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Asymptotic behavior for elliptic problems with singular coefficient and nearly critical Sobolev growth

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Abstract Let $B_R \subset \mathbb{R}^N\,(N \geq 3)$ be a ball centered at the origin with radius R. We investigate the asymptotic behavior of positive solutions for the Dirichlet problem $-\Delta u = \frac{\mu u}{|x|^2} + u^{2^*-1-\varepsilon}$, $u > 0$ in B_R, $u = 0$ on ∂B_R when $\varepsilon \to 0^+$ for suitable positive numbers μ.

Keywords Asymptotic behavior · Singularity · Critical Sobolev · Hardy exponents

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1 Introduction and main results

Consider the problem with critical Hardy terms

$$\begin{cases} -\Delta u = \mu \dfrac{u}{|x|^2} + u^p & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded, star-shaped domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $0 \leq \mu < \bar{\mu} = (\frac{N-2}{2})^2$, $p > 1$, $N \geq 3$. It is well known that the character of this

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problem changes when the exponent $p+1$ passes through the critical Sobolev exponent $2^* = \frac{2N}{N-2}$. If $p+1 < 2^*$, then problem (1.1) always has a solution, whatever the domain Ω is (see [1]), while if $p+1 \geq 2^*$ it has no solution for any star-shaped domain (see [1]).

Recently, considerable attention has been paid to problems like (1.1) in which the right-hand side term u^p is replaced by a perturbation $f(u)$ of the pure power, such as $f(u) = u^p + \lambda u^q$ with $\lambda > 0$ and $0 < q < p$, because of (1) its definite celestial mechanics and physics background (see [4] for example) and (2) its critical singular term (Hardy term) and critical (nearly critical) Sobolev growth. Many existence and nonexistence results have been given with various parameters μ, λ and q, we refer to [1, 7, 8, 12, 16] and the references therein. When $\mu = 0$, there are many interesting results on the asymptotic behavior of the positive solutions of (1.1), which can be found in [3, 5, 6, 13, 14, 19] and the references therein.

In this paper, we study the case $\mu \in [0, \bar{\mu})$. We mainly concentrate on the asymptotic behavior of the solutions to problem (1.1) when $p+1$ approaches the critical Sobolev exponent from below. Thus we set $p = 2^* - 1 - \varepsilon$. We will let $\varepsilon > 0$ tend to zero and choose a ball

$$\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}, \quad R > 0.$$

For each $0 < \varepsilon < 2^* - 1$ and $R > 0$, problem (1.1) has a solution in $H_0^1(\Omega)$ (see [1]) which we will often denote by $u_\varepsilon(x)$. We will prove that this solution is radially symmetric.

We denote $\nu = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}$ and $\Gamma(x)$ the Gamma function. Our principal results include two theorems, the first dealing with the behavior of the limit value of $u_\varepsilon(x)$ at the origin which is the singular point of $u_\varepsilon(x)$ and the second describing the shape of $u_\varepsilon(x)$ when x is away from the origin, as $\varepsilon \to 0$.

Theorem 1.1 *Let $u_\varepsilon(x) \in H_0^1(\Omega)$ be a solution of problem* (1.1) *in which $p = 2^* - 1 - \varepsilon$ and $\Omega = B_R$. Then*

$$\begin{aligned}\lim_{\varepsilon \to 0} \lim_{|x| \to 0} \varepsilon u_\varepsilon^2(x) \mid x|^{2\nu} \\ = 4N^{\frac{N-2}{2}}(N-2)^{-\frac{N+2}{2}}(2\sqrt{\bar{\mu}-\mu})^{N-1} \frac{\Gamma(N)}{[\Gamma\left(\frac{N}{2}\right)]^2} \frac{1}{R^{2\sqrt{\bar{\mu}-\mu}}}.\end{aligned}$$

Theorem 1.2 *Let $u_\varepsilon(x) \in H_0^1(\Omega)$ be a solution of problem* (1.1) *in which $p = 2^* - 1 - \varepsilon$ and $\Omega = B_R$. Then, for every $x \neq 0$,*

$$\begin{aligned}\lim_{\varepsilon \to 0} \varepsilon^{-\frac{1}{2}} u_\varepsilon(x) = \frac{1}{2}(2\sqrt{\bar{\mu}-\mu})^{\frac{N-3}{2}} N^{\frac{N-2}{4}} (N-2)^{-\frac{N-6}{4}} R^{\sqrt{\bar{\mu}-\mu}} \frac{\Gamma\left(\frac{N}{2}\right)}{[\Gamma(N)]^{\frac{1}{2}}} \\ \times \left(\frac{1}{|x|^{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}} - \frac{1}{|x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}} R^{2\sqrt{\bar{\mu}-\mu}}} \right).\end{aligned}$$

For Theorem 1.2, we have

Remark 1.1 Theorem 1.2 indicates that for every $x \neq 0$, $u_\varepsilon(x) = O(\varepsilon^{\frac{1}{2}})$ as $\varepsilon \to 0$. Further more, Theorem 1.2 also states that, when ε tends to zero, the solutions, suitably rescaled, tend to a precise function which is (apart from multiplicative constants) the fundamental solution of the linear operator $-\Delta - \mu/|x|^2$.

The main method to prove these two theorems is to transform problem (1.1) into a boundary value problem of ODE and then employ a shooting technique introduced by Atkinson and Peletier in [3]. In [3] where $\mu = 0$ and the solution $u_\varepsilon(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, so following the famous Gidas–Ni–Nirenberg Theorem in [15] directly, all solutions of (1.1) are radial symmetric. However in our case $\mu \neq 0$, due to the appearance of the critical Hardy term $\frac{\mu u}{|x|^2}$, the solution $u_\varepsilon(x)$ is singular at the origin and fails to satisfy the conditions of Gidas–Ni–Nirenberg Theorem in [15]. As we will see later, this difficulty is not so easy as we expect. To overcome this difficulty, we employ an important result in [9] which can be expressed as following:

Theorem 1.3 **(Chou and Chu)** *Let $u(x)$ be bounded $C^2(B_R \setminus \{0\}) \cap C(\bar{B_R})$ solution of*

$$\begin{cases} -\partial_i(|x|^a \partial_i u) + |x|^b u^q = 0 & x \in B_R \setminus \{0\}, \\ u > 0 & x \in B_R, \\ u = 0 & x \in \partial B_R. \end{cases}$$

Then $u(x)$ is radially symmetric in B_R provided $q \geq 1$, $a(a/2 + N - 2) \leq 0$ and $a/2 \geq b/q$.

More precisely, first using Moser iteration and a generalized comparison principle in [10], we prove that the exact singularity of $u_\varepsilon(x)$ at the origin is $|x|^{-\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}$, so $v(x) = |x|^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}} u_\varepsilon(x)$ is bounded in $L^\infty(\Omega)$ and solves a new equation. Then applying Theorem A to this new equation, we deduce that $v(x)$ is radially symmetric, continuous at the origin and satisfies an ODE. At last by dealing with the ODE satisfied by $v(x)$, we establish our main results.

