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Perturbations of foliated bundles and evolutionary equations

Dedicated to Roberto Conti on the occasion of his 80th birthday

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Abstract. In two earlier papers, we presented a perturbation theory for laminated, or foliated, invariant sets $\mathcal{K}^o$ for a given finite-dimensional system of ordinary differential equations, see [20,21]. The main objective in that perturbation theory is to show that: if the given vector field has a suitable exponential trichotomy on $\mathcal{K}^o$, then any perturbed system that is C^1-close to the given vector field near $\mathcal{K}^o$ has an invariant set $\mathcal{K}^n$, where $\mathcal{K}^n$ is homeomorphic to $\mathcal{K}^o$ and where the perturbed vector field has an exponential trichotomy on $\mathcal{K}^n$.

In this paper we present a dual-faceted extension of this perturbation theory to include: (1) a class of infinite-dimensional evolutionary equations that arise in the study of reaction diffusion equations and the Navier–Stokes equations and (2) nonautonomous evolutionary equations in both finite and infinite dimensions. For the nonautonomous problem, we require that the time-dependent terms in the problem lie in a compact, invariant set M. For example, M may be the hull of an almost periodic, or a quasiperiodic, function of time.

Key words. evolutionary equations – Navier–Stokes equations – proper negative continuation – quasiperiodic – skew product dynamics

1. Introduction

> Explain the apparent persistence of chaotic dynamics, as is seen in such real-world phenomena as (autonomous or nonautonomous) Lagrangian turbulence.

This statement describes one of the major challenges facing mathematical scientists today, but it does not raise a new issue. Like a specter, chaotic dynamics has lurked in the wings of the shadows of dynamical systems from the beginning. In the mid 1900s, chaotic behavior began to assume a central role as a principal object of study. Much of the recent work on hyperbolic dynamics can be interpreted as an attempt to find a mathematical explanation of the persistence of chaotic dynamics.

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In the context of dynamical systems, the notion of "persistence" can best be explained in terms of the compact invariant sets. For example, one begins with a compact invariant set $\mathcal{K}$ for a given system. Next one makes a small perturbation in the given system, thereby obtaining new dynamics and a new compact invariant set $\mathcal{K}_n$. A reasonable mathematical explanation of persistence of the dynamics on $\mathcal{K}$ is to show that: (1) $\mathcal{K}_n$ is "close to" $\mathcal{K}$ (perhaps even being homeomorphic to $\mathcal{K}$) and (2) the new dynamics on $\mathcal{K}_n$ is "close to" the given dynamics on $\mathcal{K}$. In this way, the explanation of the persistence of the dynamics follows from a good perturbation theory.

It is in this setting that the theory of foliated invariant sets comes on stage. The concept of a foliated invariant set is a relatively new notion in the study of dynamical systems. The family of all such invariant sets comes to us as the result of a mathematical marriage, viz., the union of the family of (normally hyperbolic) compact invariant manifolds with the family of hyperbolic sets (i.e., Anosov flows). As is well known, there are good perturbation theories for both invariant manifolds and for Anosov flows. (See [28, 9, 14] for normally hyperbolic invariant manifolds and [35, 3, 19, 25] for hyperbolic sets.) Furthermore, these theories have proven to be satisfactory when restricted to those perturbations where the perturbed problem remains within the same family. However, except for some trivial cases, neither perturbation theory could be applied to the other family. Moreover, neither theory can be used to explain the persistence of the dynamics resulting from a coupling between the dynamics of a normally hyperbolic invariant manifold and the dynamics of an Anosov flow.

As is shown here, the perturbation theory of foliated invariant sets fills this gap. In this paper, we present two important extensions of the basic theory of foliated invariant sets, as developed in [20, 21, 23, 24]. First we extend the perturbation theory to include foliated invariant sets in an infinite-dimensional setting, including the Navier–Stokes equations. (This extension unifies the earlier work on compact invariant manifolds, see [24, 5], on the one hand, and the work on hyperbolic sets, see [15], on the other.) Second, we treat both the autonomous and the nonautonomous problems in this new perturbation theory. Previous works have been restricted to the autonomous theory alone.

To put this into context, we begin with a pair of evolutionary equations

$$\begin{aligned} \partial_t u + Au &= F(u), \\ \partial_t y + Ay &= F(y) + G(y, t), \end{aligned} \tag{1.1}$$

where the first equation is the "given" equation, an autonomous equation, and $G = G(y, t)$ is a time-varying perturbation term. We are also interested in the fully nonautonomous problem

$$\begin{aligned} \partial_t u + Au &= F(u, t), \\ \partial_t y + Ay &= F(y, t) + G(y, t), \end{aligned} \tag{1.2}$$

where again $G = G(y, t)$ is the perturbation term. Note that problem (1.1), like the fully autonomous problem

$$\begin{aligned} \partial_t u + Au &= F(u), \\ \partial_t y + Ay &= F(y) + G(y), \end{aligned} \tag{1.3}$$

is a special case of (1.2). Each of the six equations in (1.1)–(1.3) is an evolutionary equation on a Banach space W, where $u,\ y \in W$. Among other things, we assume that the problems are well defined for all $t \geq 0$.

Let us turn next to the initial value problem

$$\partial_t u + Au = F(u), \qquad \text{where } u(0) = u_0 \text{ and } u_0 \in W, \tag{1.4}$$

for the autonomous equation in (1.1). We assume that the problem (1.4) is well posed and we let $S(t)u_0$ denote the maximally defined mild solution of this problem. Recall that a set $\mathcal{K}$ in W is invariant whenever one has $S(t)\mathcal{K} = \mathcal{K}$, for all $t \geq 0$. A heuristic description of a **foliated invariant set** $\mathcal{K}$ is:

- $\mathcal{K}$ is a compact, invariant set for $S(t)$;
- For each $x \in \mathcal{K}$, there is a **leaf** $\mathcal{S}(x)$ with the property that $\mathcal{S}(x)$ is invariant with $x \in \mathcal{S}(x) \subset \mathcal{K}$;
- There is an integer $k \geq 1$ and a $\rho_0 > 0$ such that for each $x \in \mathcal{K}$, the leaf $\mathcal{S}(x)$ contains a k-dimensional Lipschitz continuous disk $\mathcal{D}_{\rho_0}(x)$, of radius ρ_0 with center at x and $x \in \mathcal{D}_{\rho_0}(x) \subset \mathcal{S}(x)$; and
- $\mathcal{S}(x) = \cup_{y \in \mathcal{S}(x)} \mathcal{D}_{\rho_0}(y)$.

We will refer to k as the (local) **dimension of the leaves** of $\mathcal{K}$. The foliated invariant set $\mathcal{K}$ is said to be **smooth** if the disks $\mathcal{D}_{\rho_0}(y)$ are of class $C^{1,1}$. The two prototypical examples of foliated invariant sets are:

- A compact, connected k-dimensional invariant manifold $\mathcal{M}$, where $\mathcal{S}(x) = \mathcal{M}$, for all $x \in \mathcal{M}$, and
- A hyperbolic set, where $k = 1$ and each leaf $\mathcal{S}(x)$ is the full orbit through $x \in \mathcal{K}$. (In this case, it is typically assumed that the set $\mathcal{K}$ contains no stationary solutions, see [3].)

It follows directly that if $\mathcal{K}_i$ is a foliated invariant set for a semiflow $S_i(t)$ on a space W_i, for $i = 1, 2$, then the product $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$ is a foliated invariant set for the product semiflow on the product space $W = W_1 \times W_2$. Furthermore, if $x = (x_1, x_2) \in \mathcal{K}$, then the leaf $\mathcal{S}(x)$ satisfies $\mathcal{S}(x) = \mathcal{S}(x_1) \times \mathcal{S}(x_2)$, where $\mathcal{S}(x_i)$ is the leaf in $\mathcal{K}_i$ containing x_i, for $i = 1, 2$. By using these set-theoretic products, one quickly encounters foliated invariant sets that are not included among the prototypical examples. Other examples appear in [20].

In Section 4, we present a more detailed formulation of the concept of a foliated invariant set, for both the autonomous problem (1.4) and the nonautonomous problem

$$\partial_t u + Au = F(u, t). \tag{1.5}$$

Our interest in foliated invariant sets $\mathcal{K}$ only begins with the geometric description given above. Since our goal is to better understand the dynamical behavior in the vicinity of $\mathcal{K}$ – and especially the behavior as one adds the time-varying term G

to either (1.4) or (1.5) – we require that the given equation have an exponential trichotomy on $\mathcal{K}$. In particular, we want to show that if G is "small", see (5.5), then the perturbed equation has dynamical properties that are close to those seen on $\mathcal{K}$.

Let us turn to the fully autonomous problem (1.3) to illustrate the three theorems we will prove: the robustness property, the shadow property, and the homeomorphism property. We will let $S_1(t)u_0$ denote the mild solutions of the first equation in (1.3), the given equation, and let $S_2(t)y_0$ denote the mild solutions of the second equation, the perturbed equation.

Theorem (A: Robustness property). *Let $\mathcal{K}$ be a smooth foliated invariant set for the given equation in* (1.3) *and assume that linearized dynamics on $\mathcal{K}$ has an exponential trichotomy, where a Lipschitz property is satisfied. If the perturbation term G is "small," then there is a continuous mapping $h : \mathcal{K} \to W$ such that $\mathcal{K}^G = h(\mathcal{K})$ is a foliated invariant set for the perturbed equation and $\|h(u) - u\| \leq \epsilon_0$, for all $u \in \mathcal{K}$, where ϵ_o is "small."*

The smallness of ϵ_0 means that the mapping h is close to the identity mapping; thus $\mathcal{K}$ is close to $\mathcal{K}^G$. If there exists an induced flow $S^G(t)$ on $\mathcal{K}$, where $\mathcal{K}$ is an invariant set, for $S^G(t)$ and

$$(1.6) \qquad S_2(t)h(u_0) = h(S^G(t)u_0), \qquad \text{for all } u_0 \in \mathcal{K},\ t \geq 0,$$

then we refer to $S^G(t)$ as the **shadow flow** on $\mathcal{K}$. We now have the following result:

Theorem (B: Shadow property). *Under the hypotheses of Theorem A, there is a shadow flow $S^G(t)$ on $\mathcal{K}$, and the identity* (1.6) *holds.*

The commutivity relationship (1.6) contains some valuable dynamical information. For example, if $\mathcal{K}$ is a stable periodic orbit, then its image $\mathcal{K}^G$ is also a cycle, and the shadow flow $S^G(t)$ differs from the unperturbed dynamics $S_1(t)$ by a change in the speed, in the period, and/or in the phase along the orbit $\mathcal{K}$. When h is a homeomorphism, then equation (1.6) is a statement of the lower semicontinuity with respect to the perturbation term G. In this connection, we have the following result.

Theorem (C: Homeomorphism property). *Under the hypotheses of Theorem A, the mapping h is a homeomorphism of $\mathcal{K}$ onto $\mathcal{K}^G$.*

In Section 2 we present the function-analytic underpinnings of the theory of evolutionary equations in a Banach space. For the most part, this brief summary is taken from [33]. In Section 3 we present the basic theory of skew product semiflows that is required for the nonautonomous problem. Our main goal here is to formulate the necessary theory for quasiperiodic forcing of the equations. A more detailed concept of a foliated bundle, for both the autonomous and the nonautonomous problems, is developed in Section 4. In the case of quasiperiodic forcing, we will show that the nonautonomous problem (1.2) can be reformulated as a suitable autonomous problem, see equations (5.3)–(5.4). Some preliminary estimates of properties of solutions are presented in Section 5. Section 6 contains the main lemmas and proofs of the robustness property, the shadow property, and the homeomorphism property, for both the autonomous problem and the nonautonomous problem with quasiperiodic forcing. Finally, in Section 7 we present some extensions of our basic theory. These extensions include cases where the time dependence is not quasiperiodic.

2. Evolutionary equations

In this section we review some of the classical issues arising in the study of the dynamics of linear and nonlinear evolutionary equations. In particular, we present various features of the longtime dynamics of the evolutionary equations (1.4) and (1.5) on a Banach space W, with norm $\|w\| = \|w\|_W$. Additional information and details can be found, for example, in [12,18,33]. We begin with the linear problem

$$\partial_t u + Au = 0. \tag{2.1}$$

2.1. Linear theory

Hypothesis A: We assume here that the linear operator A is a positive sectorial operator on W. As a result, A is a closed operator on W, with a domain $\mathcal{D}(A)$ that is dense in W, and $-A$ is the infinitesimal generator of an analytic semigroup e^{-At} on W. Thus for each $u_0 \in W$, the function $u(t) = e^{-At}u_0$ is the (unique) solution of equation (2.1) with $u(0) = u_0$. Since the operator A is positive, there exist constants $a > 0$ and $M_0 \geq 1$ such that

$$\|e^{-At}u_0\| \leq M_0\|u_0\|e^{-at}, \qquad \text{for all } u_0 \in W \text{ and } t \geq 0. \tag{2.2}$$

For each $\alpha \geq 0$, we let A^α denote the fractional power of A, and we set $V^{2\alpha} = \mathcal{D}(A^\alpha)$, the domain of A^α. Each A^α is a closed, densely defined, linear operator on W, and one has the continuous imbedding $\mathcal{D}(A^\alpha) \mapsto \mathcal{D}(A^\beta)$, whenever $\alpha \geq \beta$. This means that there is a constant $c = c_{\alpha,\beta} > 0$ such that $\|u\|_{V^{2\alpha}} \leq c\|u\|_{V^{2\beta}}$, for all $u \in \mathcal{D}(A^\alpha)$, where the norm $\|u\|_{V^{2\alpha}}$ is the graph norm, $\|u\|_{V^{2\alpha}} = \|A^\alpha u\| = \|A^\alpha u\|_W$. Thus the identity mapping $I : V^{2\alpha} \to V^{2\beta}$ is in $\mathcal{L} = \mathcal{L}(V^{2\alpha}, V^{2\beta})$, the space of bounded linear transformations from $V^{2\alpha}$ into $V^{2\beta}$. For $M \in \mathcal{L}$, the operator norm $\|M\|_{\mathcal{L}}$ is defined by

$$\|M\|_{\mathcal{L}} \stackrel{\text{def}}{=} \sup\{\|A^\beta M u\| : \|A^\alpha u\| \leq 1\}.$$

There is no loss in generality in assuming that the sectorial operator A is positive. Indeed, if A is any sectorial operator, there is a real number $a \in \mathbb{R}$ such that the linear operator $B = A + aI$ is a positive, sectorial operator. In this case, the first equation in (1.1) is equivalent to the equation $\partial_t u + Bu = H(u)$, where $H(u) = F(u) + au$. The positivity of the sectorial operator A offers a convenient way of describing the fractional power spaces $V^{2\alpha}$. If A is not positive, then one can achieve the same goals indirectly by using B and its fractional powers, since the semigroups satisfy $e^{-Bt} = e^{-at}e^{-At}$, for $t \geq 0$, see [12].

In some cases, such as the Navier–Stokes equations and many reaction diffusion equations, the linear operator A appearing in (1.1)–(1.5) and (2.1) has additional properties that are useful for the analysis of the solutions of nonlinear equations. For example, W may be a Hilbert space and A is a self-adjoint operator with compact resolvent. In this case A has an orthonormal basis of eigenvectors for W. By using the Bubnov–Galerkin scheme with this basis, one can develop a family of finite-dimensional ordinary differential equations, which in turn can be used to approximate the solutions of the infinite-dimensional problem (1.1). For example, the Leray–Hopf theory of weak solutions of the Navier–Stokes equations is constructed in this way. See [33] for more details and [24] for a related application to the Couette–Taylor flow.

The fundamental theorem on sectorial operators gives some very valuable information about the connection between the analytic semigroup e^{-At} and the fractional powers A^α, for $\alpha \geq 0$. In particular, one has $A^\alpha e^{-At}u = e^{-At}A^\alpha u$, for all $u \in \mathcal{D}(A^\alpha)$ and $t \geq 0$, and e^{-At} is an analytic semigroup on V^α, for each $\alpha \geq 0$. In this setting, for $\beta \geq 0$, there is a constant $M_1 \geq 1$ such that

$$\|A^\beta e^{-At}u_0\| \leq M_1 e^{-at}\|A^\beta u_0\|, \qquad \text{for all } u_0 \in V^{2\beta}. \tag{2.3}$$

In addition, for any $\alpha \geq 0$ and $t > 0$, the semigroup e^{-At} maps W into $\mathcal{D}(A^\alpha)$, and there is a constant $M_\alpha > 0$ such that

$$\|e^{-At}\|_{\mathcal{L}(W,\mathcal{D}(A^\alpha))} = \|A^\alpha e^{-At}\|_{\mathcal{L}(W,W)} \leq M_\alpha t^{-\alpha}e^{-at}, \qquad \text{for all } t > 0, \tag{2.4}$$

where $a > 0$ is given by (2.2). Moreover, for $0 < \alpha \leq 1$, there is a constant $K_\alpha > 0$ such that

$$\|e^{-At}w - w\| \leq K_\alpha t^\alpha \|A^\alpha w\|, \qquad \text{for } t \geq 0 \text{ and } w \in \mathcal{D}(A^\alpha). \tag{2.5}$$

Furthermore, the solutions $e^{-At}w$ are Lipschitz continuous in t, for $t > 0$. More precisely, for every $\beta \geq 0$, there is a constant $C_\beta > 0$ such that

$$\|A^\beta(e^{-A(t+h)} - e^{-At})w\| \leq C_\beta |h| t^{-(1+\beta)}\|w\|, \tag{2.6}$$

for all $t > 0$ and $w \in W$, see [18,33].

The coefficients M_α, K_α, and C_β appearing above depend on properties of the spectrum of the linear operator A. Furthermore, it follows from inequality (2.4) and the definition of the Gamma function Γ that

$$\int_0^t \|A^\beta e^{-A\tau}\|_{\mathcal{L}(W,W)} \leq M_\beta \int_0^t \tau^{-\beta}e^{-a\tau}\,d\tau \leq M_\beta\, a^{\beta-1}\Gamma(1-\beta), \quad \text{for all } t \geq 0.$$

2.2. *Nonlinear theory*

In this section, we define the function spaces C_{Lip} and C^1_F, which contain the nonlinear terms F and G in equations (1.2). We also introduce the related topologies on these spaces, and we outline the basic theory concerning the properties of the solutions of these equations. For the linear problem (2.1) we assume that Hypothesis A holds.

We define the space of functions

$$C_{\mathrm{Lip}} = C_{\mathrm{Lip}}(V^{2\beta} \times \mathbb{R}, W), \qquad \text{where } 0 \le \beta < 1, \tag{2.7}$$

to be the collection of all functions $F : V^{2\beta} \times \mathbb{R} \to W$ such that, for every bounded set U in $V^{2\beta}$, there exist constants $k_0 = k_0(U) = k_0(F, U) \ge 0$ and $k_1 = k_1(U) = k_1(F, U) \ge 0$, such that

$$\|F(u, t)\| \le k_0, \qquad \text{for all } u \in U \text{ and all } t \in \mathbb{R}, \tag{2.8}$$

and

$$\|F(u_1, t) - F(u_2, t)\| \le k_1 \|A^\beta (u_1 - u_2)\|, \tag{2.9}$$

for all $u_1,\ u_2 \in U$ and all $t \in \mathbb{R}$. When $F = F(u)$ is autonomous, we say that F is in C^a_{Lip}, where

$$C^a_{\mathrm{Lip}} \stackrel{\mathrm{def}}{=} C^a_{\mathrm{Lip}}(V^{2\beta}, W) \subset C_{\mathrm{Lip}}(V^{2\beta} \times \mathbb{R}, W) \tag{2.10}$$

and the superscript a in C^a_{Lip} refers to this autonomous feature.

Also, we define

$$C^1_F = C^1_F(V^{2\beta} \times \mathbb{R}, W), \qquad \text{for some } \beta \text{ with } 0 \le \beta < 1, \tag{2.11}$$

to be the collection of all functions $F \in C_{\mathrm{Lip}}$ such that $F = F(u, t)$ is continuously Fréchet differentiable in u on $V^{2\beta}$, where the derivative $DF(u_0, t) = D_u F(u_0, t)$ satisfies $DF(u_0, t) \in \mathcal{L}(V^{2\beta}, W)$, for each $u_0 \in V^{2\beta}$ and $t \in \mathbb{R}$. Thus F satisfies

$$F(u_0 + v, t) = F(u_0, t) + DF(u_0, t)v + E(u_0, v, t), \tag{2.12}$$

for all $u_0,\ v \in V^{2\beta}$ and $t \in \mathbb{R}$, where the error term $E = E(u_0, v, t)$ satisfies (5.7)–(5.11). The continuity of $DF = D_u F$ means that the mapping $(u_0, t) \to DF(u_0, t)$ is a strongly continuous mapping of $V^{2\beta} \times \mathbb{R}$ into $\mathcal{L} = \mathcal{L}(V^{2\beta}, W)$. It follows from (2.9) that, for each bounded set U in $V^{2\beta}$, the Fréchet derivative $DF(u, t)$ satisfies

$$\|DF(u_0, t)\|_{\mathcal{L}} \le k_1, \qquad \text{for all } (u_0, t) \in U \times \mathbb{R}.$$

We are interested in the case where the Fréchet derivative DF is Lipschitz continuous, which leads us to the following definition. We say that $F \in C^{1,1}_F$ if F is in C^1_F and for each bounded set U in $V^{2\beta}$, there is a constant $k_3 = k_3(U) = k_3(F, U) \ge 0$ such that

$$\|DF(u_1, t) - DF(u_2, t)\|_{\mathcal{L}} \le k_3 \|A^\beta (u_1 - u_2)\|,$$

for all $u_1,\ u_2 \in U$ and all $t \in \mathbb{R}$. We say that $F \in C^2_F$ if F is in $C^{1,1}_F$ and F has two continuous Fréchet derivatives with respect to $u \in V^{2\beta}$. For example, the inertial term in the Navier–Stokes equations is in C^2_F, see [33], (61.32) in Section 6.1).

Let Hypothesis A be satisfied and let $F = F(u, t)$ be in C_{Lip}. We say that a function $u = u(t)$ is a **mild solution** of

$$\partial_t u + Au = F(u, t), \qquad u(\tau) \in V^{2\beta},\ \tau \in \mathbb{R} \tag{2.13}$$

on some interval $\tau \le t < T$, where $\tau < T \le \infty$, provided that

$$u(t) = e^{-A(t-\tau)}u(\tau) + \int_\tau^t e^{-A(t-s)} F(u(s), s)\, ds, \qquad \text{for } \tau \le t < T. \tag{2.14}$$

The basic theory on the existence, uniqueness, continuity, and regularity of mild solutions of (2.13) can be found in [33, Sections 4.7–4.8]; also see [13] and [18]. We present here a summary of this theory for the convenience of the reader.

First we note that for every $(u_0, t_0) \in V^{2\beta} \times \mathbb{R}$ and every $F \in C_{\text{Lip}}$, there is a unique mild solution $u = u(t)$ of (2.13) (with $t_0 = \tau$) on some interval $t_0 \le t < T$, where $t_0 < T \le \infty$, and one has $u \in C[t_0, T; V^{2\beta})$. Furthermore, this solution is maximally defined in the sense that either $T = \infty$ or one has $\lim_{t \to T^-} \|A^\beta u(t)\| = \infty$. Since $f(t) \overset{\text{def}}{=} F(u(t), t)$ is in $L^\infty_{\text{loc}}[t_0, T; W)$, it follows that the mild solution $u(t)$ is locally Hölder continuous in time t, that is, $u(t)$ is in $C^{0,\phi_0}_{\text{loc}}(t_0, T; V^{2\beta})$, for some ϕ_0 with $0 < \phi_0 \le 1$, see [33, Lemma 42.7]. If, in addition, the function $f(t)$ itself is locally Hölder continuous, then $u(t)$ is a strong solution of (2.13), and it has additional regularity properties, as noted in Section 2.4. Also see [33, Theorem 42.9].

In order that $f(t)$ be Hölder continuous, it suffices to assume that the nonlinear term $F = F(u, t)$ is in $C_{\text{Lip};\phi}$, for some ϕ with $0 < \phi \le 1$, where

$$C_{\text{Lip};\phi} = C_{\text{Lip};\phi}(V^{2\beta} \times \mathbb{R}, W),$$

and the space $C_{\text{Lip};\phi}$ is the collection of those functions $F = F(u, t)$ in C_{Lip} with the property that for any bounded sets U in $V^{2\beta}$ and J in $\mathbb{R}$, there exists a constant $k_3 = k_3(U, J) = k_3(F, U, J)$ such that

$$\|F(u_1, t_1) - F(u_2, t_2)\| \le k_3 \left(\|A^\beta (u_1 - u_2)\| + |t_1 - t_2|^\phi \right), \tag{2.15}$$

for all $(u_1, t_1),\ (u_2, t_2) \in U \times J$, with $|t_1 - t_2| \le 1$. Thus, if F satisfies (2.15), then the mild solutions $u = u(t)$ are **strong solutions** in $V^{2\beta}$, see [33, Lemma 47.2]. Notice that the space C^a_{Lip} of autonomous nonlinearities satisfies $C^a_{\text{Lip}} \subset C_{\text{Lip};\phi}$, for any ϕ with $0 < \phi \le 1$.

Next, observe that if $F \in C_{\text{Lip}}$, then every translate F_τ is in C_{Lip}, where

$$F_\tau(u, t) = F(u, \tau + t), \qquad \tau,\ t \in \mathbb{R}. \tag{2.16}$$

Furthermore, if F satisfies (2.8) and (2.9), then for each $\tau \in \mathbb{R}$, so does F_τ, with the same bounds k_0 and k_1. On the space C_{Lip}, we will use a Fréchet space topology of uniform convergence on bounded sets $U \times J$ in $V^{2\beta} \times \mathbb{R}$. For each bounded set $U \times J$ and $F \in C_{\text{Lip}}$, we define two pseudonorms:

$$\begin{aligned} \|F\|_{\{U,J\}} &= \sup_{t \in J} \sup_{u \in U} \|F(u, t)\|, \\ \|F\|_{\{A;U,J\}} &= \sup_{t \in J} \int_0^t \sup_{u \in U} \|A^\beta e^{-A(t-s)} F(u, s)\|\, ds. \end{aligned} \tag{2.17}$$

Note that (2.4), (2.8), and the definition of Gamma function imply that

$$\|F\|_{\{A;U,J\}} \le M_\beta\, a^{\beta-1} \Gamma(1-\beta)\, \|F\|_{\{U,J\}}, \qquad \text{where } J = [0, n]. \tag{2.18}$$

It follows that either family of pseudonorms in (2.17) generates a (Fréchet space) metric on C_{Lip}, see [16] or [33].

The topologies $\mathcal{T}^0_{\mathbf{bo}}$ and $\mathcal{T}^0_A$ on $C_{\mathbf{Lip}}$: The topology on C_{Lip} generated by the pseudonorms $\|\cdot\|_{\{A;U,J\}}$ will be denoted by $\mathcal{T}^0_A$. The subscript refers to the role played by the sectorial operator A in the definition of these pseudonorms. A second topology on C_{Lip} is generated by the pseudonorms $\|F\|_{\{U,J\}}$ and is denoted by $\mathcal{T}^0_{\mathrm{bo}}$, i.e., it is the topology of uniform convergence on bounded sets, i.e., the bounded-open (bo)-topology. It follows from inequality (2.18) that the topologies satisfy $\mathcal{T}^0_A \subset \mathcal{T}^0_{\mathrm{bo}}$.

Note that the space of autonomous nonlinearities C^a_{Lip} is a closed subset in both $\mathcal{T}^0_A$ and $\mathcal{T}^0_{\mathrm{bo}}$. In fact, if $F \in C^a_{\mathrm{Lip}}$, the pseudonorm $\|F\|_{\{U,J\}}$ does not depend on the bounded set J in $\mathbb{R}$. As a result, we will write $\|F\|_{\{U\}} = \|F\|_{\{U,J\}}$ in this case. Note also that the restriction of the topology $\mathcal{T}^0_{\mathrm{bo}}$ to C^a_{Lip} is generated by the pseudonorms $\|F\|_{\{U\}}$, where U is a bounded set in $V^{2\beta}$. Furthermore, when $G \in C_{\mathrm{Lip}}$, the mapping

$$t \to G(\cdot, t) : \mathbb{R} \to C^a_{\mathrm{Lip}}$$

is a continuous mapping of $\mathbb{R}$ into C^a_{Lip} and the space $C(\mathbb{R}, C^a_{\mathrm{Lip}})$ agrees with the space $C_{\mathrm{Lip}} = C_{\mathrm{Lip}}(V^{2\beta} \times \mathbb{R}, W)$, when C^a_{Lip} has the $\mathcal{T}^0_{\mathrm{bo}}$ topology, as described above.

We will use the notation $S(F,t)u_0$ to denote the mild solution of (2.13) with $\tau = 0$ and $u(0) = u_0$. By means of a simple change of variables in (2.14), one then obtains $u(\tau + t) = S(F_\tau, t)u(\tau)$, for $t \geq 0$. When $u(\tau) = S(F, \tau)u_0$, this leads to the **cocycle identity**

$$S(F, \tau + t) = S(F_\tau, t)S(F, \tau), \qquad \text{for } \tau,\ t \geq 0. \tag{2.19}$$

We let $T(F, u_0)$ satisfy $0 < T(F, u_0) \leq \infty$, where $0 \leq t < T(F, u_0)$ is the interval of definition for the maximally defined solution $S(F,t)u_0$.

The mapping $S : (F, u_0, t) \to S(F,t)u_0$ is a continuous mapping of the space

$$\Xi = \left\{(F, u_0, t) \in C_{\mathrm{Lip}} \times V^{2\beta} \times [0, \infty) : 0 \leq t < T(F, u_0)\right\} \tag{2.20}$$

into $V^{2\beta}$, and Ξ is an open set in $C_{\mathrm{Lip}} \times V^{2\beta} \times [0, \infty)$, see [33, Theorem 47.5]. For the space $V^{2\beta}$, we use the strong topology. We let $(\Xi, \mathcal{T}^0_A)$ and $(\Xi, \mathcal{T}^0_{\mathrm{bo}})$ denote the space Ξ with the respective topologies on C_{Lip}. We note that the mild solution mapping $S : (\Xi, \mathcal{T}^0_A) \to V^{2\beta}$ is continuous, from which it follows that the mapping $S : (\Xi, \mathcal{T}^0_{\mathrm{bo}}) \to V^{2\beta}$ is continuous as well, see [33, Theorem 47.5]. More precisely, let $F_i \in C_{\mathrm{Lip}}$ and let $u_i = u_i(t)$ be the mild solutions of the problems

$$\partial_t u_i + Au_i = F_i(u_i, t), \qquad u_i(0) = u_{i0} \in V^{2\beta}, \quad \text{for } i = 1, 2.$$

Set $w = u_1 - u_2$ and $w_0 = w(0) = u_{10} - u_{20}$. Let $F_i(u_i) = F_i(u_i(t), t)$, for $i = 1, 2$. Then $w = w(t)$ is a solution of

$$w(t) = e^{-At}w_0 + \int_0^t e^{-A(t-s)}[F_1(u_1) - F_2(u_2) \pm F_1(u_2)]\, ds.$$

Assume that $\|A^\beta u_i(t)\| \le a$, for $0 \le t \le \tau$. Then there are positive constants $C_{a,\tau}$ and $D_{a,\tau}$ such that

$$\|A^\beta w(t)\| \le C_{a,\tau}\|A^\beta w_0\| + D_{a,\tau}\|F_1 - F_2\|_{\{A;U_{2a},J_\tau\}}, \tag{2.21}$$

where $U_a = \{u \in V^{2\beta} : \|A^\beta u\| \le a\}$ and $J_\tau = [0,\tau]$. Thus one finds that $S(F,t)u_0$ is locally Lipschitz continuous in F and u_0, uniformly for t in compact sets.

