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Existence and stability in the α-norm for partial functional differential equations of neutral type

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Abstract In this work, we study a general class of partial neutral functional differential equations. We assume that the linear part generates an analytic semigroup and the nonlinear part is Lipschitz continuous with respect to the α-norm associated to the linear part. We discuss the existence, uniqueness, regularity and stability of solutions. Our results are illustrated by an example. This work extends previous results on partial functional differential equations (Fitzgibbon and Parrot, Nonlinear Anal., TMA **16**, 479–487 (1991), Hale, Rev. Roum. Math. Pures Appl. **39**, 339–344 (1994), Hale, Resen. Inst. Mat. Estat. Univ. Sao Paulo **1**, 441–457 (1994), Travis and Webb, Trans. Am. Math. Soc. **240** 129–143 (1978), Wu and Xia, J. Differ. Equ. **124** 247–278 (1996)).

Keywords Partial functional differential equations · Analytic semigroup · Fractional power · Mild solution · Characteristic value · Stability

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1 Introduction

Let $(X, |\cdot|)$ be a Banach space, $(\mathcal{L}(X), \|\cdot\|)$ be the space of bounded linear operators on X, and α be a constant such that $0 < \alpha < 1$. We consider the following

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class of partial neutral functional differential equations (PNFDE)

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} D(x_t) = -AD(x_t) + F(x_t), \quad \text{for } t \geq 0, \\ x_0 = \varphi \in C_\alpha, \end{cases} \tag{1.1}$$

where $A : D(A) \subseteq X \to X$ is a linear operator, $C_\alpha := C([-r, 0]; D(A^\alpha))$, $r > 0$, denotes the space of continuous functions from $[-r, 0]$ into $D(A^\alpha)$, and the operator A^α is the fractional α-power of A. This operator $(A^\alpha, D(A^\alpha))$ will be described in Sect. 2. For $x \in C([-r, b]; D(A^\alpha))$, $b > 0$, and $t \in [0, b]$, x_t denotes, as usual, the element of C_α defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. F is a continuous function from C_α with values in X and D is a bounded linear operator from $C := C([-r, 0], X)$ into X defined by $D(\varphi) = \varphi(0) - D_0(\varphi)$, for $\varphi \in C$, where the operator D_0 is given by

$$D_0(\varphi) = \int_{-r}^{0} \mathrm{d}\eta(\theta)\varphi(\theta), \quad \text{for } \varphi \in C,$$

and $\eta : [-r, 0] \to \mathcal{L}(X)$ is of bounded variation and non-atomic at zero; that is

$$\operatorname*{var}_{[-\varepsilon,0]}(\eta) \to 0 \text{ as } \varepsilon \to 0.$$

In this paper, we discuss the existence, uniqueness, regularity and stability in the α-norm for Eq. (1.1).

It is well known, that if the phase space C_α is the space of all continuous functions from $[-r, 0]$ into X (i.e. $\alpha = 0$), Eq. (1.1) has been studied by several authors. For more details, we refer to the book of Wu [17]. For example, Wu and Xia considered in [18] a system of partial neutral functional differential-difference equations defined on the unit circle S^1, which is a model for a continuous circular array of resistively coupled transmission lines with mixed initial boundary conditions. They obtained equations of the form

$$\begin{aligned} &\frac{\partial}{\partial t}[x(\cdot, t) - qx(\cdot, t - r)] \\ &\quad = K \frac{\partial^2}{\partial \xi^2}[x(\cdot, t) - qx(\cdot, t - r)] + f(x_t), \quad t \geq 0, \end{aligned} \tag{1.2}$$

where $\xi \in S^1$, K a positive constant and $0 \leq q < 1$. The space of initial data was chosen to be $C([-r, 0]; H^1(S^1))$. Motivated by this work, Hale presented, in [11, 12], the basic theory of existence and uniqueness, and properties of the solution operator, as well as Hopf bifurcation and conditions for the stability and instability of periodic orbits for a more general class of PNFDE on the unit circle S^1. For the sake of comparison, let us briefly restate the equations considered by Hale in [11, 12]. If $\varphi \in C([-r, 0]; H^1(S^1))$, we write it as $\varphi(\xi, \theta)$ for $\xi \in S^1$ and $\theta \in [-r, 0]$. For any function $\tilde{f} \in C^{k+1}(C([-r, 0]; \mathbb{R}); \mathbb{R})$, $k \geq 1$, we let $f \in C^{k+1}(C([-r, 0]; H^1(S^1)); L^2(S^1))$ be defined by $f(\varphi)(\xi) = \tilde{f}(\varphi(\xi, .))$, $\xi \in S^1$. Let $\tilde{D} \in \mathcal{L}(C([-r, 0]; \mathbb{R}); \mathbb{R})$ be defined by

$$\begin{cases} \tilde{D}\psi = \psi(0) - \tilde{g}(\psi), \\ \tilde{g}(\psi) = \displaystyle\int_{-r}^{0} \mathrm{d}\eta(\theta)\psi(\theta), \end{cases}$$

where η is of bounded variation and non-atomic at 0.

We define $D \in \mathcal{L}(C([-r, 0]; H^1(S^1))\,;\; H^1(S^1)\,)$ as

$$D(\varphi)(\xi) = \tilde{D}\,(\varphi(\xi, .))\,, \quad \text{for } \xi \in S^1. \tag{1.3}$$

Hale considered, in [11, 12], PNFDE of the form

$$\frac{\partial}{\partial t} D x_t = K \frac{\partial^2}{\partial \xi^2} D x_t + f\,(x_t)\,, \qquad t \geq 0, \tag{1.4}$$

with $C([-r, 0]; H^1(S^1))$ as the space of initial data. He considered the Laplace operator $A_0 = K\frac{\partial^2}{\partial \xi^2}$ with domain $H^2(S^1)$, which yields an operator generating an analytic semigroup.

In [1–3], we considered a natural generalization of the work of Hale [11, 12]. We extended the study to the case when the linear part of PNFDE is non-densely defined Hille-Yosida operator.

In [8] and [16], Eq. (1.1) has been studied with respect to the α-norm, but in the particular case when $D_0 \equiv 0$.

As far as we know, no general theory exists in the case of partial neutral functional differential equations in fractional power spaces. Our aim in this paper, is to develop a basic theory of existence, uniqueness, regularity and stability for this problem.

This paper is organized as follows: in the first part of Sect. 2, we recall some preliminary results about analytic semigroups and fractional power associated to its generator. After that, we start to prove our main results. We prove the existence, uniqueness and regularity of solutions in the α-norm. In Sect. 3, we use a result of Desch and Schappacher [7], to develop a principle of linearized stability for Eq. (1.1). Finally, in Sect. 4, we propose an application.

2 Existence, uniqueness and regularity of solutions

We first study the existence, uniqueness and regularity of mild solutions of Eq. (1.1), using fixed point argument. We assume the following:
($\mathbf{H_1}$) The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on the Banach space X and satisfies $0 \in \rho(A)$.

We know that there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$, for $t \geq 0$.

If the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma I)$. Then, without loss of generality, we can assume that $0 \in \rho(A)$. This remark is valuable here only for proving existence, uniqueness and regularity of solutions.

We consider, see Pazy [14], the fractional power $(A^\alpha, D(A^\alpha))$, for $0 < \alpha < 1$, and its inverse $A^{-\alpha}$. We recall the following known results.

Proposition 2.1 *([14], pp. 69–75) Let* $0 < \alpha < 1$ *and assume that* ($\mathbf{H_1}$) *holds. Then,*

(i) $D(A^\alpha)$ *is a Banach space for the norm* $|x|_\alpha := |A^\alpha x|,\ x \in D(A^\alpha)$,
(ii) $T(t) : X \to D(A^\alpha)$, *for every* $t > 0$,

(iii) $A^{\alpha}T(t)x = T(t)A^{\alpha}x$, *for every* $x \in D(A^{\alpha})$ *and* $t \geq 0$,
(iv) *for every* $t > 0$, $A^{\alpha}T(t)$ *is bounded on* X *and there exists* $M_{\alpha} > 0$ *such that*

$$\|A^{\alpha}T(t)\| \leq M_{\alpha}\frac{e^{\omega t}}{t^{\alpha}}, \quad \textit{for every } t > 0, \tag{2.1}$$

(v) $A^{-\alpha}$ *is a bounded linear operator on* X *with* $D(A^{\alpha}) = \mathrm{Im}(A^{-\alpha})$,
(vi) *there exists* $N_{\alpha} > 0$ *such that*

$$\|(T(t) - I)A^{-\alpha}\| \leq N_{\alpha}t^{\alpha}, \quad \textit{for } t > 0 \textit{ small enough}.$$

We denote by X_{α} the Banach space $(D(A^{\alpha}), |\cdot|_{\alpha})$ and by $C_{\alpha} := C([-r, 0]; X_{\alpha})$ the space of continuous functions from $[-r, 0]$ into X_{α} endowed with the norm

$$\|\varphi\|_{\alpha} := \sup_{\theta \in [-r,0]} |\varphi(\theta)|_{\alpha}, \quad \varphi \in C_{\alpha}.$$

Remark that $(C_{\alpha}, \|\cdot\|_{\alpha})$ is also a Banach space. To prove our result on existence and uniqueness, we need to assume that
($\mathbf{H}_2$) $|F(\varphi_1) - F(\varphi_2)| \leq k\|\varphi_1 - \varphi_2\|_{\alpha}$, for $\varphi_1, \varphi_2 \in C_{\alpha}$, where k is a positive constant,

($\mathbf{H}_3$) if $x \in X_{\alpha}$ and $\theta \in [-r, 0]$ then $\eta(\theta)x \in X_{\alpha}$ and $A^{\alpha}\eta(\theta)x = \eta(\theta)A^{\alpha}x$.

