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Shui-Nee Chow[1] · Wenzhang Huang[2,⋆]

On the number of spikes of solutions for a forced singularly perturbed differential equation

Dedicated to Professor Roberto Conti on the occasion of his 80th birthday

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1. Introduction

Consider a singularly perturbed system with forcing that is derived from an electrical circuit, called a Cellonics element (see the Web site of Cellonics: www.cellonics.com):

$$\begin{aligned} \dot{v} &= f(t) - i, \\ \epsilon \dot{i} &= v - \phi(i), \end{aligned} \tag{1.1}$$

where v and i denote the voltage and current, respectively, $\epsilon > 0$ is a small parameter, $f(t)$ is the time derivative of the input voltage, and ϕ is the current-voltage characteristic of the circuit and is given by the following;

$$\phi(i) = \begin{cases} K_1 i, & \text{if } i > 0; \\ K_2 i, & \text{if } i_0 < i \le 0; \\ K_2 i_0 + K_1(i - i_0), & \text{if } i \le i_0, \end{cases}$$

where $i_0 < 0$ is a constant, $K_1 > 0$, and $K_2 < 0$. The graph of the function is piecewise linear and of S-shape.

Figure 1 shows the components of the circuit and the graph, and Figure 2 shows the characteristic function ϕ.

In Figure 3, we show a typical analog input and its output signal after going through the Cellonics element.

We note that the input and output signals in Figure 1 are typical of bursting and spiking outputs in neuron models (see, for example, [1]). In this paper, we will present a detailed study of the spiking phenomena over a fixed, finite time interval,

S.-N. Chow: Georgia Institute of Technology, Center for Dynamical Systems and Nonlinear Studies, Atlanta, GA 30332, USA, e-mail: chow@math.gatech.edu

W. Huang: University of Alabama in Huntsville, Department of Mathematical Sciences, Huntsville, AL 35899, USA

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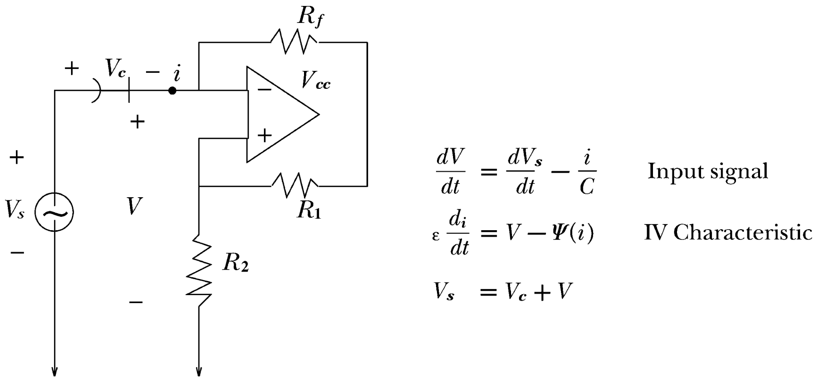


Fig. 1. A Cellonics circuit

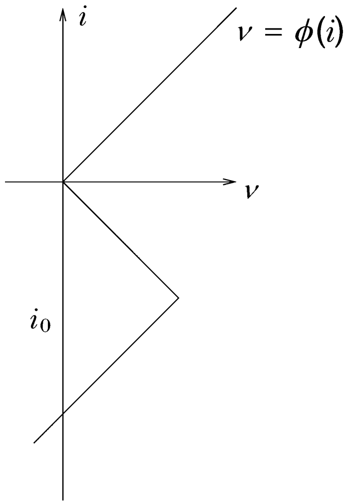


Fig. 2. Characteristic curve

in particular, the number of spikes over a finite time interval and their bifurcation. Note that we will not be concerned with asymptotic or chaotic behavior of the system (see, for example, [5,4] and references therein). In what follows, we will explain our motivation for the mathematical studies on the number of spikes over a fixed time interval.

A vector field analysis indicates that the stable slow motions of equations (1.1) consist of two sets $\hat{S}_1$ and $\hat{S}_2$ in the v-i plane with

$$\hat{S}_1 = \{(v, i) : \ i = \phi^{-1}(v) = kv, v \geq 0\},$$
$$\hat{S}_2 = \{(v, i) : \ i = \phi^{-1}(v) = -b + kv, v \leq v^*\},$$

where $k = 1/K_1$, $v^* = K_2 i_0$, and $b = -i_0 + kv^*$.

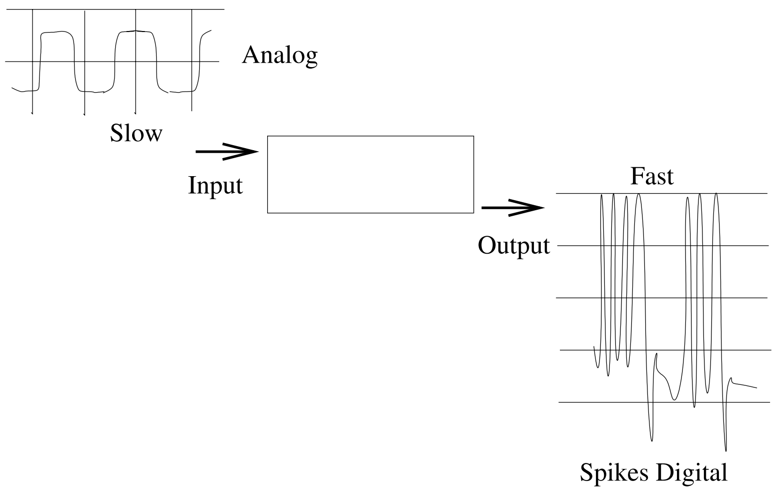


Fig. 3. Modulation

It is clear that the attracting part of the slow motions are those points (v, i) with $0 \leq v \leq v^*$, namely,

$$S_1 = \{(v, i) : \ i = kv, 0 \leq v \leq v^*\},$$
$$S_2 = \{(v, i) : \ i = -b + kv, 0 \leq v \leq v^*\}.$$

The fundamental property of equations (1.1) is the relaxation oscillation of its solutions. For sufficiently small $\epsilon > 0$, a solution starting at any point will converge quickly to a small neighborhood of one piece of stable slow motion and stays there for a relatively long time. Because of an external force $f(t)$ the solution will eventually be pushed out of this neighborhood and then quickly jump to a small neighborhood of another piece of stable motion. When time varies, the solutions repeat this pattern, which can create many interesting dynamical phenomena, such as stable periodic solutions, period doubling, chaos, horseshoes, etc. [5,4]. We also note that equations (1.1) is precisely the equation considered by Levinson in [6]. However, the reason Levinson considers the piecewise linear case is to show it has the same singular behavior as the forced van der Pol equation and for which the proof is considerably simpler. In [2,3], Levi studied in detail a forced system similar to ours but with an extra ϵ as the coefficient of i in the first equation of equations (1.1).

However, the purpose of our paper is not to further explore those properties mentioned above but to consider them in a different direction that has important applications to signal processing, analog-digital modulation, and wireless or wire communication. The circuit in Figure 1 is part of a technology recently patented by Cellonics. It is a new technology, based on nonlinear dynamical systems, of converting analog signals to digital signals. In Figure 4, we illustrate how this conversion is achieved by counting the number of spikes separated by silent periods.

One of the most important features of their demodulation technique is its carrier-rate decoding, which enables one information symbol to be carried in one RF carrier

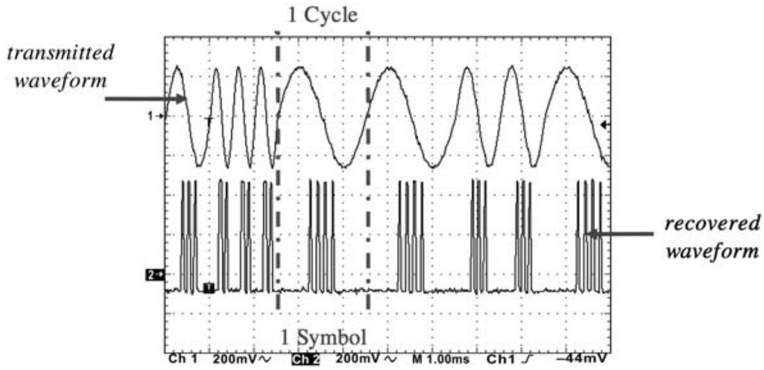


Fig. 4.

cycle. Conventional systems, which are usually linear systems, require thousands of cycles to capture one symbol. Thus, this unique carrier-rate decoding is offered throughout at maximum rate. We refer the reader to `www.cellonics.com` for more detail and its application in communication.

We will consider the mathematics behind this technology. In particular, we are interested in the number of cycles a solution completes in one time period of the periodic forced function f. Since any completion of one cycle will introduce a spike in the solution, in other words, we want to study the number of spikes for a solution in one time period. Another important issue, from the point view of applications, is the stability property (which is required for the circuit to achieve one information symbol in one RF cycle). There are two aspects regarding the stability of spike solutions:

1. Stability of the number of spikes under perturbation of initial value uniformly for all sufficiently small $\epsilon > 0$.
2. Stability of the number of spikes as time goes to infinite. That is, the number of spikes remains fixed when time runs in the first time period, second time period, third time period, etc.

Since a solution reaches either a small neighborhood V_1 of S_1 or a small neighborhood V_2 of S_2 in a very short time, we may only consider solutions that start at a point in V_1 or V_2. Let $P(z_0)$ be a period 1 map corresponding to equations (1.1). That is,

$$P(z_0) = (v(T, 0, z_0), i(T, 0, z_0)),$$

where T is the period of the forced function f and $(v(t, 0, z_0), i(t, 0, z_0))$ is the solution of equations (1.1) satisfying the initial condition $(v(0), i(0)) = z_0$. Then, for instance, in order to have the stability of $(v(t, 0, z_0), i(t, 0, z_0))$ with $z_0 \in V_1$ in the sense of the number of spikes, there must be a neighborhood U_{z_0} of z_0 such that

$$P^n(U_{z_0}) \subset V_1, \quad n = 1, 2, \ldots.$$

However, the above inclusion is not easily verified unless $P(V_1) \subset V_1$, and it may not be sufficient because it only indicates the completion of the number of cycles in one time period but does not guarantee that the number of cycles is constant at any one time period. This indeed could happen if $P(V_1) \not\subset V_1$. On the other hand, if $P(V_1) \subset V_1$, then we certainly have $P^n(V_1) \subset V_1$. Moreover, by the continuity of solution on the initial value we can conclude that any solution starting from a point in V_1 has a fixed number of spikes at any time interval $[nT, (n+1)T]$. To this end, in this paper we look for those forced functions f in equations (1.1) that ensure the existence of a small neighborhood V_1 of S_1 or V_2 of S_2 such that

$$P(V_1) \subset V_1 \quad \text{or} \quad P(V_2) \subset V_2.$$

To be specific, throughout this paper we choose the external force to be a commonly used sine function

$$f(t) = a\sin(\nu t)$$

with a the amplitude and ν the natural frequency. With the above choice of f, equations (1.1) become

$$\begin{aligned} \dot{v} &= a\sin(\nu t) - i, \\ \epsilon\dot{i} &= v - \phi(i). \end{aligned} \tag{1.2}$$

Let

$$a^* = \min\left\{\frac{K_2 i_0}{K_1}, -2i_0\right\}. \tag{1.3}$$

Moreover, for each $\nu > 0$ we let $P(\cdot, \nu)$ be the time $\frac{2\pi}{\nu}$ map introduced by the flows of equations (1.2). Then in this paper we shall establish the following result.

Theorem 1.1. *There exist small neighborhoods V_1 of S_1 and V_2 of S_2 such that for each $a \in (0, a^*)$, there exist three sequences $\{\nu_N\}$, $\{\nu_N^*\}$, and $\{\tilde{\nu}_N\}$ with*

$$\tilde{\nu}_{N+1} < \nu_N < \nu_N^* < \tilde{\nu}_N, \quad N = 1, 2, \ldots$$

such that the following hold:

For each positive integer M and $\sigma > 0$ with

$$\sigma < \frac{1}{2}\min\left\{\nu_N^* - \nu_N, \tilde{\nu}_N - \nu_N^*, \ N = 1, 2, \ldots, M\right\},$$

there is a $\epsilon^* > 0$ such that

A. *$P(V_1, \nu) \subset V_1$ for each $\nu \in \cup_{N=1}^{M}[\nu_N+\sigma, \nu_N^*-\sigma]$ and $\epsilon \in (0, \epsilon^*]$. Furthermore, $\nu \in [\nu_N + \sigma, \nu_N^* - \sigma]$ implies that any solution starting from a point in V_1 has exactly N spikes in one time period.*

B. *$P(V_2, \nu) \subset V_2$ for each $\nu \in \cup_{N=1}^{M}[\nu_N^* + \sigma, \tilde{\nu}_N - \sigma]$ and $\epsilon \in (0, \epsilon^*]$. In addition, $\nu \in [\nu_N + \sigma, \nu_N^* - \sigma]$ implies that any solution starting from a point in V_2 has exactly N spikes in one time period.*

This paper is organized as follows. In Section 2 we study the reduced system of (1.2) at the slow motions S_1 and S_2 as ϵ trends to zero, which is essential for establishing Theorem 1.1. Sections 3 and 4 are devoted to proving two key propositions needed in Section 2. We give a complete proof for conclusion A of our Theorem 1.1 in Section 5. To limit the size of the paper we shall omit the proof of conclusion B of Theorem 1.1, which is essentially the same as for the proof of conclusion A.

