Annali di Matematica 185, S293–S313 (2006)

Digital Object Identifier (DOI) 10.1007/s10231-004-0146-0

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On the order of magnitude of the baroclinic flow in the primitive equations of the ocean

Received: February 8, 2004
Published online: May 20, 2005 – © Springer-Verlag 2005

Abstract. In this article, we study the baroclinic flow in the primitive equations (PEs) of the ocean, which are known to be the fundamental equations of the ocean, [4]–[8]. We prove that the magnitude of the baroclinic flow in the L^2-norm is of order $O(\delta)$, where δ is the aspect ratio of the ocean. Some numerical simulations of the PEs of the ocean consistent with these estimates are also presented.

Mathematics Subject Classification (1991). 65N12, 49K35, 76D55

Key words. primitive equations – geostrophic – asymptotic – barotropic flow – baroclinic flow

1. Introduction

Seawater is a slightly compressible fluid; its motion and state are governed by the general hydrodynamical equations and the diffusion equations of temperature and salinity. The general hydrodynamical equations can be approximated by the Boussinesq equations (BEs) or the primitive equations (PEs), which are derived from the BEs by replacing the vertical momentum equation by the hydrostatic equation, thanks to the fact that the ratio H/L between the vertical and the horizontal scales is very small, [4–8, 13], and by assuming that the density ρ is constant except in the buoyancy term. The BEs and the PEs of the ocean are the core equations of the large-scale ocean; their mathematical analysis was conducted in [4]. However, the phenomena of ocean dynamics are very complicated. To understand in detail the structure of the ocean, scientists rely on simplified models both from the physical and mathematical points of view.

There are two essential characteristics of the ocean that are used in simplifying the PEs or the BEs of the ocean. The first one is that, for large-scale geostrophysical flows, the ratio δ between the vertical and the horizontal scales (called the aspect ratio) is very small. Another small parameter is the Rossby number ϵ, which is the

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ratio of the speed of (horizontal) wind to the speed of rotation of the earth around the poles axis. For the ocean, these numbers are of order 10^{-3}.

In numerical ocean modeling, it is very common in oceanography to treat differently the dynamics of the depth average flow $\bar{u} = \frac{1}{h}\int_{-h}^{0} u\, dz$ (called barotropic flow) and that of the departure $u^\flat = u - \bar{u}$ (called baroclinic flow). In fact, the barotropic and baroclinic adjustment processes in the ocean have quite different characteristic time scales. Barotropic adjustment occurs on the order of days, while it takes years for the adjustment of slow planetary waves in the midlatitude [2]. Motivated by this time scale split, oceanographers use the idea of differential time stepping for the barotropic and the baroclinic equations when developing algorithms for solving the PEs of the ocean. Additionally, interaction between baroclinic and the barotropic waves are thought to be unimportant from a climatic perspective, at least insofar as one needs to explicitly and accurately resolve them [2].

In this article we study the baroclinic flow of the PEs of the ocean and prove that its magnitude (in the L^2-norm) is of order $O(\delta)$ for the PEs of the ocean with both continuous density stratification and with double diffusions. Let us recall that for the mesoscale or synoptic scale ocean, $\delta = O(10^{-3})$.

The article is organized as follows. In the next section, we recall from [9] and [8] the PEs of the ocean with continuous density stratification and the geostrophic scaling. Using some a priori estimates, we prove that the magnitude (in the L^2-norm) of the baroclinic flow is of order $O(\delta)$. The third section presents a similar result for the baroclinic flow of the the PEs of the ocean with double diffusions. Some numerical simulations of the PEs of the ocean consistent with these estimates are presented in the fourth section.

2. The PEs of the ocean and the geostrophic scaling

It is well known that the ocean is made of a slightly compressible fluid subject to the Coriolis force. It is also well accepted that the Boussinesq equations govern the motion and state of seawater. Furthermore, if we take into account that the depth of the ocean is small compared to the radius of the earth, the Boussinesq equations are well approximated by the PEs. In this section, we recall the PEs of the ocean and their geostrophic scaling following [8].

2.1. The PEs of the oceans. We first recall the primitive equations of the ocean in the presence of a stratification. We write the total density ρ_{tot} and total pressure p_{tot} in the form

$$\rho_{tot} = \rho_s(z) + \tilde{\rho}, \quad p_{tot} = p_s(z) + \tilde{p}, \tag{2.1}$$

where the vertical mean pressure $p_s(z)$ and density $\rho_s(z)$ satisfy the equation

$$\frac{\partial p_s(z)}{\partial z} = -\rho_s(z) g \tag{2.2}$$

and $\tilde{p}$ and $\tilde{\rho}$ are spatially and temporally varying departures from the standard values p_s and ρ_s, respectively.

Then assuming the β-plane approximation, the PEs of the ocean in dimensional form read (see [8])

$$\text{(2.3)}\quad \begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} - \nu \dfrac{\partial^2 \mathbf{v}}{\partial z^2} + \tilde{f} k \times \mathbf{v} + \dfrac{1}{\rho_0} \operatorname{grad} \tilde{p} + (\mathbf{v}\cdot\nabla)\mathbf{v} + w\dfrac{\partial \mathbf{v}}{\partial z} = 0, \\ \dfrac{\partial \tilde{p}}{\partial z} = -g\tilde{\rho}, \\ \operatorname{div} \mathbf{v} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial \tilde{\rho}}{\partial t} - \mu_T \Delta \tilde{\rho} - \nu_T \dfrac{\partial^2 \tilde{\rho}}{\partial z^2} + (\mathbf{v}\cdot\nabla)\tilde{\rho} + w\dfrac{\partial \tilde{\rho}}{\partial z} + w\dfrac{\partial \rho_s}{\partial z} = \nu_T \dfrac{\partial^2 \rho_s}{\partial z^2} + \dfrac{\alpha\rho_0}{C_p}\tilde{Q}. \end{cases}$$

The boundary conditions are given by

$$\text{(2.4)}\quad \begin{cases} \rho_0 \nu \dfrac{\partial \mathbf{v}}{\partial z} = \tilde{\tau}_v,\ w = 0,\ C_p \rho_0 \nu_T \dfrac{\partial \tilde{\rho}}{\partial z} = \tilde{\alpha}_T(\tilde{\rho}^* - \tilde{\rho}), \ \text{ on } \tilde{\Gamma}_i, \\ \dfrac{\partial \mathbf{v}}{\partial z} = 0,\ w = 0,\ \dfrac{\partial \tilde{\rho}}{\partial z} = 0 \text{ on } \tilde{\Gamma}_b, \\ \mathbf{v} = 0,\ \dfrac{\partial \tilde{\rho}}{\partial n} = 0 \ \text{ on } \tilde{\Gamma}_l. \end{cases}$$

The initial conditions are given by

$$\text{(2.5)}\qquad (\mathbf{v}, \tilde{\rho}) = (\mathbf{v}_0, \tilde{\rho}_0) \text{ at } t = 0.$$

Here $\tilde{\rho}^*$ is a given function representing the apparent density distribution on the upper surface of the ocean; $\tilde{\tau}_v$ is the (given) wind stress, which drives the motion of the ocean; n is the unit outward normal on $\tilde{\Gamma}_l$; and k is the vertical unit vector. Let us recall that the boundary of the domain $\tilde{\mathcal{M}}$ consists of the following three parts:

$$\text{(2.6)}\quad \begin{aligned} &\tilde{\Gamma}_i(z=0) = \text{ upper boundary of the ocean (interface with air)}, \\ &\tilde{\Gamma}_l = \text{ lateral boundary}, \\ &\tilde{\Gamma}_b(z=-\tilde{h}) = \text{ bottom of the ocean}. \end{aligned}$$

In (2.3)–(2.5), the unknown functions are the horizontal velocity $\mathbf{v} = (u, v)$, the vertical velocity w, the density $\tilde{\rho}$, and the pressure $\tilde{p}$. The positive constant C_p is the heat capacity of the ocean, ρ_0 is the reference value of the density, g is the gravitational constant, $\tilde{f}$ is the Coriolis parameter, and $\tilde{Q}$ is the (given) rate of internal heating that vanishes in the ocean and is introduced here for mathematical generality. For more details on the PEs of the ocean, the reader is referred to [3, 9, 10, 14] for the physical aspect and to [4], in which the existence results for the system (2.3)–(2.5) is studied; see also [12].

Here the differential operators ∇, Δ, and div are all (2D) horizontal operators acting on the variables x and y. The positive constants ν and μ are the viscosity coefficients, and $\nu_T > 0$ and $\mu_T > 0$ are the thermal diffusivity.

The PEs (2.3)–(2.5) are derived from the Boussinesq equation using the fact that the aspect ratio δ (ratio between the horizontal and the vertical length scales) is small [4, 8, 9].

2.2. Geostrophic scaling. Here we first recall from [8] a standard scaling for the PEs of the ocean. We set

$$(x, y, z, t) = \left(Lx', Ly', Hz', \frac{L}{U}t'\right), \tag{2.7}$$

where U is the reference value of the horizontal velocity.