The topologies $\mathcal{T}^1_{\mathrm{bo}}$ and $\mathcal{T}^1_A$ on C^1_F: Two C^1-pseudonorms $\|F\|_{\{C^1;A;U,J\}}$ and $\|F\|_{\{C^1;U,J\}}$ are defined for F in C^1_F as follows:

$$\|F\|_{\{C^1;A;U,J\}} \stackrel{\text{def}}{=} \|F\|_{\{A;U,J\}} + \sup_{t\ge 0}\int_0^t \sup_{u\in U}\|A^\beta e^{-A(t-s)}DF(u,s)\|_{\mathcal{L}}\,ds,$$

$$\|F\|_{\{C^1;U,J\}} \stackrel{\text{def}}{=} \|F\|_{\{U,J\}} + \sup_{t\in J}\sup_{u\in U}\|DF(u,t)\|_{\mathcal{L}},$$

where $\mathcal{L} = \mathcal{L}(V^{2\beta}, W)$. As argued in (2.18), one has

$$\|F\|_{\{C^1;A;U,J\}} \le M_\beta\, a^{\beta-1}\Gamma(1-\beta)\|F\|_{\{C^1;U,J\}}, \qquad \text{where } J = [0,n].$$

Let $\mathcal{T}^1_A$ and $\mathcal{T}^1_{\mathrm{bo}}$ denote the topologies generated on C^1_F by the two pseudonorms $\|\cdot\|_{\{C^1;A;U,J\}}$ and $\|\cdot\|_{\{C^1;U,J\}}$, respectively. The last inequality then shows that one has $\mathcal{T}^1_A \subset \mathcal{T}^1_{\mathrm{bo}}$.

When $F \in C^a_{\mathrm{Lip}} \cap C^1_F$, then the C^1-pseudonorm $\|F\|_{\{C^1;U,J\}}$ does not depend on the bounded set J in $\mathbb{R}$. In this case we will express this pseudonorm as $\|F\|_{\{C^1;U\}} = \|F\|_{\{C^1;U,J\}}$.

2.3. The linearized equation

In this section we consider various time-varying linear operators $B(t)$, where $B(t) \in \mathcal{L} = \mathcal{L}(V^{2\beta}, W)$, for t in some interval in $\mathbb{R}$. In particular, we consider the spaces

$$\mathcal{M}^\infty(0,\infty;\mathcal{L}) \stackrel{\text{def}}{=} L^\infty(0,\infty;\mathcal{L}) \cap C[0,\infty;\mathcal{L}) \text{ and}$$

$$\mathcal{M}^\infty(\mathbb{R};\mathcal{L}) \stackrel{\text{def}}{=} L^\infty(\mathbb{R};\mathcal{L}) \cap C(\mathbb{R};\mathcal{L}).$$

More explicitly, we now assume that $F \in C^1_F$, see (2.11). Let $u = u(t)$ be a solution of the nonlinear evolutionary equation (2.13). Since $F = F(u,t)$ has a continuous Fréchet derivative in u, $DF(u,t) = D_u F(u,t)$, we linearize (2.13) along the solution $u(t)$ and thereby obtain the nonautonomous linear problem

$$\partial_t v + Av = DF(u(t),t)v.$$

Assume now that $u = u(t) = S(F,t)u_0$ is a mild solution of (2.13) that satisfies $S(F,t)u_0 \in U$, for $t \ge 0$, where U is some bounded set in $V^{2\beta}$. Then

$$B(t) = B(F,u_0;t) \stackrel{\text{def}}{=} DF(S(F,t)u_0, t) \tag{2.22}$$

is well defined, for all $(F, u_0, t) \in \Xi$, and the mapping $(F, u_0, t) \to B(F, u_0, t)$ is a mapping of Ξ into $\mathcal{L}$. Since U is a bounded set, one has $B(\cdot) \in \mathcal{M}^\infty(0, \infty; \mathcal{L})$. If, instead, the solution $u(t) = S(F, t)u_0$ is a global solution, with $u(t) \in U$, for all $t \in \mathbb{R}$, then $B(\cdot) \in \mathcal{M}^\infty(\mathbb{R}, \mathcal{L})$. In the sequel we let $\mathcal{M}^\infty = \mathcal{M}^\infty(J; \mathcal{L})$, where $J = [0, \infty)$, or $J = \mathbb{R}$.

Next we consider two topologies for the space $\mathcal{M}^\infty = \mathcal{M}^\infty(0, \infty; \mathcal{L})$. These are $\mathcal{T}^0_{\mathrm{bo}}$ and $\mathcal{T}^0_{\mathrm{A}}$. The metric topology on $(\mathcal{M}^\infty, \mathcal{T}^0_{\mathrm{bo}})$ is generated by the family of pseudonorms

$$\|B\|_{\{J_\tau\}} = \sup_{t \in J_\tau} \|B(t)\|_{\mathcal{L}}, \qquad \text{where } B \in \mathcal{M}^\infty \text{ and } J_\tau = [0, \tau].$$

Likewise, the metric topology on $(\mathcal{M}^\infty, \mathcal{T}^0_{\mathrm{A}})$ is generated by the family of pseudonorms

$$\|B\|_{\{A; J_\tau\}} = \sup_{t \in J_\tau} \int_0^t \|A^\beta e^{-A(t-s)} B(s)\|_{\mathcal{L}}\, ds, \quad \text{where } B \in \mathcal{M}^\infty \text{ and } J_\tau = [0, \tau].$$

Let $B \in \mathcal{M}^\infty$ and consider the linear problem

$$\partial_t v + Av = B(t)v, \qquad v(0) = v_0 \in V^{2\beta} \text{ and } t \in J, \tag{2.23}$$

where A satisfies Hypothesis A. As noted in [33, Section 4.4], the solution operator $\Phi(B, t)$ for (2.23) satisfies the equation

$$\Phi(B, t)v = e^{-At}v + \int_0^t e^{-A(t-s)} B(s)\, \Phi(B, s)v\, ds, \tag{2.24}$$

for $v \in V^{2\beta}$ and $t \geq 0$. The basic theory on the properties of the solution operator $\Phi(B, t)$ is contained in [33, Theorem 44.1]. Among many such properties, one has $\Phi(B, 0) = I$, the identity; $\Phi(B, t)$ is the Fréchet derivative of the solution $S(F, t)u$, with respect to u at time t, when B satisfies (2.22); and

$$\Phi(B, t)v \in C[0, \infty; V^{2\beta}) \cap C^{0,\phi}_{\mathrm{loc}}(0, \infty; V^{2r}), \qquad \text{for any } B \in \mathcal{M}^\infty,$$

for all r with $0 \leq r < 1$, where the Hölder exponent ϕ depends on r. If $B \in \mathcal{M}^\infty$ satisfies the Hölder condition

$$B \in C^{0,\phi_1}_{\mathrm{loc}}(J; \mathcal{L}), \tag{2.25}$$

for some ϕ_1 with $0 < \phi_1 \leq 1$, then the solutions $\Phi(B, t)v$ are strong solutions of (2.23), see [33, Theorem 44.4]. For example, if B is given by (2.22), where $F \in C^{1,1}_F \cap C_{\mathrm{Lip};\phi}$, then B satisfies (2.25).

Furthermore, the mapping

$$(B, w_0, t) \to \Phi(B, t)w_0$$

is a continuous mapping of $\mathcal{M}^\infty \times V^{2\beta} \times [0, \infty)$ into $V^{2\beta}$ with respect to either the $\mathcal{T}^0_{\mathrm{bo}}$ or the $\mathcal{T}^0_A$ topology on $\mathcal{M}^\infty$. For $B \in \mathcal{M}^\infty$, we define the translate B_τ by

$$B_\tau(t) = B(\tau + t), \qquad \text{for } \tau, \in J \text{ and } t \geq 0.$$

By means of a simple change of variables, one readily finds that $\Phi(B, t)$ satisfies the **cocycle identity**:

$$\Phi(B, \tau + t) = \Phi(B_\tau, t)\, \Phi(B, \tau), \qquad \text{for } \tau,\ t \geq 0. \tag{2.26}$$

Compare with (2.19). If B satisfies (2.22), for $0 \leq t < \infty$, then B_τ satisfies

$$B_\tau(t) = B(F, u_0; \tau + t) = B(F_\tau, S(F, \tau)u_0; t), \qquad \text{for } \tau,\ t \geq 0. \tag{2.27}$$

Thus one has

$$B_\tau(t) = DF\,(S(F, \tau + t)u_0, \tau + t) = DF_\tau\,(S(F_\tau, t)S(F, \tau)u_0, t)\,.$$

We define

$$\Xi^1 \overset{\text{def}}{=} \{(F, u_0, t) \in \Xi : F \in C^1_F\}.$$

For $(F, u_0, t) \in \Xi^1$, we set

$$Z(F, u_0) = B, \quad \text{where} \quad B(t) = DF(S(F, t)u_0, t), \tag{2.28}$$

for $0 \leq t < T(F, u_0)$. Thus $B = B(t)$ is the time-varying linear operator that satisfies $B(t) \in \mathcal{L}$. One can readily verify that Ξ^1 is an open set in $C^1_F \times V^{2\beta} \times [0, \infty)$ and that $Z(F, u_0, t)$ is a continuous mapping of Ξ^1 into $\mathcal{L}$, when C^1_F has either topology $\mathcal{T}^1_{\text{bo}}$ or $\mathcal{T}^1_A$. See [33, Section 4.7].

2.4. *Nonlinear theory revisited*

Next we examine the linear inhomogeneous equation

$$\partial_t v + Av = B(t)v + g(t), \tag{2.29}$$

where Hypothesis A is satisfied, $B = B(t) \in \mathcal{M}^\infty$, $g \in L^\infty_{\text{loc}}[0, T; W)$, and $0 < T \leq \infty$. We will say that $v = v(t)$ is a **mild solution** of equation (2.29) in $V^{2\beta}$ on the interval $[0, T)$, if $v(0) = v_0 \in V^{2\beta}$, $v(\cdot) \in C[0, T; V^{2\beta})$, and the variation of constants formula

$$v(t) = e^{-At}v_0 + \int_0^t e^{-A(t-s)}[B(s)v(s) + g(s)]\, ds, \qquad \text{for } 0 \leq t < T, \tag{2.30}$$

holds. We note that, for every $v_0 \in V^{2\beta}$ and every $g \in L^\infty_{\text{loc}}[0, T; W)$, there is a unique mild solution $v = v(t)$ of equation (2.29) in $V^{2\beta}$ on the interval $[0, T)$.

If $g \equiv 0$, then the mild solution v is given by $v(t) = \Phi(B, t)v_0$, for all $t \geq 0$. For the general case, the mild solution v of equation (2.29) satisfies a second variation of constants formula:

$$v(t) = \Phi(B, t)v_0 + \int_0^t \Phi(B_s, t - s)\, g(s)\, ds, \qquad \text{for } t \geq 0, \tag{2.31}$$

where $v_0 \in V^{2\beta}$ and B_s is the translate $B_s(t) = B(s + t)$, for $s,\ t \geq 0$. The proof of (2.31) can be found in [33, Theorem 45.6].

Finally, we note that, if $v_0 \in V^{2\beta}$, then the mild solution v of equation (2.29) satisfies

$$v(\cdot) \in C[0, T; V^{2\beta}) \cap C^{0,\phi}_{\text{loc}}(0, T; V^{2r}),$$

for every r with $0 \le r < 1$, where $0 < \phi < 1 - r$. If, in addition, B and g satisfy

$$B \in C^{0,\phi_1}_{\text{loc}}(0, \infty; \mathcal{L}(V^{2\beta}, W)) \quad \text{and} \quad g \in L^1_{\text{loc}}[0, T; W) \cap C^{0,\phi_2}_{\text{loc}}(0, T; W), \tag{2.32}$$

for some ϕ_i with $0 < \phi_i \le 1$, for $i = 1, 2$, then v is a strong solution of equation (2.29) in W on $0 \le t < T$, see [33, Theorem 44.6].

Let us now turn to the perturbed equation

$$\partial_t u + Au = F(u, t) + G(u, t), \tag{2.33}$$

where Hypothesis A is satisfied and F and G are in C^1_F. Let $(F, u_0, t) \in \Xi^1$ and let $B = B(t)$ satisfy (2.28). Assume further that $B \in \mathcal{M}^\infty(0, \infty; \mathcal{L})$ and let $\Phi(B, t)$ be given by (2.24). Let $u_1 = u_1(t)$ be a mild solution of (2.33) in $V^{2\beta}$ on the interval $0 \le t < T$, and define $w = u_1 - u_2$, where $u_2 = u_2(t) = S(F, t)u_0$. It then follows that w is a mild solution of the equation

$$\partial_t w + Aw = F(u_2 + w, t) - F(u_2, t) + G(u_2 + w, t).$$

Also, (2.12) implies that the last equation can be written in the form

$$\partial_t w + Aw = B(t)w + H(u_2, w, t), \tag{2.34}$$

where

$$H(u_2, w, t) = E(u_2, w, t) + G(u_2 + w, t)$$

and $E = E(u_2, w, t)$ satisfies (5.7)–(5.11), with $u_0 = u_2(t)$ and $t \ge 0$. By setting $g(t) = H(u_2(t), w(t), t)$, for $0 \le t < T$, it follows from (2.30) and (2.31) that w satisfies two variation of constants formulae:

$$w(t) = e^{-At} w_0 + \int_0^t e^{-A(t-s)} [B(s)v(s) + H(u_2(s), w(s), s)]\, ds \tag{2.35}$$

and

$$w(t) = \Phi(B, t) w_0 + \int_0^t \Phi(B_s, t - s)\, H(u_2(s), w(s), s)\, ds, \tag{2.36}$$

for $0 \le t < T$, where $w_0 = w(0) = u_1(0) - u_2(0) = u_1(0) - u_0$.

3. Skew product semiflows

We now turn to the dynamical issues underlying the perturbation theory arising in problems (1.1) and (1.2). Let us begin with the space of nonlinearities C_{Lip}, see (2.7). As noted above, we will use the Fréchet space topologies $\mathcal{T}^0_{\text{bo}}$ and $\mathcal{T}^0_A$. When

$F \in C^1_F$, we also use the C^1-topologies $\mathcal{T}^1_{\text{bo}}$ and $\mathcal{T}^1_A$. Recall that if $F \in C_{\text{Lip}}$, then the translate F_τ is in C_{Lip}, for any $\tau \in \mathbb{R}$, and the cocycle identity (2.19) is valid. Since the mapping

$$(3.1) \qquad (F, \tau) \to F_\tau$$

is a continuous mapping of $C_{\text{Lip}} \times \mathbb{R}$ into C_{Lip}, this mapping is a flow on C_{Lip} in either topology $\mathcal{T}^0_{\text{bo}}$ or $\mathcal{T}^0_A$. Furthermore, the collection of autonomous nonlinearities C^a_{Lip} is precisely the set of stationary motions for this flow, that is to say, $F_\tau = F$, for all $\tau \in \mathbb{R}$, if and only if $F \in C^a_{\text{Lip}}$. A set $\mathcal{H}$ in C_{Lip} is **invariant** if $F_\tau \in \mathcal{H}$, for all $\tau \in \mathbb{R}$, whenever $F \in \mathcal{H}$. Recall that if $\mathcal{H}$ is an invariant set in a flow, then its closure is also invariant. Because of this, we will focus on closed invariant sets $\mathcal{H}$ below. For example, if S_0 is a nonempty set in C_{Lip}, then $H(S_0)$, the **hull** of S_0, is a closed, invariant subset of C_{Lip}, where

$$(3.2) \qquad H(S_0) \stackrel{\text{def}}{=} \text{Cl}\{F_\tau : \tau \in \mathbb{R} \text{ and } F \in S_0\},$$

and the closure Cl is taken in the respective topology $\mathcal{T}^0_{\text{bo}}$ or $\mathcal{T}^0_A$.

The class of all compact, invariant subsets of C_{Lip} is of special interest in our applications. Recall that a subset $\mathcal{H}$ in C_{Lip} is compact if for every sequence $\{F^{(k)}\}$ in $\mathcal{H}$, there is a subsequence – which we relabel as $\{F^{(k)}\}$ – such that $\{F^{(k)}\}$ is convergent. The convergence is formulated in terms of the respective topology on C_{Lip}, be it $\mathcal{T}^0_{\text{bo}}$ or $\mathcal{T}^0_A$. For example, $\{F^{(k)}\}$ converges to F in $(C_{\text{Lip}}, \mathcal{T}^0_A)$ provided that for any bounded set $U \times J$ in $V^{2\beta} \times \mathbb{R}$, one has

$$\|F^{(k)} - F\|_{\{A;U,J\}} \to 0, \qquad \text{as } k \to \infty.$$

In the case of the topology $\mathcal{T}^0_{\text{bo}}$, the Ascoli–Arzelá Theorem gives a characterization of compactness, see, e.g., [32].

For our applications, the typical compact, invariant set $\mathcal{H}$ in C_{Lip} arises as the continuous image of a known flow on a compact space. More precisely, let M be a (nonempty) compact space and let $\sigma(\theta, t) = \theta \cdot t$, for $\theta \in M$ and $t \in \mathbb{R}$, denote a flow on M. Next let $h : M \to C^a_{\text{Lip}}$ be a continuous mapping, that is, for each $\theta \in M$, the image $h(\theta)$ is a point in C^a_{Lip}, which we write as

$$(3.3) \qquad h(\theta) = \widehat{F}(u, \theta), \qquad \text{for } u \in V^{2\beta} \text{ and } \theta \in M.$$

For each $\theta \in M$, we define $\hat{h}(\theta) \in C_{\text{Lip}}$ by $\hat{h}(\theta) = F$, where

$$(3.4) \qquad F(u, t) = \widehat{F}(u, \theta \cdot t) = \widehat{F}(u, \sigma(\theta, t)), \qquad \text{for } u \in V^{2\beta} \text{ and } t \in \mathbb{R}.$$

The collection $\mathcal{H} = \hat{h}(M)$, or

$$(3.5) \qquad \mathcal{H} = \{F \in C_{\text{Lip}} : F(u, t) = \widehat{F}(u, \theta \cdot t), \text{ for some } \theta \in M\},$$

is the continuous image of the compact set M, and it is a compact, invariant set in C_{Lip}. Moreover, when $F = \hat{h}(\theta)$, the translate F_τ satisfies $F_\tau = \hat{h}(\theta \cdot \tau)$, that is to say,

$$(3.6) \qquad F_\tau(u, t) = F(u, \tau + t) = \widehat{F}(u, \sigma(\theta \cdot \tau, t)), \qquad \text{for } \tau,\ t \in \mathbb{R}.$$

Thus the flow $(F, \tau) \to F_\tau$ on $\mathcal{K}$ is a **lifting** of the flow $(\theta, \tau) \to \theta \cdot \tau$ on M. We note that any lifting $\hat{h}$ maps full orbits in M onto full orbits in $\mathcal{K} = \hat{h}(M)$, see (3.5). This happens even when $\hat{h}$ is not a homeomorphism.

3.1. Nonlinear dynamics

Next we introduce a single setting that is rich enough to include all the dynamical features of the six equations appearing in (1.1)–(1.3) in a single framework. In doing this, we build upon the theory of nonautonomous dynamics used in [26] for the Navier–Stokes equations. Let $\mathcal{H}$ be any (nonempty) invariant set in C_{Lip}. Now consider the family of evolutionary equations

$$\partial_t u + Au = F(u, t), \qquad \text{where } F \in \mathcal{H}. \tag{3.7}$$

For example, $\mathcal{H}$ may be the hull $H(S_0)$, where $S_0 = \{F, G\}$ and F and G are given in (1.1)–(1.3).

Let Ξ and $T(F, u_0)$ be given by (2.19)–(2.20), and let $S(F, t)u_0$ be the mild solution of (3.7) with $S(F, 0)u_0 = u_0$. For $(F, u_0, t) \in \Xi$, we define $\pi(F, u_0; t)$ by

$$\pi(F, u_0; t) = (F_t, S(F, t)u_0). \tag{3.8}$$

Since $S(F, t)$ satisfies the cocycle identity (2.19), it follows that if

$$\tau \in [0, T(F, u_0)) \text{ and } t \in [0, T(F_\tau, S(T, \tau)u_0)),$$

then $\tau + t \in [0, T(F, u_0))$, and π satisfies

$$\pi(F, u_0; \tau + t) = \pi(F_\tau, S(F, \tau)u_0; t) = \pi(\pi(F, u_0; \tau); t).$$

When $T(F, u_0) = \infty$, for all $F \in \mathcal{H}$ and all $u_0 \in V^{2\beta}$, this implies that π is a semiflow on the product space $\mathcal{H} \times V^{2\beta}$. Since the flow $(F, \tau) \to F_\tau$ does not depend on u_0, this is a **skew product semiflow**.

The requirement that $T(F, u_0) = \infty$ for all $u_0 \in V^{2\beta}$ is too strong for some of our applications since we are interested only in the dynamical behavior of (3.7) in the vicinity of a specific compact set $\mathcal{K}$ in $\mathcal{H} \times V^{2\beta}$. To allow for this and to be more precise, we make a definition:

Definition. A set $\mathcal{K}$ in $\mathcal{H} \times V^{2\beta}$ is said to be an **invariant set** for π if the following holds:

- one has $T(F, u_0) = \infty$, for all $(F, u_0) \in \mathcal{K}$; and
- one has $\pi(t)\mathcal{K} = \mathcal{K}$, for all $t \geq 0$, where

$$\pi(t)\mathcal{K} = \{\pi(F, u_0; t) : (F, u_0) \in \mathcal{K}\}.$$

Let $\mathcal{K}$ be an invariant set for π in $\mathcal{H} \times V^{2\beta}$. Define $\mathcal{H}_1$, the **shadow** of $\mathcal{K}$, by

$$\mathcal{H}_1 \stackrel{\text{def}}{=} \{F \in C_{\text{Lip}} : \text{ there is a } u_1 \in V^{2\beta} \text{ with } (F, u_1) \in \mathcal{K}\}. \tag{3.9}$$

Thus $\mathcal{H}_1 = \mathbb{P}_1\mathcal{K}$, where $\mathbb{P}_1$ is the projection $\mathbb{P}_1 : \mathcal{H} \times V^{2\beta} \to \mathcal{H}$ given by $\mathbb{P}_1(F, u_0) = F$. Clearly, one has $\mathcal{H}_1 \subset \mathcal{H}$ and $\mathcal{H}_1$ is also an invariant set in C_{Lip}. For each $F \in \mathcal{H}_1$, we define the **fiber** $\mathcal{K}(F)$ by

$$\mathcal{K}(F) \stackrel{\text{def}}{=} \{u_0 \in V^{2\beta} : (F, u_0) \in \mathcal{K}\}. \tag{3.10}$$

As a result of the skew product structure, the invariance condition $\pi(t)\mathcal{K} = \mathcal{K}$ is equivalent to asserting that

$$S(F, \tau)\, \mathcal{K}(F) = \mathcal{K}(F_\tau), \qquad \text{for all } \tau \geq 0. \tag{3.11}$$

Our interest will be in the theory of compact invariant sets in $\mathcal{H} \times V^{2\beta}$. In this connection, we note the following result, which is easily verified. (Item 3 is a standard result, see [33, Lemma 21.2], for example.)

Lemma 3.1. *Let $\mathcal{H}$ be a nonempty invariant set in C_{Lip} and let $\mathcal{K}$ be a nonempty, compact, invariant set for π in $\mathcal{H} \times V^{2\beta}$. Then the following hold:*

1. *The shadow $\mathcal{H}_1$, see* (3.9)*, is a nonempty, compact, invariant set in $\mathcal{H}$.*
2. *Each fiber $\mathcal{K}(F)$, see* (3.10)*, is a nonempty, compact set in $V^{2\beta}$.*
3. *For every $(F, u_0) \in \mathcal{K}$, there is a negative continuation ϕ, that is, ϕ is in $C((-\infty, 0], V^{2\beta})$, and for every $\tau \leq 0$, one has $(F_\tau, \phi(\tau)) \in \mathcal{K}$ with*

$$S(F_\tau, -\tau + t)\phi(\tau) = S(F, t)u_0, \qquad \textit{for all } t \geq 0.$$

The theory we develop here for nonautonomous dynamics in infinite dimensions builds upon the properties of mild solutions, as described above. However, a crucial aspect of this theory is that the linear dynamics – which is outlined in Section 3.2 – must be a "good" approximation to the nonlinear dynamics of the problem. As will be shown, this means that, if $(F, u_0) \in \mathcal{K}$, then $S(F, t)u_0$, the solution of the nonlinear problem, needs to be Fréchet differentiable in u_0, for each $t \in \mathbb{R}$. In the case of the Navier–Stokes equations, for example, this requires that this mild solution in $\mathcal{K}$ be a strong solution of the problem. In this connection, we have the following result, which is essentially proved in [33, Lemma 47.2 and Theorem 47.6].

Lemma 3.2. *Let $\mathcal{H}$ be a nonempty, invariant set in C_{Lip} and assume that for each $F \in \mathcal{H}$, there is a $\phi \in (0, 1]$ such that $F \in C_{\mathrm{Lip};\phi}$, see* (2.15)*. Let $\mathcal{K}$ be a nonempty, compact, invariant set for π in $\mathcal{H} \times V^{2\beta}$. Then for each $(F, u_0) \in \mathcal{K}$, the globally defined mild solution $u(t) = S(F, t)u_0$ is a strong solution and a classical solution of* (3.7) *in V^{2r}, for all $t \in \mathbb{R}$, and u satisfies*

$$u(\cdot) \in C^{0,1-r}_{loc}(\mathbb{R}; V^{2r}) \cap C(\mathbb{R}; \mathcal{D}(A)), \tag{3.12}$$

for each r with $0 \leq r < 1$.

3.2. Smooth nonlinearities

The considerations described above are applicable when the nonlinear terms F have additional smoothness and lie in the space C^1_F. For example, the mapping (3.1), where $(F, \tau) \to F_\tau$ is continuous and defines a flow on C^1_F, with respect to either topology, $\mathcal{T}^1_{\mathrm{bo}}$ or $\mathcal{T}^1_A$. Also, if the set S_0 is in C^1_F, then the hull $H(S_0)$, see (3.2), is a closed invariant set in C^1_F, when the closure is taken in either topology $\mathcal{T}^1_{\mathrm{bo}}$ or $\mathcal{T}^1_A$.

When $\mathcal{H}$ is a (nonempty) invariant set in C^1_F, then the system of equations (3.7) generates a skew product flow π, see (3.8), on $\mathcal{H} \times V^{2\beta}$ in either topology $\mathcal{T}^1_{\mathrm{bo}}$ or $\mathcal{T}^1_A$. Furthermore, Lemmas 3.1 and 3.2 remain valid when $\mathcal{H}$ is a (nonempty) invariant set in C^1_F, where C^1_F has either the $\mathcal{T}^1_{\mathrm{bo}}$ or the $\mathcal{T}^1_A$ topology.

The new feature that now arises is the associate linearized equation (2.23). In order to see the role to be played by the linearized equation, let us assume that $\mathcal{H}$ is a nonempty, invariant set in C^1_F, and let $\mathcal{K}$ be a nonempty, compact, invariant set for π in $\mathcal{H} \times V^{2\beta}$. As noted above, Lemma 3.1 holds. Now let (F, u_0) be an element in $\mathcal{K}$ and let $Z(F, u_0) = B$ satisfy (2.28). Let $\Phi(B, t)$ satisfy (2.24). We now define a mapping Π by

$$\begin{aligned} &\Pi : \mathcal{K} \times V^{2\beta} \times [0, \infty) \to \mathcal{K} \times V^{2\beta} \\ &\Pi(F, u_0, w_0; t) = (F_t, S(F, t)u_0, \Phi(B, t)w_0). \end{aligned} \tag{3.13}$$

As a result of the two cocycle properties (2.19) and (2.26), the mapping Π is a semiflow on $\mathcal{E} = \mathcal{K} \times V^{2\beta}$. This mapping Π is a (level 2) skew product semiflow since F_t does not depend on (u_0, w_0) and $(F_t, S(F, t)u_0)$ does not depend on w_0. Since $\Phi(B, t)w_0$ is linear in w_0, Π is referred to as a **linear skew product semiflow**.

Remark. The considerations in the last paragraph remain valid under far greater generality than presented here. The set $\mathcal{K}$ need not be compact. For example, it may be any bounded, positively invariant set in $\mathcal{H} \times V^{2\beta}$. In fact, it need not even be bounded, but then the linear term B need not be in $L^\infty(0, \infty; \mathcal{L})$. The point is that Π is still a linear skew product semiflow, with related topologies on the spaces. We omit these details.

3.3. Quasiperiodicity

The principal application of the theory of evolutionary equations (1.1)–(1.2) to be made in this study is to those nonautonomous equations that are quasiperiodic in time. There are many equivalent formulations of quasiperiodicity in the literature, see, e.g., [6,10,34]. We opt for the formulation that fits best in the framework described above. For this purpose, we let $M = T^k$ denote the k-dimensional torus, i.e., T^k is the k-fold product $T^k = S^1 \times \cdots \times S^1$ of the circle S^1. We will use $\theta = (\theta_1, \cdots, \theta_k)$ to represent a typical point in T^k, where $\theta_j \in S^1$, for $1 \le j \le k$, and each coordinate is periodic with period 2π. The flow on T^k is the **twist flow**

$$(\theta, t) \to \theta \cdot t \overset{\text{def}}{=} \theta + \omega t \mod (2\pi), \qquad \text{for } t \in \mathbb{R}, \text{ and } \omega \in \mathbb{R}^k, \tag{3.14}$$

where toroidal arithmetic is used here. We require that ω satisfy the condition

$$\omega \cdot n = 0 \iff n = 0, \qquad \text{where } n \in \mathbb{Z}^k, \tag{3.15}$$

$\omega \cdot n = \omega_1 n_1 + \cdots + \omega_k n_k$, and the terms $n_1, \cdots, n_k$ are integers. The collection of all real numbers

$$\mathcal{M} = \mathcal{M}(\omega) = \{\omega \cdot n : n \in \mathbb{Z}^k\} \tag{3.16}$$

is called the **frequency module** generated by ω. The condition (3.15) implies that the k-frequencies $\{\omega_1, \cdots, \omega_k\}$ form a basis for the module $\mathcal{M}(\omega)$. Thus the **algebraic dimension** of $\mathcal{M}(\omega)$ is k.

Definition: A function $F = F(u, t)$ in C_{Lip} is said to be **quasiperiodic (in time)** if there exist an integer $k \geq 1$, an $\omega \in \mathbb{R}^k$, and a continuous mapping $h : T^k \to C^a_{\text{Lip}}$ such that for each $\theta \in T^k$, the function $h(\theta) = \widehat{F}(\theta, u)$ satisfies

$$F(u, t) = \widehat{F}(\theta_0 + \omega t, u), \qquad \text{for all } t \in \mathbb{R}, \tag{3.17}$$

and some $\theta_0 \in T^k$, see equations (3.3)–(3.4).