2. Reduced systems in slow motion

Since the dynamics of the singularly perturbed system (1.2) has a close connection with the dynamics of its reduced system as $\epsilon \to 0$, it is a natural way to begin our study on reduced systems in slow motion. Let us first reduce the number of parameters by introducing the following translation and time scaling:

$$x(t) = \frac{1}{K_1 a} v\left(\frac{t}{\nu}\right), \quad y(t) = \frac{1}{a} i\left(\frac{t}{\nu}\right).$$

Equation (1.2) is therefore transformed into the dimensionless form

$$\begin{aligned} \dot{x} &= \omega[\sin t - y], \\ \dot{y} &= \frac{1}{\rho}[x - g(y)], \end{aligned} \tag{2.1}$$

where

$$g(y) = \begin{cases} y, & y \geq 0 \\ Ky, & y^* \leq y \leq 0 \\ \beta + y, & y \leq y^* \end{cases} \tag{2.2}$$

and

$$\begin{aligned} \omega &= \frac{1}{K_1 \nu}, \\ \rho &= \frac{\epsilon}{\nu K_1}, \\ K &= \frac{K_2}{K_1}, \\ y^* &= \frac{i_0}{a}, \\ \beta &= \frac{(K_2 - K_1)i_0}{a\, K_1}. \end{aligned} \tag{2.3}$$

Accordingly, the slow motions S_1 and S_2 are transformed into W_1 and W_2, respectively, as

$$\begin{aligned} W_1 &= \{(x, y) : \ y = x, \ 0 \leq x \leq x^*\}, \\ W_2 &= \{(x, y) : \ y = -\beta + x, \ 0 \leq x \leq x^*\} \end{aligned}$$

with

$$x^* = \frac{K_2 i_0}{a K_1}. \tag{2.4}$$

From the first equation of system (2.1) we see that the reduced systems on W_1 and W_2 are given respectively by the differential equations

$$\dot{x} = \omega[\sin t - x], \qquad 0 \le x \le x^*, \tag{2.5}$$

$$\dot{x} = \omega[\sin t + \beta - x], \quad 0 \le x \le x^*. \tag{2.6}$$

To reflect the dynamics of original singularly perturbed system (2.1), a solution of the reduced system (2.5)–(2.6) is described as follows: If a solution starts at W_1 with $x(0) = x_0 \in [0, x^*]$, then $x(t)$ is subject to equation (2.5) before $x(t)$ reaches 0. If there is a first time $t_1 > 0$ such that $x(t_1) = 0$, then the solution is considered as jumping to W_2 immediately and so that $x(t)$ is given by equation (2.6) with the initial condition $x(t_1) = 0$ (note: if $x(0) = 0$, then the solution $x(t)$ of (2.5) is increasing for small $t > 0$ and then decreasing to 0 in a late time. This late time will be considered t_1 instead of $t_1 = 0$). If there is again a first time $t_2 > t_1$ such that $x(t_2) = x^*$, then the solution returns to W_1 and hence $x(t)$ is governed by equation (2.5) again with the initial condition $x(t_2) = x^*$. In such a way the solution is defined by equations (2.5)–(2.6) alternatively. That is,

$$\begin{aligned}
&\dot{x} = \omega[\sin t - x],\\
&\qquad t \in [0, t_1] \quad \text{with } \ x(0) = x_0 \in [0, x^*], \ \ x(t_1) = 0, \\
&\qquad t \in [t_{2k}, t_{2k+1}] \quad \text{with } \ x(t_{2k}) = x^*, \qquad x(t_{2k+1}) = 0, \quad k \ge 1,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\dot{x} = \omega[\sin t + \beta - x],\\
&\qquad t \in [t_{2k+1}, t_{2k+2}] \quad \text{with } \ x(t_{2k+1}) = 0, \ \ x(t_{2k+2}) = x^*, \ \ k \ge 0,
\end{aligned} \tag{2.8}$$

for some sequence

$$0 < t_1 < t_2 < \cdots < t_n < \cdots,$$

where the time sequence $\{t_n = t_n(x_0, \omega)\}$ depends both on the initial value x_0 and $\omega > 0$. Alternatively we can discuss solutions beginning W_2 in the same way. To reduce the length of the paper, we shall focus on the search for a solution starting at W_1 only. It is also easy to see that $t_n(x_0, \omega)$ are continuously differentiable on x_0 and ω. Since system (2.5)–(2.6) is 2π period, the period 1 map $\tilde{P} : [0, x^*] \to [0, x^*]$ is defined as usual as

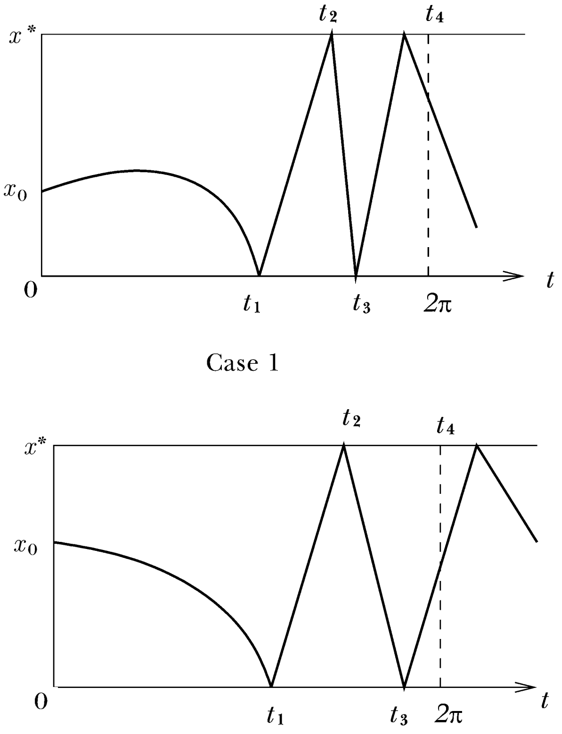
$$\tilde{P}(x_0, \omega) = x(2\pi, x_0, \omega).$$

We frequently omit the parameter ω to reduce the notation. In conjunction with our main Theorem 1.1, we are interested in the case in which a solution starting from a point in W_1 returns to W_1 at time 2π. Hence we must distinguish between two possible cases for a time sequence $\{t_n = t_n(x_0, \omega)\}$.

Case 1: There is an integer $N \ge 1$ such that $2\pi \in [t_{2N}, t_{2N+1}]$.
Case 2: There is an integer $N \ge 1$ such that $2\pi \in (t_{2N-1}, t_{2N})$.

We note that case 2 implies that a solution starting at a point in W_1 arrives at W_2 at time 2π. A solution $x(t, x_0)$ returns to W_1 if and only if

$$2\pi \in [t_{2N}, t_{2N+1}]$$



Case 1

Case 2

Fig. 5.

for some $N \geq 1$ (Figure 5). If this happens, we denote it by

$$\tilde{P}(x_0) \in [0, x^*]_{W_1};$$

here we use the lower index "$_{W_1}$" to denote that the interval corresponds to slow motion W_1. Moreover, in case 1 a solution completes in exactly N cycles around W_1 and W_2 in one time period. Also in this case we have the expression for $\tilde{P}(x_0)$ as

$$\tilde{P}(x_0) = e^{-\omega 2\pi} \left(x^* e^{\omega t_{2N}(x_0,\omega)} + \omega \int_{t_{2N}(x_0,\omega)}^{2\pi} e^{\omega\tau} \sin \tau d\tau \right). \tag{2.9}$$

Throughout the paper we suppose that

$$a < a^* = \min \left\{ \frac{K_2 i_0}{K_1}, -2i_0 \right\}. \qquad \textbf{(H1)}$$

From (2.3) and (2.4) we see that (**H1**) is equivalent to the inequalities

$$x^* > 1, \quad \beta > x^* + 2. \qquad \textbf{(H2)}$$

It is easy to verify that assumption (**H2**) guarantees that a solution of equations (2.5)–(2.6) is bounded by 0 and x^*.

Our main goal of this section is to establish the following.

Theorem 2.1. *There is a sequence of intervals* $\{I_N = (\omega_N^*, \omega_N)\}_{N=1}^{\infty}$ *with*

$$0 < \omega_1^* < \omega_1 < \omega_2^* < \omega_2 < \cdots < \omega_N^* < \omega_N \cdots ,$$

such that for each $\omega > 0$*,*

$$\tilde{P}([0, x^*], \omega) \subset (0, x^*)_{W_1}$$

if and only if $\omega \in I_N$ *for some positive integer* N*. In addition, if* $\omega \in I_N$*, then for each* $x_0 \in [0, x^*]$*, the solution* $x(t, x_0)$ *has exactly* N *spikes in* 2π*-time period.*

Remark 1. If $\tilde{P}([0.x^*], \omega) \subset (0, x^*)_{W_1}$, then we know that the period one map $\tilde{P}$ has at least a fixed point $x_0 \in (0, x^*)$ and hence $x(t, x_0)$ is a periodic solution of period 2π. In fact we can show that the fixed point is unique if $\tilde{P}([0.x^*], \omega) \subset (0, x^*)_{W_1}$. However, we should not provide a proof here because it is too tedious. On the other hand, the monotonicity of $\tilde{P}(x_0)$ on x_0 (see Property **P1** in Lemma 2.1) guarantees the convergence of any solution $x(t, x_0)$ to a periodic solution with the number of spikes independent of the initial value x_0. This is most important for the application.

Before proceeding to the proof of Theorem 2.1 let us first provide two key propositions that will be proved in Section 3 and Section 4 respectively due to the length of their proof.

Proposition 2.1. *For a fixed* $\omega > 0$*, if there is an integer* $N \geq 1$ *such that*

$$t_{2N+1}(0, \omega) = 2\pi,$$

then $t_{2N}(x^*, \omega) < 2\pi$.

Proposition 2.2. *The time sequence* $\{t_n(x_0, \omega)\}$ *is strictly monotone, decreasing with respect to* $\omega > 0$ *as long as* $t_1(x_0, \omega) \in (\pi, 2\pi)$.

Remark 2. Since $x_0 + \omega \int_0^{t_1(x_0,\omega)} e^{\omega\tau} \sin \tau d\tau = 0$, we must have $t_1(x_0, \omega) > \pi$. On the other hand, if $t_1(x_0, \omega) \geq 2\pi$, then the solution takes longer than 2π time to return to W_1. Hence only those values (x_0, ω) for which $t_1(x_0, \omega) \in (\pi, 2\pi)$ will be of interest to us.

In addition to Propositions 2.1 and 2.2, we need the following results.

Lemma 2.1. *The sequence* $\{t_n(x_0, \omega)\}$ *has the following properties:*

P1. *For each* $\omega > 0$ *and* $0 \leq x_1 < x_2 \leq x^*$*,*

$$t_n(x_1, \omega) < t_n(x_2, \omega) < t_{n+2}(x_1, \omega), \quad n = 1, 2, \ldots .$$

An immediate consequence of the above inequalities is that $\tilde{P}(x_0, \omega)$ *increases with respect to* x_0 *whenever* $\tilde{P}(x_0, \omega) \in (0, x^*)_{W_1}$.

P2. *Let $0 \le x_1 < x_2 \le x^*$ and $\omega > 0$ be fixed. If $\tilde{P}([x_1, x_2], \omega) \subset [0, x^*]_{W_1}$, then there is an $N \ge 1$ such that*

$$t_{2N}(x_0, \omega) \le 2\pi \le t_{2N+1}(x_0, \omega)$$

for all $x_0 \in [x_1, x_2]$. Hence all solutions $x(t, x_0, \omega)$ with $x_0 \in [x_1, x_2]$ have the same number of spikes in 2π period.

Proof. Property **P1** is obvious (Figure 6). Let us verify **P2**. First by assumption there is an $N \ge 1$ such that

$$t_{2N}(x_1, \omega) \le 2\pi \le t_{2N+1}(x_1, \omega).$$

We define

$$x_M = \sup\{x : \ x \in [x_1, x_2], \ t_{2N}(x, \omega) \le 2\pi\}.$$

We claim that $x_M = x_2$. Suppose, on the contrary, that $x_M < x_2$. Then by the continuity of t_{2N} we deduce that $t_{2N}(x_M, \omega) = 2\pi$. Hence $t_{2N-1}(x_M, \omega) < t_{2N}(x_M, \omega) = 2\pi$. So again by continuity there exists $\tilde{x} \in (x_M, x_2]$ such that $t_{2N-1}(\tilde{x}, \omega) < 2\pi$. However, $\tilde{x} > x_M$ implies that (by **P1**)

$$2\pi = t_{2N}(x_M, \omega) < t_{2N}(\tilde{x}, \omega),$$

so that $\tilde{P}(\tilde{x}, \omega) \notin [0, x^*]_{W_1}$, which leads to a contradiction.

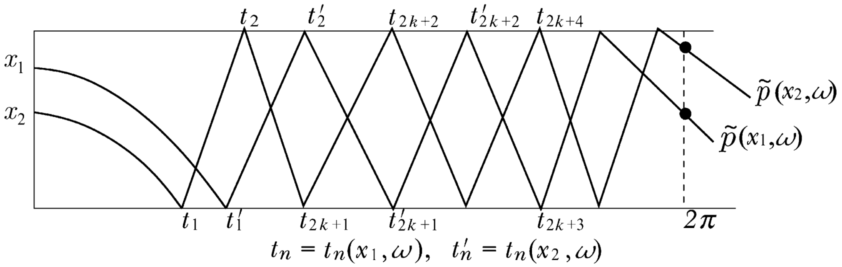


Fig. 6.

Lemma 2.2. *For each fixed integer $N \ge 1$,*

$$\lim_{\omega \to 0} t_{2N+1}(0, \omega) = \lim_{\omega \to 0} t_{2N}(x^*, \omega) = \infty,$$
$$\lim_{\omega \to \infty} t_{2N+1}(0, \omega) = \lim_{\omega \to \infty} t_{2N}(x^*, \omega) = \pi.$$

Proof. First equation (2.5) and $\sin t > 0$ for $t \in (0, \pi)$ imply that $t_1(x_0, \omega) > \pi$. Then the conclusion of Lemma 2.2 follows easily from the fact that increasing ω will fasten the oscillation of the solution $x(t, x_0, \omega)$ of equations (2.5) and (2.6), while decreasing ω will reduce the oscillation of the solution.

From Proposition 2.2 and Lemma 2.2 it follows that, for each integer $N \geq 1$, there are unique positive real numbers ω_N^* and ω_N such that

$$t_{2N}\big(x^*, \omega_N^*\big) = 2\pi, \quad t_{2N+1}(0, \omega_N) = 2\pi.$$

Lemma 2.3. *For each integer $N \geq 1$, $\omega_N^* < \omega_N < \omega_{N+1}^*$.*

Proof. For each fixed $N \geq 1$, Proposition 2.1 yields that $t_{2N}(x^*, \omega_N) < 2\pi$. Note that $t_{2N}(x^*, \omega)$ is increasing as ω decreases by Proposition 2.2. Hence, $t_{2N}(x^*, \omega_N^*) = 2\pi$ implies that $\omega_N^* < \omega_N$. Similarly, one can show that $\omega_N < \omega_{N+1}^*$.

Proof of Theorem 2.1. For each fixed $\omega > 0$, by **P2** of Lemma 2.1,

$$P([0, x^*], \omega) \subset (0, x^*)_{W_1}$$

if and only if there is an $N \geq 1$ such that

$$t_{2N}(x_0, \omega) < 2\pi < t_{2N+1}(x_0, \omega), \quad x_0 \in [0, x^*]. \tag{2.10}$$

Since $t_n(x_0, \omega)$ increases as x_0 increases, (2.10) is equivalent to

$$t_{2N}(x^*, \omega) < 2\pi < t_{2N+1}(0, \omega). \tag{2.11}$$

By Proposition 2.2, (2.11) is equivalent to

$$\omega_N^* < \omega < \omega_N,$$

that is, if and only if $\omega \in (\omega_N^*, \omega_N)$ for some positive integer N. Moreover, $\omega \in I_N$ implies (2.10). Hence $x(t, x_0, \omega)$ has exactly N spikes for all $x_0 \in [0, x^*]$.