With the same method, we also study the asymptotic behavior of solutions to the following (possibly degenerate) elliptic problem:

$$\begin{cases} -\mathrm{div}(|x|^\alpha \nabla u) = |x|^\beta u^{p(\alpha,\beta)-1-\varepsilon} & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where

$$p(\alpha, \beta) = \frac{2(N+\beta)}{N+\alpha-2}, \quad N + \beta > 0, \quad \beta - \alpha + 2 > 0, \quad 0 < \varepsilon < p(\alpha, \beta) - 1. \tag{1.3}$$

Problem (1.2) has been investigated in many papers, for example [8, 9, 11, 17]. Let $H_0^1(\Omega, |x|^\alpha)$ be the completion of $C_0^\infty(\Omega)$ under the inner product

$$\langle u, v \rangle = \int_\Omega |x|^\alpha \nabla u \cdot \nabla v \, \mathrm{d}x$$

and $L^p(\Omega, |x|^\beta)$ be the weighted L^p space with the weight $|x|^\beta$. By Mountain Pass Lemma and compactness of embedding $H_0^1(\Omega, |x|^\alpha) \hookrightarrow L^q(\Omega, |x|^\beta)$ for $q \in (1, p(\alpha, \beta))$, it is easy to prove that for $0 < \varepsilon < p(\alpha, \beta) - 1$ and $\Omega = B_R$, problem (1.2) has a radially symmetric solution denoted by $u_{\alpha,\beta,\varepsilon}(x)$ in $H_0^1(\Omega, |x|^\alpha)$ (see also [17] for the case $\alpha = 0$). We have the following asymptotic behaviors of $u_{\alpha,\beta,\varepsilon}(x)$ as $\varepsilon \to 0$:

Theorem 1.4 *Suppose* (1.3) *holds and let* $u_{\alpha,\beta,\varepsilon}(x)$ *be the solution of problem* (1.2) *in which* $\Omega = B_R$. *Then*

$$\lim_{\varepsilon\to 0} \varepsilon u^2_{\alpha,\beta,\varepsilon}(0) = \frac{2(\beta-\alpha+2)}{N+\alpha-2}(N+\alpha-2)^{\frac{N+\alpha-2}{\beta-\alpha+2}}(N+\beta)^{\frac{N+\alpha-2}{\beta-\alpha+2}} \times \frac{\Gamma\left(\frac{2(N+\beta)}{\beta-\alpha+2}\right)}{\left[\Gamma\left(\frac{N+\beta}{\beta-\alpha+2}\right)\right]^2}\frac{1}{R^{N+\alpha-2}}.$$

Theorem 1.5 *Suppose* (1.3) *holds and let* $u_{\alpha,\beta,\varepsilon}(x)$ *be the solution of problem* (1.2) *in which* $\Omega = B_R$. *Then, for every* $x \neq 0$,

$$\lim_{\varepsilon\to 0} \varepsilon^{-\frac{1}{2}} u_\varepsilon(x) = C(\alpha,\beta,N)R^{\frac{N+\alpha-2}{2}}\left(\frac{1}{|x|^{N+\alpha-2}} - \frac{1}{R^{N+\alpha-2}}\right),$$

where

$$C(\alpha,\beta,N) = \frac{1}{2}\left(\frac{N+\alpha-2}{\beta-\alpha+2}\right)^{\frac{1}{2}}[(N+\beta)(N+\alpha-2)]^{\frac{N+\alpha-2}{2(\beta-\alpha+2)}}\frac{\Gamma\left(\frac{N+\beta}{\beta-\alpha+2}\right)}{\left[\Gamma\left(\frac{2(N+\beta)}{\beta-\alpha+2}\right)\right]^{\frac{1}{2}}}.$$

To end this section, we give two remarks:

Remark 1.2 Our results are the same as those in [3, 13, 20] if $\mu = 0$ in Theorems 1.1 and 1.2 and $\alpha = \beta = 0$ in Theorems 1.4 and 1.5.

Remark 1.3 In this paper, we have obtained two other important results indeed, i.e., a unique result of solutions to problem (1.1) and (1.2) (see Remark 3.2) and a nonexistence result of solutions in $H^1(\mathbb{R}^N)$ to problem (1.1)and (1.2) (see Remark 4.1).

This paper is organized as follows: in the following section, we will estimate the singularity of the solution to problem (1.1) in a more general domain and we will give some basic estimates in Sect. 3. The last section will be devoted to the proof of the main results.

2 Estimate of the singularity

In the sequel, we fix $p = 2^* - 1 - \varepsilon > 0$ in problem (1.1) and study the singularity and radial symmetry of the solution $u_\varepsilon(x) \in H^1_0(\Omega)$. By standard elliptic regularity theory, $u_\varepsilon \in C^2(\Omega \setminus \{0\}) \cap C^1(\bar{\Omega} \setminus \{0\})$. Hence the singular point of u_ε should be the origin.

Suppose $u(x) \in H^1_0(\Omega)$ satisfies problem (1.1). Let $v(x) = |x|^\nu u(x)$, then $v(x)$ satisfies

$$\begin{cases} -\mathrm{div}(|x|^{-2\nu}\nabla v) = |x|^{-(2^*-\varepsilon)\nu} v^{2^*-1-\varepsilon} & x \in \Omega, \\ v > 0 & x \in \Omega, \\ v = 0 & x \in \partial\Omega. \end{cases} \tag{2.1}$$

By the regularity theory of elliptic equations, $v \in C^2(\Omega \setminus \{0\}) \cap C^1(\bar{\Omega} \setminus \{0\})$. Moreover, we have

Lemma 2.1 *$v(x) \in H_0^1(\Omega, |x|^{-2\nu})$ and $v(x)$ is bounded in Ω.*

Proof For simplicity, set $\alpha = -2\nu$, $\beta = -(2^* - \varepsilon)\nu$, then $p(\alpha, \beta) = 2^* + \frac{\nu\varepsilon}{\sqrt{\bar{\mu}-\mu}}$. By Caffarelli–Kohn–Nirenberg Inequalities (see [8] for example), we have

$$\left(\int_\Omega |x|^\alpha |\nabla w|^2\right)^{\frac{1}{2}} \geq A_{\alpha,\beta}\left(\int_\Omega |\, |x|^\beta |w|^{p(\alpha,\beta)}\right)^{\frac{1}{p(\alpha,\beta)}}, \quad \forall\ w \in H_0^1(\Omega, |x|^\alpha). \tag{2.2}$$