Note that if $F_0 = F_0(u, t)$ is quasiperiodic (in time), then $H(F_0)$, the hull of F_0, satisfies $H(F_0) = \hat{h}(T^k)$, see (3.4)–(3.5). Thus the flow $(F, \tau) \to F_\tau$ on $H(F_0)$ is a lifting of the twist flow (3.14) on the torus T^k. It follows from (3.17) that the hull $H(F_0)$ is homeomorphic to a torus T^m and that the **topological dimension** satisfies $\dim H(F_0) = m$, where $m \leq k$. It then follows from the Cartwright Theorem, see [7,8], that the frequency module $\mathcal{M}(H(F_0))$ of $H(F_0)$ has algebraic dimension m, and that $\mathcal{M}(H(F_0)) \subset \mathcal{M}(\omega)$. Consequently, there is a frequency vector $\widehat{\omega} = (\widehat{\omega}_1, \cdots, \widehat{\omega}_m)$ such that:

- $\widehat{\omega}_j \in \mathcal{M}(\omega)$, for $1 \leq j \leq m$;
- One has $\widehat{\omega} \cdot \widehat{n} = 0 \iff \widehat{n} = 0$, where $\widehat{n} \in \mathbb{Z}^m$; and
- $\mathcal{M}(H(F_0)) = \mathcal{M}(\widehat{\omega})$.

Thus the flow on $H(F_0)$ is (essentially) the twist flow on T^m with frequency vector $\widehat{\omega}$.

In this definition, we require that the lifting $h : T^k \to C^a_{\text{Lip}}$, or $\hat{h} : T^k \to C_{\text{Lip}}$, be only continuous, i.e., $h \in C(T^k, C^a_{\text{Lip}})$ or $\hat{h} \in C(T^k, C_{\text{Lip}})$. However, to address some of the technical issues of interest in the study of quasiperiodic dynamics, one sometimes requires that h, or $\hat{h}$, have additional smoothness. Hölder continuous liftings are of special interest. We say that h is a **Hölder continuous lifting** of T^k, i.e., that $h \in C^\alpha(T^k, C^a_{\text{Lip}})$, if there is an α with $0 < \alpha \leq 1$ and there are constants $K_\alpha > 0$ and e_α, with $0 < e_\alpha \leq 1$, such that

$$d(h(\theta), h(\widetilde{\theta})) \leq K_\alpha \, |\theta - \widetilde{\theta}|^\alpha, \qquad \text{whenever } |\theta - \widetilde{\theta}| \leq e_\alpha, \tag{3.18}$$

where d is a fixed metric on C^a_{Lip} generated by any of the pseudonorms described in Sections 2 or 3. (See [16] or [33, Appendix A] for more details.) A similar definition of Hölder continuity is is in play when $h \in C^1_F$, or when $\hat{h}$ is in either $C(T^k, C_{\text{Lip}})$, or $C(T^k, C^1_F)$.

Alternate notation: There is an alternate notation for the skew product semiflows π and Π described in (3.8) and (3.13). What is involved is to rewrite the evolutionary equation (3.7) by using $\widehat{F} = \widehat{F}(\theta, u)$ in place of $F = F(u, t)$, see (3.17). In this notation, one needs to append the differential equation for the twist flow on the torus T^k, which is given by $\partial_t \theta = \omega$, see (3.14). Thus (3.7) then becomes the system

$$\partial_t u + Au = \widehat{F}(\theta, u) \quad \text{and} \quad \partial_t \theta = \omega. \tag{3.19}$$

It is convenient to use the abbreviated notation

$$\theta \cdot t = \theta + \omega t$$

for the solutions of the θ-equation, i.e., for the twist flow, see (3.14). We then let $S(\theta, t)u_0$ denote the mild solutions of the u-equation that satisfy $S(\theta, 0)u_0 = u_0$, for $u_0 \in V^{2\beta}$. It follows from the cocycle identities (2.19) and (3.6) that $S(\theta, t)$ satisfies a related cocycle identity:

$$S(\theta, \tau + t) = S(\theta \cdot \tau, t)S(\theta, \tau), \qquad \text{for } \tau,\ t \geq 0. \tag{3.20}$$

The skew product semiflow π then assumes the form given in (3.21). In this notation, the space Ξ, see (2.20), is replaced by

$$\Xi = \{(\theta, u, t) \in T^k \times V^{2\beta} \times [0, \infty) : 0 \leq t < T(\theta, u)\},$$

where $[0, T(\theta, u))$ is the interval of definition of the maximally defined mild solution $S(\theta, t)u$. A similar change is made when Ξ^1 is used in place of Ξ. In the case of the linear skew product semiflow Π, one obtains the representation

$$\begin{aligned} \pi(\theta, u_0; t) &= (\theta \cdot t, S(\theta, t)u_0) \\ \Pi(\theta, u_0, w_0; t) &= (\theta \cdot t, S(\theta, t)u_0, \Phi(B, t)w_0), \end{aligned} \tag{3.21}$$

where

$$B = Z(\theta, u_0) \quad \text{and} \quad B(t) = D\widehat{F}(\theta \cdot t, S(\theta, t)u_0), \tag{3.22}$$

see (2.28). We now have the following result:

Lemma 3.3. *Let $h : T^k \to C^a_{Lip}$ be a Hölder continuous mapping that satisfies* (3.18). *Let $F = F(u, t)$ and $\widehat{F} = \widehat{F}(\theta, u)$ satisfy* (3.17). *Then F is Hölder continuous in time t, and the conclusions of Lemma* 3.2 *are valid.*

Proof. From (3.17) one obtains

$$F(u, t + h) - F(u, t) = \widehat{F}(\theta_0 + \omega t + \omega h, u) - \widehat{F}(\theta_0 + \omega t, u).$$

Inequality (3.18) then implies that for $(u, t) \in U \times J$, where $U \times J$ is a bounded set in $V^{2\beta} \times \mathbb{R}$, there is a constant $\widehat{e}_\alpha > 0$ such that

$$\|A^\beta(F(u, t + h) - F(u, t))\| \leq \widehat{e}_\alpha |h|^\alpha, \qquad \text{for all } (u, t) \in U \times J,$$

whenever $|h| \leq |\omega|^{-1} e_\alpha$. Hence, F is in the Hölder space $C_{\mathrm{Lip};\alpha}$. □

3.4. Global solutions

The concept of an invariant set is closely connected with the notions of a negative continuation and a global solution. Let $S(t)$ be a semiflow on a space W and let $\mathcal{K}$ be an invariant set in W, that is, one has $S(t)\mathcal{K} = \mathcal{K}$, for all $t \geq 0$. Then for any $u \in \mathcal{K}$, there is a continuous function v in $C(\mathbb{R}, \mathcal{K})$ that satisfies $v(0) = u$ and

$$S(t)v(\tau) = v(\tau + t), \qquad \text{for all } \tau \in \mathbb{R} \text{ and all } t \geq 0. \tag{3.23}$$

It follows from (3.23) that $v(t) = S(t)u$, for all $t \geq 0$. Any continuous function v satisfying (3.23) is called a **global solution**, and the restriction of v to $(-\infty, 0]$ is called a **negative continuation** of $S(\cdot)u$.

When (3.23) holds, it is convenient to define $S(t)u$, for $t \leq 0$, by setting $S(t)u = v(t)$. However, when doing this, keep in mind that, without additional assumptions, the negative continuation need not be uniquely determined. We will be using this notation for the invariant set $\mathcal{K}$, as described in Lemmas 3.1 and 3.2. When $(F, u_0) \in \mathcal{K}$, then $S(F, t)u_0$ is a global solution. Likewise, when $(F, u_0, w_0) \in \mathcal{K} \times V^{2\beta}$ and $B = Z(F, u_0) \in \mathcal{M}^\infty(\mathbb{R}, \mathcal{L})$, then $\Phi(B, t)w_0$ is also a global solution.

4. Foliated invariant sets

In this section, we present a precise statement of the concept of a foliated invariant set. This concept involves two aspects: (1) geometrical properties and (2) dynamical properties. While our goal is the study of the properties of foliated, invariant sets in the theory of nonautonomous dynamics, as in (1.2), it is convenient to begin with a definition of a foliated invariant set for the autonomous problem

$$\partial_t u + Au = F(u), \tag{4.1}$$

where A satisfies Hypothesis A and $F \in C^a_{\mathrm{Lip}} \cap C^1_F$, see (2.10)–(2.11).

4.1. Autonomous theory

Let W be a Banach space. We begin with abbreviated statements of five properties F1–F5 a set $\mathcal{K}$ in W may possess. More detailed statements of these properties are given later in this section. A set $\mathcal{K}$ in W is said to be an **F1–F2 set** if it is nonempty and there is an integer $k \geq 1$ and a $\rho_0 > 0$ such that:

- F1: $\mathcal{K}$ is a compact set; and
- F2: for each $u \in \mathcal{K}$, there is a k-dimensional uniformly, Lipschitz continuous disk $\mathcal{D}_{\rho_0}(u)$, of radius ρ_0 and center at u, with $u \in \mathcal{D}_{\rho_0}(u) \subset \mathcal{K}$.

When $\mathcal{K}$ is an F1–F2 set in W, we define $\mathcal{S}_1(u) = \mathcal{D}_{\rho_0}(u)$,

$$\mathcal{S}_{i+1}(u) = \bigcup_{v \in \mathcal{S}_i(u)} \mathcal{D}_{\rho_0}(v), \quad \text{for } u \in \mathcal{K} \text{ and } i \geq 1, \quad \text{and} \quad \mathcal{S}(u) = \cup_{i=1}^\infty \mathcal{S}_i(u).$$

The set $\mathcal{S}(u)$ is called the **leaf** through u, and one has $u \in \mathcal{D}_{\rho_0}(u) \subset \mathcal{S}(u) \subset \mathcal{K}$, for all $u \in \mathcal{K}$. A more detailed statement of property F2 is that, for any $u \in \mathcal{K}$, there exist a disk $D(u, \rho_0)$ in a k-dimensional normed linear space $X = X(u)$, with center at the origin and with radius $\rho_0 > 0$, and a Lipschitz continuous function $\nu : D(u, \rho_0) \to W$ such that the image

$$\mathcal{D}_{\rho_0}(u) \stackrel{\text{def}}{=} \{v = u + p + \nu(p) : p \in D(u, \rho_0)\} = \text{Graph } \nu \tag{4.2}$$

is an open neighborhood of u in the leaf $\mathcal{S}(u)$. The uniformity condition in property F2 refers to the feature that there is an $L_0 > 0$ such that $\nu : D(u, \rho_0) \to W$ is Lipschitz continuous with

$$\|\nu(p_1) - \nu(p_2)\|_W \leq L_0 \|p_1 - p_2\|_W, \qquad \text{for } p_1,\ p_2 \in D(u, \rho_0) \tag{4.3}$$

and all $u \in \mathcal{K}$. Since $\nu(0) = 0$, one finds that

$$\|\nu(p)\|_W \leq L_0 \|p\|_W, \qquad \text{for } p \in D(u, \rho_0) \text{ and } u \in \mathcal{K}. \tag{4.4}$$

(Note that L_0 does not depend on $u \in \mathcal{K}$.)

We let $T_v\mathcal{S}(u)$ denote the tangent space to the leaf $\mathcal{S}(u)$ at the point v, where $v \in \mathcal{S}(u)$ and $u \in \mathcal{K}$. The k-dimensional space $X(u)$ can be identified with the tangent space $T_u\mathcal{S}(u)$. Since $X(u)$ is a finite-dimensional space in a Banach space W, there is a complementary space $Y(u)$, where $X(u)$ and $Y(u)$ are closed subspaces of W that satisfy $W = X(u) + Y(u)$ and $X(u) \cap Y(u) = \{0\}$. We let $\mathbb{P}$ be the bounded linear projection on W, where the range and null space satisfy $\mathcal{R}(\mathbb{P}) = X(u)$ and $\mathcal{N}(\mathbb{P}) = Y(u)$. Let $\mathbb{Q} = \mathbb{I} - \mathbb{P}$ be the complementary projection. We assume, for now, that $\nu(p) \in Y(u)$, for all $p \in D(u, \rho_0)$, see (4.2). One then has $\mathbb{Q}\ \nu(p) = \nu(p)$ and $\mathbb{P}\, p = p$, for all $p \in D(u, \rho_0)$.

The foliated, invariant set $\mathcal{K}$ is said to be **smooth** when the function $\nu = \nu(p)$ in (4.2) is smooth on $D(u, \rho_0)$, for each $u \in \mathcal{K}$. More precisely, the set $\mathcal{K}$ is said to be $\boldsymbol{C^1}$**-smooth** when each such ν is a C^1-mapping, i.e., ν has a continuous derivative on $D(u, \rho_0)$. Likewise, $\mathcal{K}$ is said to be $\boldsymbol{C^{1,1}}$**-smooth** when each such ν is a $C^{1,1}$-mapping, i.e., ν has a continuous derivative on $D(u, \rho_0)$ <u>and</u> the derivative of ν, $D\nu(p)$, is Lipschitz continuous on $D(u, \rho_0)$.

Definition: Let $S(t)u$ denote a semiflow defined on a subset U in W. Thus $(u, t) \to S(t)u$ is continuous for $(u, t) \in U \times (0, \infty)$, $S(0)u = u$, and $S(t)S(s)u = S(t + s)u$, for all $u \in U$ and $s,\ t \geq 0$, see [33, Section 2.1]. We say that a set $\mathcal{K}$ in U is a **foliated, invariant set** if it is an F1–F2 set that satisfies F3, F4, and F5, where

- F3: $\mathcal{K}$ is a compact, connected, invariant set, and for each $u \in \mathcal{K}$, the leaf $\mathcal{S}(u)$ is an invariant set;
- F4: the induced linearized semiflow over $\mathcal{K}$ has an exponential trichotomy and the four inequalities (4.10)–(4.13) hold, with dim $\mathcal{U}^o(u) = k$, for all $u \in \mathcal{K}$; and
- F5: the tangency condition (4.15) holds.

Since the notation used in the definition of an exponential dichotomy is needed below, we now present this definition in the setting where W is replaced by $V^{2\beta}$ and the semiflow on $V^{2\beta}$ is generated by the mild solutions $S(F, t)u_0$ of (4.1). We now let $\mathcal{K}$ be an F1–F2 set in $V^{2\beta}$, and we assume that F3 holds. Thus $\mathcal{K}$ is invariant, or $S(F, t)\mathcal{K} = \mathcal{K}$, for all $t \geq 0$. For $u \in \mathcal{K}$, we let $S(F, t)u$ denote any global solution of (4.1) in $\mathcal{K}$ passing through u, see Section 3.4. We now use the linear skew product semiflow on $\mathcal{E} = \mathcal{E}(\mathcal{K}) = \mathcal{K} \times V^{2\beta}$, as described in Section 3.2. Thus for $(u, w) \in \mathcal{E}$, one has

$$\Pi(F, u, w; t) = (F, S(F, t)u, \Phi(B, t)w), \qquad \text{for } t \geq 0, \tag{4.5}$$

since F is autonomous. Recall that $B = Z(F, u)$, see (2.28), and $\Phi(B, t)$ satisfies (2.24).

The **fiber** of $\mathcal{E}$ over the point $u \in \mathcal{K}$ is $\mathcal{E}(u) = \{u\} \times V^{2\beta}$. A mapping $P : \mathcal{E} \to \mathcal{E}$ is said to be a **projector** if P is continuous and has the form $P(u, w) = (u, P(u)w)$, where $P(u)$ is a continuous (linear) projection on the fiber $\mathcal{E}(u)$. This means that $P(u) : \mathcal{E}(u) \to \mathcal{E}(u)$ is a bounded linear mapping that satisfies $P(u)P(u) = P(u)$. For any projector $P : \mathcal{E} \to \mathcal{E}$ we define the **range** and **null space** by

$$\mathcal{R} = \mathcal{R}(P) = \{(u, w) \in \mathcal{E} : P(u)w = w\} \quad \text{and}$$
$$\mathcal{N} = \mathcal{N}(P) = \{(u, w) \in \mathcal{E} : P(u)w = 0\}.$$

Note that the fibers $\mathcal{R}(u)$ and $\mathcal{N}(u)$ are closed linear subspaces of $\mathcal{E}(u)$ since $P(u)$ is a continuous linear mapping. Furthermore, these fibers vary continuously in u, which implies that $P(u)$ varies continuously in the operator norm in $\mathcal{L}(V^{2\beta}, V^{2\beta})$. The range and nullspace of a projector are subbundles of $\mathcal{E}$. Also, one has $\mathcal{R} \cap \mathcal{N} = \mathcal{K} \times \{0\}$ and $\mathcal{R} + \mathcal{N} = \mathcal{E}$, i.e.,

$$\mathcal{R}(u) \cap \mathcal{N}(u) = \{0\} \quad \text{and} \quad \mathcal{R}(u) + \mathcal{N}(u) = \mathcal{E}(u), \qquad \text{for all } u \in \mathcal{K}. \tag{4.6}$$

If P is a projector on $\mathcal{E}$, then the mapping $Q = I - P$, where $Q : \mathcal{E} \to \mathcal{E}$ is defined by $Q(u, w) = (u, (I - P(u))w)$, is also a projector on $\mathcal{E}$. The projector Q is called the **complementary projector** to P, and one has $\mathcal{R}(Q) = \mathcal{N}(P)$ and $\mathcal{N}(Q) = \mathcal{R}(P)$. A projector P on $\mathcal{E}$ is said to be **invariant** if one has

$$P(S(F, t)u)\Phi(B, t) = \Phi(B, t)P(u), \qquad \text{for all } t \geq 0 \text{ and } u \in \mathcal{K}. \tag{4.7}$$

The invariance of a projector is equivalent to the assertion that both subbundles, $\mathcal{R}$ and $\mathcal{N}$, are positively invariant under the linear skew product semiflow Π. That is to say, $\Pi(t)\mathcal{R} \subseteq \mathcal{R}$ and $\Pi(t)\mathcal{N} \subseteq \mathcal{N}$. Note that P is invariant if and only if the complementary projector Q is invariant.

The following definition is taken from [24] or [33, Section 4.5]. We say that Π has an **exponential trichotomy** over an invariant set $\mathcal{K}$ in $V^{2\beta}$, with **characteristics** $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and K, where

$$\lambda_1 < \lambda_2 \leq 0 \leq \lambda_3 < \lambda_4 \quad \text{and} \quad K \geq 1 \tag{4.8}$$

if there exist three projectors P^s, P^o, and P^u defined over $\mathcal{K}$ with the respective ranges

$$\mathcal{R}(P^s) = \mathcal{S} = \mathcal{U}^s, \quad \mathcal{R}(P^o) = \mathcal{U}^o, \quad \mathcal{R}(P^u) = \mathcal{U} = \mathcal{U}^u$$

and $\mathcal{U}^{ou} = \mathcal{R}(Q^o)$, where $Q^o = I - P^o = P^s + P^u$ satisfies[1]

$$\mathcal{U}^{ou} = \mathcal{U}^o + \mathcal{U}^u = \{(u, w^o + w^u) \in \mathcal{E} : w^o \in \mathcal{U}^o \text{ and } w^u \in \mathcal{U}^u\}.$$

Also, the following hold:

- Each of the projectors P^i, for $i = s, o, u$, is invariant on $\mathcal{E} = \mathcal{E}(\mathcal{K})$.
- For each $u \in \mathcal{K}$, the projections $P(u)$, $Q(u)$, and $R(u)$ commute and one has

$$I = P^s(u) + P^o(u) + P^u(u) \quad \text{and} \quad P^j(u)P^k(u) = 0, \text{ when } j \neq k. \tag{4.9}$$

- For each $(u, w) \in \mathcal{U}^u$, let $B = Z(F, u)$. Then there is a global solution

$$\phi^{u,w}(t) = (S(F, t)u, \Phi(B, t)w), \qquad \text{for } t \in \mathbb{R},$$

  such that $\|e^{-\lambda_4 t}\Phi(B, t)w\| \to 0$, as $t \to -\infty$. (This global solution need not be unique, and we let $(S(F, t)u, \Phi(B, t)w)$ denote any global solution through (u, w) that satisfies $\|e^{-\lambda_4 t}\Phi(B, t)w\| \to 0$, as $t \to -\infty$.)
- For each $(u, w) \in \mathcal{U}^o$, let $B = Z(F, u)$. Then there is a global solution

$$\phi^{u,w}(t) = (S(F, t)u, \Phi(B, t)w), \qquad \text{for } t \in \mathbb{R},$$

  such that $\|e^{-\lambda_2 t}\Phi(B, t)w\| \to 0$, as $t \to -\infty$. (This global solution need not be unique, and we let $(S(F, t)u, \Phi(B, t)w)$ denote any global solution through (u, w) that satisfies $\|e^{-\lambda_2 t}\Phi(B, t)w\| \to 0$, as $t \to -\infty$.)
- The following four inequalities are valid for all $(u, w) \in \mathcal{E}$:

$$\|A^\beta \Phi(B, t)P^s(u)w\| \le K\|A^\beta w\|e^{\lambda_1 t}, \qquad \text{for all } t \ge 0, \tag{4.10}$$

$$\|A^\beta \Phi(B, t)P^u(u)w\| \le K\|A^\beta w\|e^{\lambda_4 t}, \qquad \text{for all } t \le 0, \tag{4.11}$$

$$\|A^\beta \Phi(B, t)P^o(u)w\| \le K\|A^\beta w\|e^{\lambda_3 t}, \qquad \text{for all } t \ge 0, \tag{4.12}$$

$$\|A^\beta \Phi(B, t)P^o(u)w\| \le K\|A^\beta w\|e^{\lambda_2 t}, \qquad \text{for all } t \le 0, \tag{4.13}$$

  where $B = Z(F, u)$ and (4.11) and (4.13) are valid for any negative continuation, as described above.
- Since inequalities (4.10)–(4.13) hold at $t = 0$, one has

$$\|A^\beta P^i(u)w\| \le K\|A^\beta w\|, \qquad \text{for all } (u, w) \in \mathcal{E} \text{ and } i = s, o, u. \tag{4.14}$$

Property F5, the tangency condition, states that

$$\mathcal{R}(P^o(u)) = \mathcal{U}^o(u) = T_u\mathcal{S}(u), \qquad \text{for all } u \in \mathcal{K}, \tag{4.15}$$

[1] The expression $\mathcal{U}^o + \mathcal{U}^u$ defined here is referred to as the Whitney sum of the subbundles $\mathcal{U}^o$ and $\mathcal{U}^u$.

where $T_u\mathcal{S}(u)$ denotes the tangent space to the leaf $\mathcal{S}(u)$ at point u. In some cases, Property F5 is a consequence of the other four properties F1–F4, see Lemma 5.7 below.

There are a number of properties of an exponential trichotomy that follow directly from the definition, see [33, Section 4.5]. These properties include

$$\Pi(t)\mathcal{U}^s \subset \mathcal{U}^s \quad \text{and} \quad \Pi(t)\mathcal{U}^o = \mathcal{U}^o, \quad \text{and} \quad \Pi(t)\mathcal{U}^u = \mathcal{U}^u, \tag{4.16}$$

for $t \geq 0$. Consequently, the restriction of $\Phi(B,t)$ to $\mathcal{U}^o(u)$, or $\mathcal{U}^u(u)$, is an isomorphism onto $\mathcal{U}^o(S(F,t)u)$, or $\mathcal{U}^u(S(F,t)u)$, respectively, where $B = Z(F,u)$. In addition, the negative continuations $\Phi(B,t)w$ described above are uniquely determined by w, and the extended cocycle identities

$$\begin{cases} \Phi(B,\tau+t)P^o(u) = \Phi(B_\tau,t)\Phi(B,\tau)P^o(u), \\ \Phi(B,\tau+t)P^s(u) = \Phi(B_\tau,t)\Phi(B,\tau)P^s(u), \end{cases} \qquad \text{for all } \tau,\ t \in \mathbb{R}, \tag{4.17}$$

are valid, for all $u \in \mathcal{K}$.

Because of the extended cocycle identity (4.17), one can reverse time in this case. As a result, the following inequalities hold for all $(u,w) \in \mathcal{E}$:

$$\|A^\beta\Phi(B,t)P^u(u)w\| \geq K^{-1}\|A^\beta P^u(u)w\|e^{\lambda_4 t}, \qquad \text{for all } t \geq 0, \tag{4.18}$$

and

$$\|A^\beta\Phi(B,t)P^o(u)w\| \geq K^{-1}\|A^\beta P^o(u)w\|e^{\lambda_2 t}, \qquad \text{for all } t \geq 0, \tag{4.19}$$

as well as

$$\|A^\beta\Phi(B,t)P^o(u)w\| \geq K^{-1}\|A^\beta P^o(u)w\|e^{\lambda_3 t}, \qquad \text{for all } t \leq 0. \tag{4.20}$$

In the infinite-dimensional setting, one is unable to convert inequality (4.10) into a statement about the behavior of solutions in the stable bundle $\mathcal{U}^s$, for time $t \leq 0$. The reason is that, in general, the solution $\Phi(B,t)P^s(u)w$ need not exist for $t < 0$. However, a partial negative extension is possible in the following case: Let $(u,w) \in \mathcal{E}$ and let $S(F,t)u$ be a global solution in $\mathcal{K}$ through u. Let $B = Z(F,u)$, where $B \in \mathcal{M}^\infty(\mathbb{R},\mathcal{L})$. Assume that there is a $v \in V^{2\beta}$ and a $\tau > 0$ such that $\Phi(B_{-\tau},\tau)v = P^s(u)w$. Since $P^s(u)w \in \mathcal{U}^s(u)$, one has $v \in \mathcal{U}^s(S(F,-\tau)u)$. Also, (4.7) and (4.10) imply that, for $t \geq 0$, one has

$$\begin{aligned} \|A^\beta P^s(S(F,t)u)\Phi(B,t)w\| &= \|A^\beta\Phi(B_{-\tau},\tau+t)P^s(S(F,-\tau)u)v\| \\ &\leq K\|A^\beta v\|e^{\lambda_1(\tau+t)}, \end{aligned}$$

which goes to 0, as $t \to \infty$, since $\lambda_1 < 0$. We now adopt the convention of defining $\Phi(B,t)P^s(u)w$, for $-\tau \leq t \leq 0$, by the formula

$$\Phi(B,t)P^s(u)w \stackrel{\text{def}}{=} \Phi(B_{-\tau},\tau+t)v, \qquad \text{for } -\tau \leq t \leq 0. \tag{4.21}$$

Next we set $\widehat{v} = \Phi(B,t)P^s(u)w$, for some t with $-\tau \le t \le 0$. Then $\widehat{v} \in \mathcal{U}^s(S(F,t)u)$ and

$$\Phi(B_t,-t)\widehat{v} = P^s(u)w = \Phi(B_t,-t)P^s(S(F,t)u))\widehat{v}.$$

Hence (4.10) implies that

$$\|A^\beta P^s(u)w\| = \|A^\beta \Phi(B_t,-t)P^s(S(F,t)u)\widehat{v}\| \le Ke^{\lambda_1(-t)}\|A^\beta \widehat{v}\|,$$

which in turn yields

$$\|A^\beta \widehat{v}\| \ge K^{-1}e^{\lambda_1 t}\|A^\beta P^s(u)w\|, \qquad \text{for } -\tau \le t \le 0.$$

Consequently, we obtain

$$\|A^\beta \Phi(B,t)P^s(u)w\| \ge K^{-1}e^{\lambda_1 t}\|A^\beta P^s(u)w\|, \qquad \text{for } -\tau \le t \le 0. \tag{4.22}$$

It should be emphasized that (4.21) depends on points $P^s(u)w$ and v. There may be several such points v that can be used since the negative continuations are not assumed to be unique. However, in every case inequality (4.22) is valid.

Additional features: A foliated, invariant set for the semiflow π on $V^{2\beta}$ may satisfy additional properties of dynamical interest. Let us look first at properties F2 and F5. Because of the tangency condition (4.15), we fix the space $X(u)$ to be the tangent space $T_u\mathcal{S}(u) = \mathcal{R}(P^o(u)) = \mathcal{U}^o(u)$. Thus, $D(u,\rho_0) \subset \mathcal{U}^o(u)$, that is,

$$D(u,\rho_0) = \{p \in \mathcal{U}^o(u) : \|A^\beta p\| \le \rho_0\}. \tag{4.23}$$

The range of ν is assumed to be in $\mathcal{R}(Q^o(u))$, i.e., $\nu(p) \in \mathcal{R}(Q^o(u))$, when $p \in D(u,\rho_0)$. Thus, one has $Q^o(u)\nu(p) = \nu(p)$ and $P^o(u)p = p$, for all $p \in D(u,\rho_0)$.

Notice that, since $P^o(u)$ varies continuously in $u \in \mathcal{K}$, the tangency condition (4.15) implies that the function ν in (4.2) is a C^1-mapping, i.e., the leaves of a foliated, invariant set are C^1-smooth. Indeed, if ν satisfies (4.2) and the tangency condition (4.15) holds, then the derivative $D\nu$ satisfies

$$D\nu(p) = P^o(v), \qquad \text{where } v = u + p + \nu(p). \tag{4.24}$$

Since $P^o(v)$ varies continuously in v, this implies that ν is a C^1-mapping.

In addition to smoothness, we will require that the exponential trichotomy on $\mathcal{K}$ satisfy another technical condition, the Lipschitz property. In particular, we say that the foliated, invariant set satisfies the **Lipschitz property** if the following hold:

1. There is an $L_0 > 0$ such that (4.3)–(4.4) hold and

   $$\|A^\beta(P^i(u_1) - P^i(u_2))\|_{\mathcal{L}} \le L_0\|A^\beta(u_1-u_2)\|, \qquad \text{for } i = s,o,u, \tag{4.25}$$

   for all u_1 and u_2 in the disk $\mathcal{D}_{\rho_0}(u)$ and all u in $\mathcal{K}$, where $\mathcal{L} = \mathcal{L}(V^{2\beta}, V^{2\beta})$.
2. Moreover, the neutral projector P^o satisfies

   $$\|A^\beta(P^o(u_1) - P^o(u_2))\|_{\mathcal{L}} \le L_0\|A^\beta(u_1-u_2)\|, \tag{4.26}$$

   for all $u_1,\ u_2 \in \mathcal{K}$ with $\|A^\beta(u_1-u_2)\| \le 2\rho_0$.

In this setting, conditions (4.3) and (4.4) now read

$$\|A^{\beta}(\nu(p_1) - \nu(p_2))\| \le L_0 \|A^{\beta}(p_1 - p_2)\|, \qquad \text{for } p_1,\ p_2 \in D(u, \rho_0), \tag{4.27}$$

and

$$\|A^{\beta}\nu(p)\| \le L_0 \|A^{\beta} p\|, \qquad \text{for } p \in D(u, \rho_0) \text{ and } u \in \mathcal{K}. \tag{4.28}$$

The proof of the following lemma, in the finite-dimensional case, appears in [21]. Since the argument, which is based on the Gronwall inequality, extends readily to the infinite-dimensional setting we study here, we omit the details.