3. Proof of Proposition 2.1

We shall give a complete proof of Proposition 2.1 in this section. First, for $x_0 \in [0, x^*]$, it is easy to verify that the solution $x(t) = x(t, x_0)$ of reduced equations (2.5)–(2.6) has the explicit formulas

$$x(t) = e^{-\omega t}\Big[x_0 + \omega \int_0^t e^{\omega\tau} \sin \tau d\tau\Big], \quad t \in [0, t_1],$$

$$x(t) = \omega e^{-\omega t} \int_{t_{2k-1}}^t e^{\omega\tau}(\sin \tau + \beta) d\tau, \quad t \in [t_{2k-1}, t_{2k}],$$

$$x(t) = e^{-\omega(t - t_{2k})} x^* + e^{-\omega t} \omega \int_{t_{2k}}^t e^{\omega\tau} \sin \tau d\tau, \quad t \in [t_{2k}, t_{2k+1}].$$

From the above expressions and the boundary conditions we have

$$x_0 + \omega \int_0^{t_1} e^{\omega\tau} \sin \tau d\tau = 0, \tag{3.1}$$

$$x^* e^{\omega t_{2k}} = \omega \int_{t_{2k-1}}^{t_{2k}} e^{\omega\tau}(\sin \tau + \beta) d\tau, \tag{3.2}$$

$$e^{\omega t_{2k}} x^* + \omega \int_{t_{2k}}^{t_{2k+1}} e^{\omega\tau} \sin \tau d\tau = 0. \tag{3.3}$$

Lemma 3.1. *There is a unique* $\omega^* > 0$ *such that*

$$\int_0^{\frac{3\pi}{2}} e^{\omega^*\tau} \sin \tau d\tau = 0.$$

In addition, we have $t_3(0, \omega^*) > 2\pi$.

Proof. For $\omega > 0$, we have

$$\begin{aligned}\int_0^{\frac{3\pi}{2}} e^{\omega\tau} \sin \tau d\tau &= \frac{1}{1+\omega^2}(e^{\omega\tau}[\omega \sin \tau - \cos \tau])\Big|_0^{\frac{3\pi}{2}} \\ &= \frac{1}{1+\omega^2}\big(- \omega e^{\frac{3\pi\omega}{2}} + 1\big).\end{aligned}$$

Hence $\int_0^{\frac{3\pi}{2}} e^{\omega w\tau} \sin \tau d\tau = 0$ is equivalent to

$$h(\omega) = e^{\frac{3\pi\omega}{2}} - \frac{1}{\omega} = 0.$$

Since the function $h(\omega)$ increases as ω increases, and

$$h(0) = -\infty, \quad h(\infty) = \infty,$$

there is a unique $\omega^* > 0$ such that

$$h(\omega^*) = e^{\frac{3\pi\omega^*}{2}} - \frac{1}{\omega^*} = 0. \tag{3.4}$$

To claim $t_3(0, \omega^*) > 2\pi$, let us suppose its opposite, i.e., that $t_3 = t_3(0, \omega^*) \leq 2\pi$. Then $t_1(0, \omega^*) = 3\pi/2$ implies $t_2(0, \omega^*) \in (3\pi/2, 2\pi.)$. By (3.3) we have

$$x^* e^{\omega^* t_2} = -\omega^* \int_{t_2}^{t_3} e^{\omega^*\tau} \sin \tau d\tau.$$

Note that (3.4) implies $\omega^* e^{\frac{3\pi\omega^*}{2}} = 1$. It follows that

$$\begin{aligned}x^* &= -\omega^* \int_{t_2}^{t_3} e^{\omega^*(\tau - t_2)} \sin \tau d\tau \\ &< -\omega^* \int_{3\pi/2}^{2\pi} e^{\omega^*(\tau - 3\pi/2)} \sin \tau d\tau \\ &= -\omega^* \int_0^{\pi/2} e^{\omega^*\tau} \sin\left(\frac{3\pi}{2} + \tau\right) d\tau \\ &= -\frac{\omega^*}{1+(\omega^*)^2} e^{\omega^*\tau}\left[\omega^* \sin\left(\frac{3\pi}{2} + \tau\right) - \cos\left(\frac{3\pi}{2} + \tau\right)\right]\Big|_0^{\pi/2} \\ &= \frac{\omega^*}{1+(\omega^*)^2}[e^{\pi\omega^*/2} - \omega^*] \\ &< \omega^* e^{\frac{3\pi\omega^*}{2}} \\ &= 1.\end{aligned}$$

This contradicts the assumption $x^* > 1$.

Lemma 3.2. *For all* $\omega \in [\omega^*, \infty)$

$$-e^{\omega t_1(0,\omega)} \sin(t_1(0,\omega)) > 2.$$

Proof. Let $t(\omega) = t_1(0,\omega)$. By Lemma 3.1 we have $t(\omega^*) = 3\pi/2$. Proposition 2.2 implies that

$$\pi < t(\omega) < 3\pi/2, \quad \omega \in (\omega^*, \infty). \tag{3.5}$$

First suppose $\omega \in \left(\omega*, \frac{1}{\pi}\right]$ (it is true that $\omega^* < 1/\pi$). By definition of $t(\omega)$ we have

$$\begin{aligned} 0 &= \int_0^{t(\omega)} e^{\omega\tau} \sin \tau d\tau \\ &= \frac{1}{1+\omega^2}\left(e^{\omega t(\omega)}[\omega \sin t(\omega) - \cos t(\omega)] + 1\right). \end{aligned} \tag{3.6}$$

Equation (3.6) yields

$$e^{\omega t(\omega)}[\omega \sin t(\omega) - \cos t(\omega)] + 1 = 0$$

or, equivalently,

$$-e^{\omega t(\omega)} \sin t(\omega) = \frac{1 - e^{\omega t(\omega)} \cos t(\omega)}{\omega}. \tag{3.7}$$

Now $t(\omega) \in (\pi, 3\pi/2)$ implies that $-e^{\omega t(\omega)} \cos t(\omega) > 0$. Hence $\omega \leq 1/\pi$ and (3.7) give that

$$-e^{\omega t(\omega)} \sin t(\omega) \geq \pi > 2.$$

Next suppose $\omega > 1/\pi$. Again by the definition of $t(\omega)$ we have

$$\begin{aligned} -\int_\pi^{t(\omega)} e^{\omega\tau} \sin \tau d\tau &= \int_0^\pi e^{\omega\tau} \sin \tau d\tau \geq \int_0^\pi e^{\tau/\pi} \sin \tau d\tau \\ &= \frac{1+e}{1+\left(\frac{1}{\pi}\right)^2} \geq \frac{3.7}{1+\frac{1}{9}} \geq \pi. \end{aligned} \tag{3.8}$$

Noting that

$$-e^{\omega t(\omega)} \sin(t(\omega)) \geq -e^{\omega\tau} \sin \tau, \quad \tau \in [\pi, t(\omega)],$$

together with (3.8) we obtain

$$\begin{aligned} -e^{\omega t(\omega)} \sin(t(\omega))(t(\omega) - \pi) &= -\int_\pi^{t(\omega)} e^{\omega t(\omega)} \sin(t(\omega)) d\tau \\ &\geq -\int_\pi^{t(\omega)} e^{\omega\tau} \sin \tau d\tau \geq \pi. \end{aligned} \tag{3.9}$$

Recall that $t(\omega) - \pi < \pi/2$. Equation (3.7) therefore yields

$$-e^{\omega t(\omega)} \sin(t(\omega)) \geq \frac{\pi}{t(\omega) - \pi} > 2.$$

□

Lemma 3.3. *For $\omega \geq w^*$, if $t_1(x^*, \omega) < 2\pi$ and $|\sin t_1(0, \omega)| \leq |\sin t_1(x^*, \omega)|$, then*

$$\omega[t_1(x^*, \omega) - t_1(0, \omega)] < \ln\left(\frac{x^*}{2} + 1\right).$$

Proof. Let $t_1 = t_1(0, \omega)$ and $\tilde{t}_1 = t_1(x^*, \omega)$. From (3.1) and (3.2) we have

$$x^* + \omega \int_0^{\tilde{t}_1} e^{\omega\tau} \sin \tau d\tau = \omega \int_0^{t_1} e^{\omega\tau} \sin \tau d\tau = 0.$$

It follows that

$$\begin{aligned} x^* &= -\omega \int_{t_1}^{\tilde{t}_1} e^{\omega\tau} \sin \tau d\tau > -\sin(t_1)\omega \int_{t_1}^{\tilde{t}_1} e^{\omega\tau} d\tau \\ &= -e^{\omega t_1} \sin(t_1)(e^{\omega(\tilde{t}_1 - t_1)} - 1). \end{aligned}$$

The above inequality and Lemma 3.2 yield

$$e^{\omega(\tilde{t}_1 - t_1)} \leq \frac{x^*}{-e^{\omega t_1} \sin(t_1)} + 1 \leq \frac{x^*}{2} + 1.$$

That is,

$$\omega(\tilde{t}_1 - t_1) < \ln\left(\frac{x^*}{2} + 1\right).$$

□

Definition 1. Let $\omega > 0$ be fixed. For $a \in (\pi, 2\pi)$, we define

D1. $\Delta(a) > 0$ to be the first positive number such that

$$x^* e^{\omega a} + \omega \int_a^{a+\Delta(a)} e^{\omega\tau} \sin \tau d\tau = 0;$$

D2. $\Gamma(a) > 0$ such that

$$\omega \int_a^{a+\Gamma(a)} e^{-\omega(a+\Gamma(a)-\tau)}[\sin \tau + \beta] d\tau = x^*.$$

From the above definitions we see that $a + \Delta(a)$ is the first time at which the solution to the initial value problem

$$\dot{x} = \omega[\sin t - x], \quad x(a) = x^*$$

vanishes, while the number $a + \Gamma(a)$ is the first time at which the solution to the initial value problem

$$\dot{x} = \omega[\zeta(t) + \beta - x], \quad x(a) = 0$$

arrives at x^*. It should be clear that both $\Delta(a)$ and $\Gamma(a)$ are differentiable, and $a + \Delta(a)$ and $a + \Gamma(a)$ are strictly monotone increasing functions of a.

Proposition 3.1. *Suppose that there is a $\tilde{a} \in (\pi, 2\pi)$ such that $a + \Delta(a) < 2\pi$ for $a \in [\pi, \tilde{a}] \in [\pi, 2\pi]$. Let $\dot{\Delta}$ be the derivative of Δ. Then*

(a) *There is at most one value $a_1 \in [\pi, \tilde{a}]$ at which $\dot{\Delta}(a)$ changes its sign from negative to positive;*
(b) $\dot{\Delta}(a) < 0$ *if* $a + \Delta(a) \le \frac{3\pi}{2}$*;*
$\dot{\Delta}(a) > 0$ *if* $a + \Delta(a) - \frac{3\pi}{2} \ge \frac{3\pi}{2} - a$*, in particular,* $\dot{\Delta}(a) > 0$ *if* $a \ge \frac{3\pi}{2}$*;*
(c) $\omega\Delta(a) > \ln(x^* + 1)$ *for* $a \in [\pi, \tilde{a}]$.

Proof. Let us first prove (a). It is easy to check that

$$x^* e^{\omega a} + \omega \int_a^{a+\Delta(a)} e^{\omega\tau} \sin\tau d\tau = 0$$

is equivalent to

$$x^* + \omega \int_0^{\Delta(a)} e^{\omega\tau} \sin(a+\tau) d\tau = 0.$$

Differentiating the above equality with respect to a we obtain

$$\omega e^{\omega\Delta(a)} \sin(a + \Delta(a))\dot{\Delta}(a) + \omega \int_0^{\Delta(a)} e^{\omega\tau} \cos(a+\tau) d\tau = 0.$$

This yields

$$\dot{\Delta}(a) = -\frac{e^{-\omega\Delta(a)}}{\sin(a+\Delta(a))} \int_0^{\Delta(a)} e^{\omega\tau} \cos(a+\tau) d\tau. \tag{3.10}$$

Note that $\sin(a + \Delta(a)) < 0$ since $\pi < a + \Delta(a) < 2\pi$. Equation (3.10) implies that the sign of $\dot{\Delta}(a)$ is determined by the integral $\int_0^{\Delta(a)} e^{\omega\tau} \cos(a+\tau) d\tau$. Now we write

$$\begin{aligned}\int_0^{\Delta(a)} e^{\omega\tau} \cos(a+\tau) d\tau &= e^{-\omega a} \int_a^{a+\Delta(a)} e^{\omega\tau} \cos(\tau) d\tau \\ &= e^{-\omega a} \left[\int_a^{3\pi/2} e^{\omega\tau} \cos(\tau) d\tau + \int_{3\pi/2}^{a+\Delta(a)} e^{\omega\tau} \cos(\tau) d\tau \right].\end{aligned} \tag{3.11}$$

It is clear that both the integrals

$$\int_a^{3\pi/2} e^{\omega\tau} \cos(\tau) d\tau \quad \text{and} \quad \int_{3\pi/2}^{a+\Delta(a)} e^{\omega\tau} \cos(\tau) d\tau \tag{3.12}$$

increase as a increases. Hence by (3.11) we deduce that there is at most one number a_1 at which the integral $\int_0^{\Delta(a)} e^{\omega\tau} \cos(a+\tau) d\tau$ changes its sign from negative to positive. The same holds for $\dot{\Delta}(a)$. This completes the proof of Part (a).

For part (b), if $a + \Delta(a) \le 3\pi/2$, then both integrals in (3.12) are negative, so that $\dot{\Delta}(a) < 0$. Next suppose that $a + \Delta(a) > 3\pi/2$ and

$$a + \Delta(a) - \frac{3\pi}{2} \ge \frac{3\pi}{2} - a, \quad \text{equivalently,} \quad a + \Delta(a) \ge 3\pi - a.$$

Then, since

$$\int_a^{3\pi/2} e^{\omega\tau}\cos(\tau)d\tau = -\int_{3\pi/2}^{3\pi-a} e^{\omega(3\pi-\tau)}\cos(\tau)d\tau,$$

we obtain

$$\begin{aligned}\int_a^{3\pi/2} & e^{\omega\tau}\cos(\tau)d\tau + \int_{3\pi/2}^{a+\Delta(a)} e^{\omega\tau}\cos(\tau)d\tau \\ &= \int_{3\pi/2}^{a+\Delta(a)} e^{\omega\tau}\cos(\tau)d\tau - \int_{3\pi/2}^{3\pi-a} e^{\omega(3\pi-\tau)}\cos(\tau)d\tau \\ &> \int_{3\pi/2}^{a+\Delta(a)} e^{\omega\tau}\cos(\tau)d\tau - \int_{3\pi/2}^{a+\Delta(a)} e^{\omega\tau}\cos(\tau)d\tau = 0. \end{aligned} \tag{3.13}$$

Equations (3.11) and (3.13) therefore imply that $\dot{\Delta}(a) > 0$. Also it is trivial that $\dot{\Delta}(a) > 0$ for $a \geq 3\pi/2$.