For any $u \in H_0^1(\Omega)$ satisfying problem (1.1), we claim $v(x) = |x|^\nu u(x) \in H_0^1(\Omega, |x|^\alpha)$. Indeed, by Hardy inequality,

$$\begin{aligned}\int_\Omega |x|^\alpha |\nabla v|^2 &= \int_\Omega |x|^\alpha \big||x|^\nu \nabla u + \nu |x|^{\nu-2} u x\big|^2\\ &\leq 2\left(\int_\Omega |\nabla u|^2 + \nu^2 \int_\Omega \frac{u^2}{|x|^2}\right)\\ &\leq C.\end{aligned}$$

Because $v(x)$ satisfies

$$\int_\Omega |x|^\alpha \nabla v \cdot \nabla\phi = \int_\Omega |x|^\beta v^p \phi, \qquad \forall\phi \in H_0^1(\Omega, |x|^\alpha),$$

for $s, l > 1$, defining $v_l = \min\{v, l\}$ and taking $\phi = v v_l^{2(s-1)} \in H_0^1(\Omega, |x|^\alpha)$ in the above equation, we deduce

$$\int_\Omega |x|^\alpha |\nabla v|^2 v_l^{2(s-1)} + 2(s-1)\int_\Omega |x|^\alpha |\nabla v_l|^2 v_l^{2(s-1)} = \int_\Omega |x|^\beta v^{p+1} v_l^{2s-2}.$$

Hence, we can write

$$\begin{aligned}\left(\int_\Omega |x|^\beta \big(v v_l^{s-1}\big)^{p(\alpha,\beta)}\right)^{\frac{2}{p(\alpha,\beta)}} &\leq A_{\alpha,\beta}^{-2}\int_\Omega |x|^\alpha |\nabla\big(v v_l^{s-1}\big)|^2\\ &\leq 2A_{\alpha,\beta}^{-2}\bigg((s-1)^2\int_\Omega |x|^\alpha v_l^{2(s-1)} |\nabla v_l|^2\\ &\quad + \int_\Omega |x|^\alpha v_l^{2(s-1)} |\nabla v|^2\bigg)\\ &\leq 2A_{\alpha,\beta}^{-2} s \int_\Omega |x|^\beta |v|^{p+2s-1}.\end{aligned} \tag{2.3}$$

From (2.3), we see that $v \in L^{p+2s-1}(\Omega, |x|^\beta)$ implies $v \in L^{sp(\alpha,\beta)}(\Omega, |x|^\beta)$.

Define, for $j = 0, 1, 2, \ldots$,

$$\begin{cases} p - 1 + 2s_0 = p(\alpha, \beta),\\ p - 1 + 2s_{j+1} = p(\alpha, \beta) s_j,\end{cases} \tag{2.4}$$

$$\begin{cases} M_0 = C A_{\alpha,\beta}^{-2},\\ M_{j+1} = \big(2A_{\alpha,\beta}^{-2} s_j M_j\big)^{\frac{p(\alpha,\beta)}{2}},\end{cases} \tag{2.5}$$

where C is a fixed number such that $\int_\Omega |x|^\alpha |\nabla v|^2 \leq C$.

From (2.4) we derive

$$s_j = (p(\alpha,\beta)-2)^{-1}\{(2^{-1}p(\alpha,\beta))^{j+1}(p(\alpha,\beta)-p-1)+p-1\}.$$

From (2.5), similar to the computation in [18], we derive

$$\exists\, d>0 \quad \text{independent of } j, \quad \text{such that } M_j \le e^{\mathrm{d}s_{j-1}}.$$

From the fact that $2 < p+1 < p(\alpha,\beta)$, it follows that $s_j > 1$ for all $j \ge 0$, $s_j \to +\infty$ as $j \to +\infty$.

By (2.3), (2.4) and (2.5),

$$\begin{aligned}\int_\Omega |x|^\beta v^{p+2s_1-1} &\le \left(2A_{\alpha,\beta}^{-2}s_0\right)^{\frac{p(\alpha,\beta)}{2}}\left(\int_\Omega |x|^\beta v^{p+2s_0-1}\right)^{\frac{p(\alpha,\beta)}{2}}\\ &\le \left(2A_{\alpha,\beta}^{-2}s_0\right)^{\frac{p(\alpha,\beta)}{2}}\left(\int_\Omega |x|^\beta v^{p(\alpha,\beta)}\right)^{\frac{p(\alpha,\beta)}{2}}\\ &\le \left(2A_{\alpha,\beta}^{-2}s_0M_0\right)^{\frac{p(\alpha,\beta)}{2}}\\ &\le M_1.\end{aligned}$$

Similarly,

$$\int_\Omega |x|^\beta v^{p+2s_j-1} \le M_j.$$

So, by $p-1+2s_{j+1} = p(\alpha,\beta)s_j$, denoting $C(\Omega,\beta) = \max_{x\in\Omega}|x|^{-\beta}$, we obtain

$$\begin{aligned}|v|_{L^{p(\alpha,\beta)s_j}(\Omega)} &\le C(\Omega,\beta)^{\frac{1}{p(\alpha,\beta)s_j}}|v|^{\frac{1}{p(\alpha,\beta)s_j}}_{L^{p(\alpha,\beta)s_j}(\Omega,|x|^\beta)}\\ &\le C(\Omega,\beta)^{\frac{1}{p(\alpha,\beta)s_j}}M_j^{\frac{1}{p(\alpha,\beta)s_j}}\\ &\le C(\Omega,\beta)^{\frac{1}{p(\alpha,\beta)s_j}}e^{\frac{\mathrm{d}s_{j-1}}{p(\alpha,\beta)s_j}}.\end{aligned}$$

Taking limit on each side of the above inequality and using $s_{j-1} \to +\infty$, $\frac{s_j}{s_{j-1}} \to \frac{p(\alpha,\beta)}{2}$, as $j \to +\infty$, we have

$$|v|_{L^\infty(\Omega)} \le e^{2d},$$

which implies the conclusion. □

From Lemma 2.1, we deduce $u(x)|x|^\nu$ is upper bounded in Ω. For the lower bound of $u(x)|x|^\nu$ in Ω, we have

Lemma 2.2 *Suppose $u \in H_0^1(\Omega)$ satisfies problem* (1.1) *and $0 \le \mu < \bar{\mu}$, then for any $B_\rho \subset\subset \Omega$ there exists a $C(\rho) > 0$, such that*

$$u(x) \ge C(\rho)|x|^{-\nu}, \quad \forall\, x \in B_\rho \subset\subset \Omega.$$

Proof We only need to prove that there exist $\eta > 0$ and $C > 0$, such that

$$u(x) \geq C|x|^{-\nu}, \quad \forall\ x \in B_\eta \subset \Omega.$$

Let $f(x) = \min\{u^{2^*-1-\varepsilon}(x), l\}$ with $l > 0$, then $f \in L^\infty(\Omega)$. Let $u_1 \in H_0^1(\Omega)$ be the solution of the following linear problem

$$\begin{cases} -\Delta u_1 - \mu \dfrac{u_1}{|x|^2} = f & x \in \Omega, \\ u_1 = 0 & x \in \partial\Omega. \end{cases} \tag{2.6}$$

By Lemma 2.1 and a result in [10], there exists a constant $C_1 > 0$ such that $0 \leq u_1 \leq C_1|x|^{-\nu}$. Since u is a supersolution of problem (2.6), by the comparison principle proved in [10], $0 \leq u_1 \leq u$, so it suffices to prove the result for u_1.