The assumption ($\mathbf{H}_3$) implies that if $\varphi \in C_{\alpha}$ then $D_0(\varphi) \in X_{\alpha}$ and $A^{\alpha}D_0(\varphi) = D_0(A^{\alpha}\varphi)$, where the expression $A^{\alpha}\varphi$ is defined, for $\varphi \in C_{\alpha}$ and $\theta \in [-r, 0]$, by

$$(A^{\alpha}\varphi)(\theta) := A^{\alpha}(\varphi(\theta)).$$

Definition 2.2 Let $\varphi \in C_{\alpha}$. A continuous function $x : [-r, +\infty) \to X_{\alpha}$ is called a mild solution of Eq. (1.1) if

(i) $D(x_t) = T(t)D(\varphi) + \int_0^t T(t-s)F(x_s)\,\mathrm{d}s, \quad t \geq 0$,
(ii) $x_0 = \varphi$

Definition 2.3 Let $\varphi \in C_{\alpha}$. A continuous function $x : [-r, +\infty) \to X_{\alpha}$ is called a strict solution of Eq. (1.1) if

(i) $t \to D(x_t)$ is continuously differentiable on $[0, +\infty)$,
(ii) $D(x_t) \in D(A)$, for $t \geq 0$,
(iii) x satisfies (1.1).

Now, we state our first result.

Theorem 2.4 *Assume that* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) *hold. Then, for* $\varphi \in C_{\alpha}$, *Eq. (1.1) has a unique mild solution which is defined for all* $t \geq 0$.

Proof Let $T > 0$ and $\varphi \in C_\alpha$. We consider the set

$$\Lambda = \{x \in C([-r, T]; X_\alpha) : x(\theta) = \varphi(\theta), \text{ for } \theta \in [-r, 0]\}.$$

Λ is a closed subset of $C([-r, T]; X_\alpha)$ provided with the uniform norm topology. Let $\mathcal{J}$ be the operator defined on $C([-r, T]; X_\alpha)$ by

$$\mathcal{J}(x)(t) = \begin{cases} D_0(x_t) + T(t)D(\varphi) + \int_0^t T(t-s)F(x_s)\,\mathrm{d}s, & \text{if } t \in [0, T], \\ \varphi(t), & \text{if } t \in [-r, 0]. \end{cases}$$

First, we have immediately that $\mathcal{J}(\Lambda) \subseteq \Lambda$. Furthermore, for $x, y \in \Lambda$ and $t \in [0, T]$, one has

$$(\mathcal{J}(x) - \mathcal{J}(y))(t) = \quad D_0(x_t - y_t) + \int_0^t T(t-s)(F(x_s) - F(y_s))\,\mathrm{d}s.$$

Taking the α-norm $|\cdot|_\alpha$ and using (2.1), we obtain

$$|(\mathcal{J}(x) - \mathcal{J}(y))(t)|_\alpha \leq |A^\alpha D_0(x_t - y_t)| + M_\alpha k \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} \|x_s - y_s\|_\alpha \,\mathrm{d}s.$$

As η is non-atomic at zero, we can choose $\varepsilon > 0$ such that

$$\operatorname*{var}_{[-\varepsilon,0]}(\eta) < 1.$$

Let $0 < T < \varepsilon$. Then, for $t \in [0, T]$ and $\theta \in [-r, -\varepsilon]$, we have $t + \theta \leq 0$. Consequently, for $t \in [0, T]$,

$$A^\alpha D_0(x_t - y_t) = \int_{-\varepsilon}^0 \mathrm{d}\eta(\theta)(A^\alpha(x(t+\theta) - y(t+\theta))).$$

This implies that

$$|A^\alpha D_0(x_t - y_t)| \leq \operatorname*{var}_{[-\varepsilon,0]}(\eta)\|x_t - y_t\|_\alpha \leq \operatorname*{var}_{[-\varepsilon,0]}(\eta) \max_{s\in[0,T]} |x(s) - y(s)|_\alpha.$$

It follows that

$$|(\mathcal{J}(x) - \mathcal{J}(y))(t)|_\alpha \leq \left[\operatorname*{var}_{[-\varepsilon,0]}(\eta) + M_\alpha k \int_0^T \frac{e^{\omega s}}{s^\alpha}\,\mathrm{d}s\right] \max_{s\in[0,T]} |x(s) - y(s)|_\alpha.$$

Remark that

$$\lim_{T\to 0} \int_0^T \frac{e^{\omega s}}{s^\alpha}\,\mathrm{d}s = 0.$$

Then, we can choose $T \in [0, \varepsilon]$ such that

$$\operatorname*{var}_{[-\varepsilon,0]}(\eta) + M_\alpha k \int_0^T \frac{e^{\omega s}}{s^\alpha}\,\mathrm{d}s < 1.$$

Then, $\mathcal{J}$ is a strict contraction in Λ. That means that $\mathcal{J}$ has a unique fixed point in Λ. We obtain the existence and uniqueness of a mild solution of Eq. (1.1) on the interval $[0, T]$. Proceeding inductively, we extend the solution uniquely and continuously to $[-r, +\infty)$. □

If instead of assuming that D_0 is given by a function of bounded variation and Condition $(\mathbf{H}_3)$, we make the assumption that

$(\mathbf{H}_3')$ $D_0 \in \mathcal{L}(C_\alpha, X_\alpha)$ and $\|D_0\|_{\mathcal{L}(C_\alpha, X_\alpha)} < 1$,

then, we obtain the same result as in Theorem 2.4.

Proposition 2.5 *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3'})$ *hold. Then, for* $\varphi \in C_\alpha$, *Eq. (1.1) has a unique mild solution which is defined for all* $t \geq 0$.

Arguing as above, we prove that the operator $\mathcal{J}$ is a strict contraction in Λ, for $T > 0$ small enough.

We focus now our study on the regularity of the mild solution of Eq. (1.1). Under more restrictive conditions on F and φ, we obtain a strict solution of Eq. (1.1).

Theorem 2.6 *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *hold. Furthermore, assume that* $F : C_\alpha \to X$ *is continuously differentiable and* F' *is locally Lipschitz continuous. Let* $\varphi \in C_\alpha$ *be such that*

$$\varphi' \in C_\alpha, \quad D(\varphi) \in D(A) \quad \text{and} \quad D(\varphi') = -AD(\varphi) + F(\varphi).$$

Then, the mild solution x *of Eq. (1.1) belongs to* $C^1([0, +\infty); X_\alpha)$. *Consequently, it is a strict solution of Eq. (1.1).*

Proof Let $T > 0$ and x be the mild solution of Eq. (1.1). Consider the equation

$$\begin{cases} D(y_t) = T(t)\, D(\varphi') + \displaystyle\int_0^t T(t-s)\, F'(x_s)\, y_s \,\mathrm{d}s \quad \text{for } t \in [0, T], \\ y_0 = \varphi'. \end{cases} \tag{2.2}$$

Then, using the same reasoning as in the proof of Theorem 2.4, we prove that Eq. (2.2) has a unique continuous solution y on $[-r, T]$. Let $z \in C([-r, T]; X_\alpha)$ be defined by

$$z(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t y(s)\,\mathrm{d}s, & \text{for } t \in [0, T], \\ \varphi(t), & \text{for } t \in [-r, 0]. \end{cases}$$

Simple computations yield

$$z_t = \varphi + \int_0^t y_s \,\mathrm{d}s, \quad \text{for } t \in [0, T]. \tag{2.3}$$

