constants. For the function $p(t) = e^{\lambda t}$, condition (1.6) is clearly satisfied, and the functions $\varphi_i(t)$, $i = 1, 2$, given by (1.7) have the asymptotic behavior

$$\varphi_1(t) \sim \left(\frac{\alpha}{\lambda}\right)^2 e^{-\lambda/\alpha t}, \quad \varphi_2(t) \sim \left(\frac{\alpha}{\lambda}\right)^2 t^{1/\alpha} e^{-\lambda/\alpha t}. \tag{4.2}$$

It is easy to see that, for this $p(t)$ and $q(t) = ke^{\mu t}$, conditions (3.1), (3.2), (3.12) and (3.13) hold if $\mu < 0$, (3.3) and (3.14) hold if $\mu < \lambda$, and (3.22) and (3.23) hold if $\mu < (\beta/\alpha)\lambda$. Using this fact, we have the following statements for Eq. (4.1) from Theorems 3.1–3.8.

(i) If $\mu < 0$, then (4.1) possesses all types of positive solutions $x(t)$ satisfying $x(t) \sim c_i\varphi_i(t)$ for some constants $c_i > 0$, $i = 1, 2, 3, 4$. (Note that $\varphi_3(t) = 1$, $\varphi_4(t) = t$.)

(ii) If $0 \leq \mu < \min\{1, \beta/\alpha\}\lambda$, then (4.1) possesses positive type II-solutions satisfying $x(t) \sim c_i\varphi_i(t)$, $i = 1, 2$, as well as positive type IV-solutions satisfying $x(t) \sim c_i\varphi_i(t)$, $i = 3, 4$.

(iii) Let $\alpha \neq \beta$ and $\min\{1, \beta/\alpha\}\lambda \leq \mu < \max\{1, \beta/\alpha\}\lambda$. If $\alpha < \beta$, then (4.1) possesses only positive type II-solutions $x(t)$ such that $x(t) \sim c_i\varphi_i(t)$, $i = 1, 2$. If $\alpha > \beta$, then (4.1) possesses only positive IV-solutions $x(t)$ such that $x(t) \sim c_i\varphi_i(t)$, $i = 3, 4$.

(iv) Let $\mu > \max\{1, \beta/\alpha\}\lambda$, where α and β may or may not be equal. Then, (3.46) and (3.47) are satisfied, and so by Theorem 3.11 all proper solutions of (4.1) are oscillatory.

It should be remarked that (4.1) has neither of the two types of positive intermediate solutions. In fact, the nonexistence of a solution $x(t)$ such that $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$ [respectively $\varphi_3(t) \prec x(t) \prec \varphi_4(t)$] follows from the observation that conditions (3.31) and (3.32) for some $x(t)$ with $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$ [respectively (3.33) and (3.34) for some $x(t)$ with $\varphi_3(t) \prec x(t) \prec \varphi_4(t)$] are satisfied if and only if $\mu \geq (\beta/\alpha)\lambda$ and $\mu < (\beta/\alpha)\lambda$ [respectively $\mu \geq \lambda$ and $\mu < \lambda$], respectively, which obviously is an impossibility.

Example 4.2 Consider the equation

$$(t^{\lambda}|x''|^{\alpha-1}x'')'' + (\lambda - 2\alpha)(\lambda - 2\alpha - 1)t^{\lambda-2\alpha-2}(\log t)^{-\beta}|x|^{\beta-1}x = 0, \quad t \geq e, \tag{4.3}$$

where $\lambda > 2\alpha + 1$, which is a special case of (A) with $p(t) = t^{\lambda}$ and $q(t) = (\lambda - 2\alpha)(\lambda - 2\alpha - 1)t^{\lambda-2\alpha-2}(\log t)^{-\beta}$. Condition (1.6) is clearly satisfied for this equation. It is easy to verify that

$$\varphi_1(t) \sim a_1(\alpha,\lambda)t^{-(\lambda-2\alpha)/\alpha}, \quad \varphi_2(t) \sim a_2(\alpha,\lambda)t^{-(\lambda-2\alpha-1)/\alpha}, \tag{4.4}$$

$$\int_e^t (t-s)q(s)\,\mathrm{d}s \sim b_1(\alpha,\beta,\lambda)t^{\lambda-2\alpha}(\log t)^{-\beta}, \tag{4.5}$$

$$\int_e^t (t-s)s^\beta q(s)\,\mathrm{d}s \sim b_2(\alpha,\beta,\lambda)t^{\lambda+\beta-2\alpha}(\log t)^{-\beta}, \tag{4.6}$$

where a_i and b_i, $i = 1,\ 2$, are positive constants depending only on the indicated parameters.

We first note that

$$\int_e^\infty t^m q(t)\,\mathrm{d}t = \infty \quad \text{for } m = 0,\ 1,\ \beta,\ \beta+1, \tag{4.7}$$

which means that (3.1), (3.2), (3.12) and (3.13) fail to hold. Using (4.4)–(4.6), we can show that (3.3) holds if and only if $\alpha > \beta$, that (3.4) holds if and only if $\alpha < \beta$, and that (3.22) and (3.23) hold if and only if $\alpha < \beta$ or $\alpha = \beta > 1$. Applying Theorems 3.1–3.8, we conclude that:

(i) Equation (4.3) has positive solutions $x(t)$ such that $x(t) \sim c_i\varphi_i(t)$ for some $c_i > 0$, $i = 1,\ 2$, if and only if $\alpha < \beta$ or $\alpha = \beta > 1$.