Now since $u_1 \not\equiv 0$, $u_1 \geq 0$ and $-\Delta u_1 \geq 0$ in Ω, we have some $\delta > 0$ and $\eta > 0$ such that $u_1 \geq \delta$ in $B_{2\eta}$.

Choose $C > 0$ satisfying $C|x|^{-\nu} \leq \delta$ for $|x| \geq \eta$ and set $w = (u_1 - C|x|^{-\nu})^-$, then $w \in H_0^1(B_\eta)$.

By Hardy inequality and the fact that $|x|^{-\nu}$ solves the problem $-\Delta u - \frac{\mu u}{|x|^2} = 0$, we deduce

$$\begin{aligned} 0 &\geq -\int_{B_\eta} |\nabla w|^2 + \frac{\mu w^2}{|x|^2} \\ &= \int_{B_\eta} \nabla(u_1 - C|x|^{-\nu}) \cdot \nabla w - \int_{B_\eta} \frac{\mu}{|x|^2}(u_1 - C|x|^{-\nu})w \\ &= \int_{B_\eta} f w - C\left(\int_{B_\eta} \nabla|x|^{-\nu} \cdot \nabla w - \int_{B_\eta} \frac{\mu}{|x|^2}|x|^{-\nu} w\right) \\ &\geq \frac{C\nu}{\eta^{1+\nu}} \int_{\partial B_\eta} w \geq 0. \end{aligned}$$

Hence, by maximum principle, $w \equiv 0$ in B_η, which is our desired conclusion. □

Now combining Lemma 2.1 and Lemma 2.2, we have

Proposition 2.1 *Suppose $u \in H_0^1(\Omega)$ satisfies problem* (1.1) *and* $0 \leq \mu < \bar{\mu}$, *then for any $\Omega' \subset\subset \Omega$, there exist two positive constants C_1 and C_2, such that*

$$\begin{cases} u(x)|x|^\nu \geq C_1 & \forall x \in \Omega' \subset\subset \Omega, \\ u(x)|x|^\nu \leq C_2 & \forall x \in \Omega. \end{cases} \tag{2.7}$$

Moreover, $u(x)$ is radially symmetric if $\Omega = B_R$.

Proof (2.7) is obvious. The radial symmetry can be deduced from Lemma 2.1 and Theorem 1.3. □

3 Some basic estimates

Now, by Proposition 2.1, setting $r = |x|$, we can write $v(r) = |x|^{\nu} u_{\varepsilon}(x)$. Then $v(r)$ satisfies

$$\begin{cases} v'' + \frac{N-2\nu-1}{r} v' + \frac{1}{r^{(2^*-2-\varepsilon)\nu}} v^{2^*-1-\varepsilon} = 0, \\ v > 0 \quad \text{for } 0 < r < R, \\ v(R) = 0. \end{cases} \tag{3.1}$$

Set $t = (\frac{N-2\nu-2}{r})^{N-2\nu-2}$ and $y(t) = (N-2\nu-2)^{-\nu} v(r)$. Problem (3.1) can be rewritten as

$$\begin{cases} y''(t) = -t^{-k(\varepsilon)} y^{2^*-1-\varepsilon}, \\ y(t) > 0 \quad \text{for } T < t < \infty, \\ y(T) = 0, \end{cases} \tag{3.2}$$

where $m = 1+2\sqrt{\bar{\mu}-\mu} = N-2\nu-1$, $k(\varepsilon) = \frac{2m}{m-1} - \frac{(2^*-2-\varepsilon)\nu}{m-1}$, $T = (\frac{m-1}{R})^{m-1}$. The critical Sobolev exponent can now be expressed as

$$2^* - 1 = 2k(\varepsilon) - 3 - \frac{2\nu\varepsilon}{m-1}.$$

To simplify notation, we will always write $k(\varepsilon)$ as k in the sequel.

First we give

Lemma 3.1 *Suppose that $y(t)$ satisfies problem (3.2), then there exists a positive number $\gamma < +\infty$ such that*

$$\lim_{t\to+\infty} y'(t) = 0 \quad \text{and} \quad \lim_{t\to+\infty} y(t) = \gamma.$$

Proof By Proposition 2.1, $y(t)$ is bounded in $[T, +\infty)$. From (3.2) we know $y''(t) < 0$ for all $t > T$, so $y'(t)$ decreases strictly in $t \in (T, +\infty)$. Hence

$$y'(t) \to l \quad \text{as} \quad t \to +\infty.$$

In the cases $l > 0$ and $l < 0$, we deduce $y(t) \to +\infty$ and $y(t) \to -\infty$ as $t \to +\infty$, respectively, which contradicts the boundedness of $y(t)$. Therefore $y'(t) \to 0$ and $y(t) \to \gamma < +\infty$ as $t \to +\infty$. □

Remark 3.1

(i) From Lemma 3.1, if we define $v(0) = \lim_{r\to 0} v(r) = (N-2\nu-2)^{\nu}\gamma$, then $v(r) \in C[0, R]$. Furthermore, $v'(r) < 0$ for all $r \in (0, R]$.

(ii) $y'(t) > 0$ for all $t \geq T$ and $y'(t) \sim \frac{1}{k-1} t^{1-k} \gamma^{2^*-1-\varepsilon}$ as $t \to +\infty$. So from $y(t) = (N-2\nu-2)^{-\nu} v(r)$, we deduce that

$$\begin{cases} v'(0^+) = 0 & \text{if } 0 \leq \mu < \mu^*, \\ v'(0^+) = -\dfrac{1}{k-1}(N-2\nu-2)^{\nu}\gamma^{2^*-1-\varepsilon} & \text{if } \mu = \mu^*, \\ v'(0^+) = -\infty & \text{if } \mu^* < \mu < \bar{\mu}, \end{cases}$$

where

$$\mu^* = \left(\frac{N-2}{2}\right)^2 - \left(\frac{2(N-2)-(N-2)^2\varepsilon}{8-2(N-2)\varepsilon}\right)^2.$$

Consider the equation

$$\begin{cases} y''(t) + t^{-k} y^{2^*-1-\varepsilon} = 0 \quad t < \infty, \\ \lim_{t\to\infty} y(t) = \gamma, \end{cases} \tag{3.3}$$

where $\gamma > 0$.

Since $k > 2$, it follows from [2] that problem (3.3) has, for every $\gamma > 0$, a unique solution which will be denoted by $y(t,\gamma)$. Define

$$T(\gamma) = \inf\{t > 0 : y(t,\gamma) > 0 \quad \text{on} \quad (t,\infty)\}, \tag{3.4}$$

then $T(\gamma) = \gamma^{\frac{2^*-2-\varepsilon}{k-2}} T(1)$. By Lemma 4.1 in Sect. 4, $T(1) > 0$. Thus for every $\gamma > 0$, $T(\gamma) > 0$.