We will show that $x = z$ on $[0, T]$. From Eq. (2.2), we get

$$\int_0^t D(y_s)\,\mathrm{d}s = \int_0^t T(s)\, D(\varphi')\,\mathrm{d}s + \int_0^t \int_0^s T(s-\sigma)\, F'(x_\sigma)\, y_\sigma \,\mathrm{d}\sigma\,\mathrm{d}s. \tag{2.4}$$

On the other hand, we have for $t \in [0, T]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t T(t-s)\, F(z_s)\,\mathrm{d}s = T(t)\, F(\varphi) + \int_0^t T(t-s) F'(z_s) y_s \,\mathrm{d}s.$$

This implies that

$$\int_0^t T(s)\, F(\varphi)\, \mathrm{d}s = \int_0^t T(t-s)\, F(z_s)\, \mathrm{d}s - \int_0^t \int_0^s T(s-\sigma)\, F'(z_\sigma) y_\sigma \, \mathrm{d}\sigma\, \mathrm{d}s. \tag{2.5}$$

It follows that

$$\begin{aligned} D(z_t) &= D(\varphi) + \int_0^t T(s)(-AD(\varphi) + F(\varphi))\, \mathrm{d}s + \int_0^t \int_0^s T(s-\sigma) F'(x_\sigma) y_\sigma \, \mathrm{d}\sigma\, \mathrm{d}s, \\ &= T(t)D(\varphi) + \int_0^t T(s)F(\varphi)\, \mathrm{d}s + \int_0^t \int_0^s T(s-\sigma) F'(x_\sigma) y_\sigma \, \mathrm{d}\sigma\, \mathrm{d}s. \end{aligned}$$

Using (2.5), we obtain that

$$\begin{aligned} D(z_t) = T(t)D(\varphi) &+ \int_0^t T(t-s)F(z_s)\, \mathrm{d}s \\ &+ \int_0^t \int_0^s T(s-\sigma)(F'(x_\sigma) - F'(z_\sigma)) y_\sigma \, \mathrm{d}\sigma\, \mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} D(x_t - z_t) = &\int_0^t T(t-s)(F(x_s) - F(z_s))\, \mathrm{d}s \\ &- \int_0^t \int_0^s T(s-\sigma)(F'(x_\sigma) - F'(z_\sigma)) y_\sigma \, \mathrm{d}\sigma\, \mathrm{d}s. \end{aligned}$$

By Fubini's Theorem, we obtain

$$\begin{aligned} D(x_t - z_t) = &\int_0^t T(t-s)(F(x_s) - F(z_s))\, \mathrm{d}s \\ &- \int_0^t (\int_0^{t-s} T(\sigma)\, \mathrm{d}\sigma)(F'(x_s) - F'(z_s)) y_s \, \mathrm{d}s. \end{aligned}$$

To complete the proof, we need the following lemma. □

Lemma 2.7 *Assume that* ($\mathbf{H}_1$) *and* ($\mathbf{H}_3$) *hold. There exist positive constants a, b and c such that, if $w \in C\,([-r, +\infty)\,; X_\alpha)$ is a solution of the equation*

$$\begin{cases} D\,(w_t) = f(t), & t \geq 0, \\ w_0 = \varphi \in C_\alpha, \end{cases}$$

where f is a continuous function from $[0, +\infty)$ *into X_α, then*

$$\|w_t\|_\alpha \leq \Big(a\|\varphi\|_\alpha + b \sup_{0 \leq s \leq t} |f(s)|_\alpha\Big) e^{ct}, \quad t \geq 0. \tag{2.6}$$

The above lemma is an immediate consequence of the following which is the same result but for $\alpha = 0$.

Lemma 2.8 *[17] Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_3)$ *hold. There exist positive constants* a, b *and* c *such that, if* $w \in C\left([-r, +\infty); X\right)$ *is a solution of the equation*

$$\begin{cases} D\left(w_t\right) = f(t), & t \geq 0, \\ w_0 = \varphi \in C := C\left([-r, 0]; X\right), \end{cases}$$

where f *is a continuous function from* $[0, +\infty)$ *into* X*, then*

$$\|w_t\| \leq \Big(a\|\varphi\| + b \sup_{0\leq s\leq t} |f(s)|\Big)e^{ct}, \quad t \geq 0. \tag{2.7}$$

We put, for $t \in [0, T]$,

$$f(t) = \int_0^t T(t-s)\left(F(x_s) - F(z_s)\right) \mathrm{d}s - \int_0^t \left(\int_0^{t-s} T(\sigma)\, \mathrm{d}\sigma\right)(F'(x_s) - F'(z_s))y_s\, \mathrm{d}s.$$

Then we get, for $t \in [0, T]$,

$$|f(t)|_\alpha \leq K_1 \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha}\|x_s - z_s\|_\alpha\, \mathrm{d}s + K_2 \int_0^t \left(\int_0^{t-s} \frac{e^{\omega\sigma}}{\sigma^\alpha}\, \mathrm{d}\sigma\right)\|x_s - z_s\|_\alpha\, \mathrm{d}s,$$

for some positive constants K_1 and K_2 . Without loss of generality, we suppose that $\omega > 0$. Then, if we choose

$$0 < T \leq \frac{\alpha}{\omega} \quad \text{and} \quad K_3 := T^\alpha e^{-\omega T} \int_0^T \frac{e^{\omega\sigma}}{\sigma^\alpha}\, \mathrm{d}\sigma,$$

we obtain, for $t \in (0, T]$,

$$\int_0^t \frac{e^{\omega\sigma}}{\sigma^\alpha}\, \mathrm{d}\sigma \leq K_3 \frac{e^{\omega t}}{t^\alpha}.$$

This implies that, for $t \in [0, T]$

$$|f(t)|_\alpha \leq (K_1 + K_2K_3) \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha}\|x_s - z_s\|_\alpha\, \mathrm{d}s.$$

Suppose that $T \leq r$. Then, for $s \in [0, t]$,

$$\|x_s - z_s\|_\alpha \leq \max_{0\leq\sigma\leq t} |x(\sigma) - z(\sigma)|_\alpha,$$

because $x(\sigma) = z(\sigma) = \varphi(\sigma)$, for $\sigma \in [-r, 0]$. Consequently, for $t \in [0, T]$ with $T \leq \min\{\frac{\alpha}{\omega}, r\}$, we get

$$\sup_{0\leq s\leq t} |f(s)|_\alpha \leq (K_1 + K_2K_3)\left(\int_0^T \frac{e^{\omega s}}{s^\alpha}\, \mathrm{d}s\right) \max_{0\leq\sigma\leq t} |x(\sigma) - z(\sigma)|_\alpha.$$

Then,

$$\begin{aligned}\|x_t - z_t\|_\alpha &= \max_{0\leq\sigma\leq t} |x(\sigma) - z(\sigma)|_\alpha \\ &\leq (K_1 + K_2K_3)\left(\int_0^T \frac{e^{\omega s}}{s^\alpha}\,\mathrm{d}s\right)\max_{0\leq\sigma\leq t}|x(\sigma) - z(\sigma)|_\alpha.\end{aligned}$$

For $T > 0$ small enough, we have

$$(K_1 + K_2K_3)\left(\int_0^T \frac{e^{\omega s}}{s^\alpha}\,\mathrm{d}s\right) < 1.$$

It follows that $x = z$ on $[-r, T]$. Then, x is continuously differentiable on $[0, T]$ with respect to the α-norm, for $T > 0$ small enough. By steps, we prove that x is continuously differentiable on $[0, T]$ with respect to the α-norm, for every $T > 0$. Since $X_\alpha \hookrightarrow X$, we deduce that $x \in C^1([0, +\infty); X)$ and satisfies Eq. (1.1).

Remark 2.1 If the delay is discrete; that is for example for $F(\varphi) = G(\varphi(-r))$, then, the Lipschitz condition on F is not needed to obtain the existence of mild solutions. Also, the local Lipschitz condition on F' is not needed for the existence of strict solutions. In this case, we can proceed by steps.

3 The nonlinear solution semigroup and linearized stability

Define the operator $U(t)$, for $t \geq 0$, on C_α by

$$U(t)(\varphi) = x_t(., \varphi),$$

where $x(., \varphi)$ is the mild solution of Eq. (1.1) for the initial condition $\varphi \in C_\alpha$. One can prove the proposition.