We also set

$$\mathbf{v} = U\mathbf{v}',\ w = \frac{H}{L}Uw',\ \tilde{h} = Hh, \tag{2.8}$$

$$\begin{aligned} &p_{tot} = p_s(z) + L\rho_0 \tilde{f}_0 U p',\ \rho_{tot} = \rho_s(z) + \epsilon F \rho_0 \rho',\ \tilde{f}_0 = 2\Omega\cos\theta_0, \\ &\tilde{f} = \tilde{f}_0(1+\epsilon f_1),\ f_1 = \frac{1}{\epsilon}\frac{\cos\theta - \cos\theta_0}{\cos\theta_0} = O(1)^1,\quad f = 1 + \epsilon f_1, \end{aligned} \tag{2.9}$$

where $F = \tilde{f}_0^2 L^2/gH$ is the Froude number, $\delta = H/L$ is the aspect ratio, $\epsilon = U/\tilde{f}_0 L$ is the Rossby number, and Ω is the angular velocity of the earth.

Other nondimensional parameters are given by

$$\begin{cases} \dfrac{1}{R_{e_1}} = \dfrac{\mu}{LU},\ \dfrac{1}{R_{e_2}} = \dfrac{\nu}{LU}, \\ \dfrac{1}{R_{t_1}} = \dfrac{\mu_T}{LU},\ \dfrac{1}{R_{t_2}} = \dfrac{\nu_T}{LU}, \\ \rho^* = \dfrac{\tilde{\rho}^*}{\epsilon F \rho_0},\ \alpha_T = \dfrac{L\tilde{\alpha}_T}{C_p H U \rho_0},\ \tau_v = \dfrac{L}{\rho_0 U^2 H}\tilde{\tau}_v. \end{cases} \tag{2.10}$$

We introduce the nondimensional function

$$F_2 = \frac{\nu_T g}{H\tilde{f}_0 U^2 \rho_0}\frac{\partial^2 \rho_s}{\partial z^2} + \frac{gH\alpha}{\tilde{f}_0 U^2 C_p}\tilde{Q}. \tag{2.11}$$

Hereafter we assume that

$$F_2 = O(1). \tag{2.12}$$

The nondimensional space domain $\mathcal{M}$ has the form

$$\begin{aligned} &\mathcal{M} = \{(x, y, z);\ (x, y) \in \mathcal{M}_s,\ -h < z < 0\}, \\ &\mathcal{M}_s \subset \left\{(x, y);\ |x| < \frac{1}{2},\ |y| < \frac{1}{2}\right\}. \end{aligned} \tag{2.13}$$

We denote by Γ_i, Γ_b, and Γ_l the boundaries of $\mathcal{M}$ given by

$$\begin{aligned} &\Gamma_i(z=0) = \text{ upper boundary of the ocean (interface with air)}, \\ &\Gamma_l = \text{ lateral boundary}, \\ &\Gamma_b(z=-h) = \text{ bottom of the ocean}. \end{aligned} \tag{2.14}$$

Substituting these expressions of dimensional functions, variables, and parameters in the PEs (2.3), and suppressing the primes in the nondimensional variables, we

[1] That is $|\theta - \theta_0|$ is small, $O(\epsilon)$.

obtain the following nondimensional form of the PEs of the ocean (see [8] for more details):

$$
(2.15)\quad \begin{cases} \epsilon\left[\dfrac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}+w\dfrac{\partial \mathbf{v}}{\partial z}\right]-\dfrac{\epsilon}{R_{e_1}}\Delta\mathbf{v}-\dfrac{\epsilon}{\delta^2 R_{e_2}}\dfrac{\partial^2\mathbf{v}}{\partial z^2}+fk\times\mathbf{v}+\operatorname{grad} p=0,\\ \dfrac{\partial p}{\partial z}=-\rho,\\ \operatorname{div}\mathbf{v}+\dfrac{\partial w}{\partial z}=0,\\ \epsilon\left[\dfrac{\partial \rho}{\partial t}+(\mathbf{v}\cdot\nabla)\rho+w\dfrac{\partial \rho}{\partial z}\right]-\dfrac{\epsilon}{R_{t_1}}\Delta\rho-\dfrac{\epsilon}{\delta^2 R_{t_2}}\dfrac{\partial^2\rho}{\partial z^2}-\sigma w=\epsilon F_2, \end{cases}
$$

with

$$
(2.16)\qquad \sigma=\sigma(z)\equiv-\frac{1}{\rho_0 F}\frac{\partial\rho_s}{\partial z}>0 \text{ (assuming stable stratification, [8]).}
$$

The boundary conditions (2.4) become

$$
(2.17)\quad \begin{cases} \dfrac{1}{\delta^2 R_{e_2}}\dfrac{\partial \mathbf{v}}{\partial z}=\tau_v,\ w=0,\ \dfrac{1}{\delta^2 R_{t_2}}\dfrac{\partial\rho}{\partial z}=\alpha_T(\rho^*-\rho) \text{ on } \Gamma_i,\\ \dfrac{\partial \mathbf{v}}{\partial z}=0,\ w=0,\ \dfrac{\partial\rho}{\partial z}=0 \text{ on } \Gamma_b,\\ \mathbf{v}=0,\ \dfrac{\partial\rho}{\partial n}=0 \text{ on } \Gamma_l. \end{cases}
$$

The initial conditions are

$$
(2.18)\qquad (\mathbf{v},\rho)=(\mathbf{v}_0,\rho_0) \text{ at } t=0.
$$

From $(2.15)_2$, $(2.15)_3$, and (2.17) we derive

$$
(2.19)\qquad \begin{aligned} &p=p_s+\int_z^0\rho\,dz,\\ &w=\int_z^0\operatorname{div}\mathbf{v}\,dz\equiv W(\mathbf{v}),\\ &\int_{-h}^0\operatorname{div}\mathbf{v}\,dz=0. \end{aligned}
$$

Then equations (2.15) become

$$
(2.20)\quad \begin{cases} \epsilon\left[\dfrac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}+W(\mathbf{v})\dfrac{\partial \mathbf{v}}{\partial z}\right]-\dfrac{\epsilon}{R_{e_1}}\Delta\mathbf{v}-\dfrac{\epsilon}{\delta^2 R_{e_2}}\dfrac{\partial^2\mathbf{v}}{\partial z^2}+fk\times\mathbf{v}\\ \qquad\qquad+\operatorname{grad} p_s+\operatorname{grad}\displaystyle\int_z^0\rho\,dz=0,\\ \displaystyle\int_{-h}^0\operatorname{div}\mathbf{v}\,dz=0,\\ \epsilon\left[\dfrac{\partial \rho}{\partial t}+(\mathbf{v}\cdot\nabla)\rho+W(\mathbf{v})\dfrac{\partial \rho}{\partial z}\right]-\dfrac{\epsilon}{R_{t_1}}\Delta\rho-\dfrac{\epsilon}{\delta^2 R_{t_2}}\dfrac{\partial^2\rho}{\partial z^2}-\sigma W(\mathbf{v})=\epsilon F_2. \end{cases}
$$

Remark 2.1. Hereafter, for simplicity we will consider only homogeneous boundary conditions and we will assume that the functions ρ^* and τ_v in (2.17) are identically zero, that is, $\rho^* = \tau_v = 0$; see Remark 3.2 for the nonhomogeneous case at the end of Section 3. Let us mention that with the boundary data of the form $\tau_v = (c\cos 2\pi y, 0)$, which is commonly used in oceanography to simulate the double-gyre phenomenon, the boundary conditions (2.17) present a discontinuity on $\partial\Gamma_i$; since $\partial\mathbf{v}/\partial z = 0$ on $\partial\Gamma_i$ by $(2.17)_2$, and $\partial v/\partial z = \delta^2 R_{e_2}\tau_v \neq 0$ by $(2.17)_1$. This discontinuity may affect the accuracy/stability of the numerical schemes if special care is not taken. To overcome this problem, it is common in oceanography to take $\tau_v = 0$ and to compensate it with a body force $F_1 = g(z)\Gamma_v$ in the momentum equations. We will also consider the general case where the zero forcing term on the right-hand side of $(2.20)_1$ is replaced by a given function ϵF_1. To simplify the analysis, we also assume that σ given by (2.16) is constant. Some specific empirical formulations for p_s and ρ_s can be obtained from physical and numerical considerations. A typical density profile is given in [9], and a linear approximation $\rho_s(z) = \rho_0 - bz$ $(b > 0)$ for the density profile was used in [1].

2.3. Function spaces. We first introduce the following function spaces based on the space $L^2(\mathcal{M})$ and the Sobolev space $H^1(\mathcal{M})$:

$$V_1 = \left\{\mathbf{v} \in (H^1(\mathcal{M}))^2, \int_{-h}^{0} \operatorname{div} \mathbf{v} dz = 0,\ \mathbf{v} = 0 \text{ on } \Gamma_l\right\}. \tag{2.21}$$

Let $V_2 = H^1(\mathcal{M})$, $V = V_1 \times V_2$ and let H be the closure of V in $(L^2(\mathcal{M}))^3$. Hereafter we denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\mathcal{M})$ or in $(L^2(\mathcal{M}))^n$. The associated norm will be denoted by $|\cdot|_{L^2}$, and $\|\cdot\|$ will denote the norm in $H^1(\mathcal{M})$ or $(H^1(\mathcal{M}))^n$ for any positive integer n.