Hence, for any given $T > 0$ and $\varepsilon > 0$ small, there exists a unique γ such that problem (3.3) has solution $y(t,\gamma)$ such that $y(T,\gamma) = 0$.

Remark 3.2 From the above analysis we conclude that the solution to problem (1.1) is unique when Ω is a ball centered at the origin.

Now we give an upper and lower bound for $y(t,\gamma)$.

Lemma 3.2 *Suppose $\varepsilon > 0$ small, then*

$$y(t,\gamma) < z(t,\gamma) \quad for \quad T(\gamma) \le t < \infty, \tag{3.5}$$

where

$$z(t,\gamma) = \gamma\left(1 + \frac{1}{k-1}\frac{\gamma^{2^*-2-\varepsilon}}{t^{k-2}}\right)^{-\frac{1}{k-2}},$$

satisfying

$$\begin{cases} z''(t) + t^{-k}\gamma^{-\left(\frac{2\nu}{m-1}+1\right)\varepsilon} z^{2^*-1} = 0, \quad 0 < t < \infty, \\ \lim_{t\to\infty} z(t,\gamma) = \gamma. \end{cases} \tag{3.6}$$

The proof is similar to that in [19], and we omit it here.

Set

$$T_\varepsilon = \frac{\gamma^{(2^*-2-\varepsilon)/(k-2)}}{k_1(\varepsilon)} = \frac{\gamma^{2-\frac{m-1+2\nu}{(m-1)(k-2)}\varepsilon}}{k_1(\varepsilon)}, \tag{3.7}$$

where $k_1(\varepsilon) = (k-1)^{\frac{1}{k-2}}$. Then for any $\alpha > 0$,

$$z(\alpha T_\varepsilon, \gamma) = c_{\alpha,\varepsilon}\gamma, \tag{3.8}$$

where

$$c_{\alpha,\varepsilon} = \frac{\alpha}{(1+\alpha^{k-2})^{1/(k-2)}}.$$

Lemma 3.3 *Let $\alpha > 0$ and $\varepsilon > 0$ small, then for every $t \geq \alpha T_\varepsilon$,*

$$y(t,\gamma) \geq z(t,\gamma)(1 - d_{\alpha,\varepsilon}\varepsilon),$$

where

$$d_{\alpha,\varepsilon} = \frac{(1 - c_{\alpha,\varepsilon})(1 + 2\nu/(m-1))}{c_{\alpha,\varepsilon}^{2+(1+2\nu/(m-1))\varepsilon}}.$$

Proof By (3.3) and (3.6), we have

$$y(t,\gamma) = \gamma - \int_t^\infty (s-t)s^{-k} y^{2k-3-(1+2\nu/(m-1))\varepsilon}(s,\gamma)\,\mathrm{d}s,$$

$$z(t,\gamma) = \gamma - \int_t^\infty (s-t)s^{-k} \gamma^{-(1+2\nu/(m-1))\varepsilon} z^{2k-3}(s,\gamma)\,\mathrm{d}s.$$

Hence, by Lemma 3.2,

$$y(t,\gamma) > z(t,\gamma) - \int_t^\infty (s-t)s^{-k} z^{2k-3}(s,\gamma)\big(z^{-(1+2\nu/(m-1))\varepsilon}(s,\gamma) - \gamma^{-(1+2\nu/(m-1))\varepsilon}\big)\,\mathrm{d}s. \tag{3.9}$$

By the mean value theorem we deduce

$$\begin{aligned}
&\big|z^{-(1+2\nu/(m-1))\varepsilon}(s,\gamma) - \gamma^{-(1+2\nu/(m-1))\varepsilon}\big| \\
&\quad = \left(1 + \frac{2\nu}{m-1}\right)\varepsilon\theta^{-1-(1+2\nu/(m-1))\varepsilon}|z(s,\gamma) - \gamma|, \quad z(s,\gamma) \leq \theta \leq \gamma.
\end{aligned}$$

Hence, using (3.8), if $\alpha T_\varepsilon \leq t < \infty$, we have

$$\begin{aligned}
&\big|z^{-(1+2\nu/(m-1))\varepsilon}(s,\gamma) - \gamma^{-(1+2\nu/(m-1))\varepsilon}\big| \\
&\quad \leq \left(1 + \frac{2\nu}{m-1}\right)\varepsilon(c_{\alpha,\varepsilon}\gamma)^{-1-(1+2\nu/(m-1))\varepsilon}\gamma \\
&\quad \leq \left(1 + \frac{2\nu}{m-1}\right)\varepsilon c_{\alpha,\varepsilon}^{-1-(1+2\nu/(m-1))\varepsilon} \\
&\qquad \gamma^{-(1+2\nu/(m-1))\varepsilon}.
\end{aligned}$$

Inserting this estimate into (3.9), we find that, if $t \geq \alpha T_\varepsilon$, then

$$\begin{aligned}
y(t,\gamma) &> z(t,\gamma) - \frac{(1+2\nu/(m-1))\varepsilon}{c_{\alpha,\varepsilon}^{1+(1+2\nu/(m-1))\varepsilon}} \\
&\quad \times \int_t^\infty (s-t)s^{-k}\gamma^{-(1+2\nu/(m-1))\varepsilon} z^{2k-3}(s,\gamma)\,\mathrm{d}s \qquad (3.10)\\
&= z(t,\gamma) + \frac{(1+2\nu/(m-1))\varepsilon}{c_{\alpha,\varepsilon}^{1+(1+2\nu/(m-1))\varepsilon}}(z(t,\gamma) - \gamma).
\end{aligned}$$

On the other hand, by (3.8), if $t \geq \alpha T_\varepsilon$,

$$\gamma = c_{\alpha,\varepsilon}^{-1} z(\alpha T_\varepsilon, \gamma) \leq c_{\alpha,\varepsilon}^{-1} z(t,\gamma).$$

So we can deduce from (3.10) that

$$y(t,\gamma) > z(t,\gamma)\left(1+\frac{(1+2\nu/(m-1))}{c_{\alpha,\varepsilon}^{1+(1+2\nu/(m-1))\varepsilon}}\left(1-\frac{1}{c_{\alpha,\varepsilon}}\right)\varepsilon\right),$$

which is the bound we want to prove. □

Now we return to problem (3.2). We fix T and denote the solution by $y(t)$. Then $\lim_{t\to\infty} y(t) = \gamma(\varepsilon)$ and $\gamma(\varepsilon)$ depends on ε. Moreover

Lemma 3.4 $\lim_{\varepsilon\to 0}\gamma(\varepsilon) = \infty$.