Finally, for part (c), by the definition of $\Delta(a)$ we have

$$x^* e^{\omega a} = \omega \int_a^{a+\Delta(a)} e^{\omega\tau}|\sin\tau|d\tau < e^{\omega(a+\Delta(a))} - e^{\omega a}.$$

It follows that

$$x^* < e^{\omega\Delta(a)} - 1$$

or

$$x^* + 1 < e^{\omega\Delta(a)},$$

so that

$$w\Delta(a) > \ln(x^* + 1).$$

□

Arguing in the same way as above one is able to obtain the following proposition.

Proposition 3.2. *Suppose that there is a* $\tilde{a} \in (\pi, 2\pi]$ *such that* $a + \Gamma(a) < 2\pi$ *for all* $a \in (\pi, \tilde{a}]$. *Then the following hold:*

(a′) *There is at most one value* $a_1 \in [\pi, \tilde{a}]$ *at which* $\dot{\Gamma}(a_1)$ *changes its sign from positive to negative;*

(b′) $\dot{\Gamma}(a) > 0$ *if* $a + \Gamma(a) \leq \frac{3\pi}{2}$, $\dot{\Gamma}(a) < 0$ *if* $a \geq \frac{3\pi}{2}$*;*

(c′) $\ln\frac{\beta}{\beta - x^*} < \omega\Gamma(a) < \ln\frac{\beta-1}{\beta-1-x^*}$.

Now we give the proof of Proposition 2.1.

Proof of Proposition 2.1. For any positive integer N, let $\omega_N > 0$ such that

$$t_{2N+1}(0, \omega_N) = 2\pi.$$

We shall prove $t_{2N}(x^*, \omega_N) < 2\pi$. Let us introduce the following simplified notations:

$$t_n = t_n(0, \omega_N), \quad n = 0, 1, \cdots, 2N + 1$$

and

$$\tilde{t}_n = t_n(x^*, \omega_N), \quad n = 0, 1, \cdots, 2N.$$

Then, by the definitions of $\Delta(a)$ and $\Gamma(a)$, we have

$$\Delta(t_{2k}) = t_{2k+1} - t_{2k}, \quad \Delta(\tilde{t}_{2k}) = \tilde{t}_{2k+1} - \tilde{t}_{2k}, \tag{3.14}$$

$$\Gamma(t_{2k-1}) = t_{2k} - t_{2k-1}, \quad \Gamma(\tilde{t}_{2k-1}) = \tilde{t}_{2k} - \tilde{t}_{2k-1}. \tag{3.15}$$

In addition, by property **P1** of Lemma 2.1 we have

$$t_n < \tilde{t}_n < t_{n+2}, \quad n = 1, 2, \ldots, 2N - 1. \tag{3.16}$$

From Proposition 2.2 and Lemma 3.1 it follows that, for all $\omega \in (0, \omega^*)$,

$$t_3(0, \omega) > t_3(0, \omega^*) > 2\pi.$$

Hence $N \geq 1$ and $t_{2N+1}(0, \omega_N) = 2\pi$ imply that

$$\omega_N > \omega^* \quad \text{and} \quad t_1(0, \omega_N) < t_1(0, \omega^*) = \frac{3\pi}{2}.$$

Therefore, there must be an integer n^* with $0 \leq n^* \leq N - 1$ such that

$$t_{2n^*+1} < \frac{3\pi}{2} \leq t_{2(n^*+1)+1}. \tag{3.17}$$

Also we have

$$t_1 < \tilde{t}_1 < t_{2N+1} = 2\pi.$$

We will break the proof down into two parts. The first part handles the case

$$|\sin t_1| \leq |\sin \tilde{t}_1|, \tag{\textbf{C1}}$$

and the second part handles the case

$$|\sin t_1| > |\sin \tilde{t}_1|. \tag{\textbf{C2}}$$

Proof under condition (C1). Let us first suppose $2 \leq n^* < N - 2$. We will give a remark at the end of the proof of the proposition under condition (**C1**) to explain how the case of $n^* < 2$ or $n^* \geq N - 2$ can be treated analogously. By (3.16) and (3.17) we have

$$\frac{3\pi}{2} \leq t_{2k+1} < \tilde{t}_{2k}, \quad k = n^* + 1, \ldots, N - 1, \tag{3.18}$$

and

$$\frac{3\pi}{2} < \tilde{t}_{2k} < t_{2(k+1)}, \quad k = n^* + 2, \ldots, N-1. \tag{3.19}$$

Part (b′) of Proposition 3.2 implies that $\Gamma(a)$ is decreasing for $a \geq 3\pi/2$. Hence by (3.15) and (3.18) we obtain

$$\tilde{t}_{2k+2} - \tilde{t}_{2k+1} = \Gamma\left(\tilde{t}_{2k+1}\right) < \Gamma(t_{2k+1}) = t_{2k+2} - t_{2k+1}, \tag{3.20}$$
$$k = n^* + 1, \ldots, N-1.$$

Similarly, by applying part (b) of Proposition 3.1 to (3.19) and with the use of (3.14) we have

$$\tilde{t}_{2k+1} - \tilde{t}_{2k} = \Delta\left(\tilde{t}_{2k}\right) < \Delta(t_{2(k+1)}) = t_{2(k+1)+1} - t_{2(k+1)}, \tag{3.21}$$
$$k = n^* + 2, \ldots, N-1$$

(see Figure 7, to the right of point $3\pi/2$).

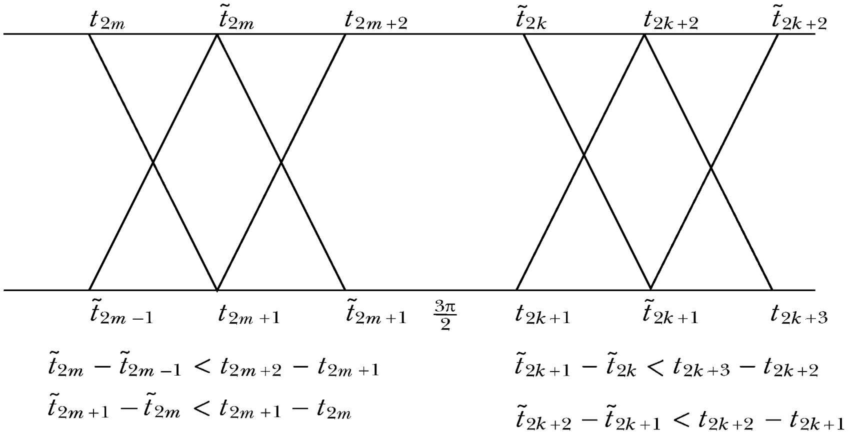


Fig. 7.

Equations (3.20) and (3.21) yield

$$\begin{aligned}
\tilde{t}_{2N} - \tilde{t}_{2n^*+3} &= \sum_{k=n^*+1}^{N-1} \left(\tilde{t}_{2k+2} - \tilde{t}k_{2k+1}\right) + \sum_{k=n^*+2}^{N-1} \left(\tilde{t}_{2k+1} - \tilde{t}_{2k}\right) \\
&< \sum_{k=n^*+1}^{N-1} \left(t_{2k+2} - t_{2k+1}\right) + \sum_{k=n^*+2}^{N-1} \left(t_{2(k+1)+1} - t_{2(k+1)}\right) \\
&= \sum_{k=n^*+1}^{N-1} \left(t_{2k+2} - t_{2k+1}\right) + \sum_{k=n^*+3}^{N} \left(t_{2k+1} - t_{2k}\right) \\
&= t_{2N+1} - t_{2n^*+5} + (t_{2n^*+4} - t_{2n^*+3}).
\end{aligned} \tag{3.22}$$

Next, again by (3.17) we have

$$\tilde{t}_{2k+1} < \tilde{t}_{2k+1} < t_{2n^*+1} < \frac{3\pi}{2}, \quad k = 1, 2, \ldots, n^* - 1 \tag{3.23}$$

and

$$\tilde{t}_{2k} < t_{2k+2} < t_{2n^*+1} < \frac{3\pi}{2}, \quad k = 1, 2, \ldots, n^* - 1. \tag{3.24}$$

Hence (3.23) and part (b) of Proposition 3.1 imply that

$$\begin{gathered}\tilde{t}_{2k+1} - \tilde{t}_{2k} = \Delta(\tilde{t}_{2k}) < \Delta(t_{2k}) = t_{2k+1} - t_{2k}, \\ k = 1, 2, \ldots, n^* - 1.\end{gathered} \tag{3.25}$$

Similarly, (3.24) and part (b$'$) of Proposition 3.2 yield that

$$\begin{gathered}\tilde{t}_{2k} - \tilde{t}_{2k-1} = \Gamma(\tilde{t}_{2k-1}) < \Gamma(t_{2k+1}) = t_{2k+2} - t_{2k+1}, \\ k = 1, 2, \ldots, n^* - 1\end{gathered} \tag{3.26}$$

(see Figure 8, to the left of point $3\pi/2$).

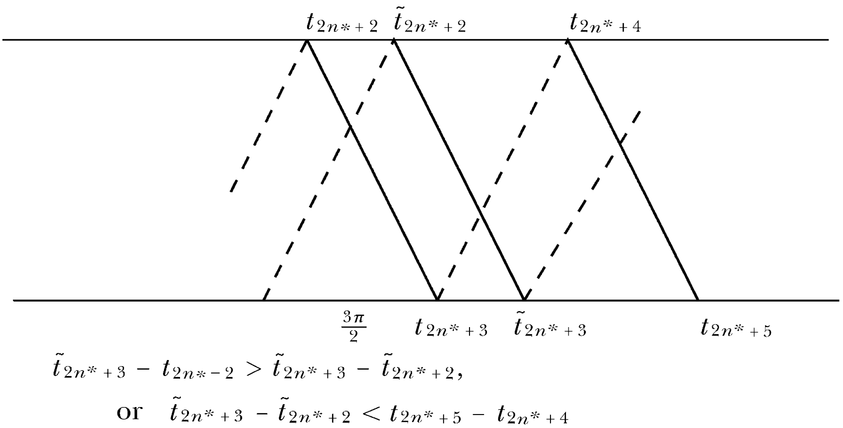


Fig. 8.

From (3.25) and (3.26) we obtain

$$\begin{aligned}\tilde{t}_{2n^*-1} - \tilde{t}_1 &= \sum_{k=1}^{n^*-1} \left(\tilde{t}_{2k+1} - \tilde{t}_{2k}\right) + \sum_{k=1}^{n^*-1} \left(\tilde{t}_{2k} - \tilde{t}_{2k-1}\right) \\ &< \sum_{k=1}^{n^*-1} \left(t_{2k+1} - t_{2k}\right) + \sum_{k=1}^{n^*-1} \left(t_{2k+2} - t_{2k+1}\right) \\ &= t_{2n^*} - t_2.\end{aligned} \tag{3.27}$$

Now we have either $\tilde{t}_{2n^*+1} < 3\pi/2$ or $\tilde{t}_{2n^*+1} \geq 3\pi/2$. If $\tilde{t}_{2n^*+1} < 3\pi/2$, then $t_{2n^*+1} < \tilde{t}_{2n^*+1}$ and part (b) of Proposition 3.1 imply

$$\tilde{t}_{2n^*+1} - \tilde{t}_{2n^*} < t_{2n^*+1} - t_{2n^*}. \tag{\textbf{D1}}$$

If $\tilde{t}_{2n^*+1} \geq 3\pi/2$, then $3\pi/2 < \tilde{t}_{2n^*+2} < t_{2n^*+4}$. Again by part (b) of Proposition 3.1 we have

$$\tilde{t}_{2n^*+3} - \tilde{t}_{2n^*+2} < t_{2n^*+5} - t_{2n^*+4}. \tag{\textbf{D2}}$$

Let us suppose that we have case (**D1**). One will be convinced that the same approach is applicable to case (**D2**). We observe that

$$t_{2n^*+2} < \tilde{t}_{2n^*+2} < t_{2n^*+4}.$$

Hence part (a) of Proposition 3.1 guarantees that the following must be true:

$$\Delta(t_{2n^*+2}) > \Delta\left(\tilde{t}_{2n^*+2}\right) \quad \text{or} \quad \Delta\left(\tilde{t}_{2n^*+2}\right) < \Delta(t_{2n^*+4}).$$

Without loss of generality we suppose the second inequality occurs. That is,

$$\tilde{t}_{2n^*+3} - \tilde{t}_{2n^*+2} = \Delta(t_{2n^*+2}) < \Delta\left(\tilde{t}_{2n^*+4}\right) = t_{2n^*+5} - t_{2n^*+4}. \tag{3.28}$$

(Indeed one will see from the following proof that the choice of the first inequality does not affect the final result.) Similarly by applying part (a′) of Proposition 3.2 to the inequality

$$\tilde{t}_{2n^*-1} < t_{2n^*+1} < \tilde{t}_{2n^*+1}$$

we conclude that

$$\Gamma\left(\tilde{t}_{2n^*-1}\right) < \Gamma(t_{2n^*+1}) \quad \text{or} \quad \Gamma(t_{2n^*+1}) > \Gamma\left(\tilde{t}_{2n^*+1}\right)$$

must hold. Specifically, suppose we have the first inequality, so that

$$\tilde{t}_{2n^*} - \tilde{t}_{2n^*-1} = \Gamma\left(\tilde{t}_{2n^*-1}\right) < \Gamma(t_{2n^*+1}) = t_{2n^*+2} - t_{2n^*+1}. \tag{3.29}$$

By adding through, respectively, the left-hand and right-hand sides of inequalities (3.22), (3.27), (3.28), (3.29), and (**D1**) we thereby obtain

$$\tilde{t}_{2N} - \tilde{t}_{2n^*+2} + \tilde{t}_{2n^*+1} - \tilde{t}_1 < t_{2N+1} - t_{2n^*+3} + t_{2n^*+2} - t_2. \tag{3.30}$$

Recall that $|\sin t_1| \leq |\sin \tilde{t}_1|$. Hence from Lemma 3.3 it follows that

$$\omega_N\left(\tilde{t}_1 - t_1\right) < \ln\left(\frac{x^*}{2} + 1\right). \tag{3.31}$$