Now we define the following forms:

$$a_1(\mathbf{v}, \tilde{\mathbf{v}}) = \int_{\mathcal{M}} \left\{\frac{1}{R_{e_1}}\nabla\mathbf{v}\nabla\tilde{\mathbf{v}} + \frac{1}{\delta^2 R_{e_2}}\frac{\partial\mathbf{v}}{\partial z}\frac{\partial\tilde{\mathbf{v}}}{\partial z}\right\} d\mathcal{M}, \tag{2.22}$$

$$a_2(\rho, \tilde{\rho}) = \int_{\mathcal{M}} \sigma\left\{\frac{1}{R_{t_1}}\nabla\rho\nabla\tilde{\rho} + \frac{1}{\delta^2 R_{t_2}}\frac{\partial\rho}{\partial z}\frac{\partial\tilde{\rho}}{\partial z}\right\} d\mathcal{M} + \int_{\Gamma_i} \alpha_T\sigma\rho\tilde{\rho}d\Gamma_i, \tag{2.23}$$

$$e_1(\mathbf{v}, \tilde{\mathbf{v}}) = \int_{\mathcal{M}} (fk \times \mathbf{v})\cdot\mathbf{v}d\mathcal{M}, \tag{2.24}$$

$$b_1(\mathbf{v}, \tilde{\mathbf{v}}, \hat{\mathbf{v}}) = \int_{\mathcal{M}} \left[(\mathbf{v}\cdot\nabla)\tilde{\mathbf{v}} + W(\mathbf{v})\frac{\partial\tilde{\mathbf{v}}}{\partial z}\right]\cdot\hat{\mathbf{v}}d\mathcal{M}, \tag{2.25}$$

$$b_2(\mathbf{v}, \tilde{\rho}, \hat{\rho}) = \int_{\mathcal{M}} \sigma\left[(\mathbf{v}\cdot\nabla)\tilde{\rho} + W(\mathbf{v})\frac{\partial\tilde{\rho}}{\partial z}\right]\cdot\hat{\rho}d\mathcal{M}, \tag{2.26}$$

$$r_1(\rho, \tilde{\mathbf{v}}) = \int_{\mathcal{M}} \left(\nabla\int_{z}^{0}\rho\, dz\right)\cdot\tilde{\mathbf{v}}d\mathcal{M},\ r_2(\mathbf{v}, \tilde{\rho}) = \int_{\mathcal{M}} W(\mathbf{v})\tilde{\rho}d\mathcal{M}, \tag{2.27}$$

$$a(\mathbf{u}, \tilde{\mathbf{u}}) = a_1(\mathbf{v}, \tilde{\mathbf{v}}) + a_2(\rho, \tilde{\rho}),\ r(\mathbf{u}, \tilde{\mathbf{u}}) = r_1(\rho, \tilde{\mathbf{v}}) + r_2(\mathbf{v}, \tilde{\rho}),$$
$$b(\mathbf{u}, \tilde{\mathbf{u}}, \hat{\mathbf{u}}) = b_1(\mathbf{v}, \tilde{\mathbf{v}}, \hat{\mathbf{v}}) + b_2(\mathbf{v}, \tilde{\rho}, \hat{\rho}),\ e(\mathbf{u}, \tilde{u}) = e_1(\mathbf{v}, \tilde{\mathbf{v}}), \tag{2.28}$$

for $\mathbf{u} = (\mathbf{v}, \rho)$, $\tilde{\mathbf{u}} = (\tilde{\mathbf{v}}, \tilde{\rho})$, and $\hat{\mathbf{u}} = (\hat{\mathbf{v}}, \hat{\rho})$.

We also define the following operator A_0 by

$$\begin{aligned} A_0 : H &\longmapsto H \\ A_0(\mathbf{v}, \rho) &= (\mathbf{v}, \sigma\rho). \end{aligned} \tag{2.29}$$

Proposition 2.1. *The forms a and $a_i (i = 1, 2)$ are coercive, continuous on $V \times V$ and $V_i \times V_i$, respectively. Moreover, we have*

$$\begin{aligned} &a_1(\mathbf{v}, \mathbf{v}) \geq c_1\|\mathbf{v}\|^2,\ \forall \mathbf{v} \in V_1,\ a_2(\rho, \rho) \geq c_2\|\rho\|^2, \forall \rho \in V_2, \\ &a(\mathbf{u}, \mathbf{u})\ \geq c_0\|\mathbf{u}\|^2,\ \forall \mathbf{u} \in V, \end{aligned} \tag{2.30}$$

where $c_1 = 1/c_1'$, $c_1' = cMax(R_{e_1}, R_{e_2})$, $c_2 = 1/c_2'$, $c_2' = cMax(R_{t_1}, R_{t_2})$, $c_0 = 1/c_0'$, $c_0' = Max(c_1', c_2')$, and c is an absolute constant independent of δ and of physically relevant constants such as α_T, R_{e_i}, R_{t_i}.

Furthermore, we have

$$\alpha_1|\mathbf{u}|_{L^2}^2 \leq \langle A_0\mathbf{u}, \mathbf{u}\rangle \leq \alpha_2|\mathbf{u}|_{L^2}^2,\ \forall \mathbf{u} \in H, \tag{2.31}$$

where $\alpha_1 = Min(1, \sigma), \alpha_2 = Max(1, \sigma)$.

Proof. The coercivity of a follows from the Poincaré inequality, while that of a_2 follows from the term $\rho\tilde{\rho}$ that appears in (2.23).

To prove $(2.30)_1$, we note that

$$\begin{aligned} a_1(\mathbf{v}, \mathbf{v}) &\geq cMin\Big(\frac{1}{R_{e_1}}, \frac{1}{\delta^2 R_{e_2}}\Big)\left(|\nabla\mathbf{v}|_{L^2}^2 + \left|\frac{\partial \mathbf{v}}{\partial z}\right|_{L^2}^2\right) \\ &\geq cMin\Big(\frac{1}{R_{e_1}}, \frac{1}{R_{e_2}}\Big)\left(|\nabla\mathbf{v}|_{L^2}^2 + \left|\frac{\partial \mathbf{v}}{\partial z}\right|_{L^2}^2\right) \text{ (since } \delta < 1), \\ &\geq c_1\|\mathbf{v}\|^2, \end{aligned} \tag{2.32}$$

and $(2.30)_1$ follows. The proof of $(2.30)_2$, $(2.30)_3$, or (2.31) is similar. The continuity of the forms a and a_i is obvious.

Hereafter we denote by c a generic constant that is independent of physically relevant constants such as α_T, R_{e_i}, R_{t_i} and c_4 will denote a generic constant independent of the aspect ratio δ.

Proposition 2.2. *For $(\mathbf{u}, \tilde{\mathbf{u}}) \in V \times V$, we have*

$$b(\mathbf{u}, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}) = 0,\ e(\mathbf{u}, \mathbf{u}) = 0,\ r(\mathbf{u}, \mathbf{u}) = 0. \tag{2.33}$$

Moreover, the forms r, r_i, e, e_i are continuous on $V \times V$, $V_i \times V_i$, $V \times V$, and $V_i \times V_i$, respectively.

Proof. The complete proof is given in [8]; therefore, we omit the details. It is clear that $(2.33)_1$ and $(2.33)_2$ are satisfied. To prove $(2.33)_3$, we replace $\tilde{\mathbf{v}}$ by $\mathbf{v}$ in $(2.27)_1$ and integrate as follows:

$$\begin{aligned} r_1(\rho, \mathbf{v}) &= \left\langle \nabla \int_z^0 \rho\, dz, \mathbf{v} \right\rangle \\ &= \left\langle - \int_z^0 \rho\, dz,\ \operatorname{div}\mathbf{v} \right\rangle \\ &= \left\langle \rho, \int_z^0 \operatorname{div} \mathbf{v} \right\rangle = -r_2(\mathbf{v}, \rho), \end{aligned} \tag{2.34}$$

and $(2.33)_3$ follows. The continuity of the forms r, r_i, e, e_i is obvious. □

Now we consider the following variational formulation of the PEs (2.20), (2.17)–(2.18), with a source term ϵF_1, $F_1 = O(1)$, added to the right-hand side of $(2.20)_1$:

Problem 1. For $\mathbf{u}_0 = (\mathbf{v}_0, \rho_0) \in H$, find $\mathbf{u} = (\mathbf{v}, \rho)$ such that

$$\begin{aligned} &\epsilon \frac{d}{dt} \langle A_0 \mathbf{u}, \tilde{\mathbf{u}} \rangle + \epsilon a(\mathbf{u}, \tilde{\mathbf{u}}) + \epsilon b(\mathbf{u}, \mathbf{u}, \tilde{\mathbf{u}}) + e(\mathbf{u}, \tilde{\mathbf{u}}) + r(\mathbf{u}, \tilde{\mathbf{u}}) \\ &= \epsilon \langle A_0(F_1, F_2), \tilde{\mathbf{u}} \rangle,\ \forall \tilde{\mathbf{u}} \in V, \mathbf{u} = \mathbf{u}_0 \text{ at } t = 0. \end{aligned} \tag{2.35}$$

The existence of weak solutions to Problem 1 was studied in [8], in which the following result was proved.

Proposition 2.3. *For any $T > 0$, there exists at least one solution $\mathbf{u} = (\mathbf{v}, \rho)$ defined on $(0, T)$ for the system (2.35), such that*

$$\mathbf{u} \in C([0, T], H_w), \tag{2.36}$$

where H_w is the space H endowed with the weak topology.