Proposition 3.1 *The family $(U(t))_{t\geq 0}$ is a nonlinear strongly continuous semigroup on C_α; that is*

(i) $U(0) = I$,
(ii) $U(t + s) = U(t)U(s)$, *for* $t, s \geq 0$,
(iii) *for all* $\varphi \in C_\alpha$, $U(t)(\varphi)$ *is a continuous function of* $t \geq 0$ *with values in* C_α,
(iv) *for all* $t \geq 0$, $U(t)$ *is continuous from* C_α *into* C_α,
(v) $(U(t))_{t\geq 0}$ *satisfies the following translation property, for* $t \geq 0$ *and* $\theta \in [-r, 0]$,

$$(U(t)(\varphi))(\theta) = \begin{cases} (U(t+\theta)(\varphi))(0), & \text{if } t + \theta \geq 0,\\ \varphi(t+\theta), & \text{if } t + \theta \leq 0.\end{cases}$$

We are now interested in the stability of the equilibriums of Eq. (1.1). By equilibrium, we mean a constant mild solution x^* of (1.1). Without loss of generality, we can assume that $x^* = 0$ and $F(0) = 0$.
We need the following assumption.

(**H**$_4$) $F : C_\alpha \to X$ is differentiable at zero.

Then, the linearized equation at zero of Eq. (1.1) is given by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} D\left(y_t\right) = -AD\left(y_t\right) + L(y_t), \quad \text{for } t \geq 0, \\ y_0 = \varphi \in C_\alpha, \end{cases} \tag{3.1}$$

where $L = F'(0)$. Let $(S(t))_{t \geq 0}$ be the solution semigroup on C_α associated to the linear Eq. (3.1).

Theorem 3.2 *Assume that the conditions* $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_3})$ *and* $(\mathbf{H_4})$ *hold. Then, for every* $t \geq 0$ *the derivative at zero of* $U(t)$ *is* $S(t)$.

Proof Let $t \geq 0$ be fixed and $\varphi \in C_\alpha$. We have

$$D\left[U(t)(\varphi) - S(t)\varphi\right] = \int_0^t T(t-s)\left(F(U(s)(\varphi)) - L(S(s)\varphi)\right) \mathrm{d}s.$$

We put

$$w_t = U(t)(\varphi) - S(t)\varphi,$$

and

$$g(t) = \int_0^t T(t-s)\left(F(U(s)(\varphi)) - L(S(s)\varphi)\right) \mathrm{d}s.$$

Then,

$$\begin{aligned} g(t) = & \int_0^t T(t-s)\left(F(U(s)(\varphi)) - F(S(s)\varphi)\right) \mathrm{d}s \\ & + \int_0^t T(t-s)\left(F(S(s)\varphi) - L(S(s)\varphi)\right) \mathrm{d}s. \end{aligned}$$

Using Lemma 2.7, we obtain

$$\|w_t\|_\alpha \leq be^{ct} \sup_{0 \leq s \leq t} |g(s)|_\alpha, \quad t \geq 0.$$

On the other hand, we have

$$\begin{aligned} |g(t)|_\alpha \leq & M_\alpha k \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} \|w_s\|_\alpha \, \mathrm{d}s \\ & + M_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} |F(S(s)\varphi) - L(S(s)\varphi)| \, \mathrm{d}s. \end{aligned}$$

Let $\varepsilon > 0$. By virtue of the continuous differentiability of F at 0, we deduce that there exists $\delta > 0$ such that

$$M_\alpha \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} |F(S(s)\varphi) - L(S(s)\varphi)| \, \mathrm{d}s \leq \varepsilon \|\varphi\|_\alpha, \quad \text{for } \|\varphi\|_\alpha \leq \delta.$$

Then, for $\|\varphi\|_\alpha \leq \delta$,

$$|g(t)|_\alpha \leq \varepsilon \|\varphi\|_\alpha + M_\alpha k \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} \|w_s\|_\alpha \, \mathrm{d}s.$$

Since $w_0 = 0$, then

$$\|w_s\|_\alpha = \max_{0\le\sigma\le s} |w(\sigma)|_\alpha \le \|w_t\|_\alpha, \quad \text{for } s \in [0, t] \text{ and } t \in [0, r].$$

Consequently, for $t \in [0, r]$ fixed

$$\sup_{0\le s\le t} |g(s)|_\alpha \le \varepsilon\|\varphi\|_\alpha + M_\alpha k \left(\int_0^t \frac{e^{\omega s}}{s^\alpha}\, \mathrm{d}s\right) \|w_t\|_\alpha.$$

Then, we obtain for $T \in (0, r]$ small enough and $t \in [0, T]$ fixed

$$\|w_t\|_\alpha \le \frac{be^{ct}}{1 - bM_\alpha ke^{ct} \displaystyle\int_0^t \frac{e^{\omega s}}{s^\alpha}\, \mathrm{d}s}\, \varepsilon\, \|\varphi\|_\alpha, \quad \text{for } \|\varphi\|_\alpha \le \delta.$$

That means that $U(t)$ is differentiable at 0, for each $t \in [0, T]$ and $D_\varphi U(t)(0) = S(t)$.

Now, suppose that $t \in [T, 2T]$ is fixed. It follows that, for $\max\{\|\varphi\|_\alpha, \|U(t - T)(\varphi)\|_\alpha\} \le \delta_0$, where $\delta_0 > 0$ is small enough

$$\begin{aligned}\|U(t)(\varphi) - S(t)\varphi\|_\alpha &\le \|U(T)\,(U(t-T)(\varphi)) - S(T)U(t-T)(\varphi)\|_\alpha \\ &\quad + \|S(T)\|\|U(t-T)(\varphi) - S(t-T)\varphi\|_\alpha, \\ &\le \varepsilon\|\varphi\|_\alpha.\end{aligned}$$

By steps, we conclude that $U(t)$ is differentiable at 0, for each $t \ge 0$ and $D_\varphi U(t)(0) = S(t)$. □

Theorem 3.3 *Assume that the conditions* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$ *hold. If the zero equilibrium of* $(S(t))_{t\ge0}$ *is exponentially stable, then the zero equilibrium of* $(U(t))_{t\ge0}$ *is locally exponentially stable, in the sense that there exist* $\delta > 0$, $\mu > 0$, $k \ge 1$ *such that*

$$\|U(t)(\varphi)\|_\alpha \le ke^{-\mu t}\|\varphi\|_\alpha, \quad \text{for } \varphi \in C_\alpha \text{ with } \|\varphi\|_\alpha \le \delta \text{ and } t \ge 0.$$

Moreover, if C_α *can be decomposed as* $C_\alpha = \mathcal{H}_1 \oplus \mathcal{H}_2$ *where* $\mathcal{H}_i$ *are S-invariant subspaces of* C_α, $\mathcal{H}_1$ *is finite-dimensional and with*

$$\omega = \lim_{h\to\infty} \frac{1}{h} \log \|S(h)/\mathcal{H}_2\|_\alpha,$$

we have

$$\inf\{|\lambda| : \lambda \in \sigma\,(S(t)/\mathcal{H}_1)\} > e^{\omega t},$$

then, the zero equilibrium of $(U(t))_{t\ge0}$ *is not stable, in the sense that there exist* $\varepsilon > 0$ *and a sequence* $(\varphi_n)_n$ *converging to* 0 *and a sequence* $(t_n)_n$ *of positive real numbers such that* $\|U(t_n)\varphi_n\|_\alpha > \varepsilon$.

The proof of this theorem is based on Theorem 3.2 and the following result.

Theorem 3.4 *[7] Let $(V(t))_{t\geq 0}$ be a nonlinear strongly continuous semigroup on a subset Ω of a Banach space $(Z, \|\cdot\|)$. Assume that $x_0 \in \Omega$ is an equilibrium of $(V(t))_{t\geq 0}$ such that $V(t)$ is differentiable at x_0 for each $t \geq 0$, with $W(t)$ the derivative at x_0 of $V(t)$, $t \geq 0$. Then, $(W(t))_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on Z and, if the zero equilibrium of $(W(t))_{t\geq 0}$ is exponentially stable, then the equilibrium x_0 of $(V(t))_{t\geq 0}$ is locally exponentially stable. Moreover, if Z can be decomposed as $Z = Z_1 \oplus Z_2$ where Z_i are W-invariant subspaces of Z and Z_1 is finite-dimensional and with*

$$\omega = \lim_{h\to\infty} \frac{1}{h} \log \|W(h)/Z_2\|,$$

we have

$$\inf\{|\lambda| : \lambda \in \sigma(W(t)/Z_1)\} > e^{\omega t},$$

then the equilibrium x_0 of $(V(t))_{t\geq 0}$ is not stable in the sense that there exist $\varepsilon > 0$ and a sequence $(x_n)_n$ converging to x_0 and a sequence $(t_n)_n$ of positive real numbers such that $\|V(t_n)x_n - x_0\| > \varepsilon$.