Lemma 4.1. *Let $\mathcal{K}$ be a foliated, invariant set that satisfies the five properties F1–F5. If* (4.25) *holds for $i = o$, then for each $u \in \mathcal{K}$, the disk $\mathcal{D}_{\rho_0}(u)$ is uniquely determined. Furthermore, one has $\mathcal{S}(u) = \mathcal{S}(v)$ if and only if $v \in \mathcal{S}(u)$.*

Next we show that the Lipschitz property implies that $\mathcal{K}$ is $C^{1,1}$-smooth.

Lemma 4.2. *Let $\mathcal{K}$ be a foliated, invariant set that satisfies the five properties F1–F5. If* (4.25) *holds for $i = o$, then the set $\mathcal{K}$ is $C^{1,1}$-smooth.*

Proof. As noted in (4.24), the function ν is a C^1-mapping. When (4.25) holds, then $P^o(v)$ is Lipschitz continuous for $v \in \mathcal{D}_{\rho_0}(u)$ and ν is a $C^{1,1}$-mapping. Hence $\mathcal{K}$ is $C^{1,1}$-smooth. □

4.2. Nonautonomous theory

Instead of the autonomous equation (4.1), we now turn to the nonautonomous problem

$$\partial_t u + Au = \widehat{F}(\theta, u) \quad \text{and} \quad \partial_t \theta = \omega, \tag{4.29}$$

where we use the alternate notation described in Section 3. As usual, we assume that A satisfies Hypothesis A.

Hypotheses B: The function $\widehat{F} = \widehat{F}(u, t)$ is assumed to be in C^1_F and to be quasi-periodic in time in the sense described in (3.14)–(3.17). Furthermore, we require that the lifting $\hat{h}_F : T^k \to C^1_F$ be Hölder continuous, see Lemma 3.3. We let π and Π denote the skew product semiflows given by (3.21).

Let $\mathcal{K}$ denote a compact, invariant set in $T^k \times V^{2\beta}$. For any set M in T^k, we define the fiber $\mathcal{K}(M)$ by

$$\mathcal{K}(M) = \{(\theta, u) \in \mathcal{K} : \theta \in M\}.$$

For $(\theta, u) \in \mathcal{K}$, we let $S(\theta, t)u$ denote a global solution of the u-equation in (4.29) that satisfies $S(\theta, 0)u = u$. Since $\mathcal{K}$ is a compact set, it follows that, for each $\theta \in T^k$, the fiber $\mathcal{K}(\theta)$ is a compact set in $V^{2\beta}$, and $S(\theta, t)\mathcal{K}(\theta) = \mathcal{K}(\theta \cdot t)$. This brings us to an important concept, the foliated bundle.

Definition: A set $\mathcal{K}$ in $T^k \times V^{2\beta}$ is said to be a **foliated bundle** provided the following hold:

- FB1: $\mathcal{K}$ is a nonempty, compact, connected, invariant set for π;
- FB2: there is an integer $k \geq 1$ and a $\rho_0 > 0$ such that, for each $\theta \in T^k$, the fiber $\mathcal{K}(\theta)$ is an F1–F2 set with disks $\mathcal{D}_{\rho_0}(\theta, u) = \mathcal{D}_{\rho_0}(u)$ and leaves $\mathcal{S}(\theta, u) = \mathcal{S}(u)$ satisfy the invariance property
$$S(\theta, t)\mathcal{S}(\theta, u) = \mathcal{S}(\theta \cdot t, S(\theta, t)u), \qquad \text{for all } t \geq 0,$$
and all $(\theta, u) \in \mathcal{K}$;
- FB3: the mappings f defined by (4.2) depend on $(\theta, u) \in \mathcal{K}$, and the Lipschitz coefficient L_0 in (4.3) does <u>not</u> depend on (θ, u) in $\mathcal{K}$;
- FB4: the induced linear skew product semiflow Π has an exponential trichotomy on $\mathcal{E} = \mathcal{K} \times V^{2\beta}$ and the four inequalities (4.10)–(4.13) hold, with $\dim \mathcal{U}^o(\theta, u) = k$, for all $(\theta, u) \in \mathcal{K}$; and
- FB5: the tangency condition (4.33) holds.

A foliated bundle $\mathcal{K}$ in $T^k \times V^{2\beta}$ is said to have a **fiber bundle structure** if, for every θ_0 in T^k, there is a neighborhood $U = U(\theta_0)$ in T^k and a homeomorphism $H : U \times \mathcal{K}(\theta_0) \to \mathcal{K}(U)$ of $U \times \mathcal{K}(\theta_0)$ onto $\mathcal{K}(U)$ such that for each $\theta \in U$, one has

$$H^{-1}(\mathcal{K}(\theta)) = \{\theta\} \times \mathcal{K}(\theta_0). \tag{4.30}$$

In particular, one has $\mathcal{K}(\theta) = H(\{\theta\} \times \mathcal{K}(\theta_0))$, for all $\theta \in U$.

For example, let $\mathcal{K}_0$ be a foliated, invariant set in $V^{2\beta}$ for the autonomous problem (4.1). Consider the equation

$$\partial_t u + Au = F(u) \quad \text{and} \quad \partial_t \theta = \omega, \tag{4.31}$$

where $\theta \in T^k$ and ω is a frequency vector that satisfies (3.15). Note that equation (4.31) is a special case of (3.19). The dynamics for (4.31) is in the product space $\mathcal{K} = T^k \times \mathcal{K}_0$, and $\mathcal{K}$ is a foliated bundle for (4.31) and has the fiber bundle structure.

Abbreviated notation: In the case of a foliated bundle, we will use the abbreviated notation described in this paragraph: $\mathcal{D}_{\rho_0}(u) = \mathcal{D}_{\rho_0}(\theta, u)$, for the disks $\mathcal{D}_{\rho_0}$; $\mathcal{S}(u) = \mathcal{S}(\theta, u)$, for the leaves $\mathcal{S}$; $\mathcal{R}(u) = \mathcal{R}(P(\theta, u))$ and $\mathcal{N}(u) = \mathcal{N}(P(\theta, u))$, for the range and null space of a projector P; $P^i(u) = P^i(\theta, u)$, for $i = s, o, u$, and $Q^o(u) = Q^o(\theta, u)$, for the projectors arising in the definition of an exponential trichotomy; and $\mathcal{U}^i(u) = \mathcal{U}^i(\theta, u)$, for $i = s, o, u$, for the ranges of the projectors P^i. The inequalities (4.10)–(4.14) should be be interpreted with this abbreviated notation. In particular, the characteristics $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and K do not depend on $(\theta, u) \in \mathcal{K}$. In the case of a foliated bundle, the invariance property (4.7) now takes on the form

$$P(\theta \cdot t, S(\theta, t)u)\Phi(B, t) = \Phi(B, t)P(\theta, u), \qquad \text{for } t \geq 0 \tag{4.32}$$

and $(\theta, u) \in \mathcal{K}$, where $B = Z(\theta, u) \in \mathcal{M}^\infty(\mathbb{R}, \mathcal{L})$. The tangency condition (4.15) now reads

$$\mathcal{R}(P^o(\theta, u)) = \mathcal{U}^o(\theta, u) = T_u\mathcal{S}(\theta, u), \qquad \text{for all } (\theta, u) \in \mathcal{K}. \tag{4.33}$$

5. Statements of theorems

We begin the section with precise statements of the four main theorems to be developed in the work: the robustness property (Theorem 5.1), the shadow property (Theorem 5.2), the fiber structure property (Theorem 5.3), and the homeomorphism property (Theorem 5.4). The proofs of these theorems are presented in Section 6. Some preliminary results, which are used in these proofs, are presented in this section.

We now turn to the fully nonautonomous problem (1.2), where

$$\partial_t u + Au = F(u,t), \qquad F \in \mathcal{H} \tag{5.1}$$

is the given equation and

$$\partial_t y + Ay = F(y,t) + G(y,t), \qquad F,\ G \in \mathcal{H} \tag{5.2}$$

is the perturbed equation. As usual, we assume that A satisfies Hypothesis A. The assumptions on F and G given below will insure that the set $\mathcal{H}$ is a suitable compact, invariant set in C^1_F. In Sections 5 and 6, we focus on the quasi periodic problem, because the main features of our theory can be seen in fully in this context, with a minimal of notation. Some extensions of this theory, to the setting where $\mathcal{H}$ is a more general compact, invariant set in C^1_F, are presented in Section 7.

Alternate notation: The alternate notation described in (3.19) is useful here in simplifying the formulation of the perturbation problem (5.1)–(5.2) when the time dependence is quasiperiodic. Thus, we rewrite these two equations as the two systems

$$\partial_t u + Au = \widehat{F}(\theta, u) \quad \text{and} \quad \partial_t \theta = \omega \tag{5.3}$$

and

$$\partial_t y + Ay = \widehat{F}(\theta, y) + \widehat{G}(\theta, y) \quad \text{and} \quad \partial_t \theta = \omega. \tag{5.4}$$

We let $S_1(\theta,t)u_0$ denote the mild solution of the u-equation in (5.3), with $S_1(\theta,0)u_0 = u_0$, and let $S_2(\theta,t)y_0$ denote the mild solution of the y-equation in (5.4), with $S_2(\theta,0)y_0 = y_0$. For $i = 1, 2$, we let π_i and Π_i denote the semiflows given by (3.21), where $S(\theta,t)$ is replaced by $S_i(\theta,t)$.

We assume throughout that both $\widehat{F}$ and $\widehat{G}$ satisfy Hypothesis B. This includes the case where $\widehat{F}$ and $\widehat{G}$ are spawned on different tori, say, T^m and T^n, with different frequency vectors, say, $\widehat{\omega}$ and $\widetilde{\omega}$. However, by using suitable liftings, say, h_F and h_G, one can do this over a single (perhaps higher dimensional) torus T^k with a single frequency vector ω. What is needed here is that the frequency modules $\mathcal{M}(\widehat{\omega})$ and $\mathcal{M}(\widetilde{\omega})$ be contained in $\mathcal{M}(\omega)$ and that h_F and h_G are chosen so that $\widehat{F} \in h_F(T^k)$ and $\widehat{G} \in h_G(T^k)$. The set $\mathcal{H}$ in (5.1)–(5.2) is then replaced by the torus T^k.

5.1. The main theorems

Hypotheses C: We now consider equations (5.3) and (5.4) as described above. We let $\mathcal{K}$ be a foliated bundle in $T^k \times V^{2\beta}$ for (5.3), and we assume that for a fixed $\sigma_0 > 0$, one has:

- $\mathcal{K}$ is $C^{1,1}$-smooth; and
- $\mathcal{K}$ satisfies the Lipschitz property, i.e., (4.3), (4.25), and (4.26) hold; and
- For $0 \leq \sigma \leq \sigma_0$, we let $U(\sigma)$ be the bounded set in $T^k \times V^{2\beta}$ consisting of all points (θ, y) with the property that $\|A^\beta(y-x)\| \leq \sigma$, for some $(\theta, x) \in \mathcal{K}$. Note that
$$U(0) = \bigcap_{0<\sigma\leq\sigma_0} U(\sigma) = \mathcal{K}.$$

With the parameter $\sigma_0 > 0$ now fixed, the constants k_0, k_1, and k_3 – as used in Section 2.2 – are then determined by using the nonlinear function $F = \widehat{F}$ and the bounded set $U = U(\sigma_0)$. W assume that the perturbation term G satisfies

$$\|G\|_{\{C^1;A;U,\mathbb{R}\}} \leq \delta, \tag{5.5}$$

where $\delta > 0$ is described in the perturbation theory presented below. The pseudonorm in (5.5) is defined by

$$\begin{aligned}\|G\|_{\{C^1;A;U,\mathbb{R}\}} = &\sup_{t\geq 0}\sup_{\tau\in\mathbb{R}}\int_0^t \sup_{u\in U}\|A^\beta e^{-A(t-s)}G_\tau(u,s)\|\,ds\\ &+\sup_{t\geq 0}\sup_{\tau\in\mathbb{R}}\int_0^t \sup_{u\in U}\|A^\beta e^{-A(t-s)}DG_\tau(u,s)\|_{\mathcal{L}}\,ds.\end{aligned}$$

With this background, we can state our main theorem for the perturbation problem (5.3)–(5.4).

Theorem 5.1 (Robustness property). *Let equations* (5.3) *and* (5.4) *be given, where A satisfies Hypothesis A and $\widehat{F}$ and $\widehat{G}$ satisfy Hypothesis B. Let $\mathcal{K}$ satisfy Hypothesis C; in particular, $\mathcal{K}$ is a $C^{1,1}$-smooth foliated bundle for* (5.1) *and $\mathcal{K}$ satisfies the Lipschitz property, see* (4.25)–(4.26)*.*

Then, for every $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that if G satisfies (5.5) *with this δ, then there is a continuous mapping $h : \mathcal{K} \to V^{2\beta}$ with the following properties:*

1. *The image $\mathcal{K}^n \stackrel{\text{def}}{=} h(\mathcal{K})$ is a compact, invariant, connected set for the perturbed equation* (5.4)*, and it satisfies the four properties FB1–FB4.*
2. *One has $\|A^\beta(h(v) - v)\| \leq 2\epsilon$, for all $(\theta, v) \in \mathcal{K}$.*
3. *For all $(\theta, v) \in \mathcal{K}$, both global solutions $S_1(\theta, t)v$ and $S_2(\theta, t)h(v)$ are strong solutions of* (5.3) *and* (5.4)*, respectively, and they both satisfy* (3.12)*. In particular, for each $\theta \in T^k$, the fibers $\mathcal{K}(\theta)$ and $\mathcal{K}^n(\theta)$ are in $\mathcal{D}(A)$, the domain of A.*

Note that item 3 in the last theorem follows from Lemmas 3.2–3.3 since $\widehat{F}$ and $\widehat{G}$ are Hölder continuous liftings of the torus T^k, see Section 3.3.

There are conditions under which $\mathcal{K}^n$ satisfies the tangency property FB5, see (4.33). These conditions are described in a remark at the end of Section 6.

The concept of a shadow flow arises in the perturbation theories in [20,21, 24]. As will be seen, the nonautonomous feature in this study results in a suitable variation. A mapping $\widehat{S}(\theta, t)$ is said to be a **shadow semiflow** on $\mathcal{K}$ (generated by the dynamics of (5.4)) provided that

$$(5.6) \quad h(\theta \cdot t, \widehat{S}(\theta, t)v) = S_2(\theta, t)h(\theta, v), \qquad \text{for all } (\theta, v) \in \mathcal{K} \text{ and all } t \geq 0.$$

Note that if $h(\theta, v_1) = h(\theta, v_2)$, for $(\theta, v_1),\ (\theta, v_2) \in \mathcal{K}$, then (5.6) implies that

$$h(\theta \cdot t, \widehat{S}(\theta, t)v_1) = h(\theta \cdot t, \widehat{S}(\theta, t)v_2), \qquad \text{for all } t \geq 0.$$

As noted in the main theorem, we will show the existence of such a mapping $h : \mathcal{K} \to V^{2\beta}$, for each G satisfying inequality (5.5), with δ sufficiently small. We will also prove the following result:

Theorem 5.2 (Shadow property). *Let Hypotheses A, B, and C of Theorem* 5.1 *be satisfied, and let $h : \mathcal{K} \to T^k \times V^{2\beta}$ satisfy the conclusions. Then there exists a shadow semiflow $\widehat{S}(\theta, t)$ on $\mathcal{K}$, for every G satisfying inequality* (5.5). *In particular, $\widehat{S}(\theta, t)$ satisfies the cocycle property* (3.20), *and $\widehat{\pi}(\theta, v; t) = (\theta{\cdot}t, \widehat{S}(\theta, t)v)$ is a semiflow on $\mathcal{K}$. Furthermore, when $\widehat{G} \equiv 0$, then $\widehat{S}(\theta, t) = S_1(\theta, t)$ on $\mathcal{K}$.*

One can show that the shadow flow $\widehat{S}$ depends continuously on $\widehat{G}$ in the $\mathcal{T}_A^1$ topology. For the finite-dimensional argument, see [21].

Theorem 5.3 (Fiber bundle property). *Let Hypotheses A, B, and C of Theorem* 5.1 *be satisfied, and let $h : \mathcal{K} \to T^k \times V^{2\beta}$ satisfy the conclusions. Assume that*

- *$\mathcal{K}$ has the fiber bundle structure and*
- *h is a homeomorphism of $\mathcal{K}$ onto $\mathcal{K}^n$.*

Then $\mathcal{K}^n$ has the fiber bundle structure.

We thus see that it becomes useful to know when the mapping $h : \mathcal{K} \to \mathcal{K}^n$ given by Theorem 5.1 is a homeomorphism. In this regard, we have the following result:

Theorem 5.4 (Homeomorphism property). *Let Hypotheses A, B, and C of Theorem* 5.1 *be satisfied, and let $h : \mathcal{K} \to T^k \times V^{2\beta}$ satisfy the conclusions. Assume further that the semiflow π_1 has a proper negative continuation on $\mathcal{K}$, see Section 5.2. Then with ϵ small, the mapping h is a homeomorphism of $\mathcal{K}$ onto $h(\mathcal{K})$.*

The class Σ: In the sequel we will use the class Σ consisting of all positive, real-valued functions $\beta_1 = \beta_1(r) = \beta_1(r_1, r_2)$, defined for $r = (r_1, r_2)$ with $0 < r_i < r_{i0}$, where $r_{i0} = r_{i0}(\beta_1) > 0$, for $i = 1, 2$, and satisfying $\beta_1(r) \to 0$, as $r \to 0$. For example, if some real-valued function $\xi(\epsilon)$ is of order $o(\epsilon)$, as $\epsilon \to 0$, then one can write this in the form $|\xi(\epsilon)| = \epsilon\beta_1(\epsilon)$, where $\beta_1 \in \Sigma$. We let $\beta_1,\ \beta_2, \cdots$ and $b_0,\ b_1^F,\ b_2^F, \cdots$ denote various elements of Σ.[2] We will use the terms $\beta_1, \beta_2, \cdots$ as local variables, which may be redefined from time to time. Global variables, which have a unique definition in this work, will be denoted by $b_0, b_1^F, b_2^F, \cdots$. The superscript F will be used to denote elements of Σ that depend on $\widehat{F}$ but that are independent of the perturbation term $\widehat{G}$.

Let us return to the function $E = E(u_0, v, t)$ arising in (2.12). In terms of the new notation used in (5.3), this function E has a representation $\widehat{E} = \widehat{E}(\theta, u_0, v)$ that satisfies

$$\widehat{F}(\theta, u_0 + v) - \widehat{F}(\theta, u_0) = D\widehat{F}(\theta, u_0)v + \widehat{E}(\theta, u_0, v), \tag{5.7}$$

where $D = \frac{\partial}{\partial u}$. Due to the Fréchet differentiability, the error term $\widehat{E}$ satisfies

$$\lim_{\|A^\beta v\| \to 0} \frac{\|\widehat{E}(\theta, u_0, v)\|}{\|A^\beta v\|} = 0, \qquad \text{for each } u_0 \in V^{2\beta}, \tag{5.8}$$

and uniformly for $\theta \in T^k$. It follows that, as a function of v, one has

$$\widehat{E}(\theta, u_0, \cdot) \in C_{\text{Lip}}(V^{2\beta}, W) \cap C_F^1(V^{2\beta}, W), \tag{5.9}$$

with $\widehat{E}(\theta, u_0, 0) = 0$ and $D\widehat{E}(\theta, u_0, 0) = 0$, where $D\widehat{E}(\theta, u_0, v) = \frac{\partial}{\partial v}\widehat{E}(\theta, u_0, v)$. Since $D\widehat{E}(\theta, u_0, v) \in \mathcal{L} = \mathcal{L}(V^{2\beta}, W)$, for each $v \in V^{2\beta}$, and since it is continuous in v, it follows from (5.8) that there is a function $\gamma = \gamma(\rho)$ in Σ such that $\|D\widehat{E}(\theta, u_0, v)\|_{\mathcal{L}} \le \gamma(\rho)$, for all $v \in V^{2\beta}$ with $\|A^\beta v\| \le \rho$ and all $\theta \in T^k$. Since one has

$$\widehat{E}(\theta, u_0, v_1) - \widehat{E}(\theta, u_0, v_2) = \int_0^1 D\widehat{E}(\theta, u_0, v_2 + s(v_1 - v_2))\, ds\, (v_1 - v_2),$$

and $\|A^\beta(v_2 + s(v_1 - v_2))\| \le \rho$, when $\|A^\beta v_1\|,\ \|A^\beta v_2\| \le \rho$ and $0 \le s \le 1$, one finds that

$$\|\widehat{E}(\theta, u_0, v_1) - \widehat{E}(\theta, u_0, v_2)\| \le \gamma(\rho)\|A^\beta(v_1 - v_2)\|, \tag{5.10}$$

whenever $\|A^\beta v_1\|,\ \|A^\beta v_2\| \le \rho$ and $\theta \in T^k$. Since $\widehat{E}(\theta, u_0, 0) = 0$, it follows from inequality (5.10) that for $\|A^\beta v\| = \|A^\beta(u - u_0)\| \le \rho$, one has

$$\|\widehat{E}(\theta, u_0, v)\| \le \gamma(\rho)\|A^\beta v\| \le \rho\,\gamma(\rho). \tag{5.11}$$

[2] Note that the subscript in β_i will distinguish an element $\beta_i \in \Sigma$ from the special parameter β used in Standing Hypothesis A and the A^β-norm on $V^{2\beta} = \mathcal{D}(A^\beta)$.

5.2. Proper negative continuation

Let $\mathcal{K}$ be a compact, invariant set in $T^k \times V^{2\beta}$ for the semiflow π generated by (5.3), see Hypothesis B. Let $(\theta, v) \in \mathcal{K}$ and let $B = Z(\theta, v)$, see (3.22). Then there is a constant $K \geq 1$ and $a_0 \geq 0$ such that one has

$$\|A^\beta \Phi(B,t)w\| \leq Ke^{a_0 t}\|A^\beta w\|, \qquad \text{for all } t \geq 0, \tag{5.12}$$

as well as

$$\|A^\beta \Phi(B,t)w\| \leq Kt^{-\beta}e^{a_0 t}\|w\|, \qquad \text{for all } t > 0, \tag{5.13}$$

see [33, Theorem 44.1]. Furthermore, if Π has an exponential trichotomy on $\mathcal{K}$, then we choose $a_0 \geq \lambda_3$, see (4.12), and we choose a larger value of K in (5.12) and (5.13), if necessary, so that the inequalities (4.10)–(4.22) also hold.

In the following results, we will treat ρ as a parameter, with $0 < \rho \leq \rho_0$. In this way, we study the behavior of solutions starting in smaller disks $\mathcal{D}_\rho(v) = \mathcal{D}_\rho(\theta, v)$. In the next lemma, we only require that $\mathcal{K}$ be a compact, invariant set. The exponential trichotomy is not used here.

Lemma 5.5. *Let* (5.3) *be given, where A satisfies Hypothesis A and F satisfies Hypotheses B. Let $\mathcal{K}$ be a compact, invariant set for* (5.3) *in $T^k \times V^{2\beta}$. Let $T > 0$ be fixed. Then there is a ρ_0 and a function $b_1^F \in \Sigma$ such that for any two points (θ, u_1), $(\theta, u_0) \in \mathcal{K}$ with $\|A^\beta(u_1 - u_0)\| \leq \rho_0$, there is a function $H_2(t)$ with the property that*

$$S(\theta,t)u_1 - S(\theta,t)u_2 = \Phi(B,t)(u_1 - u_0) + H_2(t), \quad \textit{for all } t \in [0, 2T], \tag{5.14}$$

where $B = Z(\theta, u_0)$, see (3.22)*. Furthermore, if $\|A^\beta(u_1 - u_0)\| \leq \rho \leq \rho_0$, then*

$$\|A^\beta H_2(t)\| \leq b_1^F(\rho)\|A^\beta(u_1 - u_0)\|, \qquad \textit{for all } t \in [0, 2T]. \tag{5.15}$$

Proof. Fix ρ_0 so that $0 < \rho_0 \leq 1$. Let $H_2(t)$ be defined by (5.14) and let (θ, u_1), $(\theta, u_0) \in \mathcal{K}$ with $\|A^\beta(u_1 - u_0)\| \leq \rho_0$. Set $w = w(t) = S(\theta,t)u_1 - S(\theta,t)u_0$, for $0 \leq t \leq 2T$. Then w is a mild solution of (2.34) with $G \equiv 0$ and $H \equiv E$. It follows from (2.36) that

$$w(t) = \Phi(B,t)(u_1 - u_0) + \int_0^t \Phi(B_s, t-s)\widehat{E}(\theta \cdot s, S(\theta,s)u_0, w(s))\, ds,$$

for $0 \leq t \leq 2T$. From (5.14) we obtain

$$H_2(t) = \int_0^t \Phi(B_s, t-s)\widehat{E}(\theta \cdot s, S_1(\theta,s)u_0, w(s))\, ds.$$

Now inequality (2.21), with $F = F_1 = F_2$, implies that there is a constant $C_2 = C_2(2T) \geq 1$ such that

$$\|A^\beta w(t)\| = \|A^\beta(S(\theta,t)u_1 - S(\theta,t)u_0))\| \leq C_2\|A^\beta(u_1 - u_0)\|, \tag{5.16}$$

for $0 \le t \le 2T$. If $\|A^\beta(u_1 - u_0)\| \le \rho$ and $C_2\rho = \sigma$, then $\|A^\beta w(t)\| \le \sigma$, for $0 \le t \le 2T$. Consequently, inequalities (5.10), (5.13), and (5.16) imply that there is a $\gamma \in \Sigma$ such that

$$\|A^\beta H_2(t)\| \le \gamma(\sigma)\, C_2\, \|A^\beta(u_1 - u_0)\| \int_0^t (t-s)^{-\beta} e^{a_0(t-s)}\, ds.$$

With $C = \int_0^{2T} (2T - s)^{-\beta} e^{a_0(2T-s)}\, ds$ and $b_1^F(\rho) = C\gamma(C_2\rho)$, one obtains (5.15). Since $\gamma \in \Sigma$, one has $b_1^F \in \Sigma$. (Note that b_1^F depends on T.) □

Inequality (5.16) and its various consequences, such as (5.15), play a pivotal role in our theory. As a matter of fact, we will need a stronger version of (5.15), which leads us to the following concept.

Definition. Let equations (5.3) be given, where A satisfies Hypothesis A, and F satisfy Hypotheses B. Let $\mathcal{K}$ be a compact, invariant set for (5.3) in $T^k \times V^{2\beta}$. We say that the semiflow π has a **proper negative continuation on** $\mathcal{K}$ provided that for any $T > 0$, there is a constant $C_2 = C_2(2T) \ge 1$ and there is a $\rho = \rho(T) > 0$ such that the global solutions $S(\theta, t)u_1$ and $S(\theta, t)u_0$ satisfy

(5.17) $$\|A^\beta(S(\theta,t)u_1 - S(\theta,t)u_0)\| \le C_2\|A^\beta(u_1 - u_0)\|, \quad \text{for } -2T \le t \le 2T,$$

whenever $(\theta, u_1),\ (\theta, u_0) \in \mathcal{K}$ with $\|A^\beta(u_1 - u_0)\| \le \rho$. Assume now that π has a proper negative continuation on $\mathcal{K}$. Then with a straightforward variation of the argument given above, one concludes that if $\|A^\beta(u_1 - u_0)\| \le \rho \le \rho_0$, then

(5.18) $$\|A^\beta H_2(t)\| \le b_1^F(\rho)\|A^\beta(u_1 - u_0)\|, \qquad \text{for all } t \in [-2T, 2T].$$

Remark. The PNC property, that is, the assumption that the semiflow π has a proper negative continuation on a given foliated bundle $\mathcal{K}$, is most noteworthy. One can easily construct examples where this does not occur. However, if the PNC property holds, then one has backward uniqueness of the initial value problem, see [36, Chapter III, Section 6] and [17,1,4]. The PNC property is not equivalent to backward uniqueness; it is stronger.

Nevertheless, the prototypical examples of foliated invariant sets described in Section 1, that is, the compact manifolds, the hyperbolic sets, and the finite set-theoretical products of these sets, will all have the PNC property when the nonlinear function $F(u)$, or $F(u,t)$, is in C_F^1.

More generally, in finite dimensions, the PNC property is a consequence of an assumption that the semiflow is generated by a Lipschitz continuous vector field on $\mathcal{K}$. This is also applicable in infinite dimensions, when there exists an inertial manifold for the unperturbed problem. The dynamics on the inertial manifold is the dynamics of a finite-dimensional system of ordinary differential equations and the foliated bundle lies in the inertial manifold, see, e.g., [33]. The PNC property also holds in the infinite dimensional setting when the foliated set is simply a compact, invariant manifold, see Pliss and Sell [24].

For the autonomous problem (4.1), it follows from Lemma 3.2 that any foliated, invariant set $\mathcal{K}$ satisfies $\mathcal{K} \subset \mathcal{D}(A)$, see (3.12). (Thus the mapping $u \to Au$ is a continuous mapping of $\mathcal{K}$ into W.) Consequently, one expects that the restriction of the nonlinear evolutionary equation (4.1) to $\mathcal{K}$ to be a Lipschitz continuous vector field, since $F \in C^a_{\mathrm{Lip}} \cap C^1_F$. For the nonautonomous problem (5.3), each fiber $\mathcal{K}(\theta)$ satisfies $\mathcal{K}(\theta) \subset \mathcal{D}(A)$, by Lemma 3.2. There is reason to expect that the PNC property may be somewhat common in our setting. The problem is to turn these heuristic insights into a rigorous theory. This issue remains under investigation.

In some instances in our theory, we do make the added assumption that the PNC property holds for a semiflow π and a compact, invariant set $\mathcal{K}$. When we need this assumption, this will be stated explicitly. Any lemmas or theorems without such an explicit statement are valid when the PNC property fails. (It would be desirable to drop the assumption of the PNC property in part or entirely.)

5.3. Preliminary properties

In this subsection, we examine some dynamical properties of compact, invariant sets $\mathcal{K}$ in $T^k \times V^{2\beta}$ that satisfy some, but not all, of the properties for a foliated bundle. In particular, we make the following Standing Hypotheses:

> **Standing Hypotheses:** Let equation (5.3) be given, where A satisfies Hypotheses A and $\widehat{F}$ satisfies Hypotheses B. Let $\mathcal{K}$ be a compact, invariant set in $T^k \times V^{2\beta}$ for (5.3) and assume that $\mathcal{K}$ satisfies the four properties FB1–FB4.

Lemma 5.6. *Let the Standing Hypotheses be satisfied, and assume that the semiflow π has a proper negative continuation on $\mathcal{K}$. Let $\rho_0 > 0$ and $b_1^F(\rho)$ be given by Lemma 5.5. Then the following properties hold:*

1. *For any $C_4 > 0$, there is a ρ with $0 < \rho \le \rho_0$, such that if $(\theta, v_0),\ (\theta, v_1) \in \mathcal{K}$ satisfy $\|A^\beta(v_1 - v_0)\| \le \rho$ and*

$$\|A^\beta Q^o(v_0)(v_1 - v_0)\| \ge \eta\, \|A^\beta(v_1 - v_0)\| > 0, \tag{5.19}$$

then there is a $\tau \in \mathbb{R}$ such that

$$\|A^\beta Q^o(S(\theta,\tau)v_0))w(\tau)\| \ge C_4 \|A^\beta P^o(S(\theta,\tau)v_0))w(\tau)\|, \tag{5.20}$$

where $w(t) = S(\theta,t)v_1 - S(\theta,t)v_0$.