Some a priori estimates

Replacing $\tilde{\mathbf{u}}$ by $\mathbf{u}$ in (2.35) yields

$$\epsilon \left\langle \frac{d}{dt} A_0 \mathbf{u}, \mathbf{u} \right\rangle + \epsilon \langle A\mathbf{u}, \mathbf{u} \rangle = \epsilon \langle A_0(F_1, F_2), \mathbf{u} \rangle, \tag{2.37}$$

from which we derive

$$\begin{aligned} &\left\langle \frac{d}{dt} A_0 \mathbf{u}, \mathbf{u} \right\rangle + \frac{1}{R_{e_1}} |\nabla \mathbf{v}|_{L^2}^2 + \frac{1}{\delta^2 R_{e_2}} \left| \frac{\partial \mathbf{v}}{\partial z} \right|_{L^2}^2 + \frac{1}{R_{t_1}} |\nabla \rho|_{L^2}^2 \\ &+ \frac{1}{\delta^2 R_{t_2}} \left| \frac{\partial \rho}{\partial z} \right|_{L^2}^2 + \alpha_T \int_{\Gamma_i} \rho^2 d\Gamma_i \leq |F_1|_{L^2} |\mathbf{v}|_{L^2} + |\sigma F_2|_{L^2} |\rho|_{L^2}, \end{aligned} \tag{2.38}$$

and

$$\begin{aligned}
&\left\langle \frac{d}{dt} A_0 \mathbf{u}, \mathbf{u} \right\rangle + \frac{1}{R_{e_1}} |\nabla \mathbf{v}|_{L^2}^2 + \frac{1}{\delta^2 R_{e_2}} \left| \frac{\partial \mathbf{v}}{\partial z} \right|_{L^2}^2 + \frac{c_2}{2} \|\rho\|^2 \\
&+ \frac{1}{2R_{t_1}} |\nabla \rho|_{L^2}^2 + \frac{1}{2\delta^2 R_{t_2}} \left| \frac{\partial \rho}{\partial z} \right|_{L^2}^2 \le |F_1|_{L^2} |\mathbf{v}|_{L^2} + |\sigma F_2|_{L^2} |\rho|_{L^2} \\
&\le \frac{1}{2R_{e_1}} |\nabla \mathbf{v}|_{L^2}^2 + c|F_1|_{L^2}^2 + \frac{c_2}{2} \|\rho\|^2 + \frac{c}{c_2} |\sigma F_2|_{L^2}^2.
\end{aligned} \tag{2.39}$$

Then

$$\begin{aligned}
&\left\langle \frac{d}{dt} A_0 \mathbf{u}, \mathbf{u} \right\rangle + \frac{1}{2R_{e_1}} |\nabla \mathbf{v}|_{L^2}^2 + \frac{1}{2\delta^2 R_{e_2}} \left| \frac{\partial \mathbf{v}}{\partial z} \right|_{L^2}^2 \\
&+ \frac{1}{2R_{t_1}} |\nabla \rho|_{L^2}^2 + \frac{1}{2\delta^2 R_{t_2}} \left| \frac{\partial \rho}{\partial z} \right|_{L^2}^2 \le c|F_1|_{L^2}^2 + \frac{c}{c_2} |\sigma F_2|_{L^2}^2.
\end{aligned} \tag{2.40}$$

Finally, using (2.31) we derive

$$|\mathbf{u}(t)|_{L^2}^2 \le c \left(|\mathbf{u}_0|_{L^2}^2 + \int_0^T |F_1|_{L^2}^2 dt + \frac{1}{c_2} \int_0^T |F_2|_{L^2}^2 dt \right) \equiv K_1 \tag{2.41}$$

and

$$\frac{1}{\delta^2 R_{e_2}} \int_0^T \left| \frac{\partial \mathbf{v}}{\partial z} \right|_{L^2}^2 dt \le K_1, \ \frac{1}{\delta^2 R_{t_2}} \int_0^T \left| \frac{\partial \rho}{\partial z} \right|_{L^2}^2 dt \le K_1. \tag{2.42}$$

2.4. Barotropic-baroclinic splitting. Hereafter, for a given function u we set:

$$\bar{u} = \frac{1}{h} \int_{-h}^0 u \, dz, \ u^\flat = u - \bar{u}. \tag{2.43}$$

In oceanography, the vertical average $\bar{u}$ is referred to as the barotropic flow and $u^\flat$ is called the baroclinic flow [3,14]. The following properties hold.

Proposition 2.4. *The following properties hold true:*

$$\begin{aligned}
&\langle \bar{u}, v^\flat \rangle = 0, \ \forall u, v \in L^2(\mathcal{M}), \\
&|u^\flat|_{L^2} \le 2h \left| \frac{\partial u}{\partial z} \right|_{L^2}, \ \forall u \in H^1(\mathcal{M}), \\
&\langle \Delta \bar{u}, v^\flat \rangle = 0, \ \forall u \in H^2(\mathcal{M}), \forall v \in L^2(\mathcal{M}), \\
&\left\langle \bar{u}, \Delta v^\flat + \frac{\partial^2 v^\flat}{\partial z^2} \right\rangle = 0, \ \forall u \in L^2(\mathcal{M}), \forall v \in H^2(\mathcal{M}) \\
&\qquad\qquad \textit{satisfying } \frac{\partial v}{\partial z} = 0 \ \textit{ on } \Gamma_i \cup \Gamma_b, \\
&\left\langle \Delta \bar{u}, \Delta v^\flat + \frac{\partial^2 v^\flat}{\partial z^2} \right\rangle = 0, \ \forall u \in H^2(\mathcal{M}), \forall v \in H^2(\mathcal{M}) \\
&\qquad\qquad \textit{satisfying } \frac{\partial v}{\partial z} = 0 \ \textit{ on } \Gamma_i \cup \Gamma_b.
\end{aligned} \tag{2.44}$$

Proof. $(2.44)_1$, $(2.44)_3$, and $(2.44)_4$ are clear by definition of $\bar{u}$ and $v^\flat$. For $(2.44)_2$, we notice that

$$\int_{-h}^{0} |u^\flat|_{L^2}^2 dz \le 4h^2 \int_{-h}^{0} \left(\frac{\partial u^\flat}{\partial z}\right)^2 dz = 4h^2 \int_{-h}^{0} \left(\frac{\partial u}{\partial z}\right)^2 dz \tag{2.45}$$

since

$$\int_{-h}^{0} u^\flat \, dz = 0. \tag{2.46}$$

Therefore, $(2.43)_2$ follows from (2.45).

For $(2.44)_4$, we notice that

$$\int_{\mathcal{M}} \left(-\Delta\bar{u} - \frac{\partial^2 \bar{u}}{\partial z^2}\right)\left(-\Delta v^\flat - \frac{\partial^2 v^\flat}{\partial z^2}\right) = \int_{\mathcal{M}} -\Delta\bar{u}\left(-\Delta v^\flat - \frac{\partial^2 v^\flat}{\partial z^2}\right) = 0 \tag{2.47}$$

since

$$\int_{-h}^{0} v^\flat \, dz = 0, \ \int_{-h}^{0} \frac{\partial^2 v^\flat}{\partial z^2} dz = 0 \text{ if } \frac{\partial v}{\partial z} = 0 \text{ on } \Gamma_i \cup \Gamma_b. \tag{2.48}$$

Therefore, $(2.44)_4$ follows from (2.48).

Proposition 2.5. *Let* $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}^\flat$ *be a solution of (2.35) given by Proposition 2.3. Then there exists a constant* $c_4 = c_4(F_1, F_2, R_{e_1}, R_{e_2}, R_{t_1}, R_{t_2}, \mathcal{M}, T)$ *independent of* δ *such that*

$$\int_{0}^{T} |\mathbf{u}^\flat|_{L^2}^2 dt \le c_4 \delta^2. \tag{2.49}$$

Proof. Using (2.42) and $(2.44)_2$ we derive

$$\begin{aligned} \int_{0}^{T} |\mathbf{u}^\flat|_{L^2}^2 dt &\le 4h^2 \int_{0}^{T} \left|\frac{\partial \mathbf{u}}{\partial z}\right|_{L^2}^2 dt \\ &\le cK_1 \delta^2, \end{aligned} \tag{2.50}$$

which proves (3.42), and we conclude that

$$\left(\int_{0}^{T} |\mathbf{u}^\flat|_{L^2}^2 dt\right)^{\frac{1}{2}} = O(\delta). \tag{2.51}$$

3. The PEs with double diffusions

In this section, we prove that the magnitude (in the L^2-norm) of the baroclinic flow of the PEs of the ocean with double diffusion is of order $O(\delta)$. The steps we follow are similar to those of the previous section; therefore, we will omit the details. We first recall from [8] the PEs of the ocean with double diffusions.

In order to obtain a proper geostrophic scaling of the PEs, we need to consider the standard temperature and salinity profiles, i.e., the mean temperature and salinity distributions. For the mesoscale ocean, it is legitimate to consider the vertical profile of these functions. Let $T_s(z)$ and $S_s(z)$ be the vertical stratification profiles of the temperature T_{tot} and the salinity S_{tot}, respectively, which can be considered as the mean values of T_{tot} and S_{tot} at level z. We refer the reader to [1,9] for some typical profiles of the temperature and salinity functions.

We write the temperature and salinity functions as follows:

$$T_{tot} = T_s(z) + \tilde{T}, \quad S_{tot} = S_s(z) + \tilde{S}, \tag{3.1}$$

$\tilde{T}$ and $\tilde{S}$ being the deviations of T_{tot} and S_{tot} from T_s and S_s, respectively.