In the following, we will concentrate our study on the linear Eq. (3.1). Let $(A_S, D(A_S))$ be the generator of the semigroup $(S(t))_{t\geq 0}$ on C_α. We have the result.

Theorem 3.5 *Assume that the conditions $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ hold. Then, the operator $(A_S, D(A_S))$ is given by*

$$\begin{cases} D(A_S) = \{\varphi \in C_\alpha : \varphi' \in C_\alpha,\ D(\varphi) \in D(A) \text{ and } D(\varphi') = -AD(\varphi) + L(\varphi)\}, \\ A_S\varphi = \varphi', \quad \varphi \in D(A_S). \end{cases}$$

Proof Let B be the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$ on C_α and $\varphi \in D(B)$. Then,

$$\begin{cases} \lim_{t\to 0^+} \frac{1}{t}(S(t)\varphi - \varphi) = \psi \quad \text{exists in } C_\alpha, \\ B\varphi = \psi. \end{cases}$$

The first expression yields to

$$\lim_{t\to 0^+} \frac{1}{t}(\varphi(t+\theta) - \varphi(\theta)) = \psi(\theta), \quad \text{for } \theta \in [-r, 0).$$

This means that the right derivative

$$\varphi'_+ = \psi$$

exists and is continuous on $[-r, 0)$. To complete the proof, we need the following lemma.

Lemma 3.6 *[14] Let x be a continuous and right differentiable function on $[a, b)$. If the function x'_+ is continuous on $[a, b)$, then x is continuously differentiable on $[a, b)$.*

From the above lemma, we deduce that the function φ is continuously differentiable on $[-r, 0)$ and $\varphi' = \psi$ on $[-r, 0)$. On the other hand, for $\theta = 0$, one has $\lim_{\theta \to 0} \varphi'_+(\theta)$ exists and is equal to $\psi(0)$. From this, we conclude that the function φ is continuously differentiable from $[-r, 0]$ into X_α and $\varphi' = \psi$. Moreover, one has

$$\frac{1}{t}\left(T(t)D(\varphi) - D(\varphi)\right) = \frac{1}{t}D(S(t)\varphi - \varphi) - \frac{1}{t}\int_0^t T(t-s)L(S(s)\varphi)\,\mathrm{d}s.$$

First, it is clear that

$$\frac{1}{t}\int_0^t T(t-s)L(S(s)\varphi)\,\mathrm{d}s \to L(\varphi) \quad \text{as } t \to 0$$

in X-norm and

$$\frac{1}{t}D(S(t)\varphi - \varphi) \to D(\varphi') \quad \text{as } t \to 0$$

in α-norm. Since $X_\alpha \hookrightarrow X$, we deduce that

$$\frac{1}{t}D(S(t)\varphi - \varphi) \to D(\varphi') \quad \text{as } t \to 0$$

in X-norm. This implies that

$$D(\varphi) \in D(A) \quad \text{and} \quad \frac{1}{t}\left(T(t)D(\varphi) - D(\varphi)\right) \to AD(\varphi) \quad \text{as}\, t \to 0$$

in X-norm. Consequently, we conclude that

$$\begin{cases} D(B) \subseteq \{\varphi \in C_\alpha : \varphi' \in C_\alpha,\ D(\varphi) \in D(A) \text{ and } D(\varphi') = -AD(\varphi) + L(\varphi)\}, \\ B\varphi = \varphi'. \end{cases}$$

Conversely, let $\varphi \in C_\alpha$ such that

$$\varphi' \in C_\alpha, \quad D(\varphi) \in D(A) \quad \text{and} \quad D\left(\varphi'\right) = -AD\left(\varphi\right) + L(\varphi).$$

In Theorem 2.6, it has been proved that $t \mapsto T(t)\varphi$ is continuously differentiable from $[0, +\infty)$ into X_α. Hence, $\varphi \in D(B)$. □

Let C be the space of continuous functions from $[-r, 0]$ into X provided with the uniform norm topology and let

$$C_D = \{\varphi \in C : D(\varphi) = 0\}.$$

Definition 3.7 [13] D is said to be stable if the zero solution of the difference equation

$$\begin{cases} D\left(y_t\right) = 0, & t \geq 0, \\ y_0 = \varphi \in C_D, \end{cases} \tag{3.2}$$

is exponentially stable.

Lemma 3.8 *If D is stable, then there exist positive constants a, b, c and d such that for any $\varepsilon \in (0, r]$ sufficiently small and any continuous function h from $[0, +\infty)$ into X, the solution v of the equation*

$$D(v_t) = h(t), \quad t \geq 0, \tag{3.3}$$

satisfies the inequality

$$\|v_t\| \leq e^{-a(t-\varepsilon)}\Big[b\|v_0\| + c \sup_{0\leq s\leq\varepsilon} |h(s)|\Big] + d \sup_{\max(\varepsilon,t-r)\leq s\leq t} |h(s)|, \quad t \geq \varepsilon. \tag{3.4}$$

Proof The idea of the proof comes from [5], in which there is the same estimate (3.4) but in finite-dimensional case. To prove (3.4) we will make a transformation of variables in Eq. (3.3) such that $h(\varepsilon) = 0$.

Consider the mapping $\Lambda : X \to C$ defined, for $c \in X$ and $\theta \in [-r, 0]$ by

$$\Lambda(c)(\theta) = \begin{cases} 0, & -r \leq \theta \leq -\varepsilon, \\ \left(1 + \dfrac{\theta}{\varepsilon}\right) c, & -\varepsilon < \theta \leq 0. \end{cases} \tag{3.5}$$

Since

$$-r \leq \frac{r-\varepsilon}{r}\theta - \varepsilon \leq -\varepsilon, \quad \text{for all } \theta \in [-r, 0],$$

then

$$|D(\Lambda(c))| \leq \operatorname*{var}_{[-\varepsilon,0]}(\eta)\,|c|.$$

We can choose $\varepsilon \in (0, r]$ sufficiently small such that

$$\operatorname*{var}_{[-\varepsilon,0]}(\eta) < 1.$$

We conclude that the linear operator $\tilde{D}(\Lambda) : X \to X$ defined by

$$\tilde{D}(\Lambda)(c) = D(\Lambda(c)),$$

is invertible.

We make now the following transformation of variables in Eq. (3.3)

$$z(t) = v(t) - y(t), \quad \text{for } t \geq \varepsilon - r,$$

where $y : [\varepsilon - r, +\infty[\to X$ is defined by

$$y(t) = \begin{cases} \Lambda([\tilde{D}(\Lambda)]^{-1}(h(\varepsilon)))(t-\varepsilon), & \varepsilon - r \leq t \leq \varepsilon, \\ [\tilde{D}(\Lambda)]^{-1}(h(t)), & t > \varepsilon. \end{cases}$$

Thus, we can rewrite Eq. (3.3) as

$$D(z_t) = h^*(t), \quad t \geq \varepsilon, \tag{3.6}$$

where

$$h^*(t) = h(t) - D(y_t), \quad t \geq \varepsilon.$$

Note that

$$y_\varepsilon(\theta) = y(\varepsilon + \theta) = \Lambda([\tilde{D}(\Lambda)]^{-1}(h(\varepsilon)))(\theta), \quad \text{for } \theta \in [-r, 0].$$

This gives

$$D(y_\varepsilon) = D(\Lambda([\tilde{D}(\Lambda)]^{-1}(h(\varepsilon)))) = \tilde{D}(\Lambda([\tilde{D}(\Lambda)]^{-1}))(h(\varepsilon)) = h(\varepsilon).$$

Hence

$$h^*(\varepsilon) = 0.$$

We can now start the proof of estimate (3.4).
It is immediate that

$$|h^*(t)| \leq |h(t)| + K_4\|y_t\|, \quad t \geq \varepsilon,$$

and

$$\|y_t\| = \sup_{t-r \leq s \leq t} |y(s)|, \quad t \geq \varepsilon.$$

Let $s \in [t - r, t]$. If $t - r \geq \varepsilon$

$$y(s) = [\tilde{D}(\Lambda)]^{-1}(h(s)),$$

and if $t - r < \varepsilon$

$$y(s) = \begin{cases} \Lambda([\tilde{D}(\Lambda)]^{-1}(h(\varepsilon)))(s - \varepsilon), & t - r \leq s \leq \varepsilon, \\ [\tilde{D}(\Lambda)]^{-1}(h(s)), & \varepsilon < s \leq t. \end{cases}$$

Then, we can assert that

$$\|y_t\| \leq K_5 \sup_{\max(\varepsilon, t-r) \leq s \leq t} |h(s)|, \quad t \geq \varepsilon, \tag{3.7}$$

and

$$|h^*(t)| \leq K_6 \sup_{\max(\varepsilon, t-r) \leq s \leq t} |h(s)|, \quad t \geq \varepsilon. \tag{3.8}$$