By part (c) of Proposition 3.1 we have

$$\omega_N(t_{2n^*+3} - t_{2n^*+2}) > \ln(x^* + 1). \tag{3.32}$$

Similarly, part (c′) of Proposition 3.2 yields

$$\omega_N\big(\tilde t_{2n^*+2}-\tilde t_{2n^*+1}\big) < \ln\left(\frac{\beta-1}{\beta-1-x^*}\right)$$
$$\omega_N(t_2-t_1) > \ln\left(\frac{\beta}{\beta-x^*}\right). \tag{3.33}$$

A straightforward computation shows that the assumption $\beta \geq x^*+2$ implies

$$\frac{\beta-1}{\beta-1-x^*}\left(\frac{x^*}{2}+1\right) \leq (x^*+1)\frac{\beta}{\beta-x^*}. \tag{3.34}$$

We therefore deduce from (3.31)–(3.34) that

$$\begin{aligned}
&\omega_N\big(\tilde t_{2n^*+2}-\tilde t_{2n^*+1}\big)+\omega_N\tilde t_1\\
&\quad= \omega_N\big(\tilde t_{2n^*+2}-\tilde t_{2n^*+1}\big)+\omega_N\big(\tilde t_1-t_1\big)+\omega_N t_1\\
&\quad< \ln\frac{\beta-1}{\beta-1-x^*}+\ln\left(\frac{x^*}{2}+1\right)+\omega_N t_1\\
&\quad= \ln\frac{\beta-1}{\beta-1-x^*}\left(\frac{x^*}{2}+1\right)+\omega_N t_1\\
&\quad\leq \ln(x^*+1)+\ln\frac{\beta}{\beta-x^*}+\omega_N t_1\\
&\quad\leq \omega_N(t_{2n^*+3}-t_{2n^*+2})+\omega_N(t_2-t_1)+\omega_N t_1\\
&\quad= \omega_N(t_{2n^*+3}-t_{2n^*+2})+\omega_N t_2.
\end{aligned} \tag{3.35}$$

Equation (3.35) gives

$$\tilde t_{2n^*+2}-\tilde t_{2n^*+1}+\tilde t_1 < t_{2n^*+3}-t_{2n^*+2}+t_2. \tag{3.36}$$

As a consequence of (3.30) and (3.36) we arrive at

$$\tilde t_{2N} < t_{2N+1} = 2\pi.$$

Hence we complete the proof of Proposition 2.1 for the case $|\sin t_1| \leq |\sin\tilde t_1|$.

Remark 3. In our proof we suppose $2 \leq n^* < N-2$. We shall show that the proof works for the case $2 > n^*$ or $n^* \geq N-2$ as well. For instance, if $n^* = N-2$, then we do not have the equality (3.20) and (3.22) becomes

$$\tilde t_{2N}-\tilde t_{2N-1} < t_{2N}-t_{2N-1}.$$

However, inequalities (3.30)–(3.36) are still valid, so that $\tilde t_{2N} < 2\pi$ still holds. For all other cases we can carry out a similar argument.

Proof under condition (C2). First, it is easy to see that $|\sin t_1| > |\sin\tilde t_1|$, $\pi < t_1 < \tilde t_1 < 2\pi$, and $t_1 < 3\pi/2$ imply that

$$\tilde t_1 - 3\pi/2 > 3\pi/2 - t_1. \tag{3.37}$$

Let us begin by showing the following claim.

Claim. $\tilde{t}_2 < t_3$.

Proof. Let $x_1(t, t_0, x_0)$ be the solution to the initial value problem

$$\dot{x}_1 = \omega_N[\sin t - x_1], \quad x_1(t_0, t_0, x_0) = x_0.$$

Then

$$0 < x_1(t_1, 0, x^*) < x^* \tag{3.38}$$

since $\tilde{t}_1 > t_1$ is the first time the solution vanishes. Therefore (3.38) implies that

$$x_1(t, t_1, x^*) > x_1(t, 0, x^*) \geq 0, \quad t \in [t_1, \tilde{t}_1] \tag{3.39}$$

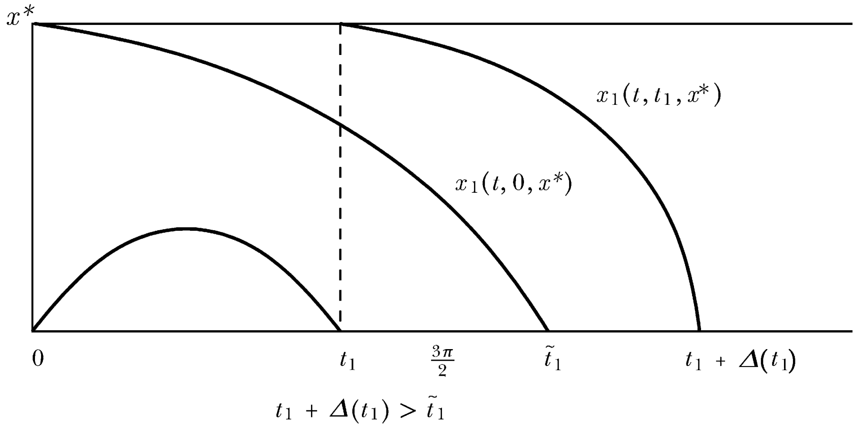


Fig. 9.

(Figure 9). It follows from the definition of $\Delta(t_1)$ and (3.39) that

$$t_1 + \Delta(t_1) > \tilde{t}_1. \tag{3.40}$$

Thus we have

$$t_1 + \Delta(t_1) - \frac{3\pi}{2} > \tilde{t}_1 - \frac{3\pi}{2} > \frac{3\pi}{2} - t_1.$$

Note that $t + \Delta(t)$ is monotone increasing, and the above inequality yields that

$$t + \Delta(t) - \frac{3\pi}{2} > \frac{3\pi}{2} - t, \quad t \geq t_1. \tag{3.41}$$

Hence part (a) of Proposition 3.1 and (3.41) imply that $\dot{\Delta}(t) > 0$ for $t \geq t_1$. Together with (3.40) we obtain

$$t_3 - t_2 = \Delta(t_2) > \Delta(t_1) > \tilde{t}_1 - t_1. \tag{3.42}$$

We therefore deduce from (3.42) that

$$t_3 - \tilde{t}_1 = t_3 - t_2 + t_2 - \tilde{t}_1 > t_2 - t_1. \tag{3.43}$$

Let $x_2(t, t_0)$ be the solution to the initial value problem

$$\dot{x}_2 = \omega_N[\sin t + \beta - x_2], \quad x_2(t_0, t_0) = 0.$$

Condition (3.43) and the assumption $|\sin t_1| > |\sin \tilde{t}_1|$ give that

$$\sin(t_1 + s) < \sin\left(\tilde{t}_1 + s\right), \quad s \in [0, t_2 - t_1] \subset \left[0, t_3 - \tilde{t}_1\right]. \tag{3.44}$$

By (3.44) and the comparison principle one easily sees that

$$x_2(t_1 + s, t_1) \leq x_2\left(\tilde{t}_1 + s, \tilde{t}_1\right), \quad s \in [0, t_2 - t_1]. \tag{3.45}$$

Note that, by the definitions t_n and $\tilde{t}_n$, $n = 1, 2$, we have

$$x_2(t_2, t_1) = x_2\left(\tilde{t}_2, \tilde{t}_1\right) = x^*.$$

The last equality and (3.45) therefore yield

$$\tilde{t}_2 - \tilde{t}_1 < t_2 - t_1. \tag{3.46}$$

As a consequence of (3.43) and (3.46) we have

$$\tilde{t}_2 = \tilde{t}_1 + \tilde{t}_2 - \tilde{t}_1 < \tilde{t}_1 + t_2 - t_1 < \tilde{t}_1 + t_3 - \tilde{t}_1 = t_3. \tag{3.47}$$

This completes the proof of our claim.

Next we have

$$\frac{3\pi}{2} < \tilde{t}_{2k} < t_{2k+2}, \quad k = 1, \ldots, N-1,$$
$$\frac{3\pi}{2} < t_{2k+1} < \tilde{t}_{2k+1}, \quad k = 1, \ldots, N-1.$$

So part (b) of Proposition 3.1 and part (b′) of Proposition 3.2 imply

$$\begin{aligned} \tilde{t}_{2k} - \tilde{t}_{2k} &< t_{2k+3} - t_{2k+2}, \quad k = 1, \ldots, N-1, \\ \tilde{t}_{2k+2} - \tilde{t}_{2k+1} &< t_{2k+2} - t_{2k+1}, \quad k = 1, \ldots, N-1. \end{aligned} \tag{3.48}$$

Equation (3.48) yields that

$$\tilde{t}_{2N} - \tilde{t}_2 < t_{2N+1} - t_3. \tag{3.49}$$

Combining (3.48) and (3.49) we get

$$\tilde{t}_{2N} < t_{2N+1} = 2\pi.$$

□

Remark 4. In our proof we cannot suppose $\tilde{t}_{2N} = t_{2N}(x^*, \omega_N) \leq 2\pi$ at the beginning, so we cannot directly apply part (b′) of Proposition 3.2 to $\Gamma(\tilde{t}_{2N-1}) = \tilde{t}_{2N} - \tilde{t}_{2N-1}$ if $\tilde{t}_{2N} > 2\pi$. However, if we replace (2.6)′ by

$$\dot{x} = \omega[\zeta(t) + \beta - x] \tag{2.6$'$}$$

with

$$\zeta(t) = \begin{cases} \sin t, & \text{if } t \leq 2\pi \\ t - 2\pi, & \text{if } t > 2\pi \end{cases}$$

and replace $\sin\tau$ in definition D2 by $\zeta(\tau)$, then Proposition 3.2 is valid for all $a > \pi$. Thus Proposition 2.1 is true for the time sequence generated by equations (2.5)′ and (2.6)′. Therefore Proposition 2.1 is also true for systems (2.5)′ and (2.6)′ because $\zeta(t) = \sin t$ for $t \in [0, 2\pi]$.

4. Proof of Proposition 2.2

We define the function $T_i : [0, \infty), 2\pi) \times (0, \infty) \to (0, \infty)$, $i = 1, 2$, such that

$$T_i(a, \omega) > a, \quad i = 1, 2$$

and

$$x^* e^{\omega a} + \omega \int_a^{T_1(a,\omega)} e^{\omega\tau} \sin\tau d\tau = 0, \tag{4.1}$$

$$\omega \int_a^{T_2(a,\omega)} e^{\omega\tau}[\sin\tau + \beta] d\tau = x^* e^{\omega T_2(a,\omega)}; \tag{4.2}$$

here $T_1(a, \omega)$ is restricted to be the first value satisfying the above equation.

Lemma 4.1. *Both functions T_1 and T_2 have the property that if $a_1 < a_2$ and $\omega_1 > \omega_2$, then*

$$T_i(a_1, \omega_1) < T_i(a_2, \omega_2), \quad i = 1, 2.$$

Proof. We shall prove the above inequality for function T_1. The proof for function T_2 is the same. First it is trivial if $T_1(a_1, \omega_1) \leq a_2$. Now suppose $T_1(a_1, \omega_1) > a_2$. By (4.1) we have

$$\frac{x^*}{\omega} e^{\omega a} + \int_a^{T_1(a,\omega)} e^{\omega\tau} \sin\tau d\tau = 0.$$

For $i = 1, 2$, let $y_i(t)$ be the solution of the equation

$$\dot{y} = \sin t - \omega_i y, \quad t > a_i$$

with initial condition $y_i(a_i) = x^*/\omega_i$. Then from the definition of T_1 and the assumption $x^* > 1$ (see (**H2**)) one sees that

$$0 < y_i(t) < x^*/\omega_i, \quad t \in (a_i, T_1(a_i, \omega_i)), \quad i = 1, 2, \tag{4.3}$$

and

$$y_1(T_1(a_1,\omega_1)) = y_2(T_1(a_2,\omega_2)) = 0. \tag{4.4}$$

Equation (4.3) and $\omega_1 > \omega_2$ imply that

$$y_1(a_2) \le \frac{x^*}{\omega_1} < \frac{x^*}{\omega_2} = y_2(a_2). \tag{4.5}$$

If $T_1(a_1,\omega_1) \ge T_1(a_2,\omega_2)$, then (4.3)–(4.5) yield that there is a time $t^* \in (a_2, T_1(a_2,\omega_2)]$ such that

$$y_1(t^*) = y_2(t^*) \quad \text{and} \quad y_1(t) < y_2(t), \qquad t \in (a_2, t^*].$$

It follows that

$$\dot{y}_2(t^*) \le \dot{y}_1(t^*) = \sin(t^*) - \omega_1 y_1(t^*) < \sin(t^*) - \omega_2 y_2(t^*) = \dot{y}_2(t^*).$$

This leads to a contradiction.

Lemma 4.2. *For $x_0 \in [0, x^*]$ and $\omega > 0$, if $t_1(x_0,\omega) \in (\pi, 2\pi)$, then*

$$\frac{\partial t_1(x_0,\omega)}{\partial \omega} < 0.$$

Proof. Recall that $t_1(x_0,\omega)$ satisfies the equation

$$x_0 + \omega \int_0^{t_1(x_0,\omega)} e^{\omega\tau} \sin\tau d\tau = 0.$$

Or equivalently

$$\int_0^{t_1(x_0,\omega)} e^{\omega\tau} \sin\tau d\tau = -\frac{x_0}{\omega}. \tag{4.6}$$

Let $t_1 = t_1(x_0,\omega)$. Differentiating equality (4.6) with respect to ω gives

$$e^{\omega t_1} \sin(t_1) \frac{\partial t_1}{\partial \omega} = \frac{x_0}{\omega^2} - \int_0^{t_1} e^{\omega\tau} \tau \sin\tau d\tau. \tag{4.7}$$

Note that $t_1 \in (\pi, 2\pi)$ implies that

$$\begin{aligned}
\int_0^{t_1} e^{\omega\tau} \tau \sin\tau d\tau &= \int_0^{\pi} e^{\omega\tau} \tau \sin\tau d\tau + \int_\pi^{t_1} e^{\omega\tau} \tau \sin\tau d\tau \\
&< \pi \int_0^{\pi} e^{\omega\tau} \sin\tau d\tau + \pi \int_\pi^{t_1} e^{\omega\tau} \sin\tau d\tau \\
&= \pi \int_0^{t_1} e^{\omega\tau} \tau \sin\tau = -\frac{x_0 \pi}{\omega} \le 0.
\end{aligned} \tag{4.8}$$

It follows from (4.7), (4.8), and the inequality $\sin(t_1) < 0$ that

$$\frac{\partial t_1(x_0,\omega)}{\omega} = \frac{1}{e^{\omega t_1} \sin(t_1)} \left[\frac{x_0}{\omega^2} - \int_0^{t_1} e^{\omega\tau} \tau \sin\tau d\tau \right] < 0.$$

□

We are now in a position to prove Proposition 2.2.