We also assume that the density profile ρ_s is such that the following equation of state is satisfied:

$$\rho_s(z) = \rho_0(1 - \beta_T(T_s - T_{ref}) + \beta_S(S_s - S_{ref})), \tag{3.2}$$

where β_T and β_S are expansion coefficients and T_{ref} and S_{ref} are the reference values of T_{tot} and S_{tot}, respectively [8].

We assume that the hydrostatic equation is satisfied. Therefore, we assume the existence of a vertical mean pressure $p_s(z)$ satisfying

$$\frac{\partial p_s(z)}{\partial z} = -\rho_s(z) g, \tag{3.3}$$

where g is the gravitational constant. We also write the total density ρ_{tot} and total pressure p_{tot} in the form (2.1).

Then, assuming the β-plane approximation, the PEs of the ocean in dimensional form read

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} - \nu \dfrac{\partial^2 \mathbf{v}}{\partial z^2} + \tilde{f} k \times \mathbf{v} + \dfrac{1}{\rho_0} \operatorname{grad} \tilde{p} + (\mathbf{v} \cdot \nabla) \mathbf{v} + w \dfrac{\partial \mathbf{v}}{\partial z} = 0, \\ \dfrac{\partial \tilde{p}}{\partial z} = -g \tilde{\rho}, \\ \operatorname{div} \mathbf{v} + \dfrac{\partial w}{\partial z} = 0, \\ \dfrac{\partial \tilde{T}}{\partial t} - \mu_T \Delta \tilde{T} - \nu_T \dfrac{\partial^2 \tilde{T}}{\partial z^2} + (\mathbf{v} \cdot \nabla) \tilde{T} + w \dfrac{\partial \tilde{T}}{\partial z} + w \dfrac{\partial T_s}{\partial z} = \nu_T \dfrac{\partial^2 T_s}{\partial z^2}, \\ \dfrac{\partial \tilde{S}}{\partial t} - \mu_S \Delta \tilde{S} - \nu_S \dfrac{\partial^2 \tilde{S}}{\partial z^2} + (\mathbf{v} \cdot \nabla) \tilde{S} + w \dfrac{\partial \tilde{S}}{\partial z} + w \dfrac{\partial S_s}{\partial z} = \nu_S \dfrac{\partial^2 S_s}{\partial z^2}, \\ \tilde{\rho} = \rho_0(-\beta_T \tilde{T} + \beta_S \tilde{S}). \end{cases} \tag{3.4}$$

The boundary conditions are given by

$$
(3.5)\qquad \begin{cases} \rho_0 \nu \dfrac{\partial \mathbf{v}}{\partial z} = \tilde{\tau}_v,\ w = 0,\ C_p \rho_0 \nu_T \dfrac{\partial \tilde{T}}{\partial z} = \tilde{\alpha}_T (\tilde{T}^* - \tilde{T}), \\ \qquad\qquad\qquad C_p \rho_0 \nu_S \dfrac{\partial \tilde{S}}{\partial z} = \tilde{\alpha}_S (\tilde{S}^* - \tilde{S}) \text{ on } \tilde{\Gamma}_i, \\ \dfrac{\partial \mathbf{v}}{\partial z} = 0,\ \ w = 0,\ \ \dfrac{\partial \tilde{T}}{\partial z} = 0,\ \ \dfrac{\partial \tilde{S}}{\partial z} = 0 \text{ on } \tilde{\Gamma}_b, \\ \mathbf{v} = 0,\ \ \dfrac{\partial \tilde{T}}{\partial n} = 0,\ \ \dfrac{\partial \tilde{S}}{\partial n} = 0 \ \text{ on } \tilde{\Gamma}_l. \end{cases}
$$

Here $\tilde{T}^*$ and $\tilde{S}^*$ are given functions representing the apparent temperature and salinity distribution on the upper surface of the ocean, while $\tilde{\tau}_v$ is the (given) wind stress, which drives the motion of the ocean; n is the outward normal on $\tilde{\Gamma}_l$, and k is the vertical unit vector.

The initial conditions are given by

$$
(3.6)\qquad (\mathbf{v}, \tilde{T}, \tilde{S}) = (\mathbf{v}_0, \tilde{T}_0, \tilde{S}_0) \text{ at } t = 0.
$$

In (3.4)–(3.6), the unknown functions are the horizontal velocity $\mathbf{v} = (u, v)$, the vertical velocity w, the temperature $\tilde{T}$, the salinity $\tilde{S}$, and the pressure $\tilde{p}$. The positive constant C_p is the heat capacity of the ocean, ρ_0 is the reference value of the density, g is the gravitational constant, and $\tilde{f}$ is the Coriolis parameter. For more details on the PEs of the ocean, the reader is referred to [3, 9, 10, 14] for the physical aspect and to [4], in which the existence and uniqueness results of the system (3.4)–(3.6) is studied; see also [12].

Here the positive constants ν and μ are the viscosity coefficients, $\nu_T > 0$ and $\mu_T > 0$ are the thermal diffusivity, and $\nu_S > 0$ and $\mu_S > 0$ are the diffusivity coefficients of the salinity.

The PEs (3.4)–(3.6) is derived from the Boussinesq equation using the fact that the aspect ratio δ (ratio between the horizontal and the vertical length scales) is small [4, 8, 9]. For the nondimensional form of the PEs (2.3)–(3.6), we consider the geostrophic scaling (2.7)–(2.9) and we scale T_s and S_s by

$$
(3.7)\qquad (T_s, S_s) = (T_{ref} T_s', S_{ref} S_s').
$$

For the temperature and salinity deviations, we set

$$
(3.8)\qquad \tilde{T} = \epsilon F T_{ref} T',\ \ \tilde{T} = \epsilon F S_{ref} S'.
$$

Other nondimensional parameters are given by

$$
(3.9)\qquad \begin{cases} \dfrac{1}{R_{e_1}} = \dfrac{\mu}{LU},\ \dfrac{1}{R_{e_2}} = \dfrac{\nu}{LU},\ \dfrac{1}{R_{t_1}} = \dfrac{\mu_T}{LU},\ \dfrac{1}{R_{t_2}} = \dfrac{\nu_T}{LU},\ \dfrac{1}{R_{s_1}} = \dfrac{\mu_S}{LU},\ \dfrac{1}{R_{s_2}} = \dfrac{\nu_S}{LU}, \\ \beta_T = \tilde{\beta}_T T_{ref},\ \beta_S = \tilde{\beta}_S S_{ref},\ T^* = \dfrac{\tilde{T}^*}{\epsilon F T_{ref}},\ S^* = \dfrac{\tilde{S}^*}{\epsilon F S_{ref}},\ \alpha_T = \dfrac{L \tilde{\alpha}_T}{C_p H U \rho_0}, \\ \alpha_S = \dfrac{L \tilde{\alpha}_S}{C_p H U \rho_0},\ \tau_v = \dfrac{L}{\rho_0 U^2 H} \tilde{\tau}_v. \end{cases}
$$

We introduce the nondimensional functions

$$F_2 = \frac{\nu_T g H}{\tilde{f}_0 U^2 T_{ref}} \frac{\partial^2 T_s}{\partial z^2}, \ F_3 = \frac{\nu_S g H}{\tilde{f}_0 U^2 S_{ref}} \frac{\partial^2 S_s}{\partial z^2}. \tag{3.10}$$

Hereafter we assume that

$$F_2 = O(1), \ F_3 = O(1). \tag{3.11}$$

Substituting these expressions of dimensional functions, variables, and parameters in the PEs (3.4), and dropping the primes in the nondimensional variables, we obtain the following nondimensional form of the PEs of the ocean with double diffusion (see [8] and above for more details):

$$\begin{cases} \epsilon \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + w \frac{\partial \mathbf{v}}{\partial z} \right] - \frac{\epsilon}{R_{e_1}} \Delta \mathbf{v} - \frac{\epsilon}{\delta^2 R_{e_2}} \frac{\partial^2 \mathbf{v}}{\partial z^2} + fk \times \mathbf{v} + \operatorname{grad} p = 0, \\ \frac{\partial p}{\partial z} = -\rho, \\ \operatorname{div} \mathbf{v} + \frac{\partial w}{\partial z} = 0, \\ \epsilon \left[\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T + w \frac{\partial T}{\partial z} \right] - \frac{\epsilon}{R_{t_1}} \Delta T - \frac{\epsilon}{\delta^2 R_{t_2}} \frac{\partial^2 T}{\partial z^2} + \sigma_1 w = \epsilon F_2, \\ \epsilon \left[\frac{\partial S}{\partial t} + (\mathbf{v} \cdot \nabla) S + w \frac{\partial S}{\partial z} \right] - \frac{\epsilon}{R_{s_1}} \Delta S - \frac{\epsilon}{\delta^2 R_{t_2}} \frac{\partial^2 S}{\partial z^2} + \sigma_2 w = \epsilon F_3, \\ \rho = -\beta_T T + \beta_S S, \end{cases} \tag{3.12}$$

where

$$\sigma_1 = \frac{1}{F T_{ref}} \frac{\partial T_s}{\partial z}, \ \sigma_2 = \frac{1}{F S_{ref}} \frac{\partial S_s}{\partial z}. \tag{3.13}$$