2. *If, instead of (5.19), one has*

$$\|A^\beta P^o(v_0)(v_1 - v_0)\| \le \frac{1}{2}\|A^\beta(v_1 - v_0)\|, \tag{5.21}$$

then for the same choice of $\rho > 0$ and $\tau \in \mathbb{R}$, one has

$$\|A^\beta(S(\theta,\tau)v_1 - S(\theta,\tau)v_0)\| \ge 2\|A^\beta(v_1 - v_0)\|, \tag{5.22}$$

provided that $\|A^\beta(v_1 - v_0)\| \le \rho$.

Proof. Item 1: Without loss of generality we assume that $\eta \le \frac{1}{2}$. Since (5.20) is true when $v_1 = v_0$, we let (θ, v_0), $(\theta, v_1) \in \mathcal{K}$ be given so that (5.19) holds. Consequently, one must have either

$$\|A^\beta P^u(v_0)(v_1 - v_0)\| \ge \frac{1}{2}\|A^\beta Q^o(v_0)(v_1 - v_0)\| \tag{5.23}$$

or

$$\|A^\beta P^s(v_0)(v_1 - v_0)\| \ge \frac{1}{2}\|A^\beta Q^o(v_0)(v_1 - v_0)\|, \tag{5.24}$$

where $P^u(v_0) = P^u(\theta, v_0)$ and $P^s(v_0) = P^s(\theta, v_0)$. When $(\theta, v) \in \mathcal{K}$, we let $S(\theta, t)v$ denote a global solution of (5.3) with $S(\theta, 0)v = v$ and $(\theta \cdot t, S(\theta, t)v) \in \mathcal{K}$, for all $t \in \mathbb{R}$.

To prove inequality (5.20), we first treat the case where inequality (5.23) holds. We define $H_2(t)$ by

$$w(t) = S(\theta, t)v_1 - S(\theta, t)v_0 = \Phi(B, t)(v_1 - v_0) + H_2(t), \qquad \text{for } t \ge 0, \tag{5.25}$$

where $B = Z(\theta, v_0) \in \mathcal{M}^\infty(\mathbb{R}, \mathcal{L})$, see (3.22). It follows from Lemma 5.5 that there is a $b_1^F \in \Sigma$ such that H_2 satisfies (5.15) whenever $\|A^\beta(v_1 - v_0)\| \le \rho$, where $v_1 = u_1$ and $v_0 = u_0$. We now fix $\tau > 0$ and, after that, fix $\rho > 0$ so that

$$\frac{\min(\eta, \frac{1}{2})}{2K} e^{(\lambda_4 - \lambda_3)\tau} \ge \max(C_4 + 1, 2 + 2K)e^{-\lambda_3\tau} + K\max(C_4 + 1, 2) \quad \text{and} \quad b_1^F(\rho) \le 1. \tag{5.26}$$

For an estimate of $\|A^\beta \Phi(B, t)(v_1 - v_0)\|$, for $0 \le t \le 2T$, we use the identity

$$v_1 - v_0 = P^s(v_0)(v_1 - v_0) + P^o(v_0)(v_1 - v_0) + P^u(v_0)(v_1 - v_0).$$

Using inequalities (5.19) and (5.23) one obtains

$$\|A^\beta P^u(v_0)(v_1 - v_0)\| \ge \frac{\eta}{2}\|A^\beta(v_1 - v_0)\|,$$

and, combining this with inequality (4.18), we find that, for all $t \ge 0$, one has

$$\begin{aligned} \|A^\beta \Phi(B, t)P^u(v_0)(v_1 - v_0)\| &\ge K^{-1}e^{\lambda_4 t}\|A^\beta P^u(v_0)(v_1 - v_0)\|, \\ &\ge \frac{\eta}{2K}e^{\lambda_4 t}\|A^\beta(v_1 - v_0)\|. \end{aligned} \tag{5.27}$$

In addition, inequalities (4.10) and (4.12) imply that, for $t \ge 0$, one has

$$\|A^\beta \Phi(B, t)P^s(v_0)(v_1 - v_0)\| \le Ke^{\lambda_1 t}\|A^\beta(v_1 - v_0)\|,$$
$$\|A^\beta \Phi(B, t)P^o(v_0)(v_1 - v_0)\| \le Ke^{\lambda_3 t}\|A^\beta(v_1 - v_0)\|.$$

Next we use Lemma 5.5 to estimate both sides of inequality (5.20). Now for $0 \le t \le \tau$, and since $\lambda_1 \le \lambda_3$, (5.25) implies

$$
\begin{aligned}
&\|A^\beta Q^o(S(\theta,t)v_0)w(t)\|\\
&\quad \geq \|A^\beta Q^o(S(\theta,t)v_0)\Phi(B,t)(v_1-v_0)\| - \|A^\beta Q^o(S(\theta,t)v_0)H_2(t)\|\\
&\qquad \text{by the invariance of } Q^o, \text{ see (4.32)},\\
&\quad \geq \|A^\beta \Phi(B,t)P^u(v_0)(v_1-v_0)\| - \|A^\beta \Phi(B,t)P^s(v_0)(v_1-v_0)\|\\
&\qquad - \|A^\beta Q^o(S(\theta,t)v_0)H_2(t)\|\\
&\qquad \text{from the inequalities above}\\
&\quad \geq \left(\frac{\eta}{2K}e^{\lambda_4 t} - Ke^{\lambda_3 t} - Kb_1^F(\rho)\right)\|A^\beta(v_1-v_0)\|.
\end{aligned}
\tag{5.28}
$$

Similarly, one obtains

$$
\begin{aligned}
&\|A^\beta P^o(S(\theta,t)v_0)w(t)\|\\
&\quad \leq \|A^\beta P^o(S(\theta,t)v_0)\Phi(B,t)(v_1-v_0)\| + \|A^\beta P^o(S(\theta,t)v_0)H_2(t)\|\\
&\quad \leq \|A^\beta \Phi(B,t)P^o(v_0)(v_1-v_0)\| + \|A^\beta P^o(S(\theta,t)v_0)H_2(t)\|\\
&\quad \leq \left(Ke^{\lambda_3 t} + Kb_1^F(\rho)\right)\|A^\beta(v_1-v_0)\|.
\end{aligned}
\tag{5.29}
$$

This then implies that

$$
\frac{\|A^\beta Q^o(S(\theta,\tau)v_0)w(\tau)\|}{\|A^\beta P^o(S(\theta,\tau)v_0)w(\tau)\|} \geq \frac{\eta\,(2K)^{-1}e^{\lambda_4\tau} - Ke^{\lambda_3\tau} - Kb_1^F(\rho)}{Ke^{\lambda_3\tau} + Kb_1^F(\rho)} \geq C_4,
$$

where the first inequality follows from the estimates immediately above, and the second from (5.26). This then implies (5.20), when (5.23) is valid.

Next we assume that inequality (5.24) is valid. While this case appears to be similar to the case where inequality (5.23) holds, there is a major difference in connection with the negative continuations of solutions of the linear problem, $\Phi(B,t)w$, for $t \leq 0$ and $w \in \mathcal{U}^s(v_0)$. When w is in $\mathcal{U}^u(v_0)$ or in $\mathcal{U}^o(v_0)$, then $\Phi(B,t)w$ admits an extension to a global solution, see Section 3.4. As we have seen, inequalities (4.18)–(4.20) contain useful information on the growth of $\Phi(B,t)w$. As noted above, when $w \in \mathcal{U}^s(v_0)$, then in general one cannot extend the linear solution for $t \leq 0$. Even the global solutions of the nonlinear dynamics in $\mathcal{K}$ are not helpful here. We must use a different approach, an approach based on the partial extension given in (4.21) and (4.22).

Assume now that inequality (5.24) holds. Next we fix $\tau < 0$ and, after that, $\rho > 0$ so that

$$
\begin{aligned}
&\frac{\min(\eta,\frac{1}{2})}{4K}e^{(\lambda_1-\lambda_2)\tau} \geq 2e^{-\lambda_2\tau} + 2K\max(C_4+1,2)\\
&\qquad\text{and}\quad b_1^F(\rho) \leq \min\left(\frac{\min(\eta,\frac{1}{2})}{4K},1\right).
\end{aligned}
\tag{5.30}
$$

Since the semiflow π has a proper negative continuation on $\mathcal{K}$, we may use inequality (5.18). We now use the global solutions $S(\theta,t)v_1$ and $S(\theta,t)v_0$ to define $\hat{v}_1 = S(\theta,\tau)v_1$ and $\hat{v}_0 = S(\theta,\tau)v_0$. Also, set $\hat{z}_1 = \hat{v}_1 - \hat{v}_0$ and $z_1 = \Phi(B_\tau,-\tau)\hat{z}_1$. Now we invoke inequality (4.22), and the terms (v,w) used there now become $(v, w) = (P^s(\widehat{v_0})\widehat{z_1}, z_1)$ in the notation used here. (Notice that $\Phi(B_\tau, -\tau)P^s(\widehat{v_0})\widehat{z_1} = P^s(v_0)z_1$). Thus (4.22), with $t = \tau$, now reads

(5.31) $$\|A^\beta \Phi(B, \tau)P^s(v_0)z_1\| \geq K^{-1}e^{\lambda_1 \tau}\|A^\beta P^s(v_0)z_1\|.$$

Next we redefine $H_2(t)$ by using

(5.32)
$$S(\theta \cdot \tau, -\tau)\hat{v}_1 - S(\theta \cdot \tau, -\tau)\hat{v}_0) = \Phi(B_\tau, t)(\hat{v}_1 - \hat{v}_0) + H_2(t), \qquad \text{for } |t| \leq |\tau|.$$

It then follows from (5.18) that

(5.33) $$\|A^\beta H_2(t)\| \leq b_1^F(\rho)\|A^\beta(v_1 - v_0)\|, \qquad \text{for } |t| \leq |\tau|,$$

whenever $\|A^\beta(v_1 - v_0)\| \leq \rho \leq \rho_0$. By setting $t = -\tau$ in (5.32), we obtain

(5.34) $$v_1 - v_0 = \Phi(B_\tau, -\tau)\hat{z}_1 + H_2(-\tau) = z_1 + H_2(-\tau).$$

From the definition of z_1, and (5.33) and (5.34), we obtain

(5.35) $$\|A^\beta z_1\| \leq \big(1 + b_1^F(\rho)\big)\|A^\beta(v_1 - v_0)\|.$$

Inequalities (5.24), (5.32), and (5.33) then yield

$$\begin{aligned}\|A^\beta P^s(v_0)z_1\| &\geq \|A^\beta P^s(v_0)(v_1 - v_0)\| - \|A^\beta P^s(v_0)\widehat{H}_2(-\tau)\| \\ &\geq \left(\frac{\eta}{2} - Kb_1^F(\rho)\right)\|A^\beta(v_1 - v_0)\|.\end{aligned}$$

Since (5.30) implies that $Kb_1^F(\rho) \leq \frac{\eta}{4}$, we get

(5.36) $$\|A^\beta P^s(v_0)z_1\| \geq \frac{\eta}{4}\|A^\beta(v_1 - v_0)\|.$$

Consequently, (5.31) and (5.36) yield

(5.37) $$\begin{aligned}\|A^\beta \Phi(B, \tau)P^s(v_0)z_1\| &\geq K^{-1}e^{\lambda_1 \tau}\|A^\beta P^s(v_0)z_1\| \\ &\geq \frac{\eta}{4K}e^{\lambda_1 \tau}\|A^\beta(v_1 - v_0)\|.\end{aligned}$$

Next we observe that inequalities (4.11), (4.13), and (5.35) imply that

(5.38) $$\begin{aligned}\|A^\beta \Phi(B, \tau)P^u(v_0)z_1\| &\leq Ke^{\lambda_2 \tau}\|A^\beta z_1\| \\ &\leq K\big(1 + b_1^F(\rho)\big)e^{\lambda_4 \tau}\|A^\beta(v_1 - v_0)\|,\end{aligned}$$

since $\lambda_2 < \lambda_4$ and $\tau < 0$, and

(5.39) $$\begin{aligned}\|A^\beta \Phi(B, \tau)P^o(v_0)z_1\| &\leq Ke^{\lambda_2 \tau}\|A^\beta z_1\| \\ &\leq K\big(1 + b_1^F(\rho)\big)e^{\lambda_2 \tau}\|A^\beta(v_1 - v_0)\|.\end{aligned}$$

From inequalities (5.37) and (5.38) we get

$$
\begin{aligned}
\|A^\beta \Phi(B,\tau)Q^o(v_0)z_1\| &\geq \|A^\beta \Phi(B,\tau)P^s(v_0)z_1\| - \|A^\beta \Phi(B,\tau)P^u(v_0)z_1\| \\
&\geq \left(\frac{\eta}{4K}e^{\lambda_1\tau} - K\big(1+b_1^F(\rho)\big)e^{\lambda_2\tau}\right)\|A^\beta(v_1-v_0)\|.
\end{aligned}
\tag{5.40}
$$

We then obtain

$$
\frac{\|A^\beta Q^o(S(\theta,\tau)v_0)w(\tau)\|}{\|A^\beta P^o(S(\theta,\tau)v_0)w(\tau)\|} \geq \frac{\eta\,(4K)^{-1}e^{\lambda_1\tau} - K\big(1+b_1^F(\rho)\big)e^{\lambda_2\tau}}{K(1+b_1^F(\rho))e^{\lambda_2\tau}} \geq C_4,
$$

where the first inequality follows from the estimates immediately above, and the second from (5.30). This then implies (5.20) when (5.23) is valid.

Item 2: Since $v_1 - v_0 = Q^o(v_0)(v_1-v_0) + P^o(v_0)(v_1-v_0)$, it follows from (5.21) that

$$
\|A^\beta Q^o(v_0)(v_1-v_0)\| \geq \|A^\beta(v_1-v_0)\| - \|A^\beta P^o(v_0)(v_1-v_0)\| \geq \frac{1}{2}\|A^\beta(v_1-v_0)\|.
$$

Hence, (5.19) holds with $\eta = \frac{1}{2}$. Let $\rho > 0$ and $\tau \in \mathbb{R}$ be given by Item 1, where $\eta = \frac{1}{2}$. We now return to the proof of Item 1 by treating the two cases $\tau > 0$ and $\tau < 0$ separately. We set $w(\tau) = S(\theta,\tau)v_1 - S(\theta,\tau)v_0$.

For $\tau > 0$ we use (5.28) and (5.29) to obtain

$$
\begin{aligned}
\|A^\beta w(\tau)\| &\geq \|A^\beta Q^o(S(\theta,\tau)v_0)w(\tau)\| - \|A^\beta P^o(S(\theta,\tau)v_1)w(\tau)\| \\
&\geq \left(\frac{\eta}{2K}e^{\lambda_4\tau} - 2Ke^{\lambda_3\tau} - 2Kb_1^F(\rho)\right)\|A^\beta(v_1-v_0)\|.
\end{aligned}
$$

Since $\eta = \frac{1}{2}$, it follows from (5.26) that

$$
\left(\frac{\eta}{2K}e^{\lambda_4\tau} - 2Ke^{\lambda_3\tau} - 2Kb_1^F(\rho)\right) \geq 2,
$$

which in turn implies (5.22).

For $\tau < 0$ we use instead (5.39) and (5.40) to obtain

$$
\|A^\beta w(\tau)\| \geq \left(\frac{\eta}{4K}e^{\lambda_1\tau} - 2K\big(1+b_1^F(\rho)\big)e^{\lambda_2\tau}\right)\|A^\beta(v_1-v_0)\|.
$$

Since $\eta = \frac{1}{2}$, it follows from (5.30) that

$$
\left(\frac{\eta}{4K}e^{\lambda_1\tau} - 2K\big(1+b_1^F(\rho)\big)e^{\lambda_2\tau}\right) \geq 2,
$$

which in turn implies (5.22). □

Theorem 5.7. *Let the Standing Hypotheses be satisfied. Assume further that the semiflow π has a proper negative continuation on $\mathcal{K}$. Then $\mathcal{K}$ is a C^1-smooth foliated bundle that satisfies the tangency property FB5, see* (4.33).

Proof. As noted above, it suffices to verify that $\mathcal{R}(P^o(u)) = T_u\mathcal{S}(u)$, where $P^o(u) = P^o(\theta, u)$, $\mathcal{S}(u) = \mathcal{S}(\theta, u)$, and $(\theta, u) \in \mathcal{K}$, see (4.33). In particular, we will show that every $(\theta, v_0) \in \mathcal{K}$, the space $\mathcal{U}^o(v_0) = \mathcal{U}^o(\theta, v_0)$ is tangent to the leaf $\mathcal{S}(\theta, v_0)$ at the point v_0. Observe that this tangency condition holds at v_0 if and only if, for every convergent sequence $v_i \to v_0$, where $v_i \in \mathcal{D}_{\rho_0}(v_0) = \mathcal{D}_{\rho_0}(\theta, v_0)$, the projector $Q^o(v_0) = Q^o(\theta, v_0)$ satisfies

$$\lim_{i\to\infty} \|A^\beta Q^o(v_0)(v_i - v_0)\| \, \|A^\beta(v_i - v_0)\|^{-1} = 0.$$

Assume on the contrary that the space $\mathcal{U}^o(v_0)$ is not tangent to $\mathcal{S}(\theta, v_0)$ at the point v_0. Then there exist an $\eta > 0$ and a sequence of points $v_i \in \mathcal{D}_{\rho_0}(v_0)$, for $i = 1, 2, \ldots$, such that $v_i \to v_0$, as $i \to \infty$, and

$$\|A^\beta Q^o(v_0)(v_i - v_0)\| \geq \eta\|A^\beta(v_i - v_0)\| > 0, \qquad \text{for all } i \geq 1. \tag{5.41}$$

Now set $C_4 = L_0+1$, where L_0 satisfies (4.3)–(4.26). Now choose $i_0 \geq 1$ so that $\|A^\beta(v_{i_0} - v_0)\| \leq \rho$, where ρ is given by Lemma 5.6. Next let $\tau \in \mathbb{R}$ be chosen such that (5.20) holds with $v_1 = v_{i_0}$. Now define $\widehat{v}_0 = S(\theta, \tau)v_0$ and $\widehat{v}_{i_0} = S(\theta, \tau)v_{i_0}$. Then $w_{i_0}(\tau) = \widehat{v}_{i_0} - \widehat{v}_0$ and $\|A^\beta(\widehat{v}_{i_0} - \widehat{v}_0)\| \leq \rho_0$. Thus $\widehat{v}_{i_0} \in \mathcal{D}_{\rho_0}(\widehat{v}_0))$. From (4.2) one has

$$w_{i_0}(\tau) = \widehat{v}_{i_0} - \widehat{v}_0 = \widehat{p} + \nu(\widehat{p}), \qquad \text{where } \widehat{p} \in D(\widehat{v}_0, \rho_0).$$

Since $\widehat{p} \in \mathcal{R}(P^o(\widehat{v}_0))$ and $\nu(\widehat{p}) \in \mathcal{R}(Q^o(\widehat{v}_0))$, one has $P^o(\widehat{v}_0)w_{i_0}(\tau) = \widehat{p}$ and $Q^o(\widehat{v}_0)w_{i_0}(\tau) = \nu(\widehat{p})$. Therefore, inequality (5.20) implies that

$$\|A^\beta \nu(\widehat{p})\| \geq (L_0 + 1)\|A^\beta \widehat{p}\|,$$

which contradicts (4.28). □

Remark. It should be noted that a weaker form of the PNC property is required for Theorem 5.7. Since $v_i \in \mathcal{D}_{\rho_0}(v_0)$, it suffices to assume only that (5.17) holds when $u_1 \in \mathcal{D}_\rho(u_0)$. The stronger version of the PNC, which is stated in the definition in Section 5.2, is needed in the proof of Theorem 5.4, see Section 6.5.

6. Proofs of main theorems

This section contains the proofs of most of the main theorems. For these proofs, we have two parameters to use. The first is the parameter ρ, which is used for the disks $\mathcal{D}_\rho(\theta, v)$ with $0 < \rho \leq \rho_0$. The other is the parameter σ, with $0 < \sigma \leq \sigma_0$, which is used to describe the neighborhoods $U(\sigma)$ arising in the definition of an exponential trichotomy. As these parameters get small, the nonlinear dynamics gets closer to the linear theory, and it is here that we can exploit the hyperbolicity in the exponential trichotomy.

6.1. Unperturbed dynamics

We now return to the function $\nu = \nu(p)$ used to describe the disk $\mathcal{D}_{\rho_0}(v)$. Because of the Lipschitz property for $\mathcal{K}$, see (4.25)–(4.28), the tangent space to the curve $r = \nu(p)$ is Lipschitz continuous, which implies that the function ν in (4.2) is of class $C^{1,1}$, see Lemma 4.2. Since the space $r = 0$ coincides with the neutral space $\mathcal{U}^o(v_0) = \mathcal{R}(P^o(v_0))$, see (4.15), it follows that $\nu(0) = 0$ and the derivative $D_p\nu = \frac{\partial \nu}{\partial p}$ satisfies $D_p\nu(0) = 0$. Furthermore, the derivative $D_p\nu$ satisfies the Lipschitz property

$$\|D_p\nu(p+\hat{p}) - D_p\nu(p)\|_{\mathcal{L}} \le \hat{L}\,\|A^\beta \hat{p}\|, \qquad \text{for } \hat{p} \in D(v_0, \rho).$$

The constant $\hat{L}$ depends on the Lipschitz coefficient for the mapping $v \to P^o(v)$, for $v \in \mathcal{K}$, and is independent of the base point v. By using a larger value for L_0, if necessary, one then has the validity of

(6.1)
$$\|A^\beta(\nu(p_1) - \nu(p_2))\| \le L_0\rho\|A^\beta(p_1 - p_2)\|, \qquad \text{for } \|A^\beta p_1\|,\ \|A^\beta p_2\| \le \rho$$

and

(6.2) $$\|A^\beta \nu(p)\| \le L_0\|A^\beta p\|^2 \le L_0\rho\|A^\beta p\| \le L_0\rho^2, \qquad \text{for } \|A^\beta p\| \le \rho,$$

as well as inequalities (4.25)–(4.26).

In the four Lemmas 6.1–6.3, we look at some of the consequences of choosing a specific number ρ_1 with $0 < \rho_1 \le \rho_0$ and satisfying

(6.3)
$$\begin{gathered} 4K^2L_0\rho_1 < 1, \quad KC_2L_0\rho_1 \le 1, \quad 48KC_2C_4\rho_1 e^{-\lambda_2 T} \le 1, \\ \rho_1 \le \frac{3}{10}\rho_0, \quad C_2\rho_1 \le \rho_0, \\ 10L_0\rho_1 \le e^{(\lambda_2 - a_0)t}, \quad \text{for } 0 \le t \le 2T, \text{ and} \\ b_1^F(\rho_1) \le \frac{1}{48K}e^{\lambda_2 t}, \quad \text{for } 0 \le t \le 2T. \end{gathered}$$

The quantities T, L_0, C_2, a_0, and b_1^F are given by (6.9), (6.1)–(6.2), (5.16), (5.12)–(5.13), and Lemma 5.5, respectively. Note that when (6.3) holds, then one has

(6.4) $$3L_0\rho_1 \le e^{(\lambda_3 - a_0)t}, \qquad \text{for } 0 \le t \le 2T,$$

since $\lambda_2 \le \lambda_3$. Also, $K \ge 1$ implies that

(6.5) $$\frac{1}{48K}e^{\lambda_2 t} < 2K^2 e^{\lambda_3 t}, \qquad \text{for } t \ge 0.$$

In addition to (6.3), we require that if $\|A^\beta(v_1 - v_0)\| \le \rho_1$, where $v_1 \in \mathcal{D}_{\rho_0}(\theta, v_0)$, then the global solutions $S_1(\theta, t)v_1$ and $S_1(\theta, t)v_0$ satisfy

(6.6) $$\|A^\beta(S_1(\theta, t)v_1 - S_1(\theta, t)v_0)\| \le \rho_0, \qquad \text{for } -2T \le t \le 2T.$$

In Lemmas 6.1 and 6.2, we treat the radius ρ in $\mathcal{D}_\rho(v_0)$ as a parameter. Since the proofs are straightforward, we omit the details.

Lemma 6.1. *Let the Standing Hypotheses be satisfied. Let* $P^i(v_0) = P^i(\theta, v_0)$ *and* $Q^o(v_0) = Q^o(\theta, v_0)$*, for* $(\theta, v_0) \in \mathcal{K}$ *and* $i = s, o, u$*. Let* ρ_1 *satisfy* (6.3)–(6.6)*. Then for all* $(\theta, v_0) \in \mathcal{K}$ *and all* u_1 *and* u_2 *in the disk* $\mathcal{D}_\rho(\theta, v_0)$*, with* $\|A^\beta(u_1 - u_2)\| \leq \rho$*, where* $0 < \rho \leq \rho_1$*, inequalities* (4.25)*,* (6.1)*, and* (6.2) *are valid, and one has*

$$\begin{aligned}\frac{3}{4}\|A^\beta P^o(v_0)(u_1 - v_0)\| &\leq (1 - L_0\rho)\|A^\beta P^o(v_0)(u_1 - v_0)\| \\ &\leq \|A^\beta(u_1 - v_0)\| \\ &\leq (1 + L_0\rho)\|A^\beta P^o(v_0)(u_1 - v_0)\| \\ &\leq \frac{5}{4}\|A^\beta P^o(v_0)(u_1 - v_0)\|.\end{aligned}$$

In addition, one obtains

(6.7)
$$\left\{\begin{matrix}\|A^\beta P^s(v_0)(u_1 - u_2)\| \\ \|A^\beta P^u(v_0)(u_1 - u_2)\| \\ \|A^\beta Q^o(v_0)(u_1 - u_2)\|\end{matrix}\right. \leq K^2 L_0\|A^\beta(u_1 - u_2)\|^2 \leq K^2 L_0 \rho\|A^\beta(u_1 - u_2)\|.$$

We will denote a typical point $v \in \mathcal{D}_\rho(\theta, v_0)$ in the form $v = v_0 + p + \nu(p)$, where $\|A^\beta p\| < \rho \leq \rho_1$. One then has

$$\begin{aligned}&\|A^\beta(v - v_0)\| \leq (1 + L_0\rho)\rho, \\ &\|A^\beta(v_1 - v_2)\| \leq (1 + L_0\rho)\|A^\beta(p_1 - p_2)\| \leq \frac{5}{4}\|A^\beta(p_1 - p_2)\|,\end{aligned} \tag{6.8}$$

where $v_i = v_0 + p_i + \nu(p_i) \in \mathcal{D}_\rho(v_0)$ and $\|A^\beta p_i\| \leq \rho$, for $i = 1, 2$. By combining Lemma 6.1 with the properties of the exponential trichotomy (4.10), (4.12), and (4.19) and inequalities (5.12), (4.25), and (6.7), we obtain the following result:

Lemma 6.2. *Let the Standing Hypotheses be satisfied. For* $(\theta, v_0) \in \mathcal{K}$*, let* $B = Z(\theta, v_0)$ *and* $\Phi(B, t)$ *satisfy* (3.22)*. Then for all* $(\theta, v_0) \in \mathcal{K}$*, all* $v_1 \in \mathcal{D}_\rho(v_0)$*, where* $0 < \rho \leq \rho_1$*, the following properties hold:*

$$\begin{aligned}\|A^\beta \Phi(B, t)P^o(v_0)(v_1 - v_0)\| &\leq K^2\|A^\beta(v_1 - v_0)\|e^{\lambda_3 t}, \\ \|A^\beta \Phi(B, t)P^s(v_0)(v_1 - v_0)\| &\leq K^3 L_0\rho\|A^\beta(v_1 - v_0)\|e^{\lambda_1 t}, \\ \|A^\beta \Phi(B, t)P^u(v_0)(v_1 - v_0)\| &\leq K^3 L_0\rho\|A^\beta(v_1 - v_0)\|e^{a_0 t}, \\ \|A^\beta \Phi(B, t)(v_1 - v_0)\| &\leq K^2\|A^\beta(v_1 - v_0)\|e^{\lambda_3 t} \\ &\quad + K^3 L_0\rho\|A^\beta(v_1 - v_0)\|e^{a_0 t}, \\ \|A^\beta \Phi(B, t)P^o(v_0)(v_1 - v_0)\| &\geq K^{-1}\|A^\beta P^o(v_0)(v_1 - v_0)\|e^{\lambda_2 t}, \\ \|A^\beta \Phi(B, t)(v_1 - v_0)\| &\geq \left(4(5K)^{-1}e^{\lambda_2 t} - K^2 L_0\rho e^{a_0 t}\right)\|A^\beta(v_1 - v_0)\|,\end{aligned}$$

for all $t \geq 0$*, where* ρ_1 *satisfies* (6.3)–(6.6).

While the inequalities in this lemma are valid for all $t \geq 0$, we will be using them when t is restricted to a finite interval $0 \leq t \leq 2T$, where $T > 0$ is fixed as follows: With the characteristics $K, \lambda_1, \lambda_2, \lambda_3$, and λ_4 of $\mathcal{K}$ given by the exponential trichotomy on $\mathcal{K}$, we seek real numbers $\tau > 0$ and $T > 0$ such that

$$(6.9) \qquad \begin{cases} 4K^2 e^{\lambda_1 \tau} & < 1, \\ 96K^2 e^{(\lambda_1 - \lambda_2)\tau} & < 1, \\ 16K^2 e^{-\lambda_4 \tau} & < 1, \\ 48K^3 e^{(\lambda_3 - \lambda_4)\tau} & < 1, \end{cases} \qquad \text{for } T \leq \tau \leq 2T.$$

Note that each of the exponents in inequalities (6.9) is negative. Consequently, there does exist a time $T > 0$ such that, for all $\tau \geq T$, these inequalities are satisfied. We fix one such T for the sequel. We will use the fact that (6.9) is valid for all τ, with $T \leq \tau \leq 2T$. Note that one has

$$(6.10) \qquad \begin{cases} 16K^3 C_2 L_0 \rho_1 e^{-\lambda_4 T} \\ 48KC_4 \rho_1 e^{-\lambda_2 T} \end{cases} \leq 1.$$

The first inequality in (6.10) follows from (6.3) and (6.9), and the second follows from (6.3).

6.2. *Local coordinates*

For $0 < \rho \leq \rho_0$ and $0 \leq \sigma \leq \sigma_0$, we define $b_3^F = b_3^F(\rho, \sigma)$ by

$$(6.11) \qquad b_3^F(\rho, \sigma) \stackrel{\text{def}}{=} \sup \left\{ \| D\widehat{F}(\theta, y_1) - D\widehat{F}(\theta, y_2) \|_{\mathcal{L}} : \| A^\beta (y_1 - y_2) \| \leq \rho \right\},$$

where the supremum is taken over all y_1, y_2 in the fiber $U(\sigma)(\theta)$ and all $\theta \in T^k$. Also, one has $\mathcal{L} = \mathcal{L}(V^{2\beta}, W)$. Since $D\widehat{F} = D_u \widehat{F}$ is bounded on $U(\sigma_0)$, it follows that $b_3^F(\rho, \sigma)$ is finite-valued. Since $D\widehat{F}$ is continuous and $\mathcal{K}$ is compact, one has $b_3^F(\rho, \sigma) \to 0$, as $(\rho, \sigma) \to 0$. That is to say, $b_3^F \in \Sigma$. In addition to (6.3)–(6.6), we require that ρ_1 and σ_1 satisfy

$$(6.12) \qquad C_3\, b_3^F(\rho_1, \sigma_1) \leq \frac{1}{144K^2} e^{2\lambda_2 T},$$

where $C_3 = C_3(2T) > 0$ is defined in Lemma 6.4.