Proof From Lemma 3.3, there exists some $C(T) > 0$ such that $\gamma(\varepsilon) \geq C(T)$ when ε is small. Suppose, in contrast, there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ as $n \to \infty$, and a number $M > 0$ such that $\gamma(\varepsilon_n) \leq M$ for all n. Then we can choose a number $\alpha > 0$ satisfying

$$\alpha T_{\varepsilon_n} = \alpha k_1(\varepsilon)^{-1}\gamma(\varepsilon_n)^{2-\frac{1+2\nu/(m-1)}{k-2}\varepsilon} \leq \alpha\left(\frac{N-2}{N}\right)^{\frac{N-2}{2}} M^2 + o(1) \leq T, \text{ for large } n.$$

So by Lemma 3.3, for large n, we have $z(t,\gamma(\varepsilon_n))(1-d_{\alpha,\varepsilon}\varepsilon_n) < 0$, which is impossible. □

Finally, we give two formulae for us to use later. Define

$$B(\xi,a,b) = \int_\xi^\infty x^{a-1}(1+x)^{-a-b}\,\mathrm{d}x$$

with positive parameters a and b. Then

$$B(0,a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{3.11}$$

By direct calculation, we obtain

Lemma 3.5 *Suppose $k > 2$, $p = 2k-3-(1+2\nu/(m-1))\varepsilon$ and ε small, then*
(i) $\int_t^\infty s^{-k}z^p(s,\gamma)\mathrm{d}s = k_1(\varepsilon)k_2(\varepsilon)\gamma^{-1+\kappa(\varepsilon)}B(\tau(\varepsilon), 1-\kappa(\varepsilon), k_2(\varepsilon))$,
(ii) $\int_t^\infty s^{-k}z^{p+1}(s,\gamma)\mathrm{d}s = k_1(\varepsilon)k_2(\varepsilon)\gamma^{\kappa(\varepsilon)}B(\tau(\varepsilon), k_2(\varepsilon)-\kappa(\varepsilon), k_2(\varepsilon))$,
where

$$\kappa(\varepsilon) = \frac{m-1+2\nu}{(m-1)(k-2)}\varepsilon, \quad k_1(\varepsilon) = (k-1)^{1/(k-2)}, \quad k_2(\varepsilon) = \frac{k-1}{k-2},$$
$$\tau(\varepsilon) = \left(\frac{t}{T_\varepsilon}\right)^{k-2}.$$

We end this section by giving

$$\begin{aligned}
&\lim_{\varepsilon\to 0} k = \frac{2N-2}{N-2} \triangleq k_0, &&\lim_{\varepsilon\to 0} k_1(\varepsilon) = \left(\frac{N}{N-2}\right)^{\frac{N-2}{2}} \triangleq k_1,\\
&\lim_{\varepsilon\to 0} k_2(\varepsilon) = \frac{N}{2} \triangleq k_2, &&\lim_{\varepsilon\to 0} c_{\alpha,\varepsilon} = \frac{\alpha}{\left(1+\alpha^{\frac{2}{N-2}}\right)^{\frac{N-2}{2}}} \triangleq c_\alpha,\\
&\lim_{\varepsilon\to 0} d_{\alpha,\varepsilon} = \frac{1-c_\alpha}{c_\alpha^2} \triangleq d_\alpha, &&\lim_{\varepsilon\to 0}\kappa(\varepsilon) = \lim_{\varepsilon\to 0}\tau(\varepsilon) = 0.
\end{aligned} \tag{3.12}$$

4 Proof of the main results

Note that if $u_\varepsilon(x)$ is a solution of problem (1.1) with $\Omega = B_R$, then from the previous analysis, we know that

$$\lim_{|x|\to 0} u_\varepsilon(x)|x|^\nu = (N-2\nu-2)^\nu \gamma(\varepsilon)$$

if $R = (m-1)T^{-1/(m-1)}$. Thus we need to understand how $\gamma(\varepsilon)$ tends to infinity as $\varepsilon \to 0$.

We define the following Pohozaev functional which was introduced in [3, 19], etc.

$$H(t) = ty'^2 - yy' + 2t^{1-k}\frac{y^p}{p+1}, \quad p = 2k-3-\left(1+\frac{2\nu}{m-1}\right)\varepsilon = 2^*-1-\varepsilon. \tag{4.1}$$

If $y(t)$ solves problem (3.3), then

$$H'(t) = -\frac{(1+2\nu/(m-1))\varepsilon}{p+1} t^{-k} y^{p+1} \tag{4.2}$$

and $y'(t) = O(t^{1-k})$ as $t \to \infty$ (see Remark 3.1). So

$$\lim_{t\to\infty} H(t) = 0.$$

Hence, since $H(T) = Ty'^2(T)$, integrating (4.2) over (T, ∞), we deduce that

$$Ty'^2(T) = \frac{(1+2\nu/(m-1))\varepsilon}{p+1} \int_T^\infty t^{-k} y^{p+1}(t)\,\mathrm{d}t. \tag{4.3}$$

This equation is crucial for us to obtain the desired results.

Lemma 4.1 *Let $T(\gamma)$ be defined as (3.4), then $T(1) > 0$.*

Proof By Lemma 3.2, $y(t,1) \le z(t,1)$ for $t \ge T(1)$. Suppose in contrast that $T(1) = 0$, then

$$y'(0,1) \le z'(0,1) = k_1^{k-1}(\varepsilon).$$

So

$$y(t,1) \le k_1^{k-1}(\varepsilon)t, \quad 0 \le t,$$

which means $H(0) = 0$.

On the other hand, combination of (4.2) and the fact $\lim_{t\to\infty} H(t) = 0$ yields $H(t) > 0$ for $T(1) \le t < \infty$.

Hence, we get a contradiction, and our conclusion follows. □

Remark 4.1 By Lemma 4.1 and the previous analysis, we deduce that for $2 < p+1 < 2^*$ and $0 \le \mu < (\frac{N-2}{2})^2$, the following elliptic problem has no solution in $H^1(\mathbb{R}^N)$.

$$\begin{cases} -\Delta u = \mu \dfrac{u}{|x|^2} + u^p & x \in \mathbb{R}^N, \\ u > 0 & x \in \mathbb{R}^N, \\ u \to 0 & |x| \to \infty. \end{cases}$$

Lemma 4.2 $\lim_{\varepsilon\to 0}\gamma^{1-\kappa(\varepsilon)}y'(T) = k_1$, *where* $\gamma = \gamma(\varepsilon)$.