Using (2.19), the PEs (3.12) can be written in the form

$$\begin{cases} \epsilon \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + W(\mathbf{v}) \frac{\partial \mathbf{v}}{\partial z} \right] - \frac{\epsilon}{R_{e_1}} \Delta \mathbf{v} - \frac{\epsilon}{\delta^2 R_{e_2}} \frac{\partial^2 \mathbf{v}}{\partial z^2} + fk \times \mathbf{v} + \operatorname{grad} p_s \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \operatorname{grad} \int_z^0 \rho \, dz = 0, \\ \int_{-h}^0 \operatorname{div} \mathbf{v} \, dz = 0, \\ \epsilon \left[\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T + W(\mathbf{v}) \frac{\partial T}{\partial z} \right] - \frac{\epsilon}{R_{t_1}} \Delta T - \frac{\epsilon}{\delta^2 R_{t_2}} \frac{\partial^2 T}{\partial z^2} + \sigma_1 W(\mathbf{v}) = \epsilon F_2, \\ \epsilon \left[\frac{\partial S}{\partial t} + (\mathbf{v} \cdot \nabla) S + W(\mathbf{v}) \frac{\partial S}{\partial z} \right] - \frac{\epsilon}{R_{s_1}} \Delta S - \frac{\epsilon}{\delta^2 R_{t_2}} \frac{\partial^2 S}{\partial z^2} + \sigma_2 W(\mathbf{v}) = \epsilon F_3, \\ \rho = -\beta_T T + \beta_S S. \end{cases} \tag{3.14}$$

The boundary conditions (3.5) become

(3.15)
$$\begin{cases} \dfrac{1}{\delta^2 R_{e_2}}\dfrac{\partial \mathbf{v}}{\partial z} = \tau_v,\, w = 0,\, \dfrac{1}{\delta^2 R_{t_2}}\dfrac{\partial T}{\partial z} = \alpha_T(T^* - T),\, \dfrac{1}{\delta^2 R_{s_2}}\dfrac{\partial S}{\partial z} = \alpha_S(S^* - S) \text{ on } \Gamma_i, \\ \dfrac{\partial \mathbf{v}}{\partial z} = 0,\ w = 0,\ \dfrac{\partial T}{\partial z} = 0,\ \dfrac{\partial S}{\partial z} = 0 \text{ on } \Gamma_b, \\ \mathbf{v} = 0,\ \dfrac{\partial T}{\partial n} = 0,\ \dfrac{\partial S}{\partial n} = 0 \text{ on } \Gamma_l. \end{cases}$$

The initial conditions are

$$(\mathbf{v}, T, S) = (\mathbf{v}_0, T_0, S_0) \text{ at } t = 0. \tag{3.16}$$

The mathematical study of the PEs of the ocean is given in [4, 8], in which the existence of weak solutions was proved.

Remark 3.1. Hereafter, for simplicity we will consider only homogeneous boundary conditions and we will assume that the functions T^*, S^*, and τ_v in (3.15) are identically zero. We will also consider the more general case where the zero forcing term in $(3.14)_1$ is replaced by a given function ϵF_1; see Remark 3.2 for the nonhomogeneous case (T^*, S^*, $\tau_v \neq 0$). We also assume for simplicity that the temperature and the salinity profiles are given such that

$$k_1 \equiv \beta_T \sigma_1^{-1} > 0,\ k_2 \equiv -\beta_S \sigma_2^{-1} > 0 \tag{3.17}$$

are constants.

Hereafter we set

$$V = V_1 \times V_2 \times V_2, \tag{3.18}$$

and we define H as the closure of V in $(L^2(\mathcal{M}))^4$.

We also define the following forms:

$$a_1(\mathbf{v}, \tilde{\mathbf{v}}) = \int_{\mathcal{M}} \left\{ \frac{1}{R_{e_1}} \nabla \mathbf{v} \nabla \tilde{\mathbf{v}} + \frac{1}{\delta^2 R_{e_2}} \frac{\partial \mathbf{v}}{\partial z} \frac{\partial \tilde{\mathbf{v}}}{\partial z} \right\} d\mathcal{M}, \tag{3.19}$$

$$a_2(T, \tilde{T}) = \int_{\mathcal{M}} k_1 \left\{ \frac{1}{R_{t_1}} \nabla T \nabla \tilde{T} + \frac{1}{\delta^2 R_{t_2}} \frac{\partial T}{\partial z} \frac{\partial \tilde{T}}{\partial z} \right\} d\mathcal{M} + \int_{\Gamma_i} \alpha_T k_1 T \tilde{T} d\Gamma_i, \tag{3.20}$$

$$a_3(S, \tilde{S}) = \int_{\mathcal{M}} k_2 \left\{ \frac{1}{R_{s_1}} \nabla S \nabla \tilde{S} + \frac{1}{\delta^2 R_{s_2}} \frac{\partial S}{\partial z} \frac{\partial \tilde{S}}{\partial z} \right\} d\mathcal{M} + \int_{\Gamma_i} \alpha_S k_2 S \tilde{S} d\Gamma_i, \tag{3.21}$$

$$e_1(\mathbf{v}, \tilde{\mathbf{v}}) = \int_{\mathcal{M}} (fk \times \mathbf{v}) \cdot \mathbf{v} d\mathcal{M}, \tag{3.22}$$

$$b_1(\mathbf{v}, \tilde{\mathbf{v}}, \hat{\mathbf{v}}) = \int_{\mathcal{M}} \left[(\mathbf{v} \cdot \nabla) \tilde{\mathbf{v}} + W(\mathbf{v}) \frac{\partial \tilde{\mathbf{v}}}{\partial z} \right] \cdot \hat{\mathbf{v}} d\mathcal{M}, \tag{3.23}$$

$$b_2(\mathbf{v},\tilde{T},\hat{T}) = \int_{\mathcal{M}} k_1\left[(\mathbf{v}\cdot\nabla)\tilde{T} + W(\mathbf{v})\frac{\partial\tilde{T}}{\partial z}\right]\cdot\hat{T}d\mathcal{M}, \tag{3.24}$$

$$b_3(\mathbf{v},\tilde{S},\hat{S}) = \int_{\mathcal{M}} k_2\left[(\mathbf{v}\cdot\nabla)\tilde{S} + W(\mathbf{v})\frac{\partial\tilde{S}}{\partial z}\right]\cdot\hat{S}d\mathcal{M}, \tag{3.25}$$

$$r_1(\mathbf{v},\tilde{T},\tilde{S}) = \int_{\mathcal{M}}\left(\nabla\int_z^0\tilde{\rho}\,dz\right)\cdot\mathbf{v}d\mathcal{M},\ \ \tilde{\rho} = -\beta_T\tilde{T}+\beta_S\tilde{S}, \tag{3.26}$$

$$r_2(\mathbf{v},\tilde{T}) = \int_{\mathcal{M}} W(\mathbf{v})\tilde{T}d\mathcal{M},\ r_3(\mathbf{v},\tilde{S}) = \int_{\mathcal{M}} W(\mathbf{v})\tilde{S}d\mathcal{M}, \tag{3.27}$$

$$\begin{aligned} &a(\mathbf{u},\tilde{\mathbf{u}}) = a_1(\mathbf{v},\tilde{\mathbf{v}}) + a_2(T,\tilde{T}) + a_3(S,\tilde{S}),\\ &r(\mathbf{u},\tilde{\mathbf{u}}) = r_1(\mathbf{v},\tilde{T},\tilde{S}) + r_2(\mathbf{v},\tilde{T}) + r_3(\mathbf{v},\tilde{S}),\\ &b(\mathbf{u},\tilde{\mathbf{u}},\hat{\mathbf{u}}) = b_1(\mathbf{v},\tilde{\mathbf{v}},\hat{\mathbf{v}}) + b_2(\mathbf{v},\tilde{T},\hat{T}) + b_3(\mathbf{v},\tilde{S},\hat{S}),\ e(\mathbf{u},\tilde{\mathbf{u}}) = e_1(\mathbf{v},\tilde{\mathbf{v}}), \end{aligned} \tag{3.28}$$

for $\mathbf{u} = (\mathbf{v},T,S)$, $\tilde{\mathbf{u}} = (\tilde{\mathbf{v}},\tilde{T},\tilde{S})$ and $\hat{\mathbf{u}} = (\hat{\mathbf{v}},\hat{T},\hat{S})$.

Here the operator A_0 is defined by:

$$\begin{aligned} &A_0 : H \longmapsto H\\ &\quad A_0(\mathbf{v},T,S) = (\mathbf{v},k_1T,k_2S). \end{aligned} \tag{3.29}$$

Proposition 3.1. *The forms* $a, a_i(i=1,2)$ *are coercive, continuous on* $V\times V$, $V_i\times V_i$, *respectively. Moreover, we have*

$$\begin{aligned} &a_1(\mathbf{v},\mathbf{v}) \geq c_1\|\mathbf{v}\|^2,\ \forall\mathbf{v}\in V_1,\ a_2(T,T)\geq c_2\|T\|^2,\ \forall T\in V_2,\\ &a_3(S,S)\geq c_3\|S\|^2,\ \forall S\in V_2, \end{aligned} \tag{3.30}$$

$$a(\mathbf{u},\mathbf{u}) \geq c_0\|\mathbf{u}\|^2,\ \forall\mathbf{u}\in V, \tag{3.31}$$

where $c_1 = c_1(R_{e_1},R_{e_2},\mathcal{M}) > 0$, $c_2 = c_2(R_{t_1},R_{t_2},\mathcal{M}) > 0$, $c_3 = c_3(R_{s_1},R_{s_2},\mathcal{M}) > 0$, $c_0 = c_0(R_{e_1},R_{e_2},R_{t_1},R_{t_2},R_{s_1},R_{s_2},\mathcal{M}) > 0$ *are constants independent of* δ. *Furthermore, we have*

$$\alpha_1|\mathbf{u}|^2_{L^2} \leq \langle A_0\mathbf{u},\mathbf{u}\rangle \leq \alpha_2|\mathbf{u}|^2_{L^2},\ \forall\mathbf{u}\in H, \tag{3.32}$$

where $\alpha_1 = Min(1,k_1,k_2), \alpha_2 = Max(1,k_1,k_2)$.