For any set S in $V^{2\beta}$ and any $\sigma > 0$, we define the σ-neighborhood $N(S, \sigma)$ by

$$N(S, \sigma) = \{ y \in V^{2\beta} : \text{ there is an } x \in S \text{ with } \| A^\beta (y - x) \| \leq \sigma \}.$$

We will use this notation when $S = \mathcal{D}_\rho(\theta, v)$ and $(\theta, v) \in \mathcal{K}$. The next step is to construct a local coordinate system near $\mathcal{K}$ by restricting this coordinate system to a suitable neighborhood of each disk $\mathcal{D}_\rho(\theta, v_0)$, where $(\theta, v_0) \in \mathcal{K}$. In addition to (6.3)–(6.6), we require that ρ_1 and σ_1 satisfy $0 < \sigma_1 \leq \sigma_0$, and for each $(\theta, v) \in \mathcal{K}$ one has

$$(6.13) \qquad \text{Convex Hull} \left(N \left(\mathcal{D}_{\rho_1}(\theta, v), \sigma_1 \right) \right) \subset N \left(\mathcal{D}_{\rho_0}(\theta, v), \sigma_0 \right).$$

In addition, we ask that ρ_1 and σ_1 satisfy (6.20) and (6.29).

We will require that σ_1 and (especially) ρ_1 satisfy a few auxiliary properties. In particular, by using the Lipschitz property, one can show that if the radius ρ of the disks $\mathcal{D}_\rho(v_0)$ and the number σ_1 are replaced by smaller values, if necessary, then one can construct a new (local) coordinate system in the vicinity of each disk $\mathcal{D}_\rho(v_0)$. In addition to ρ_1 and σ_1 satisfying (6.3)–(6.6) and (6.13), we require that these parameters satisfy (6.16) and the conclusions of the three Lemmas 6.1–6.3. This can be accomplished by using smaller values of ρ_1 and σ_1 in the references cited above.

The relationship (6.13) is important because it enables us to get a good estimate for the effective Lipschitz coefficient for the nonlinear perturbation term $\widehat{G}$, as required in (5.5). In particular, let $w_i = w_i(t)$ denote two continuous functions with

$$w_i(t) \in N\left(\mathcal{D}_{\rho_1}(\theta \cdot t, S_1(\theta, t)v), \sigma_1\right), \qquad \text{for } 0 \le t < t_0,$$

where $(\theta, v) \in \mathcal{K}$ and $0 < t_0 \le \infty$. Assume that $G \in C^1_{\mathrm{Lip}}$ satisfies inequality (5.5). Since $\lambda w_1(s) + (1-\lambda)w_2(s)$ is in the convex hull, see (6.13), for $0 \le s < t_0$ and $0 \le \lambda \le 1$, one obtains

$$G(\theta, w_1) - G(\theta, w_2) = \int_0^1 DG(\theta, w_2 + \phi(w_1 - w_2))\, d\phi\, (w_1 - w_2),$$

for $0 \le s < t_0$, which implies that

$$\begin{aligned} &e^{-A(t-s)}[G(\theta, w_1) - G(\theta, w_2)] \\ &\qquad = \int_0^1 e^{-A(t-s)} DG(\theta \cdot s, w_2 + \phi\,(w_1 - w_2))\, d\phi\, (w_1 - w_2), \end{aligned} \tag{6.14}$$

for $0 \le s < t_0$. We claim that

$$\begin{aligned} &\int_0^t \|A^\beta e^{-A(t-s)}[G(\theta \cdot s, w_1(s)) - G(\theta \cdot s, w_2(s))]\|\, ds \\ &\qquad \le \delta \sup_{0 \le s \le t} \|A^\beta(w_1(s) - w_2(s))\|. \end{aligned} \tag{6.15}$$

Indeed from (6.14) one has

$$\begin{aligned} &\int_0^t \|A^\beta e^{-A(t-s)}[G(\theta \cdot s, w_1(s)) - G(\theta \cdot s, w_2(s))]\|\, ds \\ &\qquad \le \int_0^t \int_0^1 \|A^\beta e^{-A(t-s)} DG\|_{\mathcal{L}}\, d\phi\, \|A^\beta(w_1 - w_2)\|\, ds \\ &\qquad \le \|G\|_{\{C^1; A; U, \mathbb{R}\}} \sup_{0 \le s \le t} \|A^\beta(w_1(s) - w_2(s))\|; \end{aligned}$$

thus inequality (6.15) now follows from (5.5).

Lemma 6.3. *Let the hypotheses of Theorem 5.1 be satisfied, and let ρ_1 and σ_1 satisfy* (6.3)–(6.6), (6.13), *and* (6.20). *Let $y \in N(\mathcal{D}_{\rho_1}(\theta, v_0), \sigma_1)$, where $(\theta, v_0) \in \mathcal{K}$. Then the following properties are valid:*

1. *There is a unique point* $v \in \mathcal{D}_{\rho_0}(\theta, v_0)$ *such that* $y - v \in \mathcal{U}^s(v) \oplus \mathcal{U}^u(v)$. *Furthermore, the mapping* $\psi : y \to v \overset{\text{def}}{=} \psi(y) = \psi(v_0, y)$ *is of class* $C^{1,1}$ *on* $N(\mathcal{D}_{\rho_1}(\theta, v_0), \sigma_1)$, *with* $\psi(v_0, v_1) = \psi(v_1) = v_1$, *for all* $v_1 \in \mathcal{D}_{\rho_1}(\theta, v_0)$.
2. *If, in addition, one has* $\|A^\beta(y - v_0)\| < 2\sigma_1$, *then* $v = \psi(y) \in \mathcal{D}_{\rho_1}(v_0)$.
3. *Moreover, the Fréchet derivative* $D\psi(y)$ *of* $\psi(y)$ *with respect to* y, *where* $y \in N(\mathcal{D}_{\rho_1}(\theta, v_0), \sigma_1)$, *satisfies*

   (6.16) $$D\psi(y) = P^o(v) = P^o(\psi(y)).$$
4. *The mapping* ψ *satisfies* $\psi(y) = \psi(v_0, y) = y - \phi(v_0, y) = y - \phi(y)$, *for* $y \in N(\mathcal{D}_{\rho_1}(\theta, v_0), \sigma_1)$, *where* $D\phi(y) = Q^o(v) = Q^o(\psi(y))$. *The mapping* ϕ *has the property that for all* $v \in \mathcal{K}$, *one has*

   (6.17) $$\phi(v + n) = n, \qquad \text{for } n \in \mathcal{R}(Q^o(v)).$$
5. *Let* $y_i \in N(\mathcal{D}_{\rho_1}(\theta, v_0), \sigma_1)$, *for some* $v_0 \in \mathcal{K}$, *and set* $v_i = \psi(y_i)$, *for* $i = 1, 2$. *Then one has*

   (6.18) $$v_1 - v_2 = \psi(y_1) - \psi(y_2) = P^o(v_2)(y_1 - y_2) + e_3,$$

   and there is a $b_2^F \in \Sigma$ *such that* $e_3 = e_3(y_2, y_1 - y_2)$ *satisfies*

   (6.19) $$\|A^\beta e_3\| \le b_2^F(\rho)\|A^\beta(y_1 - y_2)\|$$

   whenever $\|A^\beta(y_1 - y_2)\| \le \rho \le \rho_1$.

In the sequel, we will require that

(6.20) $$C_2\, b_2^F(\rho_1) \le \frac{1}{144K^2} e^{2\lambda_2 T} \quad (\le K),$$

in which case one has $\|v_1 - v_2\| \le 2K\|y_1 - y_2\|$. The constant C_2 is defined in Lemma 6.4, and it satisfies $C_2 \ge 1$.

The proof of this lemma is essentially the same as the argument used in [24, pp. 444–446] and [33, pp. 501–502]. In each case, one is concerned with the local geometry near a disk $\mathcal{D}_{\rho_1}(v_0)$. We omit the details here.

Notation: We will denote the new (nonlinear) coordinates of the point y by

(6.21) $$y = v + s + u = v + n,$$

where $\|y - v\| < \sigma_1$, $v \in \mathcal{K}$, $s \in \mathcal{U}^s(v)$, $u \in \mathcal{U}^u(v)$, and $n = s + u$. By (6.17) one then has $\phi(y) = n = s + u$. In the sequel we fix ρ_1 and σ_1 so that (6.3)–(6.6) (6.13), (6.20), and the conclusions of Lemmas 6.1–6.3 hold. This new coordinate system for the point y depends on the base point $(\theta, v_0) \in \mathcal{K}$. If two disks $\mathcal{D}_{\rho_1}(\theta, v_0)$ and $\mathcal{D}_{\rho_1}(\theta, v_1)$ on the same leaf have a nontrivial intersection, say, that $y \in N(\mathcal{D}_{\rho_1}(\theta, v_0), \sigma_1) \cap N(\mathcal{D}_{\rho_1}(\theta, v_1), \sigma_1)$, then $\psi(v_0, y) = \psi(v_1, y)$, i.e., the coordinate representations agree. More generally, let $y = y(t) = S_2(\theta, t)y_0$ be a solution of the perturbed equation in (5.4), where $y_0 = v_0 + n_0$, $v_0 \in \mathcal{K}$, and $n_0 \in \mathcal{R}(Q^o(v_0))$. Assume that one has $y(t) \in N(\mathcal{D}_{\rho_1}(\theta \cdot t, S_1(\theta, t)v_0), \sigma_1)$ for t in some interval I. Then the local coordinate representation

(6.22) $$y(t) = v(t) + n(t), \qquad \text{for all } t \in I,$$

where $v(t) \in \mathcal{D}_{\rho_0}(\theta \cdot t, S_1(\theta, t)v_0)$, $P^o(v(t))n(t) = 0$, and $Q^o(v(t))n(t) = n(t)$, is well defined, for all $t \in I$.

6.3. *Perturbed dynamics near $\mathcal{K}$*

Let $C(\mathcal{K}, V^{2\beta})$ denote the Banach space of continuous functions $f : \mathcal{K} \to V^{2\beta}$, where $f = f(v) = f(\theta, v)$, with the sup-norm

$$\|f\|_\infty = \sup\{\|A^\beta f(\theta, v)\| : (\theta, v) \in \mathcal{K}\}.$$

Next we define two function classes: $\mathcal{F} = \mathcal{F}(\epsilon, \ell)$ and $\mathcal{G} = \mathcal{G}(\epsilon, \ell)$ in $C(\mathcal{K}, V^{2\beta})$, where the parameters $\epsilon > 0$ and $\ell > 0$ will be chosen later. A vector-valued function f is said to belong to $\mathcal{F}(\epsilon, \ell)$ if $f \in C(\mathcal{K}, V^{2\beta})$ and, for each $(\theta, v) \in \mathcal{K}$, one has $f(v) \in \mathcal{U}^s(v)$ with $\|A^\beta f(v)\| \le \epsilon$, and the restriction of f to each disk $\mathcal{D}_{\rho_1}(\theta, v_0)$ in $\mathcal{K}$ is Lipschitz continuous with Lipschitz coefficient ℓ. Similarly, a vector-valued function $g = g(v) = g(\theta, v)$ is said to belong to $\mathcal{G}(\epsilon, \ell)$ if $g \in C(\mathcal{K}, V^{2\beta})$ and, for each $(\theta, v) \in \mathcal{K}$, one has $g(v) \in \mathcal{U}^u(v)$ with $\|A^\beta g(v)\| \le \epsilon$, and the restriction of $g = g(v)$ to each disk $\mathcal{D}_{\rho_1}(\theta, v)$ in $\mathcal{K}$ is Lipschitz continuous in v with Lipschitz coefficient ℓ. Since

$$\mathcal{U}^s = \{(\theta, v, n) : (\theta, v) \in \mathcal{K},\ n \in \mathcal{U}^s(v)\} \text{ and}$$
$$\mathcal{U}^u = \{(\theta, v, n) : (\theta, v) \in \mathcal{K},\ n \in \mathcal{U}^u(v)\}$$

are closed subsets of $\mathcal{K} \times V^{2\beta}$, see [29–31], it follows that, for every $\epsilon > 0$ and $\ell > 0$, the spaces $\mathcal{F}(\epsilon, \ell)$ and $\mathcal{G}(\epsilon, \ell)$ are closed sets in $C(\mathcal{K}, V^{2\beta})$. Consequently, the product space $\mathcal{F}(\epsilon, \ell) \times \mathcal{G}(\epsilon, \ell)$ is a complete metric space with the metric

$$\|(f_1, g_1) - (f_2, g_2)\|_\infty \overset{\text{def}}{=} \|f_1 - f_2\|_\infty + \|g_1 - g_2\|_\infty,$$

where $(f_i, g_i) \in \mathcal{F} \times \mathcal{G}$, for $i = 1, 2$.

In the argument given below, our objective will be to find $(f, g) \in \mathcal{F} \times \mathcal{G}$ so that the mapping h, which is defined by

(6.23)
$$h(\theta, u) = h(u) = u + f(u) + g(u) = u + f(\theta, u) + g(\theta, u), \qquad \text{for } (\theta, u) \in \mathcal{K},$$

satisfies the conclusions of the four Theorems 5.1–5.4. The pair (f, g) will be found as a fixed point of a suitable mapping A_τ.

Let $(\theta, v_0) \in \mathcal{K}$ and $y_0 \in N(v_0, \sigma_1)$ be given, and set

$$w(t) = S_2(\theta, t)y_0 - S_1(\theta, t)v_0, \qquad \text{with } w_0 = y_0 - v_0.$$

It follows that $w(t)$ satisfies (2.34). Assume that $\|A^\beta w_0\| \le \epsilon$ and G satisfies inequality (5.5), where $\delta > 0$. Then with $F_1 = F$ and $F_2 = F + G$, inequality (2.21) implies that, for $t_0 > 0$, there is a constant $K_1 = K_1(t_0) > 0$ such that

$$\|A^\beta w(t)\| \le K_1(t_0)(\epsilon + \delta), \qquad \text{for } 0 \le t \le t_0.$$

If ϵ and δ satisfy $K_2(\epsilon + \delta) \le \sigma_1 \le \sigma_0$, where $K_2 = K_1(2T)$, then one can choose t_0 such that $t_0 \ge 2T$ and

$$\|A^\beta w(t)\| = \|A^\beta(S_2(\theta, t)y_0 - S_1(\theta, t)v_0)\| \le K_2(\epsilon + \delta), \tag{6.24}$$

for $0 \le t \le 2T$. Next we define

$$e = e(t) = e(t, y_0) = w(t) - \Phi(B, t)w_0, \tag{6.25}$$

where $w(0) = w_0 = y_0 - v_0$ and $B = Z(\theta, v_0)$, see (3.22). Since $w(0) = \Phi(B, 0) = w_0$, it follows that $e(0) = 0$ and $e(t)$ is the mild solution of

$$\partial_t e(t) + Ae(t) = B(t)e(t) + E(S(t)v_0, w(t)) + G(S(t)v_0 + w(t)).$$

As a result of (2.34) and (2.35), the solution e satisfies

$$\begin{aligned} e(t) = &\int_0^t e^{-A(t-s)} B(s)e(s)\, ds + \int_0^t e^{-A(t-s)} E(S(s)v_0, w(s))\, ds \\ &+ \int_0^t e^{-A(t-s)} G(S(s)v_0 + w(s))\, ds. \end{aligned} \tag{6.26}$$

Now inequalities (5.5), (5.10), and (5.13) imply that there is a $\gamma \in \Sigma$, such that, for $0 \le t \le 2T$, one has

$$\begin{aligned} \|A^\beta e(t)\| \le M_\beta \|B\|_\infty &\int_0^t (t-s)^{-\beta} e^{-a(t-s)} \|A^\beta e(s)\|\, ds \\ &+ \int_0^t (t-s)^{-\beta} e^{-a(t-s)} \|A^\beta w(s)\| \gamma(\|A^\beta w(s)\|)\, ds + \delta. \end{aligned}$$

From inequality (6.24) one obtains a $\beta_1 \in \Sigma$, where

$$\|A^\beta e(t)\| \le (\epsilon + \delta)\beta_1(\epsilon, \delta) + \delta + M_\beta \|B\|_\infty \int_0^t (t-s)^{-\beta} e^{-a(t-s)} \|A^\beta e(s)\|\, ds,$$

for $0 \le t \le 2T$. It then follows from the Gronwall–Henry inequality, see Lemma 7.1, that there is a constant $C_1 > 0$ and a $b_0 \in \Sigma$ such that

$$\|A^\beta e(t)\| \le (\epsilon + \delta)b_0(\epsilon, \delta) + C_1\delta, \qquad \text{for } 0 \le t \le 2T. \tag{6.27}$$

In addition to the functions b_1^F, b_2^F, and b_3^F introduced above, we define b_4^F and b_5^F by

$$\begin{aligned} b_4^F &= b_4^F(\rho, \sigma) = K\big(C_3 b_3^F(\rho, \sigma) + C_2 b_2^F(\rho)\big), \\ b_5^F &= b_5^F(\rho, \sigma) = 3b_4^F(\rho, \sigma) + b_1^F(\rho), \end{aligned} \tag{6.28}$$

where the coefficients C_2 and C_3 are defined below in Lemma 6.4. We make additional requirements on ρ and σ so that b_4^F and b_5^F are "small." Detailed estimates can be found in [24, pp. 449–474] or [33, pp. 505–528]. For example, we will use the following below:

$$b_4^F \le b_5^F \quad \text{and} \quad 144K^2 b_4^F(\rho_1, \sigma_1) \le 1. \tag{6.29}$$

Special notation: The following notation will be used from time to time in the sequel: For $i = 1, 2$, we define $y_i = y_i(t) = S_2(\theta, t)y_{i0}$, $v_i = v_i(t) = v(t, y_{i0})$, $n_i = n_i(t) = n(t, y_{i0})$, $s_i = s_i(t) = s(t, y_{i0})$, $u_i = u_i(t) = u(t, y_{i0})$, and $S_{1,i} =$

$S_{1,i}(t) = S_1(\theta, t)v_{i0}$, where $\psi(y_{i0}) = v_{i0}$ with $v_{i0} \in \mathcal{D}_{\rho_1}(\theta, v_0)$, for some $(\theta, v_0) \in \mathcal{K}$, and $n_{i0} = y_{i0} - v_{i0} = s_{i0} + u_{i0}$. Also, we define $\triangle$ by:

$$\begin{aligned} \triangle y &= \triangle y(t) = y_1 - y_2, \quad \triangle v = \triangle v(t) = v_1 - v_2, \\ \triangle n &= \triangle n(t) = \triangle y - \triangle v, \quad \triangle w = \triangle w(t) = \triangle y - \triangle S_1, \\ \triangle S_1 &= \triangle S_1(t) = S_{1,1} - S_{1,2}, \quad \triangle z = \triangle z(t) = \triangle v - \triangle S_1, \\ \triangle s &= \triangle s(t) = s_1 - s_2, \quad \text{and} \quad \triangle u = \triangle u(t) = u_1 - u_2. \end{aligned} \tag{6.30}$$

We let the functions $E_{S_1} = E_{S_1}(t)$, $E_y = E_y(t)$, $E_v = E_v(t)$, and $E_s = E_s(t)$ be defined, for $0 \le t \le 2T$, by

$$\begin{aligned} \triangle S_1(t) &= \Phi(B, t)\triangle S_1(0) + E_{S_1}(t), \\ \triangle y(t) &= \Phi(B, t)\triangle y(0) + E_y(t), \\ \triangle v(t) &= \Phi(B, t)\triangle v(0) + E_v(t), \\ \triangle s(t) &= \Phi(B, t)P^s(v_{20})\triangle n(0) + E_s(t), \end{aligned} \tag{6.31}$$

where $B = Z(\theta, v_{20})$, see (3.22). This special notation is used in the following result and in the sequel. We also use the abbreviated notation for the skew product semiflows π and Π, see Section 4.2. This permits us to draw a curtain over the θ-variable. But never forget, while θ is out of view, its influence is omnipresent!

In the argument below, we will construct a series $\{\epsilon_i\}$ so that $0 < \epsilon_{i+1} \le \epsilon_i \le \epsilon_0$, for $i = 4, 5, \cdots$. We require that ϵ_0 be positive and satisfy

$$2\epsilon_0 \le \sigma_1, \quad \epsilon_0 \le \min\left(\rho_0, \frac{1}{3}\sigma_0\right), \quad 2L_0\epsilon_0 \le 1. \tag{6.32}$$

Lemma 6.4. *Let the hypotheses of Theorem* 5.1 *be satisfied. Then there exists an* $\epsilon_0 > 0$ *and a* $\delta_0 = \delta_0(\epsilon) > 0$*, and nonnegative constants* C_0*,* C_1*,* C_2*, and* C_3*, which depend on the characteristics of the exponential trichotomy on* $\mathcal{K}$ *and the time* T *such that* (6.32) *holds:*

$$C_2 \ge 1, \quad C_0(\epsilon_0 + \delta_0) \le \min(\rho_1, \sigma_1);$$

additionally, whenever $0 < \epsilon \le \epsilon_0$*,* $0 < \delta \le \delta_0$*, and* G *satisfies* (5.5) *with* $\delta = \delta_0(\epsilon)$*, the conclusions of Lemmas* 6.1–6.3 *hold, and the following are valid:*

1. *For any* $(\theta, v_0) \in \mathcal{K}$ *and* $y_0 = v_0 + n_0$*, where* $n_0 \in \mathcal{U}^s(v_0) + \mathcal{U}^u(v_0)$ *and* $\|A^\beta n_0\| \le 2\epsilon \le \sigma_1$*, one has*

$$\left.\begin{cases} \|A^\beta(S_2(\theta, t)y_0 - S_1(\theta, t)v_0)\| \\ \|A^\beta(v(t, y_0) - S_1(\theta, t)v_0)\| \\ \frac{1}{2}\|A^\beta n(t, y_0)\| \end{cases}\right. \le C_0(\epsilon + \delta), \qquad \text{for } 0 \le t \le 2T. \tag{6.33}$$

Furthermore, $w(t) = S_2(\theta, t)y_0 - S_1(\theta, t)v_0$ *satisfies equation* (2.35)*, with* $w(0) = n_0 = y_0 - v_0$*. Also,* $y(t) = S_2(\theta, t)y_0 = v(t) + n(t)$ *satisfies* (6.22)*, for* $0 \le t \le 2T$*, and inequality* (6.27) *holds.*

2. *Assume that* $(\theta, v_{i0}) \in \mathcal{K}$ *with* $\|A^\beta \triangle v(0)\| \le \rho_1$ *and* $\|A^\beta(y_{i0} - v_{i0})\| \le 2\epsilon \le \sigma_1$, *for* $i = 1, 2$. *Then one has*

$$y_i(t) = S_2(\theta, t) y_{i0} \in N(\mathcal{D}_{\rho_1}(\theta \cdot t, S_1(\theta, t) v_{10}), \sigma_1),$$

and

$$\left.\begin{cases} \|A^\beta \triangle y(t)\| \\ \|A^\beta \triangle v(t)\| \\ \|A^\beta \triangle n(t)\| \end{cases}\right. \le C_2 \|A^\beta \triangle y(0)\|, \qquad \text{for } 0 \le t \le 2T. \tag{6.34}$$

3. *Whenever* $\|A^\beta(y_{i0} - v_{i0})\| \le 2\epsilon \le \sigma_0$, *with* $(\theta, v_{i0}) \in \mathcal{K}$, *for* $i = 1, 2$, *and* $\|A^\beta \triangle y(0)\| \le C_2^{-1} \rho$, *with* $0 < \rho \le \rho_0$, *then*

$$I = \int_0^t \Phi(B_s, t - s) \int_0^1 [D\widehat{F}(y_2 + \phi(y_1 - y_2)) - D\widehat{F}(y_2)]\, d\phi \,\triangle y(s)\, ds,$$

when $D\widehat{F}(y_2) = D_u \widehat{F}(\theta \cdot s, y_2(s))$, *etc. satisfies*

$$\|A^\beta I\| \le C_3\, b_3^F(\rho, 2\epsilon) \|A^\beta \triangle y(0)\|, \qquad \text{for } 0 \le t \le 2T, \tag{6.35}$$

where $B = Z(\theta, v_{20})$, *see* (3.22), *and* b_3^F *is given by* (6.11).

4. *There exist* $\beta_1, \beta_2 \in \Sigma$ *such that, for* $0 \le t \le 2T$, *one has*

$$\begin{aligned} &\|A^\beta E_{S_1}(t)\| \le b_1^F(\rho) \|A^\beta \triangle v(0)\|, \\ &\|A^\beta E_y(t)\| \le \left(C_3 b_3^F(\rho, 2\epsilon) + \beta_1(\epsilon, \delta)\right) \|A^\beta \triangle y(0)\|, \\ &\|A^\beta E_v(t)\| \le \left(b_4^F(\rho, 2\epsilon) + \beta_2(\epsilon, \delta)\right) \|A^\beta \triangle y(0)\|. \end{aligned} \tag{6.36}$$

5. *Let* $y_i = y_{i0}$ *and* $v_i = v_{i0}$, $i = 1, 2$, *be given as in item 2, and* $\|A^\beta \triangle y(0)\| \le 3\|A^\beta \triangle v(0)\|$. *Then for* $0 < \epsilon \le \epsilon_0$ *and* $0 < \delta \le \delta_0$, *one has*

$$\frac{1}{2}\|A^\beta \triangle S_1(t)\| \le \|A^\beta \triangle v(t)\| \le \frac{3}{2}\|A^\beta \triangle S_1(t)\|, \qquad \text{for } 0 \le t \le 2T. \tag{6.37}$$

Define $H_5(t)$ *by* $\triangle v(t) = \triangle S_1(t) + H_5(t)$, *for* $t \in [0, 2T]$. *Then there is a* $\beta_3 \in \Sigma$ *such that, for* $0 \le t \le 2T$, *one has*

$$\|A^\beta H_5(t)\| = \|A^\beta \triangle z(t)\| \le \left(b_5^F(\rho, 2\epsilon) + \beta_3(\epsilon, \delta)\right) \|A^\beta \triangle v(0)\|. \tag{6.38}$$

6. *Under the conditions stated in item 5, for* $0 \le t \le 2T$, *one has*

$$\frac{1}{8K} e^{\lambda_2 t} \|A^\beta \triangle v(0)\| \le \|A^\beta \triangle v(t)\| \le 6K^2 e^{\lambda_3 t} \|A^\beta \triangle v(0)\|. \tag{6.39}$$

Since the proof of this lemma is essentially the same as the argument used in [24, pp. 453–457] and [33, pp. 508–512], we do not present the details here.

For any pair $(f, g) \in \mathcal{F} \times \mathcal{G} = \mathcal{F}(\epsilon, \ell) \times \mathcal{G}(\epsilon, \ell)$, we define $h : \mathcal{K} \to V^{2\beta}$ by (6.23). We assume that (6.32) holds. Since $\|A^\beta(y_0 - v_0)\| \le 2\epsilon \le \sigma_0$, one has $S_2(\theta, t) y_0 \in N(\mathcal{D}_{\rho_1}(\theta \cdot t, S_1(\theta, t) v_0), \sigma_1)$, for $0 \le t \le 2T$. The mapping

$v(t, y_0) = \psi(S_2(\theta, t)y_0) = \psi(v_0, S_2(\theta, t)y_0)$, which is defined by the local coordinate representation and which is valid for $0 \le t \le 2T$, admits a well-defined extension

$$v(t, y_0) = \psi^e(S_2(\theta, t)y_0) = \psi(S_1(\theta, t)v_0, S_2(\theta, t)y_0) \tag{6.40}$$

on a larger time interval, as long as $\|A^\beta(S_2(\theta, t)y_0 - S_1(\theta, t)v_0)\| \le \rho_1$.

Let $(f, g) \in \mathcal{F} \times \mathcal{G}$ be given, where $\ell \le 1$, and consider the collection of all solutions $S_2(\theta, t)y_0$ of the perturbed equation in (5.4) with $y_0 = h(\theta, v_0) = h(v_0)$, for $(\theta, v_0) \in \mathcal{K}$, see (6.23). For $0 < \epsilon \le \epsilon_0$ and $0 \le \delta \le \delta_0$, one has $\|A^\beta(S_2(\theta, t)y_0 - S_1(\theta, t)v_0)\| \le \sigma_1$, for $0 \le t \le 2T$. We claim that

$$\mathcal{K}_t \stackrel{\text{def}}{=} \{(\theta \cdot t, v(t, h(v_0))) : (\theta, v_0) \in \mathcal{K}\} = \mathcal{K}, \qquad \text{for each } t \in [0, 2T]. \tag{6.41}$$

Since $v(0, h(v_0)) = v_0$, it follows that $\mathcal{K}_0 = \mathcal{K}$. For $t > 0$ we note that the mapping $(v_0, t) \to v(t, h(v_0))$ is a continuous mapping of $\mathcal{K} \times [0, 2T]$ into $\mathcal{K}$. Since $\mathcal{K}$ is compact, it follows that $\mathcal{K}_t$ is compact. Furthermore, inequality (6.39) implies that the mapping $v_0 \to v(t, h(v_0))$ is an open mapping. Thus $\mathcal{K}_t$ is open. Since $\mathcal{K}$ is connected, this implies that $\mathcal{K}_t = \mathcal{K}$, for all $t \in [0, 2T]$. Due to the choice of ρ_1, see (6.13), we see that the mapping $(\theta, v_0) \to (\theta \cdot t, v(t, h(v_0))$ is a homeomorphism of $\mathcal{K}$ onto $\mathcal{K}_t$, for each $t \in [0, 2T]$.

6.4. The mapping A_τ

The basic idea in the proofs of the four main theorems 5.1–5.4 is to construct a family of mappings $A_\tau : (f, g) \to (\bar{f}_\tau, \bar{g}_\tau)$, which are defined on $\mathcal{F} \times \mathcal{G}$, for $T \le \tau \le 2T$. We will show that for ϵ and δ small, and for $T \le \tau \le 2T$, the following hold:

1. One has $\|A^\beta \bar{f}_\tau(v)\| \le \frac{3}{4}\epsilon$. (Lemma 6.5.)
2. The function $\bar{f}_\tau = \bar{f}_\tau(v)$ is Lipschitz continuous in v, for v in the disk $\mathcal{D}_\rho(\theta, v_0)$, for any $(\theta, v_0) \in \mathcal{K}$, and $\bar{f}_\tau \in \mathcal{F}$. (Lemma 6.6.)
3. The mapping $(f, g) \to \bar{f}_\tau$ is contracting on $\mathcal{F}$. (Lemma 6.7.)
4. The function $\bar{g}_\tau$ satisfies the same three properties. (Lemmas 6.8–6.10).
5. Let (f, g) denote the fixed point of the mapping $A_\tau : (f, g) \to (\bar{f}_\tau, \bar{g}_\tau)$, and set $h(v) = v + f(v) + g(v)$, for $(\theta, v) \in \mathcal{K}$. Then $\mathcal{K}^n = h(\mathcal{K})$ is an invariant set for the perturbed equation in (1.1), and the other properties of Theorems 5.1–5.4 are valid, see Theorem 6.11 and Section 6.5.