Proof of Proposition 2.2. For each fixed $x_0 \in [0, x^*]$ and $\omega > 0$, let $\{t_k(\omega) = t_k(x_0, \omega)\}$ be the corresponding time sequence. Suppose that $\omega_1 > \omega_2$ and $t_i(\omega_i) \in (\pi, 2\pi)$. $i = 1, 2$. First Lemma 4.2 gives that $t_1(\omega_1) < t_1(\omega_2)$. Moreover, by the definition of functions T_i we have

$$T_2(t_{2k-1}(\omega_i), \omega_i) = t_{2k}(\omega_i), \quad k = 1, \dots, \quad i = 1, 2, \tag{4.9}$$

and

$$T_1(t_{2k}(\omega_i), \omega_i) = t_{2k+1}(\omega_i), \quad k = 1, \dots, \quad i = 1, 2. \tag{4.10}$$

Beginning with the inequality $t_1(\omega_1) < t_1(\omega_2)$ and with the repeat use of Lemma 4.1 we are therefore able to inductively obtain

$$t_n(\omega_1) < t_n(\omega_2), \quad n = 1, \cdots .$$

□

5. Singularly perturbed systems

Now we return to the singularly perturbed system (1.1). Let us begin with the transformed singularly perturbed system (2.1), which we write here again for convenience:

$$\begin{aligned} \dot{x} &= \omega[\sin t - y], \\ \dot{y} &= \frac{1}{\rho}[x - g(y)], \end{aligned} \tag{5.1}$$

with

$$g(y) = \begin{cases} y, & y \geq 0 \\ Ky, & y^* \leq y \leq 0 \\ \beta + y, & y \leq y^* \end{cases} \tag{5.2}$$

where $K < 0$, $y^* < 0$, and $\beta > 0$.

Let Ω_i, $i = 1, 2$ be the regions given in Figure 10. Moreover, for $\gamma > 0$ let Σ_i^γ and l_i^γ, $i = 1, 2$ be the region and line segment as shown in Figure 11.

A solution $(x(t, t_0, x_0, y_0, \omega, \rho), y(t, t_0, x_0, y_0, \omega, \rho))$ of system (5.1) depends on the initial time t_0, initial date $z_0 = (x_0, y_0)$, and the parameters ω and ρ. For notational simplicity we often omit the parameters ω and ρ and simply write them as $z(t, t_0, z_0) = (x(t, t_0, x_0, y_0), y(t, t_0, x_0, y_0))$ if no confusion arises from this.

In the sequel we always let $I = [\omega_m, \omega_M]$ denote a closed interval with $0 < \omega_m < \omega_M$.

Lemma 5.1. *Given an interval I, the following hold:*

(1) *There is a $\bar{\rho}_1 > 0$ such that for all $\rho \in (0, \bar{\rho}_1]$ and $\omega \in I$, if $(x_0, y_0) \in \Omega_1$ and $x_0 > 0$, then $x_0 - \sqrt{1 - 4\omega\rho}\, y_0 > 0$.*

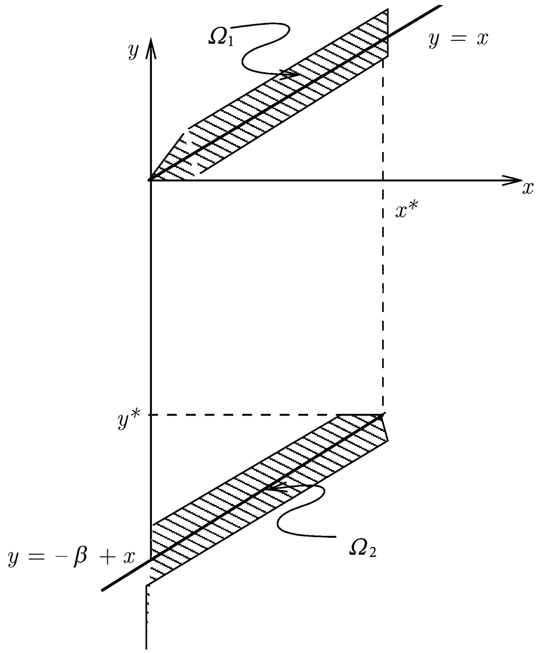


Fig. 10.

(2) *For each* $\delta > 0$ *there is a* $\rho_\delta > 0$ *such that for* $i = 1, 2$, $\omega \in I$, *and* $\rho \in (0, \rho_\delta]$, *if* $z_0 \in \partial\Sigma_i^\delta \setminus l_i^\delta$, *then the solution* $z(t, t_0, z_0)$ *enters the interior of* Σ_i^δ *and can leave* Σ_i^δ *only from* l_i^δ.
(3) *For each* $\delta > 0$, *if* $z_0 \in l_1^\delta$ *and* $t_0 \in (\pi, 2\pi)$, *then* $z(t, t_0, \omega)$ *leaves* Σ_1^δ. *If* $z_0 \in l_2^\delta$, *then for any* t_0, $z(t, t_0, \omega)$ *leaves* Σ_2^δ.

Proof. (1) can be verified algebraically. (2) and (3) follow the analysis of the vector field defined by the right-hand side of functions of (5.1). □

Before proceeding to the proof of our main theorem, let us first give an explicit formula for the solution to the following second order equation:

$$\ddot{y} = -\frac{1}{\rho}(r\dot{y} + \omega y) + \frac{\omega}{\rho}\sin t, \tag{5.3}$$

with initial condition

$$y(t_0) = y_0, \qquad \dot{y}(t_0) = \frac{y_1}{\rho}, \tag{5.4}$$

where $r \neq 0$. The corresponding linear homogeneous equation of (5.3) has the characteristic equation

$$\lambda^2 + \frac{r}{\rho}\lambda + \frac{\omega}{\rho} = 0,$$

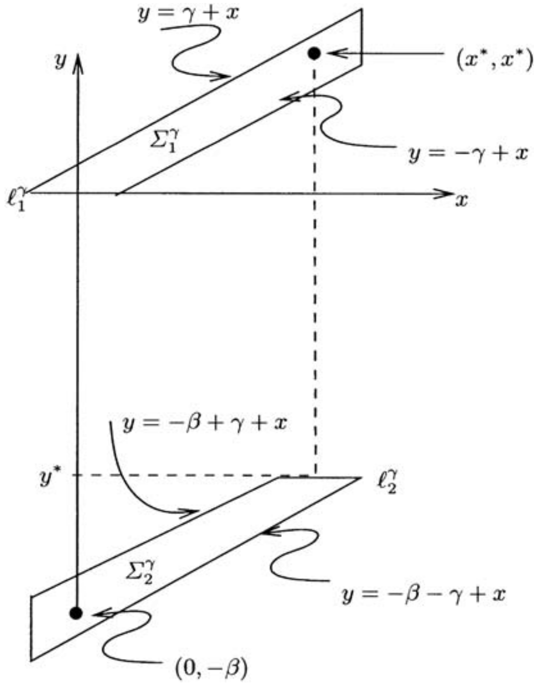


Fig. 11.

which has two roots

$$\lambda_1(\rho) = -\frac{r}{2\rho}\left[1 + \sqrt{1 - 4\omega\rho/r^2}\right],$$
$$\lambda_2(\rho) = -\frac{r}{2\rho}\left[1 - \sqrt{1 - 4\omega\rho/r^2}\right]. \tag{5.5}$$

By using the variation-of-constant formula one is able to verify that the solution to the initial value problem (5.3)–(5.4) can be expressed as

$$\begin{aligned} y(t) = &-\frac{1}{2r\sqrt{1 - 4\omega\rho/r^2}}\left[r\left(1 - \sqrt{1 - 4\omega\rho/r^2}\right)y_0 + 2y_1\right]e^{\lambda_1(\rho)(t-t_0)} \\ &+ \frac{1}{2r\sqrt{1 - 4\omega\rho/r^2}}\left[r\left(1 + \sqrt{1 - 4\omega\rho/r^2}\right)y_0 + 2y_1\right]e^{\lambda_2(\rho)(t-t_0)} \\ &- \frac{\omega}{\sqrt{1 - 4\omega\rho/r^2}}\int_{t_0}^{t}\left[e^{\lambda_1(\rho)(t-\tau)} - e^{\lambda_2(\rho)(t-\tau)}\right]\sin\tau d\tau. \end{aligned} \tag{5.6}$$

From the expressions in (5.5) one easily sees that, as $\rho \to 0$,

$$\lambda_1(\rho) \to -\mathrm{sign}(r)\infty, \quad \lambda_2(\rho) \to -\frac{\omega}{r}$$

uniformly for ω in any bounded set of $\mathbb{R}$.

Let $t_1(x_0, \omega)$ be the function defined in Section 2.

Lemma 5.2. *Suppose that there is a $d > 0$ with $\pi + d \leq t_1(x_0, \omega) \leq 2\pi - d$ for all $(x_0, \omega) \in [0, x^*] \times I$. Then for each $\gamma > 0$, there is a $\rho_\gamma > 0$ such that for each $z_0 = (x_0, y_0) \in \Omega_1$, $\omega \in I$, and $\rho \in (0, \rho_\gamma]$, the solution $z(t, 0, z_0)$ has a first time $s_1(z_0, \omega, \rho) \in (\pi, 2\pi)$ with*

$$|s_1(z_0, \omega, \rho) - t_1(x_0, \omega)| \leq \gamma, \quad z(s_1(z_0, \omega, \rho), 0, z_0) \in l_1^\gamma.$$

Proof. Let $(x(t), y(t)) = z(t, 0, z_0)$. It is sufficient to show that $(x(t), y(t))$ has a time $s_1 = s_1(z_0, \omega, \rho) \in (\pi, 2\pi)$ such $y(t) > 0$ for $t \in (0, s_1)$ and $y(s_1) = 0$, and in addition, $s_1(z_0, \omega, \rho) \to t_1(x_0, \omega)$ and $x(s_1) \to 0$ as $\rho \to 0$ uniformly for $\omega \in I$ and $z_0 = (x_0, y_0) \in \Omega_1$. Let us first suppose that $z_0 = (x_0, y_0) \in \Omega_1 \setminus (0, 0)$. Then by part (2) of Lemma 5.1, $y(t) > 0$ for small $t > 0$, so that $g(y(t)) = y(t)$. It follows that

$$\ddot{y} = \frac{1}{\rho}(\dot{x} - \dot{y}) = -\frac{1}{\rho}(\dot{y} + \omega y) + \frac{\omega}{\rho}\sin t.$$

with

$$y(0) = y_0, \quad \dot{y}(0) = \frac{x_0 - y_0}{\rho}.$$

Using (5.6) for $r = 1$ and $t_0 = 0$ we obtain

$$\begin{aligned} y(t) = &-\frac{1}{2\sqrt{1 - 4\omega\rho}}\left[2x_0 - \left(1 + \sqrt{1 - 4\omega\rho}\right)y_0\right]e^{\lambda_1(\rho)t} \\ &+ \frac{1}{2\sqrt{1 - 4\omega\rho}}\left[2x_0 - \left(1 - \sqrt{1 - 4\omega\rho}\right)y_0\right]e^{\lambda_2(\rho)t} \\ &- \frac{\omega}{\sqrt{1 - 4\omega\rho}}\int_0^t \left[e^{\lambda_1(\rho)(t-\tau)} - e^{\lambda_2(\rho)(t-\tau)}\right]\sin\tau d\tau. \end{aligned} \tag{5.7}$$

Note that $\lambda_2(\rho) > \lambda_1(\rho)$ and for $(x_0, y_0) \in \Omega_1$, $\omega \in I$, and $\rho \leq \bar{\rho}_1$ we have, by part (1) of Lemma 5.1,

$$2x_0 - \left(1 - \sqrt{1 - 4\omega\rho}\right)y_0 \geq 0.$$

With the above inequality one is able to verify that the sum of the first two terms of the right-hand side of (5.7) is positive. Also, it is trivial that the integral in (5.7) is positive for all $t \in (0, \pi]$. It follows that

$$y(t) > 0, \quad t \in (0, \pi]. \tag{5.8}$$

By the continuity of the solution to the initial value we see that (5.7) is valid for $(x_0, y_0) = (0, 0)$ as well. Since

$$\lambda_1(\rho) \to -\infty, \quad \lambda_2(\rho) \to -\omega$$

as $\rho \to 0$ uniformly for $\omega \in I$, we deduce from (5.7) that

$$y(t) \to y^*(t, x_0, \omega) = x_0 e^{-\omega t} + \omega \int_0^t e^{-\omega(t-\tau)} \sin \tau d\tau \tag{5.9}$$

as $\rho \to 0$ uniformly for $(x_0, y_0) \in \Omega_1$, $\omega \in I$, and $t \in [\epsilon, 2\pi]$ with $y(t) \geq 0$, where ϵ is any positive number. By the definition of $t_1(x_0, \omega)$ and (5.9) we have $y^*(t_1(x_0, \omega), x_0, \omega) = 0$. Hence the assumption of $t_0(x_0, \omega) \leq 2\pi - d$ for $x_0 \in [0, x^*]$ and $\omega \in I$ implies that there is a $d_1 > 0$ such that

$$\begin{aligned} y^*(2\pi, x_0, \omega) &= e^{-\omega 2\pi}\left[x_0 + \omega \int_0^{2\pi} e^{\omega(\tau)} \sin \tau d\tau\right] \\ &= e^{-\omega 2\pi} \omega \int_{t_1(x_0,\omega)}^{2\pi} e^{\omega(\tau)} \sin \tau d\tau \\ &\leq e^{-\omega 2\pi} \omega \int_{2\pi - d}^{2\pi} e^{\omega(\tau)} \sin \tau d\tau \\ &=< -d_1 \end{aligned} \tag{5.10}$$

for all $(x_0, \omega) \in [0, x^*] \times I$. It therefore follows from from (5.8)–(5.10) that there is a first time $s_1 = s_1(z_0, \omega, \rho) \in (\pi, 2\pi)$ such that $y(s_1) = 0$. In addition, since $t_1(x_0, \omega)$ is the only zero of $y^*(t, x_0, \omega)$ for $t \in [\pi, 2\pi]$ and $0 = y(s_1(z_0, \omega, \rho)$ converges to $y^*(s_1(z_0, \omega, \rho), x_0, \omega)$ as $\rho \to 0$ uniformly for $(z_0, \omega) \in \Omega_1 \times I$, we conclude that

$$s_1(z_0, \omega, \rho) \to t_1(x_0, \omega)$$

as $\rho \to 0$ uniformly for $(z_0, \omega) \in \Omega_1 \times I$. Next, let

$$h^*(t) = h^*(t, z_0, \omega, \rho) = y(t) - y^*(t, x_0, \omega), \quad t \in [0, s_1]. \tag{5.11}$$