Proof Integrating Eq. (3.2) over (T, ∞), we derive

$$y'(T) = \int_T^\infty t^{-k}y^p(t)\mathrm{d}t < \int_T^\infty t^{-k}z^p(t)\,\mathrm{d}t. \tag{4.4}$$

Hence, by Lemma 3.5 (i) and Lemma 3.4, as $\varepsilon \to 0$,

$$\gamma^{1-\kappa(\varepsilon)}y'(T) \le k_1(\varepsilon)k_2(\varepsilon)B\left(\left(\frac{T}{T_\varepsilon}\right)^{k-2}, 1-\kappa(\varepsilon), k_2(\varepsilon)\right) \to k_1k_2B(0, 1, k_2).$$

By (3.11) and the fact that $\Gamma(x+1) = x\Gamma(x)$, we deduce

$$k_1k_2B(0, 1, k_2) = k_1k_2/k_2 = k_1.$$

Therefore

$$\limsup_{\varepsilon\to 0}\gamma^{1-\kappa(\varepsilon)}y'(T) \le k_1. \tag{4.5}$$

Next, we shall show that for any $\delta > 0$,

$$\liminf_{\varepsilon\to 0}\gamma^{1-\kappa(\varepsilon)}y'(T) \ge k_1 - \delta, \tag{4.6}$$

which completes the proof of this lemma.

For a given $\alpha > 0$, by (3.2) and Lemma 3.4, we can choose a suitable small $\varepsilon > 0$ such that $\alpha T_\varepsilon > T$. Thus (4.4) can be written as

$$\begin{aligned}\gamma^{1-\kappa(\varepsilon)}y'(T) &= \gamma^{1-\kappa(\varepsilon)}\left(\int_T^{\alpha T_\varepsilon} + \int_{\alpha T_\varepsilon}^\infty\right)t^{-k}y^p(t)\,\mathrm{d}t\\ &= J_1(\varepsilon,\alpha) + J_2(\varepsilon,\alpha).\end{aligned} \tag{4.7}$$

Because $z(t) \le (\gamma/T_\varepsilon)t$ for $t > 0$, using Lemma 3.2, we have

$$\begin{aligned}J_1(\varepsilon,\alpha) &\le \gamma^{1-\kappa(\varepsilon)}\left(\frac{\gamma}{T_\varepsilon}\right)^p\int_T^{\alpha T_\varepsilon} t^{p-k}\,\mathrm{d}t\\ &< \gamma^{1-\kappa(\varepsilon)}\left(\frac{\gamma}{T_\varepsilon}\right)^p\frac{(\alpha T_\varepsilon)^{p-k+1}}{p-k+1}\\ &= \frac{k_1^{k-1}(\varepsilon)}{(k-2)(1-\kappa(\varepsilon))}\alpha^{(k-2)(1-\kappa(\varepsilon))}.\end{aligned} \tag{4.8}$$

On the other hand, by Lemma 3.3 and (i) of Lemma 3.5, for $\varepsilon > 0$ small,

$$\begin{aligned}J_2(\varepsilon,\alpha) &> \gamma^{1-\kappa(\varepsilon)}(1-d_{\alpha,\varepsilon}\varepsilon)^p\int_{\alpha T_\varepsilon}^\infty t^{-k}z^p(t)\,\mathrm{d}t\\ &= (1-d_{\alpha,\varepsilon}\varepsilon)^pk_1(\varepsilon)k_2(\varepsilon)B(\alpha^{k-2}, 1-\kappa(\varepsilon), k_2(\varepsilon)).\end{aligned} \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), we derive

$$\liminf_{\varepsilon\to 0}\gamma^{1-\kappa(\varepsilon)}y'(T) \ge k_1k_2B(\alpha^{k_0-2}, 1, k_2) - L\alpha^{k_0-2},$$

where

$$L = \lim_{\varepsilon \to 0} \frac{k_1^{k-1}(\varepsilon)}{(k-2)(1-\kappa(\varepsilon))}.$$

Hence, given any $\delta > 0$, we can choose $\alpha > 0$ such that (4.6) is satisfied.

This completes the proof. □

By (ii) of Lemma 3.5, similar computation also gives

Lemma 4.3

$$\lim_{\varepsilon \to 0} \gamma(\varepsilon)^{-\frac{(m-1+2\nu)\varepsilon}{(m-1)(k-2)}} \int_T^\infty t^{-k} y^{p+1}(t, \gamma(\varepsilon))\, \mathrm{d}t = k_1 k_2 [\Gamma(k_2)]^2 / \Gamma(2k_2).$$

Now we are ready to analyze the behavior of $\gamma(\varepsilon)$ as $\varepsilon \to 0$.

Theorem 4.1 *Let $y(t)$ be the solution of problem (3.2) and denote*

$$\gamma(\varepsilon) = \lim_{t \to \infty} y(t).$$

Then

$$\lim_{\varepsilon \to 0} \varepsilon \gamma^2(\varepsilon) = \frac{4N\sqrt{\bar{\mu} - \mu}}{(N-2)^2} \frac{k_1}{k_2} \frac{\Gamma(2k_2)}{[\Gamma(k_2)]^2} T,$$

where k_1 and k_2 are defined by (3.12).

Proof Noting (4.3), we have

$$\left(1 + \frac{2\nu}{m-1}\right) \varepsilon \gamma^{2-\kappa(\varepsilon)} = (p+1) T \frac{[\gamma^{1-\kappa(\varepsilon)} y'(T)]^2}{\gamma^{-\kappa(\varepsilon)} \int_T^\infty t^{-k} y^{p+1}(t)\, \mathrm{d}t}. \tag{4.10}$$

Combination of (4.10), Lemma 4.2 and Lemma 4.3 yields

$$\lim_{\varepsilon \to 0} \varepsilon \gamma(\varepsilon)^{2-\kappa(\varepsilon)} = \frac{4N\sqrt{\bar{\mu} - \mu}}{(N-2)^2} \frac{k_1}{k_2} \frac{\Gamma(2k_2)}{[\Gamma(k_2)]^2} T,$$

which also means

$$\gamma(\varepsilon)^{2-\kappa(\varepsilon)} = O(\varepsilon^{-1}).$$

As a consequence $\lim_{\varepsilon \to 0} \gamma(\varepsilon)^{\kappa(\varepsilon)} = 1$. Hence

$$\lim_{\varepsilon \to 0} \varepsilon \gamma^2(\varepsilon) = \frac{4N\sqrt{\bar{\mu} - \mu}}{(N-2)^2} \frac{k_1}{k_2} \frac{\Gamma(2k_2)}{[\Gamma(k_2)]^2} T.$$ □

Proof of Theorem 1.1 If $y(t)$ is the solution of (3.2), then

$$v(x) = (N - 2\nu - 2)^\nu y((m-1)^{m-1} |x|^{1-m})$$

is the solution of problem (2.1) in B_R with $R = (m-1) T^{-\frac{1}{m-1}}$, hence

$$u_\varepsilon(x) = |x|^{-\nu} v(x) = (N - 2\nu - 2)^\nu |x|^{-\nu} y((m-1)^{m-1} |x|^{1-m}).$$

Therefore, Theorem 4.1 yields

$$\lim_{\varepsilon\to 0}\lim_{|x|\to 0}\varepsilon u_\varepsilon^2(x)|x|^{2\nu} = (N-2\nu-2)^{2\nu}\frac{4N\sqrt{\bar\mu-\mu}}{(N-2)^2}\frac{k_1}{k_2}\left(\frac{m-1}{R}\right)^{m-1}\frac{\Gamma(2k_2)}{[\Gamma(k_2)]^2}$$
$$= 4N^{\frac{N-2}{2}}(N-2)^{-\frac{N+2}{2}}(2\sqrt{\bar\mu-\mu})^{N-1}\frac{\Gamma(N)}{[\Gamma(\frac{N}{2})]^2}\frac{1}{R^{2\sqrt{\bar\mu-\mu}}},$$

which is the desired conclusion of Theorem 1.1.