For each pair $(f, g) \in \mathcal{F}(\epsilon, \ell) \times \mathcal{G}(\epsilon, \ell)$, where $\ell \le 1$, we define a new function $\bar{f}_\tau = \bar{f}_\tau(v(\tau, y_0)) = \bar{f}_\tau(\theta \cdot \tau, v(\tau, y_0))$ by

$$\bar{f}_\tau(v(\tau, y_0)) = P^s(v(\tau, y_0))(S_2(\theta, t)y_0 - v(\tau, y_0)), \qquad \text{for } T \le \tau \le 2T, \tag{6.42}$$

where $y_0 = v_0 + f(v_0) + g(v_0)$. Note that for each $(\theta, v_0) \in \mathcal{K}$, it follows from (6.21) that

$$\bar{f}_\tau(v(\tau, y_0)) = P^s(v(\tau, y_0))n(\tau, y_0) = s(\tau, y_0) \in \mathcal{U}^s(v(\tau, y_0)),$$

for $T \le \tau \le 2T$, since $\mathcal{U}^s(u)$ is the range of the projection $P^s(u)$. Equation (6.42) gives the value of $\bar{f}_\tau$ at the point $(\theta \cdot \tau, v(\tau, y_0)) \in \mathcal{K}$. Due to Lemma 6.3 and (6.41), we see that $\bar{f}_\tau$ is well defined everywhere on $\mathcal{K}$, and the mapping $(\theta, v_0, \tau) \to \bar{f}_\tau(\theta, v_0)$ is a continuous mapping of $\mathcal{K} \times [T, 2T]$ into $V^{2\beta}$.

Lemma 6.5. *Let the hypotheses of Theorem* 5.1 *be satisfied, and let* ϵ_0 *and* δ_0 *be given by Lemma* 6.4. *Then there is an* ϵ_4, *with* $0 < \epsilon_4 \le \epsilon_0$, *such that for all* ϵ *with* $0 < \epsilon \le \epsilon_4$, *there is a* $\delta_4 = \delta_4(\epsilon)$, *with* $0 < \delta_4 \le \delta_0$, *such that if* G *satisfies* (5.5) *with* $\delta = \delta_4$, *and if* $f \in \mathcal{F}(\epsilon, \ell)$ *and* $g \in \mathcal{G}(\epsilon, \ell)$, *where* $0 < \ell \le 1$, *then one has*

$$\|A^\beta \bar{f}_\tau(v)\| \le \frac{3}{4}\epsilon, \qquad \text{for all } (\theta, v) \in \mathcal{K} \text{ and } T \le \tau \le 2T. \tag{6.43}$$

Since the proof of this lemma is essentially the same as the argument used in [24, pp. 459–460] and [33, pp. 514–515], we do not present the details here.

In the following result, we argue that $\bar{f}_\tau(v) = \bar{f}_\tau(\theta, v)$ is (locally) Lipschitz continuous in v, on each disk $\mathcal{D}_{\rho_1}(\theta, v_0)$ in $\mathcal{K}$, provided that $T \le \tau \le 2T$.

Lemma 6.6. *Let the hypotheses of Theorem* 5.1 *be satisfied. Then there is an* ϵ_5 *with* $0 < \epsilon_5 \le \epsilon_4$ *such that for all* ϵ *with* $0 < \epsilon \le \epsilon_5$, *there exist* $\delta_5 = \delta_5(\epsilon)$ *with* $0 < \delta_5 \le \delta_4$, *where* ϵ_4 *and* δ_4 *are given in Lemma* 6.5, *and for all* ρ *with* $0 < \rho \le \rho_1$, *there is an* $\ell_5 = \ell_5(\rho, \epsilon)$ *such that if* G *satisfies* (5.5) *with* $\delta = \delta_5$ *and if* $f \in \mathcal{F} = \mathcal{F}(\epsilon, \ell)$ *and* $g \in \mathcal{G} = \mathcal{G}(\epsilon, \ell)$ *with* $\ell \le 1$ *and* $\epsilon \le \epsilon_5$, *then the restriction of* $\bar{f}_\tau$ *to the disk* $\mathcal{D}_\rho(\theta, v_0)$, *for* $0 < \rho \le \rho_1$, *is Lipschitz continuous with Lipschitz coefficient* ℓ_5, *for every* $v_0 \in \mathcal{K}$ *and for* $T \le \tau \le 2T$. *Moreover, one has* $\bar{f}_\tau \in \mathcal{F}$, *and* $\ell_5(\rho, \epsilon) \in \Sigma$.

Proof. The main idea here is to show that there is a suitable $\ell_5 = \ell_5(\rho, \epsilon)$ in Σ such that

$$\frac{\|A^\beta(\bar{f}_\tau(v_1(\tau)) - \bar{f}_\tau(v_2(\tau)))\|}{\|A^\beta(v_1(\tau) - v_2(\tau))\|} \le \ell_5, \qquad \text{for } T \le \tau \le 2T. \tag{6.44}$$

Since the proof of this lemma is essentially the same as the argument used in [24, pp. 461–463] and [33, pp. 515–517], we present only an outline here. In the course of doing this proof, one uses $\|A^\beta \triangle y(0)\| \le 3\|A^\beta \triangle v(0)\|$ and shows that there exist functions β_4, β_5, and β_6 in Σ such that $C_2\beta_4 + 3\beta_5 \le \beta_6$ and

$$\begin{aligned} \|A^\beta E_s\| &\le (C_4\rho + \beta_4(\epsilon, \delta))\|A^\beta \triangle v(0)\| \\ &\qquad \big(b_4^F(\rho, 2\epsilon) + \beta_5(\epsilon, \delta)\big)\|A^\beta \triangle y(0)\| \\ &\le \big(C_4\rho + 3b_4^F(\rho, 2\epsilon) + \beta_6(\epsilon, \delta)\big)\|A^\beta \triangle v(0)\|, \end{aligned} \tag{6.45}$$

for $0 \le t \le 2T$, with

$$\|A^\beta \Phi(B, t)P^s(v_{20})\triangle n(0)\| \le \begin{cases} Ke^{\lambda_1 t}\|A^\beta P^s(v_{20})\triangle n(0)\|, \\ Ke^{\lambda_1 t}\|A^\beta \triangle n(0)\|, \end{cases} \tag{6.46}$$

for $t \ge 0$, where $B = Z(\theta, v_{20})$, see (3.22). □

Next we consider the two inequalities:

$$8KC_2\,\ell_5(\rho_1,\sigma_1) \le 1, \quad 48K\beta_6(\epsilon_5,\epsilon_5) \le 1. \tag{6.47}$$

Since $\ell_5 \in \Sigma$, the first inequality can be achieved by choosing smaller values of ρ_1 and σ_1, if necessary. The second inequality is merely an added restriction on ϵ_5, see Lemma 6.6.

Lemma 6.7. *Let the hypotheses of Theorem* 5.1 *be satisfied. Then for every* ϵ *with* $0 < \epsilon \le \epsilon_5$ *and* $\delta \le \delta_5(\epsilon)$*, if* G *satisfies* (5.5) *with* $\delta = \delta_5$*, then for all* $(f_i, g_i) \in \mathcal{F}(\epsilon,\ell) \times \mathcal{G}(\epsilon,\ell)$*, where* $\ell = 1$*, one has*

$$\|A^\beta(\bar f_{\tau,1}(u) - \bar f_{\tau,2}(u))\| \le \frac{5}{8}\|(f_1,g_1)-(f_2,g_2)\|_\infty, \tag{6.48}$$

for all $(\theta,u) \in \mathcal{K}$*, where* $\bar f_{\tau,i}$ *are given by* (6.42)*, for* $i = 1, 2$.

Proof. Because of the importance of this lemma in proving the main theorems, we include an outline of the argument here. We note that this follows the methodology of [24, pp. 463–464]. Also see [33, p. 518].

Let τ be fixed, where $T \le \tau \le 2T$. Once again we will use the special notation, where, for $(\theta, v_0) \in \mathcal{K}$, one now has

$$v_{10} = v_{20} = v_0 = v_0 \quad \text{and} \quad y_{i0} = v_0 + f_i(v_0) + g_i(v_0), \qquad \text{for } i = 1, 2.$$

Recall that $\bar f_i(v_i(\tau)) = s_i(\tau) = P^s(v_i(\tau))n_i(\tau)$, where $\bar f_i = \bar f_{\tau,i}$, for $i = 1, 2$.

By Lemma 6.4, the functions $y_i(t)$, $v_i(t)$ and $n_i(t)$, $s_i(t)$, and $u_i(t)$ are Lipschitz continuous functions of the initial data y_{i0}. Since $\triangle v(0) = 0$, and $\triangle y(0) = \triangle n(0)$, one has

$$\|A^\beta\triangle y(0)\| \le \|A^\beta\triangle f(v_0)\| + \|A^\beta\triangle g(v_0)\| \le \|(f_1,g_1)-(f_2,g_2)\|_\infty, \tag{6.49}$$

where $\triangle f(v_0) = f_1(v_0) - f_2(v_0)$ and $\triangle g(v_0) = g_1(v_0) - g_2(v_0)$. Also note that

$$P^s(v_2)\triangle s = \triangle\bar f(v_2) + P^s(v_2)[\bar f_1(v_1) - \bar f_1(v_2)], \qquad \text{at } t = \tau, \tag{6.50}$$

where $\triangle\bar f(v_0) = \bar f_1(v_0) - \bar f_2(v_0)$. From the first inequality in (6.45) and (6.34), one finds that

$$\|A^\beta E_s(t)\| \le \left(C_2C_4\rho + b_4^F(\rho,2\epsilon) + \beta_5(\epsilon,\epsilon)\right)\|A^\beta\triangle y(0)\|, \tag{6.51}$$

for $0 \le t \le 2T$. Since $P^s(v_0)\triangle n(0) = \triangle f(v_0)$, inequality (6.46) becomes

$$\|A^\beta\Phi(B,t)P^s(v_{20})\triangle n(0)\| \le Ke^{\lambda_1 t}\|A^\beta\triangle f(v_0)\|, \qquad \text{for } t \ge 0,$$

where $B = Z(\theta, v_{20})$, see (3.22). Therefore, from (6.51) one obtains

$$\begin{aligned}\|A^\beta\triangle s(t)\| \le{}& Ke^{\lambda_1 t}\|A^\beta\triangle f(v_0)\| \\ &+ \left(C_2C_4\rho + b_4^F(\rho,2\epsilon) + \beta_5(\epsilon,\delta)\right)\|A^\beta\triangle y(0)\|,\end{aligned} \tag{6.52}$$

for $0 \le t \le 2T$.

Since $K \geq 1$, $\lambda_2 \leq 0$, and $\beta_5 \leq \beta_6$, it then follows from inequalities (6.50), (6.44), (6.34), (6.52), and (6.49) – in that order – that, for $t = \tau$, one has:

$$\begin{aligned}
\|A^\beta \triangle \bar{f}(v_2)\| &\leq \|A^\beta P^s(v_2)\triangle s\| + \|A^\beta P^s(v_2)[\bar{f}_1(v_1) - \bar{f}_1(v_2)]\| \\
&\leq K\|A^\beta \triangle s\| + KC_2\,\ell_5(\rho,\epsilon)\|A^\beta \triangle y(0)\| \\
&\leq K^2 e^{\lambda_1 \tau}\|A^\beta \triangle f(v_0)\| \\
&\qquad + \left(KC_2\ell_5(\rho,\epsilon) + KC_2C_4\rho + Kb_4^F(\rho,2\epsilon) + K\beta_5(\epsilon,\epsilon)\right)\|A^\beta \triangle y(0)\| \\
&\leq \left(K^2 e^{\lambda_1 \tau} + KC_2\,\ell_5(\rho,\epsilon) + KC_2C_4\rho + Kb_4^F(\rho,2\epsilon) + K\beta_5(\epsilon,\delta)\right) \\
&\qquad \times \|(f_1,g_1) - (f_2,g_2)\|_\infty .
\end{aligned}$$

From inequalities (6.3), (6.9), (6.29), and (6.47), we see that each of the terms $K^2 e^{\lambda_1 \tau}$, $KC_2C_4\rho_1$, $KC_2\,\ell_5(\rho,\epsilon)$, $Kb_4^F(\rho,2\epsilon)$, and $K\beta_5(\epsilon,\delta)$ on the right-side of the last inequality is $\leq \frac{1}{8}$, when $0 < \rho \leq \rho_1$, $2\epsilon \leq \sigma_1$, and $\delta \leq \epsilon$, which implies (6.48). □

Let $(f,g) \in \mathcal{F} \times \mathcal{G}$. We now seek to define a new function $\bar{g}_\tau$, which is a companion to the function $\bar{f}_\tau$ given by (6.42). Among other things, we want $\bar{g}_\tau(v_0)$ to be in $\mathcal{U}^u(\theta, v_0)$, for every $(\theta, v_0) \in \mathcal{K}$. Let $(\theta, v_0) \in \mathcal{K}$ be given, and define $y_0 = y_0(V) = v_0 + f(v_0) + V$, where $V \in \mathcal{U}^u(\theta, v_0)$ will be treated as a parameter. Consider the equation

$$(6.53) \quad g(\theta \cdot \tau, v(\tau, y_0(V))) = P^u(v(\tau, y_0(V)))(S_2(\theta,\tau)y_0(V) - v(\tau, y_0(V))),$$

where $T \leq \tau \leq 2T$. Our objective is to show that if ϵ and δ are sufficiently small, then (6.53) has a unique solution $V \in \mathcal{U}^u(\theta, v_0)$. In this case we will denote this solution by $V = \bar{g}_\tau(\theta, v_0)$, thereby defining $\bar{g}_\tau$. Before proving this property, it is convenient to write (6.53) in the abbreviated form

$$g(v) = P^u(v)(y - v) = P^u(v)n,$$

where $y = S_2(\theta,\tau)y_0(V)$, $v = v(\tau, y_0(V))$, $n = y - v$, $P^u(v) = P^u(\widehat{\theta}, v)$, and $\widehat{\theta} = \theta \cdot \tau$. Note that (6.53) holds in the subspace $\mathcal{R}(P^u(v)) = \mathcal{U}^u(\widehat{\theta}, v)$. By adding and subtracting the three terms $P^u(S)y$, $P^u(S)S$, and $P^u(S)v$ in (6.53), where $S = S_1(\theta,\tau)v_0$, we see that (6.53) takes on the equivalent form

$$(6.54) \qquad g(v) = P^u(S)(y - S) - P^u(S)(v - S) + [P^u(v) - P^u(S)](y - v),$$

where $P^u(S) = P^u(\widehat{\theta}, S)$. Notice that each of the terms y, v, g, and $P^u(v)$ are Lipschitz continuous functions of the parameter V, while the term S does not depend on V.

Lemma 6.8. *Let the hypotheses of Theorem* 5.1 *be satisfied. Then there is an* $\epsilon_7 > 0$ *such that* $\epsilon_7 \leq \epsilon_5$, *and for all* ϵ *with* $0 < \epsilon \leq \epsilon_7$, *there is a* $\delta_7 = \delta_7(\epsilon) \in \Sigma$ *with* $0 < \delta_7 \leq \delta_5$, *where* ϵ_5 *and* δ_5 *are given in Lemma* 6.6 *such that if* G *satisfies* (5.5) *with* $\delta = \delta_7$, *and if* $f \in \mathcal{F}(\epsilon, \ell)$ *and* $g \in \mathcal{G}(\epsilon, \ell)$ *for any* ℓ *with* $0 < \ell \leq 1$, *then for each* $(\theta, v_0) \in \mathcal{K}$ *and* $\tau \in [T, 2T]$, *there is a unique solution*

$$V = \bar{g}_\tau(\theta, v_0) \stackrel{\text{def}}{=} V(v_0) \in \mathcal{U}^u(\theta, v_0)$$

of (6.53)*, and* $\|A^\beta V\| \le \frac{3}{4}\epsilon$*. Moreover,* $\bar{g}_\tau(\theta, v_0)$ *is continuous, for* $(\theta, v_0) \in \mathcal{K}$ *and* $T \le \tau \le 2T$.

Proof. We begin by rewriting (6.54) in the form

$$V = \Gamma(V) = \Gamma(V, v_0) = \Gamma(V, \theta, v_0), \tag{6.55}$$

where $(\theta, v_0) \in \mathcal{K}$, $V \in \mathcal{U}^u(v_0)$, $(f, g) \in \mathcal{F} \times \mathcal{G}$, and

(6.56)

$$\Gamma(V, v_0) = \Gamma(V, v_0; f, g)$$
$$\stackrel{\text{def}}{=} \Phi(B_\tau, -\tau) P^u(S) \left[-\hat{e}(V, \tau) + (v - S) + \left(P^u(S) - P^u(v)\right) n + g(v)\right],$$

where $B = Z(\theta, v_0)$ and $\Phi(B, t)$ are given by (3.21)–(3.22). Also, it follows from an application of (2.36) that

$$\hat{e}(V, t) = \int_0^t \Phi(B_s, t - s) H(S_1(\theta, s)v_0, w(s), s)\, ds.$$

We note that the function $\Gamma(V, \theta, v_0)$ defined by (6.56) is well defined on the domain

$$\mathcal{D}(\Gamma) \stackrel{\text{def}}{=} \{(V, \theta, v_0) \in V^{2\beta} \times \mathcal{K} : V \in \mathcal{U}^u(\theta, v_0) \text{ and } \|A^\beta V\| \le \epsilon\}.$$

Also note that the range of Γ lies in $\mathcal{U}^u(\theta, v_0)$, for each $(\theta, v_0) \in \mathcal{K}$. Indeed, one obtains $P^u(\theta, v_0)\Phi(B_\tau, -\tau) = \Phi(B_\tau, -\tau)P^u(\widehat{\theta}, S)$ from the invariance property (4.7), which in turn implies that $P^u(\theta, v_0)\Gamma(V, v_0) = \Gamma(V, v_0)$, for all $(V, v_0) \in \mathcal{D}(\Gamma)$. Note that one has $V = \Gamma(V, v_0)$ if and only if $g(v) = P^u(v)n$.

Since the proof of this lemma is essentially the same as the argument used in [24, pp. 465–468] and [33, pp. 520–523], we omit some of the intermediate steps leading to the following conclusion: The mapping $V \to \Gamma(V)$ is a contraction. In particular, there is a $\beta_5 \in \Sigma$ such that

$$\|A^\beta(\Gamma(V_1, v_0) - \Gamma(V_2, v_0))\| \le \left(b_4^F(\rho, 2\epsilon) + \beta_5(\epsilon, \delta)\right)\|A^\beta(V_1 - V_2)\|.$$

Next we let ϵ_7 be chosen so that $0 < \epsilon_7 \le \bar{\epsilon}$ and $\beta_5(\epsilon_7, \epsilon_7) \le \frac{3}{8}$. Then set $\delta_7(\epsilon) = \bar{\delta}(\epsilon))$, for $0 < \epsilon \le \epsilon_7$. Since $K \ge 1$, inequality (6.29) implies that $b_4^F(\rho, 2\epsilon) \le \frac{1}{144}$. Hence

$$\|A^\beta(\Gamma(V_1, v_0) - \Gamma(V_2, v_0))\| \le \frac{1}{2}\|A^\beta(V_1 - V_2)\|,$$

for all $(V_i, v_0) \in \mathcal{D}(\Gamma)$, for $i = 1, 2$.

Finally, the function $\bar{g}_\tau(\theta, v_0)$ is a continuous function of $(\theta, v_0) \in \mathcal{K}$, because: (1) the mapping $(V, \theta, v_0) \to \Gamma(V, \theta, v_0)$ given by equation (6.56) is continuous on $\mathcal{D}(\Gamma)$; and (2) the value $\bar{g}_\tau(\theta, v_0)$ is the unique fixed point of a contraction mapping, and $\bar{g}_\tau(\theta, v_0)$ is jointly continuous in (θ, v_0) and τ. □

In the next two lemmas, we show that $\bar{g}_\tau(\theta, v)$ is locally Lipschitz continuous in v on disks $\mathcal{D}_{\rho_1}(\theta, v_0)$ in $\mathcal{K}$ and that the mapping $(f, g) \to \bar{g}_\tau$ is contracting.

Lemma 6.9. *Let the hypotheses of Theorem* 5.1 *be satisfied. Then there is an* ϵ_8 *with* $0 < \epsilon_8 \le \epsilon_7$ *such that for all* ϵ *with* $0 < \epsilon \le \epsilon_8$, *there exists a* $\delta_8 = \delta_8(\epsilon) \in \Sigma$, *with* $0 < \delta_8 \le \delta_7$, *where* ϵ_7 *and* δ_7 *are given in Lemma* 6.8, *and, for* $0 < \rho \le \rho_1$, *there exists an* $\ell_8 = \ell_8(\rho, \epsilon) > 0$ *such that if* G *satisfies* (5.5) *with* $\delta = \delta_8$, *and if* $f \in \mathcal{F} = \mathcal{F}(\epsilon, \ell)$ *and* $g \in \mathcal{G} = \mathcal{G}(\epsilon, \ell)$ *with* $\ell \le 1$ *and* $\epsilon \le \epsilon_8$, *then the restriction of* $\bar{g}_\tau = \bar{g}_\tau(v)$ *to the disk* $\mathcal{D}_\rho(\theta, v_0)$ *is Lipschitz continuous in* v *with Lipschitz coefficient* ℓ_8, *for every* $(\theta, v_0) \in \mathcal{K}$. *Moreover, one has* $\bar{g}_\tau \in \mathcal{G}$ *and* $\ell_8(\rho, \epsilon) \in \Sigma$.

Since the proof of this lemma is essentially the same as the argument used in [24, pp. 469–472] and [33, pp. 523–527], we do not present the details here. By choosing smaller values for ρ_1 and σ_1, if necessary, we may assume that

$$16 K C_2 \ell_8(\rho_1, \sigma_1) \le 1. \tag{6.57}$$

Lemma 6.10. *Let the hypotheses of Theorem* 5.1 *be satisfied. Then there is an* ϵ_9 *with* $0 < \epsilon_9 \le \epsilon_8$, *and for every* ϵ *with* $0 < \epsilon \le \epsilon_9$, *there is a* $\delta_9 = \delta_9(\epsilon) \le \delta_8(\epsilon)$, *where* ϵ_8 *and* δ_8 *are given in Lemma* 6.9 *such that if* G *satisfies* (5.5) *with* $\delta = \delta_9$, *then for all* $(f_i, g_i) \in \mathcal{F}(\epsilon, \ell) \times \mathcal{G}(\epsilon, \ell)$, *where* $\ell \le 1$, *and* $\bar{g}_{\tau,i}$ *is given by* (6.53), *for* $i = 1, 2$. *One then has*

(6.58)
$$\|A^\beta(\bar{g}_{\tau,1}(u) - \bar{g}_{\tau,2}(u))\| \le \frac{1}{2}\|(f_1, g_1) - (f_2, g_2)\|_\infty \qquad \textit{for all } (\theta, u) \in \mathcal{K}.$$

Proof. We will use the special notation (6.30)–(6.31). Let $(f_i, g_i) \in \mathcal{F} \times \mathcal{G}$ be given, for $i = 1, 2$, and fix τ so that $T \le \tau \le 2T$. Let $(\hat{\theta}, \hat{v}_0) \in \mathcal{K}$ be fixed, and set $\hat{v}_{10} = \hat{v}_{20} = \hat{v}_0$. Define $\hat{y}_{i0} = \hat{v}_{i0} + f_i(\hat{\theta}, \hat{v}_0) + V_i$, for $i = 1, 2$, where $V_i = g_{\tau,i}(\hat{\theta}, \hat{v}_0)$ is the solution of equation (6.55) given by Lemma 6.8, with (f, g) being replaced by (f_i, g_i), for $i = 1, 2$. Let $\bar{f}_i = \bar{f}_{\tau,i}$ be given by Lemma 6.5, for $i = 1, 2$.

Define y_{i0} by $y_{i0} = S_2(\hat{\theta}, \tau)\hat{y}_{i0}$, for $i = 1, 2$, and set $\theta = \hat{\theta} \cdot \tau$ so that $\hat{\theta} = \theta \cdot (-\tau)$. Note that the solution $S_2(\theta, t)y_{i0}$ of equation (5.4) has a partial negative continuation

$$y_i(t) = S_2(\theta, t)y_{i0} \stackrel{\text{def}}{=} S_2(\hat{\theta}, \tau + t)\hat{y}_{i0}, \qquad \text{for } -\tau \le t \le 0,$$

for $i = 1, 2$. When $\epsilon \le \epsilon_0$ and $\delta \le \delta_0$, Lemma 6.4 implies that

$$\left\| A^\beta\big(S_2(\hat{\theta}, s)\hat{y}_{i0} - S_1(\hat{\theta}, s)\hat{v}_0\big)\right\| \le C_0(\epsilon + \delta) \le \sigma_1, \qquad \text{for } 0 \le s \le \tau.$$

Therefore, Lemma 6.3 implies that, for $i = 1, 2$, the functions $y_i(t)$ have the local coordinate representation

$$y_i(t) = v_i(t) + n_i(t), \qquad \text{for } -\tau \le t \le 0, \tag{6.59}$$

see (6.21)–(6.22), and $n_i(t) = s_i(t) + u_i(t)$, for $-\tau \le t \le 0$.

Next we invoke the standard notation (6.30) to define $\triangle y(t)$, $\triangle v(t)$, $\triangle n(t)$, $\triangle s(t)$, and $\triangle u(t)$, for $-\tau \le t \le 0$. We set $v_{i0} = v_i(0)$, for $i = 1, 2$. Note that one has $\triangle n(t) = \triangle s(t) + \triangle u(t)$, and

$$\text{(6.60)} \qquad \begin{cases} \triangle y(-\tau) = \hat{y}_{10} - \hat{y}_{20}, & \triangle y(0) = y_{10} - y_{20}, \\ \triangle v(-\tau) = 0, & \triangle v(0) = v_{10} - v_{20}, \\ \triangle s(-\tau) = \triangle f(\hat{v}_0), & \triangle s(0) = \bar{f}_1(v_{10}) - \bar{f}_2(v_{20}), \\ \triangle u(-\tau) = V_1 - V_2, & \triangle u(0) = g_1(v_{10}) - g_2(v_{20}), \end{cases}$$

where $\triangle f(\hat{v}_0) = f_1(\hat{v}_0) - f_2(\hat{v}_0)$. Let $P^u(v_{20}) = P^u(\theta, v_{20})$. It follows from the definitions of $\bar{f}_i$ and g_i that $P^u(v_{20})\bar{f}_i(v_{20}) = 0$ and $P^u(v_{20})g_i(v_{20}) = g_i(v_{20})$. Hence, one obtains

$$\begin{aligned} P^u(v_{20})\triangle n(0) &= P^u(v_{20})\big[\bar{f}_1(v_{10}) - \bar{f}_2(v_{20}) + g_1(v_{10}) - g_2(v_{20})\big] \\ &= \big[P^u(v_{20}) - P^u(v_{10})\big]\,\big[\bar{f}(v_{10}) + g_1(v_{10})\big] \\ &\qquad + [g_1(v_{10}) - g_1(v_{20})] + \triangle g(v_{20}), \end{aligned}$$

where $\triangle g(v_{20}) = g_1(v_{20}) - g_2(v_{20})$. It then follows from (4.25), (6.43), and Lemma 6.9 that

$$\text{(6.61)} \qquad \|A^\beta P^u(v_{20})\triangle n(0)\| \le \|A^\beta \triangle g(v_{20})\| + (\ell_8 + 2L_0\epsilon)\|A^\beta \triangle v(0)\|.$$

Let $B = Z(\theta, v_{20})$, see (3.22). Since $-\tau \le 0$, inequality (4.11) implies that

$$\text{(6.62)} \qquad \|A^\beta \Phi(B, -\tau)P^u(v_{20})\triangle n(0)\| \le Ke^{-\lambda_4\tau}\|A^\beta P^u(v_{20})\triangle n(0)\|.$$

Next we will use Lemma 6.4 to estimate the value $\|A^\beta \triangle v(0)\|$. For this purpose, we use the change of variables $s = \tau - t$, for $0 \le s \le \tau$, and let $\hat{y}_i(s) = S_2(\hat{\theta}, s)\hat{y}_{i0}$. Let $\hat{v}_i$ and $\hat{n}_i$ satisfy $\hat{y}(s) = \hat{v}_i(s) + \hat{n}_i(s)$, for $0 \le s \le \tau$, see (6.59). Then $\hat{y}_i(\tau) = y_{i0}$, for $i = 1, 2$, and $\triangle\hat{v}(\tau) = \triangle v(0)$. Also, one has $\triangle\hat{v}(0) = \triangle v(-\tau) = 0$. Next let $E_{\hat{v}} = E_{\hat{v}}(s)$ be given by (6.31). Since $\triangle\hat{v}(0) = 0$, one has $\triangle\hat{v}(s) = E_{\hat{v}}(s)$, for $0 \le s \le \tau$. Since $\triangle v(0) = \triangle\hat{v}(\tau) = E_{\hat{v}}(\tau)$, it follows from Lemma 6.4, item 4, that there is a $\beta_2 \in \Sigma$ such that

$$\text{(6.63)} \qquad \|A^\beta \triangle v(0)\| \le \big(b_4^F(\rho, 2\epsilon) + \beta_2(\epsilon, \delta)\big)\,\|A^\beta \triangle\hat{y}(-\tau)\|.$$

Next we examine the behavior of $\triangle u(t)$, for $-\tau \le t \le 0$. We define $E_u(t)$, for $-\tau \le t \le 0$, by the equation

$$\triangle u(t) = \Phi(B, t)P^u(v_{20})\triangle n(0) + E_u(t), \qquad -\tau \le t \le 0.$$

By using the argument of [24, pp. 470–471] or [33, pp. 524–525], one finds that there is a $\beta_3 \in \Sigma$ such that

$$\text{(6.64)} \qquad \begin{aligned} \|A^\beta E_u(-\tau)\| &\le K^3 L_0\rho e^{-\lambda_4\tau}\|A^\beta \triangle v(0)\| \\ &\qquad + \big(b_4^F(\rho, 2\epsilon) + \beta_3(\epsilon, \delta)\big)\|A^\beta \triangle y(-\tau)\|. \end{aligned}$$

By combining the four inequalities (6.61)–(6.64), we find that

$$\begin{aligned} \|A^\beta \triangle u(-\tau)\| &\le Ke^{-\lambda_4\tau}\|A^\beta \triangle g(v_{20})\| + \big(b_4^F(\rho, 2\epsilon) + \beta_3(\epsilon, \delta)\big)\|A^\beta \triangle y(-\tau)\| \\ &\qquad + Ke^{-\lambda_4\tau}(2L_0\epsilon + \ell_8 + KL_0\rho)\|A^\beta \triangle v(0)\|. \end{aligned}$$

Since $K \geq 1$ and $C_2 \geq 1$, it follows from (6.3), (6.9), (6.32), and (6.57) that $Ke^{-\lambda_4 \tau} \leq \frac{1}{16}$, $2L_0\epsilon \leq 1$, $\ell_8 \leq \frac{1}{16}$, and $KL_0\rho \leq 1$, and hence

$$Ke^{-\lambda_4 \tau}(2L_0\epsilon + \ell_8 + KL_0\rho) \leq 1.$$

Consequently, we obtain

$$\|A^\beta \triangle u(-\tau)\| \leq Ke^{-\lambda_4\tau}\|A^\beta \triangle g(v_{20})\| + \left(2b_4^F(\rho, 2\epsilon) + \beta_6(\epsilon, \delta)\right)\|A^\beta \triangle y(-\tau)\|,$$

where $\beta_6 = \beta_2 + \beta_3$. Since $\triangle u(-\tau) = V_1 - V_2$ and $\triangle y(-\tau) = \triangle f(\hat{v}_0) + V_1 - V_2$, we obtain

$$\|A^\beta (V_1 - V_2)\| \leq \frac{1}{4}\|A^\beta \triangle g(v_{20})\| + \left(2b_4^F + \beta_6\right)\left(\|A^\beta \triangle f(\hat{v}_0)\| + \|A^\beta (V_1 - V_2)\|\right). \tag{6.65}$$

If $(2b_4^F + \beta_6) \leq \frac{1}{4}$, then inequality (6.65) implies (6.58).