Our next objective is to estimate $\|z_t\|$, for $t \geq \varepsilon$.
By the superposition principle of solutions of linear systems, we have

$$z(t) = z^1(t) + z^2(t), \quad t \geq \varepsilon - r,$$

where

$$\begin{cases} D(z_t^1) = 0, & t \geq \varepsilon, \\ z_\varepsilon^1 = z_\varepsilon, \end{cases}$$

and

$$\begin{cases} D(z_t^2) = h^*(t), & t \geq \varepsilon, \\ z_\varepsilon^1 = 0. \end{cases}$$

Since D is stable, it follows that

$$\left\|z_t^1\right\| \leq \beta e^{-\alpha(t-\varepsilon)}\|z_\varepsilon\|.$$

As $z_\varepsilon = v_\varepsilon - y_\varepsilon$, we obtain

$$\|z_\varepsilon\| \le \|v_\varepsilon\| + K_5|h(\varepsilon)|.$$

Applying (2.7), we conclude that

$$\|v_\varepsilon\| \le K_7\|v_0\| + K_8 \sup_{0\le s\le\varepsilon} |h(s)|. \tag{3.9}$$

This gives

$$\|z_t^1\| \le \beta e^{-\alpha(t-\varepsilon)}(K_7\|v_0\| + (K_5 + K_8) \sup_{0\le s\le\varepsilon} |h(s)|).$$

We also have

$$\|z_t^2\| \le K_9 \sup_{\varepsilon\le s\le t} |h^*(s)|,$$

(see, for example, Theorem 2.1 in [10] which is easy to extend to infinite-dimensional case). Then, (3.8) implies

$$\|z_t^2\| \le K_6K_9 \sup_{\varepsilon\le s\le t}\Big(\sup_{\max(\varepsilon,s-r)\le\sigma\le s} |h(\sigma)|\Big) \le K_{10} \sup_{\max(\varepsilon,t-r)\le s\le t} |h(s)|.$$

Consequently, for $t \ge \varepsilon$

$$\begin{aligned}\|z_t\| \le{}& e^{-\mu(t-\varepsilon)}\Big(\beta K_7\|v_0\| + \beta(K_5 + K_8) \sup_{0\le s\le\varepsilon} |h(s)|\Big)\\ &+ K_{10} \sup_{\max(\varepsilon,t-r)\le s\le t} |h(s)|.\end{aligned}$$

Finally, using (3.7) we obtain

$$\begin{aligned}\|v_t\| \le{}& e^{-\mu(t-\varepsilon)}\Big(\beta K_7\|v_0\| + \beta(K_5 + K_8) \sup_{0\le s\le\varepsilon} |h(s)|\Big)\\ &+ (K_5 + K_{10}) \sup_{\max(\varepsilon,t-r)\le s\le t} |h(s)|.\end{aligned}$$

As the interval $(0, r]$ is bounded, the constants K_i can be chosen independent of ε. This completes the proof. □

Estimate (3.4) is very interesting because, if $|h(s)|$ is bounded on $[0, +\infty)$, then the ultimate bound on v_t as $t \to +\infty$ is determined by the bound on $|h(s)|$ for s in the delay interval $[t - r, t]$ as $t \to +\infty$.

Proposition 3.9 *[12] Let $D(\varphi) = \sum_{k=0}^{p} a_k\varphi(-r_k)$. Then, D is stable iff $\sum_{k=0}^{p} |a_k| < 1$.*

In the sequel, we assume the followings.

(**H**$_5$) The operator D is stable.

(**H**$_6$) The semigroup operators $T(t)$ are compact for every $t > 0$.

Theorem 3.10 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_5)$ *and* $(\mathbf{H}_6)$ *hold. Then the semigroup* $(U(t))_{t\geq 0}$ *can be decomposed as*

$$U(t) = U_1(t) + U_2(t), \quad \textit{for } t \geq 0,$$

where $U_1(t)$ *is an exponentially stable semigroup on* C_α *and* $U_2(t)$ *is compact on* C_α *for every* $t > 0$.

Proof Without loss of generality, we can assume that there exist positive constants M_0 and γ such that the semigroup $(T(t))_{t\geq 0}$ satisfies

$$\|T(t)\| \leq M_0 e^{-\gamma t}, \quad \text{for } t \geq 0. \tag{3.10}$$

Let $U_1(t)$ be defined by

$$(U_1(t)\varphi)(\theta) = \begin{cases} \varphi(t+\theta), & \text{if } t+\theta \leq 0, \\ v(t+\theta), & \text{if } t+\theta \geq 0, \end{cases}$$

where v is the unique solution of the problem

$$\begin{cases} D(v_t) = T(t)D(\varphi), & \text{for } t \geq 0, \\ v(t) = \varphi, & \text{for } t \in [-r, 0]. \end{cases}$$

On the other hand, the operator D is stable. We deduce after applying the operator A^α that

$$\|v_t\|_\alpha \leq e^{-a(t-\varepsilon)}\left[b\|\varphi\|_\alpha + c \sup_{0\leq s\leq \varepsilon} |T(s)D(\varphi)|_\alpha\right] + d \sup_{\max(\varepsilon, t-r)\leq s\leq t} |T(s)D(\varphi)|_\alpha.$$

So, from (3.10) we get, for some constants N and ν, that

$$\|U_1(t)\varphi\|_\alpha \leq N e^{-\nu t}\|\varphi\|_\alpha, \quad \text{for } t \geq 0.$$

Let $U_2(t)\varphi := w_t = u_t - v_t$. Then,

$$D(U_2(t)\varphi) = D(u_t) - D(v_t) = \int_0^t T(t-s)F(U(s)\varphi)\,\mathrm{d}s.$$

Consequently,

$$\begin{cases} D(w_t) = h(t,\varphi) := \displaystyle\int_0^t T(t-s)F(U(s)\varphi)\,\mathrm{d}s, & \text{for } t \geq 0, \\ w_0 = 0. \end{cases} \tag{3.11}$$

Let $(\varphi_k)_{k\geq 0}$ be a bounded sequence in C_α. We will show that the family $\{h(.,\varphi_k) : k \geq 0\}$ is equicontinuous and bounded on $C([0,\sigma]; X_\alpha)$, for any $\sigma > 0$ fixed. Let $\beta \in (\alpha, 1)$. Since $A^{-\beta} : X \to X_\alpha$ is compact, it is enough to prove that $\left\{A^\beta h(t,\varphi_k) : k \geq 0\right\}$ is bounded in X, for each $t \geq 0$. Since $(U(t))_{t\geq 0}$ is locally bounded in t and φ, it follows that there exists a positive constant λ such that

$$|A^\beta h(t,\varphi_k)| \leq M_\beta \lambda \int_0^t \frac{e^{\omega s}}{s^\beta}\,\mathrm{d}s, \quad \text{for every } k \geq 0.$$

We get that $\{h(t,\varphi_k) : k \geq 0\}$ is compact in X_α, for each $t \geq 0$. Its remains to prove the equicontinuity property in α-norm. Let $t > t_0$. Then,

$$A^\alpha h(t,\varphi_k) - A^\alpha h(t_0,\varphi_k) = \int_0^{t_0} A^\alpha \left(T\,(t-s) - T\,(t_0-s)\right) F\,(U(s)\varphi_k)\,\mathrm{d}s + \int_{t_0}^{t} A^\alpha T\,(t-s)\,F\,(U(s)\varphi_k)\,\mathrm{d}s.$$

Consequently,

$$\left|\int_{t_0}^{t} A^\alpha T(t-s)F(U(s)\varphi_k)\,\mathrm{d}s\right| \leq M_\alpha \lambda \int_{t_0}^{t} \frac{e^{\omega s}}{s^\alpha}\,\mathrm{d}s \to 0 \quad \text{as } t \to t_0 \text{ uniformly in } k.$$

Moreover,

$$\int_0^{t_0} A^\alpha \left(T\,(t-s) - T\,(t_0-s)\right) F\,(U(s)\varphi_k)\,\mathrm{d}s = (T(t-t_0) - I)\int_0^{t_0} A^\alpha T(t_0-s)F(U(s)\varphi_k)\,\mathrm{d}s.$$

There is a compact set K in X such that

$$\int_0^{t_0} A^\alpha T(t_0-s)F(U(s)\varphi_k)\,\mathrm{d}s \in K, \quad \text{for all } k \geq 0.$$

It is well known, form Banach–Steinhaus's theorem, that

$$\lim_{t\to t_0} \sup_{x\in K} |(T((t-t_0) - I)x| = 0.$$

This implies that

$$\lim_{t\to t_0^+} |h(t,\varphi_k) - h(t_0,\varphi_k)|_\alpha = 0, \quad \text{uniformly in } k \geq 0.$$