Proof. The proof is similar to that of Proposition 2.1.

Proposition 3.2. *For* $(\mathbf{u},\tilde{\mathbf{u}})\in V\times V$, *we have*

$$b(\mathbf{u},\tilde{\mathbf{u}},\tilde{\mathbf{u}}) = 0,\ e(\mathbf{u},\mathbf{u}) = 0,\ r(\mathbf{u},\mathbf{u}) = 0. \tag{3.33}$$

Moreover, the operators $r, r_i(i=1,2), e, e_i(i=1,2)$ *are continuous in H.*

Proof. The proof is similar to that of Proposition 2.2 and is given in [8].

We now consider the following variational formulation of the PEs (3.12)–(3.15).

Problem 2. For $\mathbf{u}_0 = (\mathbf{v}_0, T_0, S_0) \in H$, find $\mathbf{u} = (\mathbf{v}, T, S)$ such that

(3.34)
$$\epsilon \frac{d}{dt} \langle A_0 \mathbf{u}, \tilde{\mathbf{u}} \rangle + \epsilon a(\mathbf{u}, \tilde{\mathbf{u}}) + b(\mathbf{u}, \mathbf{u}, \tilde{\mathbf{u}}) + e(\mathbf{u}, \tilde{\mathbf{u}}) + r(\mathbf{u}, \tilde{\mathbf{u}}) = \epsilon \langle A_0(F_1, F_2, F_3), \tilde{\mathbf{u}} \rangle,$$
$$\forall \tilde{\mathbf{u}} \in V, \mathbf{u} = \mathbf{u}_0 \text{ at } t = 0.$$

The existence of weak solutions to Problem 2 was studied in [8], in which the following result was proved.

Proposition 3.3. *For any $T > 0$ there exists a least one solution $\mathbf{u} = (\mathbf{v}, \rho)$ for the system (3.34) defined on $(0, T)$ such that*

$$\mathbf{u} \in C([0, T], H_w), \tag{3.35}$$

where H_w is the space H endowed with weak topology.

Some a priori estimates

Replacing $\tilde{\mathbf{u}}$ by $\mathbf{u}$ in (3.34) yields

$$\epsilon \left\langle \frac{d}{dt} A_0 \mathbf{u}, \mathbf{u} \right\rangle + \epsilon \langle A \mathbf{u}, \mathbf{u} \rangle = \epsilon \langle A_0(F_1, F_2, F_3), \mathbf{u} \rangle, \tag{3.36}$$

from which we derive

$$\begin{aligned} &\left\langle \frac{d}{dt} A_0 \mathbf{u}, \mathbf{u} \right\rangle + \frac{1}{R_{e_1}} |\nabla \mathbf{v}|_{L^2}^2 + \frac{1}{\delta^2 R_{e_2}} \left| \frac{\partial \mathbf{v}}{\partial z} \right|_{L^2}^2 + \frac{k_1}{R_{t_1}} |\nabla T|_{L^2}^2 \\ &+ \frac{k_1}{\delta^2 R_{t_2}} \left| \frac{\partial T}{\partial z} \right|_{L^2}^2 + \alpha_T \int T^2 d\Gamma_i + \frac{k_2}{R_{s_1}} |\nabla S|_{L^2}^2 + \frac{k_2}{\delta^2 R_{s_2}} \left| \frac{\partial S}{\partial z} \right|_{L^2}^2 \\ &+ \alpha_S \int S^2 d\Gamma_i \le |F_1|_{L^2} |\mathbf{v}|_{L^2} + |k_2 F_2|_{L^2} |T|_{L^2} + |k_2 F_3|_{L^2} |S|_{L^2}, \end{aligned} \tag{3.37}$$

and (3.30) gives

(3.38)
$$\begin{aligned} &\left\langle \frac{d}{dt} A_0 \mathbf{u}, \mathbf{u} \right\rangle + \frac{1}{R_{e_1}} |\nabla \mathbf{v}|_{L^2}^2 + \frac{1}{\delta^2 R_{e_2}} \left| \frac{\partial \mathbf{v}}{\partial z} \right|_{L^2}^2 + \frac{c_2}{2} \|T\|^2 + \frac{k_1}{2R_{t_1}} |\nabla T|_{L^2}^2 \\ &+ \frac{k_1}{2\delta^2 R_{t_2}} \left| \frac{\partial T}{\partial z} \right|_{L^2}^2 + \frac{c_3}{2} \|S\|^2 + \frac{k_2}{2R_{s_1}} |\nabla S|_{L^2}^2 + \frac{k_2}{2\delta^2 R_{s_2}} \left| \frac{\partial S}{\partial z} \right|_{L^2}^2 \\ &\le |F_1|_{L^2} |\mathbf{v}|_{L^2} + |k_2 F_2|_{L^2} |T|_{L^2} + |k_2 F_3|_{L^2} |S|_{L^2} \\ &\le \frac{1}{2R_{e_1}} |\nabla \mathbf{v}|_{L^2}^2 + c|F_1|_{L^2}^2 + \frac{c_2}{2} \|T\|^2 + \frac{c}{c_2} |k_1 F_2|_{L^2}^2 + \frac{c_3}{2} \|S\|^2 + \frac{c}{c_3} |k_2 F_3|_{L^2}^2. \end{aligned}$$

Therefore,

(3.39)

$$\left\langle \frac{d}{dt} A_0 \mathbf{u}, \mathbf{u} \right\rangle + \frac{1}{2R_{e_1}} |\nabla \mathbf{v}|_{L^2}^2 + \frac{1}{2\delta^2 R_{e_2}} \left| \frac{\partial \mathbf{v}}{\partial z} \right|_{L^2}^2 + \frac{k_1}{2R_{t_1}} |\nabla T|_{L^2}^2 + \frac{k_1}{2\delta^2 R_{t_2}} \left| \frac{\partial T}{\partial z} \right|_{L^2}^2$$
$$+ \frac{k_2}{2R_{s_1}} |\nabla S|_{L^2}^2 + \frac{k_2}{2\delta^2 R_{s_2}} \left| \frac{\partial S}{\partial z} \right|_{L^2}^2 \le c |F_1|_{L^2}^2 + \frac{c}{c_2} |k_1 F_2|_{L^2}^2 + \frac{c}{c_3} |k_2 F_3|_{L^2}^2.$$

Finally, using (3.32) and (3.39) we derive

(3.40)

$$|\mathbf{u}(t)|_{L^2}^2 \le c \left(|\mathbf{u}_0|_{L^2}^2 + \int_0^T |F_1|_{L^2}^2 dt + \frac{1}{c_2} \int_0^T |F_2|_{L^2}^2 dt + \frac{1}{c_3} \int_0^T |F_3|_{L^2}^2 dt \right) \equiv K_2$$

and

(3.41)

$$\frac{1}{\delta^2 R_{e_2}} \int_0^T \left| \frac{\partial \mathbf{v}}{\partial z} \right|_{L^2}^2 dt \le K_2, \ \frac{1}{\delta^2 R_{t_2}} \int_0^T \left| \frac{\partial T}{\partial z} \right|_{L^2}^2 dt \le K_2, \ \frac{1}{\delta^2 R_{s_2}} \int_0^T \left| \frac{\partial S}{\partial z} \right|_{L^2}^2 dt \le K_2.$$

Proposition 3.4. *Let* $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}^\flat$ *be a solution of (3.34) given by Proposition 3.3. Then there exists a constant* $c_5 = c_5(F_1, F_2, F_3, R_{e_1}, R_{e_2}, R_{t_1}, R_{t_2}, R_{s_1}, R_{s_2}, \mathcal{M}, T)$ *independent of* δ *such that*

$$\int_0^T |\mathbf{u}^\flat|_{L^2}^2 dt \le c_5 \delta^2. \tag{3.42}$$

Proof. The proof is similar to that of Proposition 2.5.

Remark 3.2. If the wind stress τ_v in the boundary conditions (2.17) or (3.15) is not zero, we can "homogenize" the boundary condition as in [4] (assuming some regularity on τ_v) and write $\mathbf{v}$ into the form

$$\mathbf{v} = \mathbf{v}_1 + \tilde{\mathbf{v}}, \tag{3.43}$$

where $\tilde{\mathbf{v}}$ is known and depends on τ_v, and $\mathbf{v}_1$ satisfies a homogeneous boundary condition. Assuming enough regularity on $\tilde{\mathbf{v}}$ we can use the idea presented above to show that the magnitude (in the L^2-norm) of the baroclinic component $\mathbf{v}_1^\flat$ of $\mathbf{v}_1$ is of order δ, provided we make a consistent assumption on the magnitude of τ_v. The same remark holds for T and S in this section if $T^*, S^* \neq 0$.

4. Numerical results

In this section, we present some numerical simulations of the PEs of the ocean with continuous density stratification. First, we numerically check the order of magnitude of the baroclinic flow, and in the second part we present some simulations of a wind-driven flow in an idealized ocean model.