Finally, since $2\epsilon \leq \sigma_1$, see (6.32), (6.29) implies that $2b_4^F \leq \frac{1}{8}$. Fix ϵ_9 so that $0 < \epsilon_9 \leq \epsilon_8$ and $\beta_6(\epsilon_9, \epsilon_9) \leq \frac{1}{8}$, and we set $\delta_9(\epsilon) = \min(\epsilon, \delta_8(\epsilon))$, for $0 < \epsilon \leq \epsilon_9$. One then has $\beta_6(\epsilon, \delta) \leq \frac{1}{8}$, for $0 < \delta \leq \delta_9(\epsilon)$, and the lemma now follows from inequality (6.65). □

In the next result, we give a more precise formulation of Theorem 5.1.

Theorem 6.11. *Let the hypotheses of Theorem* 5.1 *be satisfied. Then for each* $\epsilon > 0$ *there exists a* $\delta = \delta(\epsilon) > 0$, *and, for* $0 < \rho \leq \rho_1$, *there is an* $\ell = \ell(\rho, \epsilon) \leq \frac{1}{8}$, *where* $\delta(\epsilon)$, $\ell(\rho, \epsilon) \in \Sigma$ *such that if G satisfies* (5.5) *with* $\delta = \delta(\epsilon)$, *then there is a continuous mapping* $h : \mathcal{K} \to T^k \times V^{2\beta}$ *such that the following properties hold:*

1. *The image* $\mathcal{K}^n = h(\mathcal{K})$ *is a compact, connected, invariant set for the perturbed equation* (5.4)*;*
2. *For each* $(\theta, v) \in \mathcal{K}$, *one has* $\phi(\theta, v) \in \mathcal{U}^s(\theta, v) \oplus \mathcal{U}^u(\theta, v)$, *where* $h(\theta, v) = v + \phi(\theta, v)$*;*
3. *The restriction of h to any disk* $\mathcal{D}_\rho(\theta, v_0)$, *where* $(\theta, v_0) \in \mathcal{K}$ *and* $0 < \rho \leq \rho_1$, *is Lipschitz continuous in v, that is,*

$$\|A^\beta(\phi(\theta, v_1) - \phi(\theta, v_2))\| \leq 2\ell\|A^\beta(v_1 - v_2)\|, \tag{6.66}$$

for all $v_1, v_2 \in \mathcal{D}_{\rho_1}(\theta, v_0)$, *and this restriction is a one-to-one mapping of* $\mathcal{D}_\rho(\theta, v_0)$ *into* $V^{2\beta}$*;*
4. *And one has* $\|A^\beta \phi(\theta, v)\| = \|A^\beta(h(\theta, v) - v)\| \leq 2\epsilon$, *for all* $(\theta, v) \in \mathcal{K}$.
5. *The perturbed set* $\mathcal{K}^n$ *satisfies the conditions FB1, FB2, FB3, and FB4, see Section* 4.2.

Proof of Theorem 6.11 and the Shadow Theorem 5.2. Let T be given by (6.9), and for $T \leq \tau \leq 2T$, let A_τ be the mapping on $\mathcal{F} \times \mathcal{G} = \mathcal{F}(\epsilon, \ell_1) \times \mathcal{G}(\epsilon, \ell_1)$ defined by

$$A_\tau : (f, g) \to (\bar{f}_\tau, \bar{g}_\tau),$$

where $\ell_1 = 1$, $\bar{f}_\tau$ is given by equation (6.42), and $V = \bar{g}_\tau(v_0)$ is given by Lemma 6.8, see equations (6.53) and (6.55). Let ϵ_9 and δ_9 be given by Lemma 6.10, and let ℓ_5 and ℓ_8 be given by Lemmas 6.6 and 6.9. Define $\ell = \ell_9(\rho, \epsilon)$ by

$$\ell = \ell_9(\rho, \epsilon) = \max(\ell_5(\rho, \epsilon), \ell_8(\rho, \epsilon)), \qquad \text{for } 0 < \epsilon \leq \epsilon_9 \text{ and } 0 < \rho \leq \rho_1.$$

Since $8KC_2\ell_5 \leq 1$ and $16KC_2\ell_8 \leq 1$, see (6.47) and (6.57), and since $K \geq 1$ and $C_2 \geq 1$, we see that $\ell_5 \leq \frac{1}{8}$ and $\ell_8 \leq \frac{1}{16}$. Hence, $\ell_9(\rho, \epsilon) \leq \frac{1}{8}$ and $\ell_9(\rho, \epsilon) \in \Sigma$. Let ϵ be fixed with $0 < \epsilon \leq \epsilon_9$, and set $\delta = \delta_9(\epsilon)$.

Assume that G satisfies (5.5) with $\delta = \delta_9$. From Lemmas 6.6 and 6.10 we see that A_τ maps $\mathcal{F} \times \mathcal{G}$ into itself, for each τ with $T \leq \tau \leq 2T$. Also, Lemmas 6.7 and 6.10 imply that A_τ is a strict contraction on $\mathcal{F} \times \mathcal{G}$. Since $\mathcal{F} \times \mathcal{G}$ is a complete metric space, the mapping A_τ has a unique fixed point.

It is at this point we raise the curtain and invite the θ-variable to join the dance. In order to appreciate the significance that (f, g) is a fixed point of A_τ, we let h be defined by (6.23), for the fixed point (f, g). Note that $f(\theta, v) \in \mathcal{U}^s(\theta, v)$ and $g(\theta, v) \in \mathcal{U}^u(\theta, v)$. Also, one has $A_\tau(f, g) = (f, g)$ if and only if

$$S_2(\theta, \tau)y_0 = v_\tau + f(\theta \cdot \tau, v_\tau) + g(\theta \cdot \tau, v_\tau), \qquad \text{for all } (\theta, v_0) \in \mathcal{K}, \tag{6.67}$$

where $y_0 = h(\theta, v_0)$ and $v_\tau = v(\tau, h(v_0)) = \psi(\theta \cdot \tau, S_2(\theta, \tau)y_0)$, see (6.40). (Note that $(\theta \cdot \tau, v_\tau) \in \mathcal{K}$.) Observe that (6.67) can be written in the form

$$S_2(\theta, \tau)h(\theta, v_0) = h(\theta \cdot \tau, v_\tau), \qquad \text{for } (\theta, v_0) \in \mathcal{K}. \tag{6.68}$$

Next we define $\mathcal{K}^n = h(\mathcal{K})$ by

$$\mathcal{K}^n \stackrel{\text{def}}{=} \{(\theta, y_0) \in T^k \times V^{2\beta} : y_0 = h(\theta, v_0), \text{ for some } (\theta, v_0) \in \mathcal{K}\}. \tag{6.69}$$

We claim that, since $A_\tau(f, g) = (f, g)$, for $T \leq \tau \leq 2T$, the semiflow π_2 satisfies

$$\pi_2(\tau)\mathcal{K}^n = \mathcal{K}^n. \tag{6.70}$$

Indeed, with $y_0 = h(\theta, v_0)$, $\hat{\theta} = \theta \cdot \tau$, $\hat{y} = S_2(\theta, \tau)y_0$, and $\hat{v} = \psi^e(\hat{y})$, one has

$$\begin{aligned} \pi_2(\tau)\mathcal{K}^n &= \{(\theta \cdot \tau, S_2(\theta, \tau)y_0)\} && \text{for some } (\theta, v_0) \in \mathcal{K}, \\ &= \{(\theta \cdot \tau, h(\theta \cdot \tau, v_\tau))\} && \text{by (6.68)}, \\ &= \{(\hat{\theta}, h(\hat{\theta}, \hat{v}))\} = \mathcal{K}^n && \text{by (6.69)}. \end{aligned}$$

It follows from (6.70) that $\pi_2(2\tau)\mathcal{K}^n = \mathcal{K}^n$ and $\pi_2(m\tau)\mathcal{K}^n = \mathcal{K}^n$, for every integer $m \geq 0$. Consequently, for each $(\theta, y_0) \in \mathcal{K}^n$, there is a global solution $S_2(\theta, t)y_0$ of equation (5.4) with

$$S_2(\theta, m\tau)y_0 \in \mathcal{K}^n, \qquad \text{for all } m \in \mathbb{Z} \text{ and for } T \leq \tau \leq 2T.$$

It needs to be noted that the fixed points of A_τ, (f, g), may depend on $\tau \in [T, 2T]$. We have not included this dependence in the notation because we will now show that these fixed points do not depend on τ. For this purpose, we now fix $\tau = T$ and let (f, g) be the fixed point of A_T. Set $(f_0, g_0) = (f, g)$, and let $h_0 = h$

be given by (6.23), using (f_0, g_0). For $0 \le t \le T$, we define (f_t, g_t) and h_t by $f_t(\theta \cdot t, v_t) = s_t$ and $g_t(\theta \cdot t, v_t) = u_t$, where

$$(6.71) \quad y_t \stackrel{\text{def}}{=} S_2(\theta, t)h_0(\theta, v_0) = v_t + s_t + u_t = h_t(\theta \cdot t, v_t), \qquad \text{for } (\theta, v_0) \in \mathcal{K},$$

where $v_t = \psi^e(\theta \cdot t, S_2(\theta, t)h_0(\theta, v_0))$ and $y_t = v_t + s_t + u_t$ is the representation of y_t given by (6.21). (Recall that ψ^e is the extension of ψ, see Lemma 6.3.) Let t be fixed with $0 \le t < T$ and redefine $\hat{y} = y_t$, $\hat{v} = v_t$, and $\hat{\theta} = \theta \cdot t$. Then (6.71) becomes

$$\hat{y} = \hat{v} + f_t(\hat{\theta}, \hat{v}) + g_t(\hat{\theta}, \hat{v}) = h_t(\hat{\theta}, \hat{v}).$$

Let $y_0 = h_0(\theta, v_0)$, where $(\theta, v_0) \in \mathcal{K}$, and set $\hat{y}_T = S_2(\hat{\theta}, T)\hat{y}$ and $\hat{v}_T = \psi^e(\hat{y}_T)$. Then the cocycle identity (3.20) implies that

$$S_2(\theta, T + t)y_0 = S_2(\theta \cdot t, T)S_2(\theta, t)y_0 = S_2(\hat{\theta}, T)\hat{y}$$
$$= \hat{y}_T = \hat{v}_T + h_t(\hat{\theta} \cdot T, \hat{v}_T).$$

This implies that (f_t, g_t) is a fixed point of A_T, see (6.67).

The question that now arises is: which space does (f_t, g_t) reside in? Is it in the space $\mathcal{F} \times \mathcal{G}$? If so, then the uniqueness of the fixed point for A_T implies that $h = h_T = h_t$ and $(f_t, g_t) = (f_T, g_T)$. We show next that (f_t, g_t) does indeed lie in $\mathcal{F} \times \mathcal{G}$.

We claim that there is an ϵ_{10}, with $0 < \epsilon_{10} \le \epsilon_9$, such that, for $0 < \epsilon \le \epsilon_{10}$, there is a $t_0 > 0$ where $(f_t, g_t) \in \mathcal{F} \times \mathcal{G}$, for $0 \le t \le t_0$. Indeed, from Lemmas 6.5, 6.6, 6.8, and 6.9, and since the fixed point (f_0, g_0) is in the range of A_T, one has $\|A^\beta f_0(\theta, u)\| \le \frac{3}{4}\epsilon$ and $\|A^\beta g_0(\theta, u)\| \le \frac{3}{4}\epsilon$, for all $(\theta, u) \in \mathcal{K}$. By continuity, there is a $t_1 > 0$ such that $\|A^\beta f_t(\theta, u)\| \le \epsilon$ and $\|A^\beta g_t(\theta, u)\| \le \epsilon$, for all $(\theta, u) \in \mathcal{K}$ and $0 \le t \le t_1$. Since $0 \le \ell < 1$, it follows from Lemmas 6.6 and 6.9 that there is a $t_2 > 0$ such that both f_t and g_t are Lipschitz continuous on each disk $\mathcal{D}_\rho(v_0)$, for $0 < \rho \le \rho_1$, with Lipschitz coefficient 1, for $0 \le t \le t_2$. By setting $t_0 = \min(t_1, t_2)$, we conclude that $(f_t, g_t) \in \mathcal{F} \times \mathcal{G}$, for $0 \le t \le t_0$.

Since the fixed point of A_T is unique, we have $h_t = h$, for all t, with $0 \le t \le t_0$. By iteration of this argument, we conclude that $h_t = h$, for all $t \ge 0$. Since $(f_t, g_t) = (f, g)$, for all $t \ge 0$, one has

(6.72)
$$S_2(\theta, t)h(\theta, v_0) = h(\theta \cdot t, v_t) = v_t + f(\theta \cdot t, v_t) + g(\theta \cdot t, v_t), \qquad \text{for all } t \ge 0,$$

where $v_t = \psi^e(S_2(\theta, t)h(\theta, v_0))$. Thus, ψ^e is a (local) inverse of h. This completes the proof of item 1 in Theorem 6.11. Thus, $\mathcal{K}^n$ satisfies the property FB1.

Items 2, 3, and 4 of Theorem 6.11 follow from the properties of (f, g) proved above, with the observation that $\phi(\theta, v) = f(\theta, v) + g(\theta, v)$. For example, since $h(u) = u + \phi(u)$, it follows from (6.66) and $\ell \le \frac{1}{8}$ that

$$\|A^\beta(h(v_1) - h(v_2))\| \ge \|A^\beta(v_1 - v_2)\| - \|A^\beta(\phi(v_1) - \phi(v_2))\| \ge \frac{3}{4}\|A^\beta(v_1 - v_2)\|,$$

which implies that the restriction of h to any disk $\mathcal{D}_{\rho_1}(\theta, v_0)$ is one-to-one.

Next we will prove the Shadow Theorem 5.2. In terms of the identity (6.72), we now define $\widehat{S}$ by

$$\widehat{S}(\theta, t)v_0 = v_t, \qquad \text{for } t \geq 0.$$

Then equation (6.72) becomes

$$S_2(\theta, t)h(\theta, v_0) = h(\theta \cdot t, \widehat{S}(\theta, t)v_0), \qquad \text{for } t \geq 0,$$

which is the shadow property (5.6). Let us now show that $\widehat{S}$ satisfies the cocycle identity. By using the cocycle property (3.20) for S_2 with (5.6), we get

$$\begin{aligned} h(\theta\cdot(\tau + t), \widehat{S}(\theta, \tau + t)v_0) &= S_2(\theta, \tau + t)h(\theta, v_0) = S_2(\theta \cdot t, \tau)S_2(\theta, t)h(\theta, v_0) \\ &= S_2(\theta \cdot t, \tau)h(\theta \cdot t, \widehat{S}(\theta, t)v_0) = h(\theta \cdot (\tau + t), \widehat{S}(\theta \cdot t, \tau)\widehat{S}(\theta \cdot t)v_0). \end{aligned}$$

By applying ψ^e to the last identity, we obtain

$$\widehat{S}(\theta, \tau + t)v_0 = \widehat{S}(\theta \cdot t, \tau)\widehat{S}(\theta \cdot t)v_0, \tag{6.73}$$

which is the cocycle identity. This completes the proof of the shadow property Theorem 5.2.

Next we turn to items 5 and 6 in Theorem 6.11. Property FB3 follows from the fact that the composition $R = M \circ N$ of two Lipschitz continuous functions M and N is itself Lipschitz continuous and the Lipschitz constant satisfies Lip $R \leq$ Lip $M \times$ Lip N. In our case, we apply this with $M = h$ and $N = \nu$, see (6.23) and (4.2). Since $h(u) = u + \phi(u)$, one has Lip $h \leq 1 + 2\ell \leq \frac{5}{4}$, by (6.66). It then follows from (4.3) that $R = h \circ \nu$ maps $D(u, \rho_1)$ into $\mathcal{K}^n$ with Lip $R \leq \frac{5}{4}L_0$. Note that $D(u, \rho_1)$ is the k-dimensional disk in $X(u) = \mathcal{U}^o(\theta, u)$, where $(\theta, u) \in \mathcal{K}$. Thus $\mathcal{K}^n$ has property FB3. Property FB2 now follows from FB1 and FB3.

This completes the proof of items 1–5 in Theorem 6.11 and the proof of the shadow property Theorem 5.2, and we have also shown that $\mathcal{K}^n$ satisfies properties FB1, FB2, and FB3. Let us turn next to property FB4.

In order to show that the linearized semiflow Π_2 has an exponential trichotomy on $\mathcal{K}^n$ when δ is sufficiently small, we will use the RET Theorem, i.e., the robustness of exponential trichotomies (RET) theorem from [22]. For all practical purposes, the argument used here is identical to that used in [24]. Even though in this earlier work we considered only the autonomous problem, where the foliated bundle $\mathcal{K}$ is reduced to a compact manifold M, it does not matter. The RET theorem, which is based on the Henry characterization of an exponential dichotomy in terms of time-discrete dynamics, see [12], is applicable in widely diverse settings. Two general features need to be established for an application of the RET theorem:

- There is a linear skew product semiflow Π over a compact invariant set $\mathcal{K}$, and Π has an exponential trichotomy.
- The perturbed linear skew product semiflow Π^λ varies continuously in a parameter λ, where λ is in a metric space Λ, and $\Pi^\lambda \to \Pi^{\lambda_0}$, as $\lambda \to \lambda_0$ in Λ, where $\Pi^{\lambda_0} = \Pi$.

The convergence $\Pi^\lambda \to \Pi^{\lambda_0}$ needs special commentary. Let Π^λ satisfy

$$\Pi^\lambda(\theta, u_0, w_0; t) = (\theta \cdot t, S^\lambda(\theta, t)u_0, \Phi(B^\lambda, t)w_0), \qquad \text{for } t \geq 0,$$

see (3.21). It is required that $S^\lambda(\theta, t)u_0$ and $\Phi(B^\lambda, t)w_0$ vary continuously in λ in the sense that

$$\begin{aligned} &S^\lambda(\theta, t)u_0 \to S^{\lambda_0}(\theta, t)u_0 \\ &\Phi(B^\lambda, t)w_0 \to \Phi(B^{\lambda_0}, t)w_0 \\ &\text{as } \lambda \to \lambda_0, \text{ uniformly for } (u_0, w_0, t) \text{ in bounded sets.} \end{aligned} \tag{6.74}$$

When the two features described above are satisfied, and when $\mathrm{dist}(\lambda, \lambda_0)$ is small, then Π^λ has an exponential trichotomy with characteristics which are close to the characteristics of Π, see [22].

For the application at hand, we let $\lambda = \widehat{G}$ and $\lambda_0 = 0$, with $\Lambda = C^1_F$ with either the topology $\mathcal{T}^1_{\mathrm{bo}}$ or $\mathcal{T}^1_{\mathrm{A}}$, see Section 2.2. (Recall that C^1_F is a Fréchet space under either of these topologies and that $\mathcal{T}^1_{\mathrm{bo}} \subset \mathcal{T}^1_{\mathrm{A}}$.) In our application, the semiflow $S^\lambda(\theta, t)$ is the shadow semiflow $\widehat{S}(\theta, t)$ on $\mathcal{K}$. The convergence required in (6.74) is satisfied, since h converges to the identity, in the $C^{1,1}$-norm, as $\epsilon \to 0$ and $\delta \to 0$, see (5.5), and items 3 and 4 of Theorem 6.11.

As a direct application of the RET theorem, we obtain an exponential trichotomy for the linear skew product semiflow Π_2 over $(\mathcal{K}, \widehat{S})$, the foliated bundle $\mathcal{K}$ with the shadow semiflow $\widehat{S}$, when $\epsilon > 0$ is small enough. Due to the shadow property (5.6), the latter exponential trichotomy lifts to an exponential trichotomy on $(\mathcal{K}^n, S_2)$, with the same characteristics. This completes the proof of FB4 and item 6. □

Remarks. 1. The question of whether or not the perturbed dynamics on $\mathcal{K}^n$ satisfies property FB5, the tangency condition (4.33), remains an open matter. If the semiflow π_2 has a proper negative continuation on $\mathcal{K}^n$, then the tangency property follows from Theorem 5.7. Except for the remarks given in Section 5.2, this issue remains open. It would be desirable to find conditions on the unperturbed semiflow that would guarantee that the perturbed dynamics on $\mathcal{K}^n$ satisfy the tangency property.

2. One can readily verify that $v(t) = \widehat{S}(\theta, t)v_0$ is the unique mild solution of the evolutionary equation

$$\begin{aligned} \partial_t v + Av &= P^o(\theta, v)[F(\theta, v + \phi(\theta, v)) + G(\theta, v + \phi(\theta, v))] \\ &= P^o(\theta, v)[F(\theta, h(\theta, v)) + G(\theta, h(\theta, v))] \\ \partial_t \theta &= \omega, \end{aligned}$$

with $v(0) = v_0$ and $(\theta, v_0) \in \mathcal{K}$. Furthermore, $v(t)$ is a strong solution of this equation, see Lemma 3.2.

Proof of Theorem 5.3. Let $h : \mathcal{K} \to \mathcal{K}^n$ be given as in Theorem 6.11. Assume that h is a homeomorphism, that $\mathcal{K}$ has a fiber bundle structure, and that (4.30) holds. Let $H : U \times \mathcal{K}(\theta_0) \to \mathcal{K}(U)$ be the associate homeomorphism, where U is a neighborhood of θ_0 in $\mathcal{K}$. Define $h_0 : U \times \mathcal{K}(\theta_0) \to U \times \mathcal{K}^n(\theta_0)$ by

$$h_0(\theta, v) = (\theta, h(\theta_0, v)), \qquad \text{for } (\theta, v) \in U \times \mathcal{K}(\theta_0).$$

Next define $H^n : U \times \mathcal{K}^n(\theta_0) \to \mathcal{K}^n(U)$ by $H^n = h\, H\, h_0^{-1}$, as in

$$U \times \mathcal{K}^n(\theta_0) \xrightarrow{h_0^{-1}} U \times \mathcal{K}(\theta_0) \xrightarrow{H} \mathcal{K}(U) \xrightarrow{h} \mathcal{K}^n(U).$$

Since the factors h, H, and h_0^{-1} are each homeomorphisms, we see that H^n is a homeomorphism. Also, with a direct computation one verifies that $(H^n)^{-1} = h_0\, H^{-1}\, h^{-1}$ satisfies

$$(H^n)^{-1}(\mathcal{K}^n(\theta)) = \{\theta\} \times \mathcal{K}^n(\theta_0).$$

Thus, $\mathcal{K}^n$ has a fiber bundle structure. □

6.5. Homeomorphism property

A proof of the following homeomorphism property, for the autonomous problem in the finite-dimensional setting, appears in [20,21]. The argument we present here, for the full nonautonomous infinite-dimensional problem, uses Lemma 5.6, and it is much shorter. Because of inequalities (6.66) and $\ell_9 \le \frac{1}{8}$, see Theorem 6.11, we see that the restriction of the mapping h to any disk $\mathcal{D}_\rho(\theta, x_0)$ in $\mathcal{K}$ is one-to-one. We will now show that, for small ϵ, the mapping h is globally one-to-one.

Proof of Theorem 5.4. We assume that the conclusions of Theorem 6.11 are in place, and we will use the notation derived above. In particular, we fix ϵ_{10} and δ_{10} as given above. Let L_0 be fixed so that the four inequalities (4.25)–(4.28) hold. We restrict ϵ and $\delta = \delta(\epsilon)$ so that $0 < \epsilon \le \epsilon_{10}$ and $0 < \delta \le \delta_{10}$. We also ask that ϵ satisfy $4L_0\epsilon \le 1$. Since the semiflow π_1 has a proper negative continuation on $\mathcal{K}$, there is a $\rho > 0$ and a $\tau \in \mathbb{R}$ so that item 2 of Lemma 5.6 holds. We require further that ϵ satisfy $4\epsilon \le \rho$.

Assume by contradiction that the mapping h is not globally one-to-one. Then there exist (θ, v_1), $(\theta, v_0) \in \mathcal{K}$ such that $v_1 \ne v_0$ and $h(v_0) = h(v_1)$. Set $y_0 = h(v_1) = h(v_0)$. From Item 4 of Theorem 6.11 we have $\|A^\beta(y_0 - v_0)\| \le 2\epsilon$ and $\|A^\beta(y_0 - v_1)\| \le 2\epsilon$. Hence one has $\|A^\beta(v_1 - v_0)\| \le 4\epsilon \le \rho$.

Next we show that inequality (5.21) is satisfied. From (6.71) we have

$$\begin{aligned} h(v_0) - v_0 = y_0 - v_0 \in \mathcal{U}^s(v_0) \oplus \mathcal{U}^u(v_0) \quad \text{and} \\ h(v_1) - v_1 = y_0 - v_1 \in \mathcal{U}^s(v_1) \oplus \mathcal{U}^u(v_1). \end{aligned}$$

Therefore, one has $P^o(v_0)(y_0 - v_0) = P^o(v_1)(y_0 - v_1) = 0$. Consequently,

$$\begin{aligned} P^o(v_0)(v_1 - v_0) &= P^o(v_0)(y_0 - v_0) + \big(P^o(v_1) - P^o(v_0)\big)(y_0 - v_1) \\ &= \big(P^o(v_1) - P^o(v_0)\big)(y_0 - v_1). \end{aligned}$$

It then follows from (4.26) that

$$\|A^\beta P^o(v_0)(v_1 - v_0)\| \le L_0 \|A^\beta(v_1 - v_0)\|\, \|A^\beta(y_0 - v_1)\|.$$

Since $\|A^\beta(y_0 - v_1)\| \le 2\epsilon$ and $2L_0\epsilon \le \frac{1}{2}$, we obtain inequality (5.21).

It then follows from Lemma 5.6, Item 2 that there exist $(\theta^1, v_1^1),\ (\theta^1, v_0^1) \in \mathcal{K}$ such that $\theta^1 = \theta \cdot \tau$, for some $\tau \in \mathbb{R}$, and

$$\left\| A^\beta \left(v_1^1 - v_0^1\right) \right\| \geq 2 \| A^\beta (v_1 - v_0) \|.$$

Since $v_i^1 = \widehat{S}(\theta, \tau) v_i$, for $i = 0, 1$, it follows from (5.6) that $h(v_1^1) = h(v_0^1)$. Thus, $\| A^\beta (v_1^1 - v_0^1) \| \leq 4\epsilon$. By applying Lemma 5.6 repeatedly, we obtain sequences $(\theta^m, v_1^m),\ (\theta^m, v_0^m) \in \mathcal{K}$ such that $h(v_1^m) = h(v_0^m)$ and

$$\left\| A^\beta \left(v_1^m - v_0^m\right) \right\| \geq 2^m \| A^\beta (v_1 - v_0) \|. \tag{6.75}$$

Since $\| A^\beta (v_1^m - v_0^m) \| \leq 4\epsilon$, inequality (6.75) implies that $\| A^\beta (v_1 - v_0) \| = 0$, which contradicts the assumption that $v_1 \neq v_0$. Hence the mapping $h : \mathcal{K} \to \mathcal{K}^n$ is one-to-one. □

7. Concluding remarks

1. General nonautonomous equations: It should be noted that the assumption of quasiperiodicity used in Section 3, and in the sequel, can be relaxed significantly. The toroidal arithmetic was used to derive the formula (3.14). This formula in turn was used to prove Lemma 3.3 and the connection with the crucial Lemma 3.2. The latter lemma is important, for example, in applications to the Navier–Stokes equations. In order to obtain a good linearization theory for these equations, one needs to show that the inertial term – that is, the nonlinear term – is Fréchet differentiable. This requires that the mild solutions $u(t) = S(F, t) u_0$ be strong solutions. One does not have a good linearization theory for the weak solutions of the Leray-Hopf class, see [33, Chapter 6, esp. (61.32)].

On the other hand, if one is willing to accept the added assumption that $F \in C_{\mathrm{Lip};\phi}$, for some $\phi > 0$, into the theory, then the torus T^k, with the twist flow $\theta \cdot t$, can be replaced by any compact, invariant set M with a flow $\sigma(m, t)$. In this case, one then replaces $\theta \cdot t$ with $\sigma(m, t)$, for $m \in M$ and $t \in \mathbb{R}$, and proceeds with no other changes.

2. Open issues: We note that the $C^{1,1}$-property for the perturbed foliated bundle is an open issue, as is the Lipschitz property for the exponential trichotomy on the perturbed foliated bundle. Also, it should be noted that the Lipschitz property for the perturbed foliated bundle need not always hold. See [2,27,25], for example.

3. The Gronwall–Henry inequality: We let r and c be positive real numbers and define

$$E_{r,c}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(c)}{\Gamma(nr + c)} z^{nr}, \quad E'_{r,c}(z) = \frac{d}{dz} E_{r,c}(z) = \sum_{n=1}^{\infty} \frac{nr\,\Gamma(c)}{\Gamma(nr + c)} z^{nr-1},$$

where Γ is the Gamma function. The proof of the following result can be found in [12, p. 188] and [33, p. 625].

Lemma 7.1 (Gronwall–Henry inequality). *Let $v(t)$ be a nonnegative function in $L^{\infty}_{loc}[0, \tau; R)$ and $h(\cdot) \in L^{1}_{loc}[0, \tau; R)$ satisfy*

$$v(t) \leq h(t) + M \int_0^t (t-s)^{r-1} v(s)\, ds, \qquad t \in (0, \tau),$$

where $0 < \tau \leq \infty$ and $r > 0$. Then one has

$$v(t) \leq h(t) + \mu \int_0^t E'_{r,1}(\mu(t-s))h(s)\, ds, \qquad t \in (0, \tau), \tag{7.1}$$

where $\mu^r = M\Gamma(r)$. If, in addition, one has $h(t) \equiv at^{c-1}$, where a and c are positive constants, then

$$v(t) \leq at^{c-1}\, E_{r,c}(\mu t), \qquad t \in (0, \tau). \tag{7.2}$$

Moreover, if $h(t) = ae^{\lambda t}$, where $\lambda > \mu$, then

$$v(t) \leq a(1-\theta)^{-1} e^{\lambda t}, \qquad \text{for } t \in (0, \tau), \tag{7.3}$$

where $\theta = \mu^r \lambda^{-r}$. Finally, one has

$$\lim_{t\to\infty} = \frac{1}{t} \log(E_{r,c}(\mu t)) = \mu, \qquad \text{whenever } r > 0 \text{ and } c > 0. \tag{7.4}$$

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