Then $h^*(t) \to 0$ as $\rho \to 0$ uniformly for $(z_0, \omega) \in \Omega_1 \times I$ and $t \in [\epsilon, s_1]$ for any $\epsilon > 0$. This implies that

$$\int_0^{s_1} h^*(s) ds \to 0$$

as $\rho \to 0$ uniformly for $(z_0, \omega) \in \Omega_1 \times I$. Recall that $\dot{x}(t) = \omega[\sin t - y(t)]$. Hence (5.9) and (5.11) yield that

$$\begin{aligned} x(s_1) &= x_0 + \int_0^{s_1} [\sin s - y(s)] ds \\ &= x_0 + \int_0^{s_1} [\sin s - h^*(s) - y^*(s, x_0, \omega)] ds \\ &= x_0 + \int_0^{s_1} \left[\sin s ds - \int_0^{s_1} h^*(s)\right] ds \\ &\quad - \int_0^{s_1} x_0 e^{-\omega s} ds - \omega \int_0^{s_1} \int_0^s e^{-\omega(s-\tau)} \sin \tau d\tau ds \\ &= x_0 e^{-\omega s_1} + \omega \int_0^{s_1} e^{-\omega(s_1-\tau)} \sin \tau d\tau - \omega \int_0^{s_1} e^{-\omega(s_1-\tau)} h^*(s) ds. \end{aligned} \tag{5.12}$$

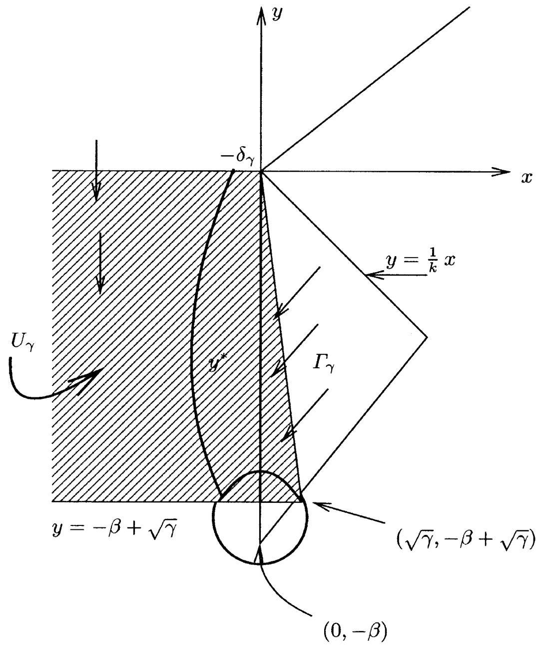


Fig. 12.

Since $s_1 \to t_1(x_0, \omega)$ and $\int_0^{s_1} h^*(s)ds \to 0$ as $\rho \to 0$ uniformly for $(z_0, \omega) \in \Omega_1 \times I$, from (5.12) it follows that

$$x(s_1) \to x_0 e^{-\omega t_1(x_0,\omega)} + \omega \int_0^{t_1(x_0,\omega)} e^{-\omega(t_1(x_0,\omega)-\tau)} \sin \tau d\tau = 0$$

as $\rho \to 0$ uniformly for $(z_0, \omega) \in \Omega_1 \times I$. □

For any point (x_0, y_0) in the x-y plane and any $\gamma > 0$ we let $B_\gamma(x_0, y_0)$ denote a disk of center (x_0, y_0) and radius γ.

Lemma 5.3. *Let $\pi < T_1 < T_2 < 2\pi$ be fixed. Then, for each $\gamma > 0$, there are $\rho_\gamma > 0$ and $\delta_\gamma > 0$ such that for each $\omega \in I$, $\rho \in (0, \rho_\gamma)$, $t_0 \in [T_1, T_2]$, and $z_0 \in l_1^{\delta_\gamma}$ (respectively if $z_0 \in l_2^{\delta_\gamma}$ and $t_0 \geq 0$), there is a $t_\gamma = t_\gamma(t_0, z_0, \omega, \rho) \in (0, \gamma]$ such that*

$$z(t_0 + t_\gamma, t_0, z_0) \in B_\gamma(0, -\beta).$$
$$\big(\text{respectively} \quad z(t_0 + t_\gamma, t_0, z_0) \in B_\gamma(x^*, x^*).\big)$$

Proof. We give the proof only for the case $z_0 \in l_1^{\delta_\gamma}$. The proof of the second case is similar. Let Γ_γ be the line segment connecting points $(0, 0)$ and $(\sqrt{\gamma}, -\beta + \sqrt{\gamma})$

(see Figure 12, where we suppose that Γ_γ is on the left-hand side of the line $y = 1/K$. This can be done if γ is small,) and let

$$\delta_\gamma = \frac{\sqrt{\gamma}}{3}.$$

Then a straightforward computation shows that there is a $\rho_1 > 0$ such that for all $\omega \in I$, $\rho \in (0, \rho_1]$, $t \in (\pi, 2\pi)$, and $(x, y) \in \Gamma_\gamma$, the vector field

$$\overrightarrow{(\omega[\sin t - y), (x - g(y))/\rho)}$$

defined by the right hand side of the system (5.1) points to the left side of Γ_γ. Hence the solution $z(t, t_0, z_0)$ cannot leave region U_γ (Figure 12) from Γ_γ if $\pi < t_0 \le t < 2\pi$. When the solution $(x(t), y(t)) = z(t, t_0, z_0)$ enters U_γ and stays in the region $U_\gamma \cap \{y^* \le y \le 0\}$, $g(y(t)) = Ky(t)$. Hence $y(t)$ satisfies the second order equation

$$\ddot{y} = -\frac{1}{\rho}(K\dot{y} + \omega y) + \frac{\omega}{\rho}\sin t \tag{5.13}$$

with initial condition

$$y(t_0) = 0, \qquad \dot{y}(t_0) = \frac{x(t_0)}{\rho} = \frac{x_0}{\rho}, \quad x_0 \in [-\delta_\gamma, 0]. \tag{5.14}$$

Applying (5.6) to (5.13) and (5.14) we obtain

$$\begin{aligned} y(t) = &-\frac{x_0}{K\sqrt{1 - 4\omega\rho/K^2}}\left(e^{\lambda_1(\rho)(t-t_0)} + e^{\lambda_2(\rho)(t-t_0)}\right) \\ &- \frac{\omega}{K\sqrt{1 - 4\omega\rho/K^2}}\int_{t_0}^{t}\left[e^{\lambda_1(\rho)(t-\tau)} - e^{\lambda_2(\rho)(t-\tau)}\right]\sin\tau d\tau, \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} \lambda_1(\rho) &= -\frac{K}{2\rho}\left[1 + \sqrt{1 - 4\omega\rho/K^2}\right], \\ \lambda_2(\rho) &= -\frac{K}{2\rho}\left[1 - \sqrt{1 - 4\omega\rho/K^2}\right]. \end{aligned} \tag{5.16}$$

Since $K < 0$, $\lambda_1(\rho) \to +\infty$ and $\lambda_2(\rho) \to -\omega/K$ uniformly for $\omega \in I$. Let $T_2 < T_2^* < 2\pi$ and

$$s^* = \max\left\{\sin t : \ t \in \left[T_1, T_2^*\right]\right\}.$$

Then $s^* < 0$. Thus (5.15) yields that (recall $I = [\omega_m, \omega_M]$)

$$\begin{aligned} y(t) &\le \frac{\omega}{|K|\sqrt{1 - 4\omega\rho/k^2}}\int_{t_0}^{t}\left[e^{\lambda_1(\rho)(t-\tau)} - e^{\lambda_2(\rho)(t-\tau)}\right]\sin\tau d\tau \\ &\le \frac{s^*\omega_m}{|K|}\int_{t_0}^{t}\left[e^{\lambda_1(\rho)(t-\tau)} - e^{\lambda_2(\rho)(t-\tau)}\right]d\tau \\ &= \phi^*(t_0, t, \rho), \end{aligned} \tag{5.17}$$

whenever $0 \le y(t) \le y^*$ and $t \in [T_1, T_2^*]$. Let

$$\sigma = \min\left\{\frac{\delta_\gamma}{\omega_M(1+\beta)}, T_2^* - T_2, \frac{\gamma}{2}\right\}.$$

Then $\sigma > 0$ and $t_0 + \sigma \in [T_1, T_2^*]$ for $t_0 \in [T_1, T_2]$. Hence

$$\phi^*(t_0, t_0+\sigma, \rho) = \frac{s^*\omega_m}{|K|}\int_{t_0}^{t_0+\sigma}\left[e^{\lambda_1(\rho)(t-\tau)} - e^{\lambda_2(\rho)(t-\tau)}\right]d\tau \to -\infty$$

as $\rho \to 0$ uniformly for all $\omega \in I$. Consequently there is a $0 < \rho_2 \le \rho_1$ such that

$$\phi^*(t_0, t_0+\sigma, \rho) < y^*$$

for all $\rho \in (0, \rho_2]$, $t_0 \in [T_1, T_2]$, and $\omega \in I$. It follows from (5.17) that for each $\rho \in (0, \rho_2]$ and each $\omega \in I$, there is a $t' = t'(x_0, \omega, \rho) \in (0, \sigma]$ such that

$$0 \le y(t) < y^*, \quad t \in [t_0, t_0 + t'), \quad y(t_0 + t') = y^*.$$

It is obvious that the solution $(x(t), y(t))$ enters the region

$$U_\gamma^* = U_\gamma \cap \{y^* \le y \le -\beta + \sqrt{\gamma}\}$$

when the time t passes $t_0 + t'$ and when there is a positive number d such that

$$x(t) - g(y(t)) < -d, \quad (x(0, y(t)) \in U_\gamma^*.$$

Hence

$$\dot{y}(t) = \frac{1}{\rho}(x - g(y(t)) \le -\frac{d}{\rho},$$

so that

$$y(t) \le y^* - \frac{d(t - t_0 - t')}{\rho} \tag{5.18}$$

whenever $(x(t), y(t)) \in U_\gamma^*$. Let

$$\rho_\gamma = \min\left\{\rho_2, \frac{\sigma d}{\beta + y^*}\right\}. \tag{5.19}$$

Then (5.19) implies that, for all $\rho \in (0, \rho_\gamma)$,

$$y^* - \frac{d\sigma}{\rho} \le y^* - \frac{d\sigma}{\rho_\gamma} \le y^* - (\beta + y^*) = -\beta < -b + \sqrt{\gamma}.$$

It therefore follows from (5.18) and the last inequality that, for all $\rho \in (0, \rho_\gamma]$, there is a $t'' \in (0, \sigma)$ such that

$$y(t_0 + t' + t'') = -\beta + \sqrt{\gamma}. \tag{5.20}$$

Now for $t \in [t_0, t_0 + t_\gamma]$ with $t_\gamma = t' + t''$ and $\omega \in I = [\omega_m, \omega_M]$, we have

$$|\dot{x}(t)| = \omega |\sin t - y(t)| \leq \omega_M(1 + \beta).$$

This yields that

$$|x(t_0 + t_\gamma)| \leq |x_0| + t_\gamma \omega_M(1 + \beta) \leq \delta_\gamma + 2\sigma\omega_M(1 + \beta) \leq \delta_\gamma + 2\delta_\gamma \leq \sqrt{\gamma}. \tag{5.21}$$

Equations (5.20) and (5.21) imply that

$$z(t_0 + t_\gamma, t_0, z_0) \in B_\gamma(0, -\beta)$$

with $t_\gamma = t' + t'' \leq 2\sigma \leq \gamma$. □

Definition 2. For $\theta \geq 0$ and $\omega > 0$ we define $\xi_i = \xi_i(\theta, \omega) > \theta, i = 1, 2$ such that

D3. $x^* e^{-\omega(\xi_1 - \theta)} + \omega \int_\theta^{\xi_1} e^{-\omega(\xi_1 - \tau)} \sin \tau d\tau = 0,$

D4. $\omega \int_\theta^{\xi_2} e^{-\omega(\xi_2 - \tau)}[\sin \tau + \beta] d\tau = x^*.$

Note that $\xi_1 > \theta$ the first time at which the solution $x_1(t, \theta, x^*)$ to the initial value problem

$$\dot{x}_1 = \omega[\sin t - x_1], \quad x_1(\theta) = x^*$$

vanishes. The number $\xi_2 > \theta$ is the first time at which the solution $x_2(t, \theta, 0)$ to the initial value problem

$$\dot{x}_2 = \omega[\sin t + \beta - x_2], \quad x_2(\theta) = 0$$

reaches x^*. The assumption of $\beta - 1 > x^*$ implies that $\xi_2(\theta, \omega)$ is uniquely defined for all $\theta \geq 0$ and $\omega > 0$.

Lemma 5.4. *Let $I = [\omega_m, \omega_M]$ with $\omega_m > \omega_m^*$ and let $D \subset \mathbb{R}_+ \times I$ be a closed set. Then the following are true.*

(a) *Suppose that there is a $\zeta > 0$ and an integer $M = 1$ or $M = 2$ such that $\xi_1(\theta, \omega) \in [(2M - 1)\pi + \zeta, 2M\pi - \zeta]$ for all $(\theta, \omega) \in D$. Then for each $\gamma > 0$, there are $\delta_\gamma > 0$ and $\rho_\gamma > 0$ such that for each $(\theta, \omega) \in D$, $|t_0 - \theta| \leq \delta_\gamma$, and $z_0 \in B_{\delta_\gamma}(x^*, x^*)$, the solution $z(t, t_0, z_0, \omega, \rho)$ with $\rho \in (0, \rho_\gamma]$ has a first time $\eta_1 = \eta_1(t_0, z_0, \omega, \rho)$ such that*

$$z(\eta_0, t_0, z_0) \in l_1^\gamma, \quad \text{and} \quad |\eta_1(t_0, z_0, \omega, \rho) - \xi_1(\theta, \omega)| \leq \gamma.$$

(b) *For each $\gamma > 0$, there is a $\delta_\gamma > 0$ and $\rho_\gamma > 0$ such that for each $(\theta, \omega) \in D$, $|t_0 - \theta| \leq \delta_\gamma$, and $z_0 \in B_{\delta_\gamma}(0, -\beta)$, the solution $z(t, t_0, z_0, \omega, \rho)$ with $\rho \in (0, \rho_\gamma]$ has a first time $\eta_2 = \eta_2(t_0, z_0, \omega, \rho)$ such that*

$$z(\eta_0, t_0, z_0) \in l_2^\gamma, \quad \text{and} \quad |\eta_2(t_0, z_0, \omega, \rho) - \xi_2(\theta, \omega)| \leq \gamma.$$

Remark 5. In part (a) of Lemma 5.4 we require that $\xi_1(\theta, \omega) \notin \{(2M - 1)\pi, 2M\pi\}$ because if $\xi_1(\theta_0, \omega_0) = (2M - 1)\pi$ or $\xi_1(\theta_0, \omega_0) = 2M\pi$, then $\xi_1(\theta, \omega)$ may not be continuous at (θ_0, ω_0), while the function $\xi_2(\theta, \omega)$ is always continuous under the condition $\beta > x^* + 1$.