Proof of Theorem 1.2 Firstly, using the same method as that in [3], we can generalize Lemma 4.2 as follows:
Let $M > 0$ and $0 < \sigma < 2$, then

$$\limsup_{\varepsilon\to 0}\left\{|\gamma(\varepsilon)^{1-\kappa(\varepsilon)}y'(t)-k_1| : T < t < M\gamma(\varepsilon)^\sigma\right\} = 0. \tag{4.11}$$

On the other hand, by the concavity of $y(t)$, we deduce

$$y'(t) \le \frac{y(t)-y(T)}{t-T} \le y'(T). \tag{4.12}$$

So, there exists a $\theta \in [T, t]$ such that

$$y(t) = y'(\theta)(t-T), \quad t \ge T. \tag{4.13}$$

Combining (4.11), (4.13) and noting the fact that $\lim_{\varepsilon\to 0}\gamma(\varepsilon)^{\kappa(\varepsilon)} = 1$, we obtain

$$\lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{2}}y(t) = \lim_{\varepsilon\to 0}\varepsilon^{-\frac{1}{2}}\gamma(\varepsilon)^{-1+\kappa(\varepsilon)}\lim_{\varepsilon\to 0}y(t)\gamma(\varepsilon)^{1-\kappa(\varepsilon)}$$
$$= [A(k_1,k_2,T)]^{-\frac{1}{2}}\lim_{\varepsilon\to 0}\gamma(\varepsilon)^{1-\kappa(\varepsilon)}y'(\theta)(t-T)$$
$$= k_1[A(k_1,k_2,T)]^{-\frac{1}{2}}(t-T), \tag{4.14}$$

where

$$A(k_1,k_2,T) = \frac{4N\sqrt{\bar\mu-\mu}}{(N-2)^2}\frac{k_1}{k_2}\frac{\Gamma(2k_2)}{[\Gamma(k_2)]^2}T,$$

and the convergence is uniform on bounded intervals.

For the solution $u_\varepsilon(x)$ of problem (1.1), (4.14) means that as $\varepsilon\to 0$

$$\varepsilon^{-\frac{1}{2}}u_\varepsilon(x) = K(N,R,\mu)\left(\frac{1}{|x|^{\sqrt{\bar\mu}+\sqrt{\bar\mu-\mu}}} - \frac{1}{|x|^{\sqrt{\bar\mu}-\sqrt{\bar\mu-\mu}}R^{2\sqrt{\bar\mu-\mu}}}\right)$$

uniformly on any annuli $\rho \le |x| \le R$, where

$$K(N,R,\mu) = \frac{1}{2}(2\sqrt{\bar\mu-\mu})^{\frac{N-3}{2}}N^{\frac{N-2}{4}}(N-2)^{-\frac{N-6}{4}}R^{\sqrt{\bar\mu-\mu}}\frac{\Gamma\left(\frac{N}{2}\right)}{[\Gamma(N)]^{\frac{1}{2}}}.$$

Hence we obtain the desired result.

Proof of Theorem 1.4 and Theorem 1.5 With the same method as that in Lemma 2.1, we can prove that solution $u_{\alpha,\beta,\varepsilon}(x)$ of problem (1.2) is in $C^2(B_R)\cap C^1(\bar{B}_R)$ and is radially symmetric. So we can transform problem (1.2) to the following ODE by setting $|x| = r$ (for simplicity, we write $u(r) = u_{\alpha,\beta,\varepsilon}(x)$):

$$\begin{cases} u''(r) + \frac{N-\alpha-1}{r}u'(r) + r^{\beta-\alpha}u^{p(\alpha,\beta)-1-\varepsilon} = 0 \\ u > 0 \quad \text{for } 0 < r < R, \\ u(R) = 0. \end{cases} \tag{4.15}$$

Set

$$t = \left(\frac{N+\alpha-2}{r}\right)^{N+\alpha-2}, \quad y(t) = (N+\alpha-2)^{\frac{\beta-\alpha}{p(\alpha,\beta)-1-\varepsilon}}u(r),$$

then, problem (4.15) can be rewritten as

$$\begin{cases} y''(t) + t^{-k}y^{p(\alpha,\beta)-1-\varepsilon}(t) = 0, \quad t \in (T,\infty), \\ y(t) > 0 \quad \text{for} \quad T < t < \infty, \\ y(T) = 0, \end{cases} \tag{4.16}$$

where

$$k = \frac{2N+\alpha+\beta-2}{N+\alpha-2}, \quad 2k-3 = p(\alpha,\beta)-1, \quad T = \left(\frac{N+\alpha-2}{R}\right)^{N+\alpha-2}.$$

Similar to Lemma 3.1, we can prove $y'(t) \to 0$ and there exists a positive $\gamma < +\infty$ such that $y(t) \to \gamma$ as $t \to +\infty$.

Denote

$$\bar{z}(t,\gamma) = \gamma\left(1 + \frac{1}{k-1}\frac{\gamma^{p(\alpha,\beta)-2-\varepsilon}}{t^{k-2}}\right)^{-\frac{1}{k-2}}.$$

We can check that $\bar{z}(t,\gamma)$ satisfies

$$\begin{cases} \bar{z}''(t) + t^{-k}\gamma^{-\varepsilon}\bar{z}^{p(\alpha,\beta)-1} = 0, \\ \lim_{t\to\infty}\bar{z}(t,\gamma) = \gamma. \end{cases}$$

Hence $\bar{z}(t,\gamma)$ plays the same role as $z(t,\gamma)$ in Sects. 3 and 4. Theorems 1.4 and 1.5 can be proved by direct calculation.

At last, we give the following remark

Remark 4.2 The method used to prove Theorem 1.1 and Theorem 1.2 is applicable to analyze the asymptotic behavior of solutions to the following more general problem:

$$\begin{cases} -\Delta u = \mu\frac{u}{|x|^2} + \frac{u^{p(\beta)-1-\varepsilon}}{|x|^\beta} & x \in \Omega, \\ u > 0 & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where

$$0 \le \mu \le \left(\frac{N-2}{2}\right)^2, \quad 0 \le \beta < 2, \quad p(\beta) = \frac{2(N-\beta)}{N-2}, \quad 0 < \varepsilon < p(\beta)-1.$$

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