4.1. Numerical accuracy check. The focus of this numerical experiment is to numerically check the order of magnitude of the baroclinic flow and compare it to the estimates (3.42). In our experiments, the basin configuration is the (non-dimensional) cube $[0, 1] \times [0, 1] \times [-1, 0]$. Let us simply recall that this is a two-gyre, wind-driven ocean problem with a steady sinusoidal wind stress (maximum $\tau_0 = 1$ dyne cm^{-4}) in a basin that is $L \times L \times H$ km (east-west $\times$ north-south $\times$ bottom-surface extent). The Coriolis parameter is given by $f = f_0 + \beta y$, $f_0 = 9.3 \times 10^{-5} s^{-1}$, $\beta = 2. \times 10^{-11}$ m^{-1}s^{-1}. The model does not include bottom topography. The ocean is forced by a steady wind stress $\tau_0 = (\tau_0^x, \tau_0^y) = (-10^{-4}\cos(2\pi y/L), 0)$ and a density variation. Other dimensional quantities are given by $U = 10^{-1}$ ms^{-1}, $g = 9.8$ ms^{-2}, and $L = 2.10^6$ m. The initial condition is given by $\mathbf{v} = \rho = 0$ at $t = 0$.

The details on the numerical method are given in [11]. Let us simply recall that all the operators in (2.20), (2.17), (2.18) are discretized using a second order central differencing scheme. The Jacobian operator appearing in (2.20) is approximated using Arakawa's method [14]. For the time integration of the model, we use a fourth order Adams–Bashforth method. In all the computations presented in this article, the (nondimensional) time step is $\Delta t = 10^{-4}$. For the space discretization, we take 100×100 points in the x-y plane and 10 points in the vertical direction. For the boundary condition (2.17), we take $\tau_v = \rho^* = 0$ and compensate with a forcing term F_1 in $(2.20)_1$ defined by

$$F_1 = c(g(z)\tau_0, 0), \tag{4.1}$$

where $g(z)$ is defined by

$$g(z) = 0.5(1 + \tanh((z/H + z_1)/\epsilon_1)), \tag{4.2}$$

where z_1 and ϵ_1 are very small constants chosen such that the forcing F_1 is nonzero only on the first couple layers from the surface of the ocean.

The forcing term F_2 in $(2.20)_2$ has the form

$$F_2(z) = c\frac{\partial^2 \rho_s}{\partial z^2}/\rho_s, \tag{4.3}$$

where ρ_s is given by

$$\rho_s(z) = 1028 - 3\exp(10z/H) \tag{4.4}$$

and c is a constant.

The following table shows the order of magnitude (in the L^2-norm) of the baroclinic flow defined by

$$\text{order} = \left(\int_0^T |\mathbf{v}^\flat|_{L^2}^2 dt\right)^{\frac{1}{2}} / \left(\int_0^T |F_1|_{L^2}^2 dt\right)^{\frac{1}{2}} \tag{4.5}$$

for different values of H and for the (nondimensional) integration time $T = 20$.

H (meters)	1000	2000	2500	3000	3500	4000	5000
$\delta = H/L$	5.10^{-4}	10^{-3}	$1.25.10^{-3}$	$1.5.10^{-3}$	$1.75.10^{-3}$	2.10^{-3}	$2.5.10^{-3}$
order	$9.28.10^{-4}$	$1.86.10^{-3}$	$2.28.10^{-3}$	$2.66.10^{-3}$	3.10^{-3}	$3.3.10^{-3}$	$3.84.10^{-3}$

The table above clearly shows that the order of magnitude (4.5) decreases with δ. Moreover, for $H = 1000\,\mathrm{m}$, the order of magnitude is almost half its value when $H = 2000\,\mathrm{m}$. Comparing the order of magnitude for $H = 2000\,\mathrm{m}$ and $H = 4000\,\mathrm{m}$ as well as for $H = 2500\,\mathrm{m}$ and $H = 5000\,\mathrm{m}$, the table seems to indicate that as δ decreases, the order of magnitude (4.5) of the baroclinic flow is (at most) of order $O(\delta)$.

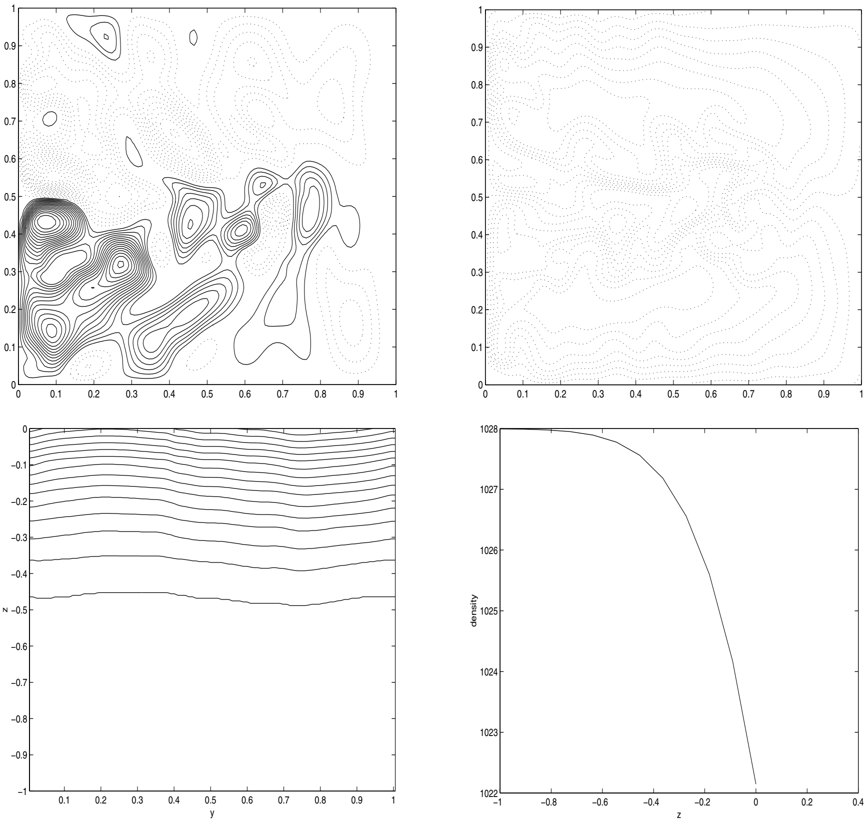

Fig. 1. Snapshot at the (nondimensional) time $t = 32.99$ of the barotropic streamfunction, the surface density deviation, the total density at $x = 0.25$, and the total density at $(x, y) = (0.5, 0.5)$.

4.2. Double-gyre simulations. In this subsection, we present some simulations of the wind-driven flow in an idealized ocean model. In our experiments, the basin configuration is as described above with $H = 4000\,\mathrm{m}$. The following figures show a time sequence of the surface density deviation (that is $\rho(x, y, 0)$), the barotropic

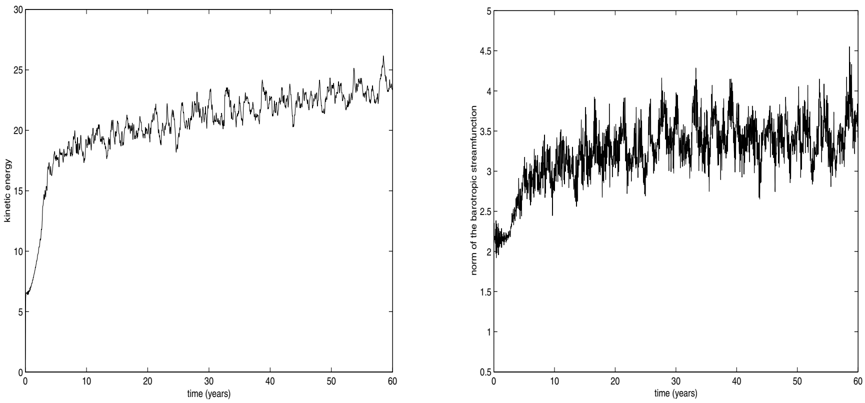

Fig. 2. Kinetic enegy and norm of the barotropic streamfunction with respect to time (in years)

streamfunction , the total density at $x = 0.25$ (that is, $(\rho_s + \rho)(0.25, y, z)$) and the total density at $(x, y) = (0.5, 0.5)$ (that is, $(\rho_s + \rho)(0.5, 0.5, z)$) for the Reynolds numbers $R_{e_1} = R_{e_2} = R_{t_1} = R_{t_2} = 10^3$. For these values of the Reynolds number, the flow remains time dependent. This is confirmed by Figure 2, which represents the energy $\left(\int_\Omega |\mathbf{v}|^2 dxdy\right)^{1/2}$ of the flow and the norm of the barotropic steamfunction $\left(\int_\Omega |\bar{\psi}|^2 dxdy\right)^{1/2}$ with respect to time (in years), where $\bar{\psi}$ is the barotropic streamfunction. A thorough analysis of the double-gyre circulation with the PEs of the ocean will appear elsewhere.

5. Conclusion

In this article, we have studied the baroclinic flow of the PEs of the ocean, and we have derived some estimates of its order of magnitude in the L^2-norm. We have presented some numerical tests that agree with the estimates. The article ends with some simulations of the wind-driven flow using the PE model.

Acknowledgements. This work was supported in part by NSF Grant DMS0305110, DOE Grant DE-FG02-01ER63251:A000, and by the Research Fund of Indiana University.

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