The proof is similar for $t < t_0$. Then, for any $\sigma > 0$, there exists a subsequence $(\varphi_k)_{k\geq 0}$ such that $h(t,\varphi_k)$ converges as $k \to +\infty$ uniformly on $[0,\sigma]$ to some function $h(t)$ in α-norm. Let w_t^k be the solution of Eq. (3.11) with $\varphi = \varphi_k$. Then,

$$D\left(w_t^j - w_t^k\right) = h(t,\varphi_j) - h(t,\varphi_k).$$

Consequently, there is a positive constant c such that

$$\left\|w_t^j - w_t^k\right\|_\alpha \leq c \sup_{0\leq s\leq t} |h(t,\varphi_j) - h(t,\varphi_k)|_\alpha.$$

This implies that the sequence $\left(w_t^k\right)_{k\geq 0}$ is a Cauchy sequence, which proves that $U_2(t)$ is compact in C_α. □

Let $(Y, \|\cdot\|)$ be a Banach space. For a bounded linear operator B in Y, we define

$$\|B\|_{\text{ess}} := \inf\{c > 0 : \chi(B(H)) \leq c\chi(H), \text{ for every bounded set } H \text{ of } Y\},$$

where $\chi(.)$ denotes the measure of noncompactness in Y.

The essential growth bound of $(S(t))_{t\geq 0}$ in C_α is given by

$$\omega_{\text{ess}}(S) := \inf_{t>0} \frac{1}{t} \log \|S(t)\|_{\text{ess}}.$$

It follows from Theorem 3.10, that

$$\omega_{\text{ess}}(S) < 0.$$

Let

$$\omega_0(S) := \inf_{t>0} \frac{1}{t} \log \|S(t)\|_\alpha$$

be the growth bound of $(S(t))_{t\geq 0}$ in C_α. Then, it is well known (see [9]) that

$$\omega_0(S) = \max\{\omega_{\text{ess}}(S), s'(A_S)\},$$

where

$$s'(A_S) = \sup\{\text{Re}\,\lambda : \lambda \in \sigma(A_S)\backslash\sigma_{\text{ess}}(A_S)\}$$

and $\sigma_{\text{ess}}(A_S)$ is the essential spectrum of A_S. Consequently, the stability of $(S(t))_{t\geq 0}$ is completely determined by $s'(A_S)$. Note that $\sigma(A_S)\backslash\sigma_{\text{ess}}(A_S)$ contains a finite number of eigenvalues of A_S.

We say that $\lambda \in \mathbb{C}$ is a characteristic value of Eq. (3.1) if there exists a nonzero $x \in D(\Delta(\lambda))\backslash\{0\}$ such that $\Delta(\lambda)x = 0$, where $\Delta(\lambda)$ is defined by

$$\Delta(\lambda) := \lambda D(e^{\lambda\cdot}I) + AD(e^{\lambda\cdot}I) - L(e^{\lambda\cdot}I),$$

and the domain $D(\Delta(\lambda))$ is given by

$$D(\Delta(\lambda)) := \left\{x \in X_\alpha : D(e^{\lambda\cdot}x) \in D(A) \text{ and } AD(e^{\lambda\cdot}x) - L(e^{\lambda\cdot}x) \in X_\alpha\right\}.$$

Consequently, we deduce the following theorem.

Theorem 3.11 *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, $(\mathbf{H}_5)$ *and* $(\mathbf{H}_6)$ *hold. Then, the following assertions hold.*

(i) *λ is an eigenvalue of A_S iff λ is a characteristic value of Eq. (3.1).*

(ii) *If $s'(A_S) < 0$, then $(S(t))_{t\geq 0}$ is exponentially stable and consequently, the zero equilibrium of $(U(t))_{t\geq 0}$ is locally exponentially stable.*

(iii) *If $s'(A_S) = 0$, then there exists $\varphi \in C_\alpha$, $\varphi \neq 0$, such that $\|S(t)\varphi\|_\alpha = \|\varphi\|_\alpha$, for $t \geq 0$.*

(iv) *If $s'(A_S) > 0$, then there exists $\varphi \in C_\alpha$ such that $\|S(t)\varphi\|_\alpha \to +\infty$ as $t \to +\infty$ and consequently, the zero equilibrium of $(U(t))_{t\geq 0}$ is instable.*

(v) *Assume that $s'(A_S) \leq 0$ and let $s_0(A_S) := \{\lambda \in P\sigma(A_S) : \text{Re}\,\lambda = 0\}$. If each λ in $s_0(A_S)$ is a pole of order 1 of the resolvent operator of A_S, then $(S(t))_{t\geq 0}$ is stable in the sense that there exists a positive constant M such that $\|S(t)\|_\alpha \leq M$, for all $t \geq 0$.*

Proof This result is an immediate consequence of (Theorem 3.7, p. 333, [9]). □

4 Application

As an application of our abstract result, we consider the following partial neutral functional differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}\left[v(t,x)-qv(t-r,x)\right]=\dfrac{\partial^2}{\partial x^2}\left[v(t,x)-qv(t-r,x)\right] \\ \quad +f\left(v(t,x),\ v(t-r,x)\,,\ \dfrac{\partial}{\partial x}\left[v(t,x)-qv(t-r,x)\right]\right), \\ \qquad \text{for } x\in[0,\pi]\,,\ t\geq 0, \\ v(t,0)=qv(t-r,0) \text{ and } \quad v(t,\pi)=qv(t-r,\pi), \quad \text{for } t\geq 0, \\ v(\theta,x)=v_0(\theta,x), \quad \text{for } \theta\in[-r,0]\,,\ x\in[0,\pi]\,, \end{cases} \tag{4.1}$$

where $v_0 \in C\left([-r,0]\times[0,\pi]\,;\mathbb{R}\right)$, q is a positive constant and $f:\mathbb{R}^3\to\mathbb{R}$ is a Lipschitz continuous function.

Let A be the operator defined on $X := L^2([0,\pi];\mathbb{R})$ by

$$\begin{cases} D(A)=H^2(0,\pi)\cap H_0^1(0,\pi), \\ Ag=-g'', \quad g\in D(A). \end{cases}$$

Then, $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover, $T(t)$ is compact on X for every $t>0$. The spectrum $\sigma(-A)$ of $-A$ is equal to the point spectrum $P\sigma(-A)$ and is given by $\sigma(-A)=\{-n^2 : n\geq 1\}$ and the associated eigenfunctions $(\phi_n)_{n\geq 1}$ are given by $\phi_n(x)=\sin(nx)$, $x\in[0,\pi]$. Actually, the semigroup $T(t)$ is explicitly defined by

$$T(t)y=\sum_{n=1}^{\infty} e^{-n^2 t}\,\langle y,\phi_n\rangle\,\phi_n, \quad t\geq 0,\ y\in X.$$

Let $\alpha=\frac{1}{2}$. From [16], we have for $t\geq 0$

$$\begin{cases} A^{\frac{1}{2}}T(t)y=\sum_{n=1}^{\infty} ne^{-n^2 t}\,\langle y,\phi_n\rangle\,\phi_n, & \text{for } y\in X, \\ A^{-\frac{1}{2}}y=\sum_{n=1}^{\infty}\dfrac{1}{n}\,\langle y,\phi_n\rangle\,\phi_n, & \text{for } y\in X, \\ A^{\frac{1}{2}}y=\sum_{n=1}^{\infty} n\,\langle y,\phi_n\rangle\,\phi_n, & \text{for } y\in D(A^{\frac{1}{2}}). \end{cases}$$

Lemma 4.1 *[16] If $\phi\in D(A^{\frac{1}{2}})$ then ϕ is absolutely continuous and $\phi'\in X$.*

Let $F : C_{\frac{1}{2}} \to X$ be the mapping defined by

$$\left(F(\varphi)\right)(x)=f\left(\varphi(0)(x)\,,\ \varphi(-r)(x)\,,\ \frac{\partial}{\partial x}\left[\varphi(0)(x)-q\varphi(-r)(x)\right]\right), \quad \text{for } x\in[0,\pi]\,,$$

$D : C := C\left([-r,0]\,,X\right)\to X$ be the bounded linear operator defined by

$$D(\varphi)(x)=\varphi(0)(x)-q\varphi(-r)(x), \quad \text{for } x\in[0,\pi]\,,$$