(ii) Equation (4.3) has neither type II- nor type III-solutions $x(t)$ such that $x(t) \sim c_3 = c_3\varphi_3(t)$ for any $c_3 > 0$; such a solution of type IV exists if and only if $\alpha < \beta$.

(iii) Equation (4.3) has neither type I- nor type III-solutions $x(t)$ such that $x(t) \sim c_4 t = c_4\varphi_4(t)$ for any $c_4 > 0$; such a solution of type IV exists if and only if $\alpha > \beta$.

From the above statements it follows that (4.3) with $\alpha = \beta \le 1$ admits none of the positive solutions that are asymptotic to constant multiples of $\varphi_i(t)$, $i = 1,\ 2,\ 3,\ 4$, which are the particular solutions of $(t^\lambda|x''|^{\alpha-1}x'')'' = 0$.

Turning our attention to the intermediate solutions of (4.3), we first note that Theorem 3.9 is not applicable, since (3.23) and (3.34) are inconsistent. But this by no means implies the nonexistence of solutions $x(t)$ of (4.3) such that $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$. On the other hand, conditions (3.3) and (3.35) are satisfied if $\alpha > \beta$, and it follows that (4.3) with $\alpha > \beta$ possesses a solution $x(t)$ such that $\varphi_3(t) = 1 \prec x(t) \prec t = \varphi_4(t)$. One such solution $x_0(t) = \log t$. The fact that $x_0(t)$ satisfies (4.3) for any values of $\alpha > 0$ and $\beta > 0$ suggests that improving Theorem 3.10 to some degree could be possible.

Example 4.3 Suppose that $\alpha > 1$, $\beta \ge 1$ and $3\alpha < \lambda < 3\alpha + 1$, and consider the equation

$$(t^\lambda|x''|^{\alpha-1}x'')'' + 2^\alpha(\lambda-3\alpha)(3\alpha+1-\lambda)t^{\lambda+\beta-3\alpha-2}|x|^{\beta-1}x = 0, \quad t \ge 1. \tag{4.8}$$

Clearly, the function $p(t) = t^\lambda$ satisfies (1.6) and (4.4), and the function $q(t) = \text{const.}t^{\lambda+\beta-3\alpha-2}$ satisfies

$$\int_1^t (t-s)q(s)\,\mathrm{d}s \sim c_1(\alpha,\beta,\lambda)t^{\lambda+\beta-3\alpha}, \tag{4.9}$$

$$\int_1^t (t-s)s^\beta q(s)\,\mathrm{d}s \sim c_2(\alpha,\beta,\lambda)t^{\lambda+2\beta-3\alpha}, \tag{4.10}$$

for some positive constants $c_i(\alpha,\beta,\lambda)$, $i = 1,\ 2$, depending only on $\alpha,\ \beta,\ \lambda$.

It can be shown that, for this $q(t)$, (4.7) (with e replaced by 1) holds, that (3.3), (3.14) and (3.23) hold if and only if $\alpha > \beta$, and that (3.22) holds if and only if $\alpha < \beta$. From Theorems 3.1–3.8 it follows that:

(i) Equation (4.8) has a positive solution $x(t)$ such that $x(t) \sim c_1\varphi_1(t)$ for some $c_1 > 0$ if and only if $\alpha < \beta$;
(ii) Equation (4.8) has a positive solution $x(t)$ such that $x(t) \sim c_2\varphi_2(t)$ for some $c_2 > 0$ if and only if $\alpha > \beta$;
(iii) Equation (4.8) has neither of type II- nor type III-solutions $x(t)$ such that $x(t) \sim c_3 = c_3\varphi_3(t)$ for any $c_3 > 0$; such a solution of type IV exists only if $\alpha > \beta$;
(iv) Equation (4.8) has neither of type I- nor type III-solutions $x(t)$ such that $x(t) \sim c_4 t = c_4\varphi_4(t)$ for any $c_4 > 0$; such a solution of type IV exists only if $\alpha > \beta$.

From the above statements it follows in particular that Eq. (4.3) with $\alpha = \beta$ possesses none of the positive solutions that are asymptotic to the functions $\varphi_i(t)$, $i = 1,\ 2,\ 3,\ 4$, as $t \to \infty$.

Since conditions (3.23) and (3.34) are satisfied if $\alpha > \beta$, Theorem 3.9 implies that (3.8) with $\alpha > \beta$ has an intermediate positive solution $x(t)$ such that $\varphi_1(t) \prec x(t) \prec \varphi_2(t)$.

The function $x_0(t) = 1/t$ is actually a solution of (4.8). Note that $x_0(t)$ satisfies (4.8) for any values of $\alpha > 0$ and $\beta > 0$. On the other hand, Theorem 3.10 does not apply to (4.8), since (3.3) and (3.35) are easily seen to be inconsistent. Nothing definite can be said about the existence of an intermediate solution $x(t)$ of (4.8) satisfying $1 = \varphi_3(t) \prec x(t) \prec t = \varphi_4(t)$.

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