$y : [-r, +\infty) \to X$ be the function defined by

$$y(t) = v(t, \cdot), \quad \text{for } t \geq -r,$$

and $\varphi(\theta) = v_0(\theta, \cdot)$, for $\theta \in [-r, 0]$. Then, Eq. (4.1) takes the abstract form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} D(y_t) = -AD(y_t) + F(y_t), & \text{for } t \geq 0, \\ y_0 = \varphi \in C_{\frac{1}{2}}. \end{cases} \tag{4.2}$$

Lemma 4.2 *F is Lipschitz continuous from $C_{\frac{1}{2}}$ into X.*

Proof Let $\varphi_1, \varphi_2 \in C_{\frac{1}{2}}$. Then, for $x \in [0, \pi]$, one has

$$\begin{aligned}(F(\varphi_1) - F(\varphi_2))(x) = {} & f\left(\varphi_1(0)(x),\ \varphi_1(-r)(x),\ \frac{\partial}{\partial x}[\varphi_1(0)(x) - q\varphi_1(-r)(x)]\right) \\ & - f\left(\varphi_2(0)(x),\ \varphi_2(-r)(x),\ \frac{\partial}{\partial x}[\varphi_2(0)(x) - q\varphi_2(-r)(x)]\right).\end{aligned}$$

Since f is Lipschitz continuous, then there exists a positive constant k such that

$$\begin{aligned}|F(\varphi_1) - F(\varphi_2)(x)| \leq k\Bigg(& |\varphi_1(0)(x) - \varphi_2(0)(x)| + |\varphi_1(-r)(x) - \varphi_2(-r)(x)| \\ & + \left|\frac{\partial}{\partial x}[\varphi_1(0)(x) - \varphi_2(0)(x) - q(\varphi_1(-r)(x) - \varphi_2(-r)(x))]\right|\Bigg).\end{aligned}$$

Which implies that

$$\begin{aligned}|F(\varphi_1) - F(\varphi_2)| \leq k\Bigg(& \sqrt{\int_0^\pi |\varphi_1(0)(x) - \varphi_2(0)(x)|^2 \,\mathrm{d}x} \\ & + \sqrt{\int_0^\pi |\varphi_1(-r)(x) - \varphi_2(-r)(x)|^2 \,\mathrm{d}x} \\ & + \sqrt{\int_0^\pi \left|\frac{\partial}{\partial x}(\varphi_1(0) - \varphi_2(0))(x)\right|^2 \mathrm{d}x} \\ & + q\sqrt{\int_0^\pi \left|\frac{\partial}{\partial x}(\varphi_1(-r) - \varphi_2(-r))(x)\right|^2 \mathrm{d}x}\Bigg).\end{aligned}$$

By [16], p. 141, we have for every $\tau \in [0, r]$

$$\sqrt{\int_0^\pi |\varphi_1(-\tau)(x) - \varphi_2(-\tau)(x)|^2 \,\mathrm{d}x} \leq \|\varphi_1 - \varphi_2\|_{\frac{1}{2}}$$

and

$$\sqrt{\int_0^\pi \left|\frac{\partial}{\partial x}(\varphi_1(-\tau) - \varphi_2(-\tau))(x)\right|^2 \mathrm{d}x} \leq \|\varphi_1 - \varphi_2\|_{\frac{1}{2}}.$$

Which means that F is Lipschitz continuous from $C_{\frac{1}{2}}$ into X. □

Consequently, we have the existence and uniqueness of mild solutions of Eq. (4.1). Let $v_0 \in C_{\frac{1}{2}}$ such that

(a) $v_0(0,\cdot) - qv_0(-r,\cdot) \in H^2(0,\pi) \cap H_0^1(0,\pi)$ and $\frac{\partial v_0}{\partial \theta} \in C_{\frac{1}{2}}$,

(b) $\frac{\partial v_0}{\partial \theta}(0,x) - q\frac{\partial v_0}{\partial \theta}(-r,x) = \frac{\partial^2}{\partial x^2}\left[v_0(0,x) - qv_0(-r,x)\right]$
$+f\left(v_0(0)(x)\,,\ v_0(-r)(x)\,,\ \frac{\partial}{\partial x}\left[v_0(0)(x) - qv_0(-r)(x)\right]\right)$, for $x \in [0,\pi]$.

We deduce that

$$\varphi \in C_{\frac{1}{2}},\ \ \varphi' \in C_{\frac{1}{2}},\ \ D(\varphi) \in D(A)\ \ \text{and}\ \ D\left(\varphi'\right) = -AD(\varphi) + F(\varphi).$$

Moreover, if we assume that f is continuously differentiable, then all assumptions of Theorem 2.6 are satisfied. We conclude that every mild solution of Eq. (4.2) is a strict solution which is also the solution of the partial differential Eq. (4.1). In the sequel, we assume that

$$0 < q < 1.$$

This means that the operator D is stable. We also assume that f is continuously differentiable and zero is a solution of (4.1), i.e. $f(0,0,0) = 0$. The linearized equation at zero of Eq. (4.1) has the following form

$$\begin{cases} \dfrac{\partial}{\partial t}\left[u(t,x) - qu(t-r,x)\right] = \dfrac{\partial^2}{\partial x^2}\left[u(t,x) - qu(t-r,x)\right] \\ \quad +au(t,x) + bu(t-r,x) + c\dfrac{\partial}{\partial x}\left[u(t,x) - qu(t-r,x)\right], \\ \quad \text{for } x \in [0,\pi],\ t \ge 0, \\ u(t,0) = qu(t-r,0), \quad u(t,\pi) = qu(t-r,\pi), \qquad \text{for } t \ge 0, \\ u(\theta,x) = u_0(\theta,x), \\ \qquad \text{for } \theta \in [-r,0],\ \ x \in [0,\pi]. \end{cases} \tag{4.3}$$

We obtain a region of stability of Eq. (4.3) as a function of parameters $a,\ b,\ c$ and q.

Theorem 3.3 *Suppose that*

$$b + qa < 0 \quad \textit{and} \quad 1 + \frac{c^2}{4} + \frac{b}{q} \ge 0.$$

Then, for every $r > 0$, all characteristic values of Eq. (4.3) have negative real parts.

Proof First, it is not difficult to prove that the spectrum $\sigma(\tilde{A})$ of the operator $\tilde{A} = \frac{\partial^2}{\partial x^2} + c\frac{\partial}{\partial x}$ is equal to the point spectrum $P\sigma(\tilde{A})$ and is given by $\{-n^2 - \frac{c^2}{4} : n \ge 1\}$. To justify this, consider

$$v'' + cv' = \lambda v, \quad v(0) = v(\pi) = 0,$$

whose nontrivial solutions can be obtained if and only if λ is one of the eigenvalues

$$\lambda_n = -n^2 - \frac{c^2}{4}, \quad n \ge 1.$$

Suppose that $b + qa < 0$. Then, the characteristic values of Eq. (4.3) are determined by the expression

$$\lambda - \frac{a + be^{-\lambda r}}{1 - qe^{-\lambda r}} = -n^2 - \frac{c^2}{4}, \quad n \geq 1. \tag{4.4}$$

We put

$$K_n = n^2 + \frac{c^2}{4}, \quad n \geq 1.$$

Then, Eq. (4.4) becomes

$$e^{\lambda r} (\lambda + K_n - a) = \lambda q + K_n q + b.$$

This implies that

$$e^{2\,\mathrm{Re}(\lambda) r}((\mathrm{Re}(\lambda) + K_n - a)^2 + (\mathrm{Im}(\lambda))^2) = q^2 \left(\left(\mathrm{Re}(\lambda) + K_n + \frac{b}{q} \right)^2 + (\mathrm{Im}(\lambda))^2 \right).$$

On the other hand, under the conditions

$$b + qa < 0 \quad \text{and} \quad 1 + \frac{c^2}{4} + \frac{b}{q} \geq 0,$$

we have, for all $n \geq 1$ and $\lambda \in \mathbb{C}$,

$$\mathrm{Re}(\lambda) + K_n - a \;>\; \mathrm{Re}(\lambda) + K_n + \frac{b}{q} \;\geq\; \mathrm{Re}(\lambda) + 1 + \frac{c^2}{4} + \frac{b}{q} \;\geq\; \mathrm{Re}(\lambda).$$

Then, if we assume that $\mathrm{Re}(\lambda) \geq 0$, we obtain that

$$e^{2\,\mathrm{Re}(\lambda) r} < q^2,$$

which is a contradiction. Then, $\mathrm{Re}(\lambda) < 0$. □

Remark that the stability result is independent of the delay. Finally, as an immediate consequence of the last theorem, we have the local stability of the zero equilibrium of Eq. (4.1).

Corollary 4.4 *Under the same assumptions as in Theorem 3.3, zero equilibrium of Eq. (4.1) is locally exponentially stable.*

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