Proof. We shall prove part (b) of Lemma 5.4 only. The proof for part (a) is similar to the proof of Lemma 5.2. Let $\delta_\gamma > 0$ be small enough such that $B_{\delta_\gamma}(0, -\beta) \subset \Sigma_2^\gamma \setminus l_2^\gamma$. By part (2) of Lemma 5.1 there is a $\rho_\gamma > 0$ such that for $\rho \in (0, \rho_\gamma]$, $\omega \in I$, and $z_0 = (x_0, y_0) \in B_{\delta_\gamma}(0, -\beta)$, the solution $z(t, t_0, z_0) = (x(t), y(t))$ stays in the interior of Σ_2^γ before it reaches l_2^γ. Let

$$h(t) = y(t) - (-\beta + x(t)).$$

Then $(x(t), y(t)) \in \Sigma_2^\gamma$ implies that $|h(t)| \le \gamma$ and

$$\dot{x}(t) = \omega[\sin t - y(t)] = \omega[\sin t + \beta - h(t) - x(t)].$$

It follows that

$$\begin{aligned} x(t) &= x_0 e^{-\omega(t-t_0)} + \omega \int_{t_0}^{t} e^{-\omega(t-\tau)}[\sin\tau + \beta - h(\tau)]d\tau \\ &\ge x_0 e^{-\omega(t-t_0)} + \omega \int_{t_0}^{t} e^{-\omega(t-\tau)}[\beta - 1 - \gamma]d\tau \\ &= x_0 e^{-\omega(t-t_0)} + [\beta - 1 - \gamma](1 - e^{-\omega(t-t_0)}) \end{aligned} \tag{5.22}$$

whenever $(x(t), y(t)) \in \Sigma_2^\gamma$. Note that, by our assumption (see (**H2**) in Section 2), $\beta - 1 > x^* + 1$. Without loss of generality we can suppose γ is small such that

$$\beta - 1 - \gamma > x^* + \gamma.$$

It therefore follows that, for sufficiently small $\delta_\gamma > 0$, there is a sufficiently large T such that

$$x_0 e^{-\omega T} + [\beta - 1 - \gamma](1 - e^{-\omega T}) > x^* + \gamma \tag{5.23}$$

for all $|x_0| \le \delta_\gamma$ and $\omega \in I$. Since $z(t, t_0, z_0)$ can leave Σ_2^γ at l_2^γ and $x(t) > x^* + \gamma$ implies $(x(t), y(t)) \notin \Sigma_2^\gamma$, (5.22) and (5.23) therefore yield that there is a first time $\eta_2 = \eta_2(t_0, z_0, \omega, \rho) \le t_0 + T$ such that

$$z(\eta_2, t_0, z_0) \in l_2^\gamma.$$

Next we claim that, if δ_γ and ρ_γ are small enough, then $|t_0 - \theta| \le \delta_\gamma$, $z_0 \in B_\delta(0, -\beta)$, $\omega \in I$, and $\rho \in (0, \rho_\gamma]$ imply that

$$|\eta_2(t_0, z_0, \omega, \rho) - \xi_2(\theta, \omega)| \le \gamma$$

for all $(\theta, \omega) \in D$. Suppose, on the contrary, that the above statement is false. For $n = 1, 2, 3, \ldots$, let $\delta_n > 0$ such that $B_{\delta_n}(0, -\beta) \subset \Sigma_2^{1/n} \setminus l_2^{1/n}$ and let $\rho^n > 0$ (with $\rho^{n+1} \le \rho^n$) be the corresponding number defined in part (2) of Lemma 5.1. Then for each n, there are $(\theta^n, \omega^n) \in D$, t_0^n with $|t_0^n - \theta^n| \le \delta_n$, $z_0^n = (x_0^n, y_0^n) \in B_{\delta_n}(0, -\beta)$, and $\rho^n \in (0, \rho_{1/n}]$ such that

$$\left|\eta_2\left(t_0^n, z_0^n, \omega^n, \rho^n\right) - \xi_2(\theta^n, \omega^n)\right| > \gamma. \tag{5.24}$$

Note that both the sequences $\{(t_0^n, z_0^n, \omega^n, \rho^n)\}$ and $\{\eta_2^n = \eta_2(t_0^n, z_0^n, \omega^n, \rho^n)\}$ are bounded. Without loss of generality we can suppose that

$$\left(t_0^n, z_0^n, \omega_n, \rho^n\right)\} \to (t^*, z^*, \omega^*, \rho^*) \qquad \eta_2^n \to \eta^* \tag{5.25}$$

as $n \to \infty$. It is obvious that $z^* = (0, -\beta)$, $\theta^n \to \theta^* = t_0^*$, and $(\theta^*, \omega^*) \in D$. Moreover, let $(x_n(t), y_n(t)) = z(t, t_0^n, z_0^n, \omega^n, \rho^n)$ for $t \in [t_0^n, \eta_2^n]$. Then we have

$$x_n\left(\eta_2^n\right) = x_0^n e^{-\omega_n\left(\eta_2^n - t_0^n\right)} + \omega_n \int_{t_o^n}^{\eta_2^n} e^{-\omega_n\left(\eta_2^n - \tau\right)}[\sin\tau + \beta - h_n(\tau)]d\tau \tag{5.26}$$

with $h_n(t) = y_n(t) - (-\beta + x_n(t))$. Since $(x_n(t), y_n(t)) \in \Sigma_2^{1/n}$ for $t \in [t_0^n, \eta_2^n]$ and $(x_n(\eta_2^n), y_n(\eta_2^n)) \in l_2^{1/n}$ imply that

$$|h_n(t)| \leq \frac{1}{n}, \quad \left[t_0^n, \eta_2^n\right] \tag{5.27}$$

and

$$x_n\left(\eta_2^n\right) \to 0 \quad \text{as} \quad n \to \infty. \tag{5.28}$$

by letting $n \to \infty$ in (5.26) and with the use of (5.27)–(5.28), we therefore obtain

$$0 = \omega^* \int_{\theta^*}^{\eta^*} e^{-\omega^*(\eta^* - \tau)}[\sin\tau + \beta]d\tau.$$

It follows from the definition of $\xi_2(\theta, \omega)$ that $\eta^* = \xi_2(\theta^*, \omega^*)$. This leads to a contradiction to (5.24) since $\xi_2(\theta, \omega)$ is continuous on (θ, ω). □

With all the results obtained above we are ready to prove conclusion A of Theorem 1.1. Since (1.1) is equivalent to (5.1), it is sufficient to prove the following theorem for (5.1).

Theorem 5.1. *Let the sequences $\{\omega_N\}$ and $\{\omega_N^*\}$ be given as in Theorem* 2.1. *Then for each positive integer M and $\mu > 0$ with*

$$\mu < \frac{1}{2} \min\left\{\omega_N^* - \omega_N,\ N = 1, 2, \ldots, M\right\},$$

there is a $\rho^* > 0$ such that $P_{(5.1)}(\Omega_1, \omega) \subset \Omega_1$ for each $\omega \in \cup_{N=1}^M [\omega_N + \mu, \omega_N^* - \mu]$ and $\rho \in (0, \rho^*]$. Furthermore, $\omega \in [\omega_N + \mu, \omega_N^* - \mu]$ implies that any solution starting from a point in Ω_1 has exactly N spikes in 2π time period. Here $P_{(5.1)}(\cdot, \omega)$ is the time 2π map introduced by the flows of (5.1).

Proof. Let $I_N^\mu = [\omega_N^* + \mu, \omega_N - \mu]$, $N = 1, 2, \ldots, M$. It is sufficient to show that for each I_N^μ there is a $\rho^* > 0$ such that

$$z(2\pi, 0, \Omega_1, \omega, \rho) \subset \Omega_1, \quad \rho \in (0, \rho_N], \quad \omega \in I_N^\mu$$

and $z(t, z_0)$ has N spikes in 2π time period. To this end, we define

$$D_n = \left\{(t_n(x_0, \omega), \omega) :\ x_0 \in [0, x^*], \omega \in I_N^\mu\right\}, \quad n = 1, 2, \ldots 2N,$$

where $t_n(x_0, \omega)$ is the time sequence defined in Section 2. Then by the definitions of $\xi_1(\theta, \omega)$ and $\xi_2(\theta, \omega)$ we see that the following equalities hold:

$$\begin{aligned} \xi_1(t_{2k}(x_0, \omega), \omega) &= t_{2k+1}(x_0, \omega), \quad k = 1, \ldots, N-1, \\ \xi_2(t_{2k-1}(x_0, \omega), \omega) &= t_{2k}(x_0, \omega), \quad\ \ k = 1, \ldots, N. \end{aligned} \tag{5.29}$$

Moreover, $\omega \in I_N^\mu$ implies that for all $x_0 \in [0, x^*]$,

$$\pi < t_1(x_0, \omega) < t_{2N}(x_0, \omega) < 2\pi < t_{2N+1}(x_0, \omega). \tag{5.30}$$

Recall that $\tilde{P}(x_0, \omega)$, the time 2π map for the reduced system (2.5)–(2.6) is defined as

$$\tilde{P}(x_0, \omega) = e^{-\omega 2\pi}\left[x^* e^{\omega t_{2N}(x_0, \omega)} + \int_{t_{2N}(x_0, \omega)}^{2\pi} e^{\omega\tau} \sin \tau d\tau\right]. \tag{5.31}$$

Hence (5.30) implies

$$0 < \tilde{P}(x_0, \omega) < x^*, \quad (x_0, \omega) \in [0, x^*] \times I_N^\mu. \tag{5.32}$$

Since $[x_0, x^*] \times I_N^\mu$ is a compact set, (5.30) and (5.32) yield that there is a $\zeta > 0$ such that for all $(x_0, \omega) \in [0, x^*] \times I$

P1. $t_n(x_0, \omega) \in [\pi + \zeta, 2\pi - \zeta]$, $1 \le n \le 2N$;
P2. $3\pi + \zeta \le t_{2N+1}(x_0, \omega) \ge 4\pi - \zeta$. (Note: We have $t_{2N+1}(x_0, \omega) = 2\pi + t_1(\tilde{P}(x_0, \omega), \omega)$ by definition. Hence P1 implies P2.)
P3. $\zeta \le \tilde{P}(x_0, \omega) \le x^* - \zeta$.

By Lemma 5.2, for each $\delta_1 > 0$, there is a $\rho_1 > 0$ such that for all $\rho \in (0, \rho_1]$ and $\omega \in I_N^\mu$, if $z_0 = (x_0, y_0) \in \Omega_1$, then there is a $s_1(z_0, \omega, \rho)$ with

$$z(s_1, z_0, \omega, \rho) \in l_1^{\delta_1}, \quad |s_1(z_0, \omega, \rho) - t_1(x_0, \omega)| \le \delta_1. \tag{5.33}$$

Let $d_1' = s_1(z_0, \omega, \rho)$ and $z_1' = z(s_1, z_0, \omega, \rho)$. Then by Lemma 5.3, for any given $\gamma_1 > 0$, if the above ρ_1 and δ_1 are small enough, then (5.33) implies there is $0 < t_{\gamma_1}(d_1', z_0') \le \gamma_1$ such that

$$z_1 = z\big(d_1' + t_{\gamma_1}\big(d_1', z_1'\big), z_1'\big) \in B_{\gamma_1}(0, -\beta). \tag{5.34}$$

Moreover, for $d_1 = d_1' + t_{\gamma_1}(d_1', z_1')$, we have

$$\begin{aligned} |d_1 - t_1(x_0, \omega)| &\le \left|d_1' - t_1(x_0, \omega)\right| + t_{\gamma_1}\big(d_1', z_1'\big) \\ &= |s_1(z_0, \omega, \rho) - t_1(x_0, \omega)| + t_{\gamma_1}\big(d_1', z_1'\big) \\ &\le \delta_1 + \gamma_1. \end{aligned} \tag{5.35}$$

It follows from (5.34), (5.35), and Lemma 5.4 that for any small number $\gamma_2 > 0$,

$$z_2' = z(\eta_2(d_1, z_1, \omega), d_1, z_1) \in l_2^{\gamma_2}$$

and

$$|\eta_2(d_1, z_1, \omega) - \xi_2(t_1(x_0, \omega), \omega)| \le \gamma_2$$

as long as $\rho \in (0, \rho_1]$ and $\rho_1, \delta_1, \gamma_1$ are sufficiently small, where $\eta_2(d_1, z_1, \omega)$ and $\xi_2(t_1(x_0, \omega), \omega)$ are defined as in Lemma 5.4. Let $d_2' = \eta_2(d_1, z_1, \omega)$. Notice that $t_2(x_0, \omega) = \xi_2(t_1(x_0, \omega))$, and we thus have

$$\left|d_2' - t_2(x_0, \omega)\right| \le \gamma_2 \quad \text{and} \quad z_2' \in l_2^{\gamma_2}.$$

Thus, with the repeat use of Lemmas 5.3–5.4 we can inductively show that for any given $\gamma > 0$ there is a $\rho^* > 0$ such that for any $\rho \in (0, \rho^*]$, $\omega \in I_N^\mu$ and $z_0 \in \Omega_1$, there are $d_1 < d_2 < \cdots < d_{2N+1}$, where d_i depends on (z_0, ω, ρ) such that

$$z(d_{2k-1}, z_0) \in B_\gamma(0, -\beta), \quad |d_{2k-1} - t_{2k-1}(x_0, \omega)| \le \gamma, \quad k = 1, \ldots, N+1, \tag{5.36}$$

$$z(d_{2k}, z_0) \in B_\gamma(x^*, x^*), \quad |d_{2k} - t_{2k}(x_0, \omega)| \le \gamma, \quad k = 1, \ldots, N. \tag{5.37}$$

In addition,

$$(x(t), y(t)) = z(t, z_0) \in \Sigma_1^\gamma, \quad t \in [d_{2N}, d_{2N+1}]. \tag{5.38}$$

Define

$$\begin{aligned} R(k_1, k_2, k_3, t) = {} & \left|(x^* - k_1)e^{-\omega(2\pi - t + k_2)} - x^* e^{-\omega(2\pi - t)}\right| \\ & + \omega\left|\int_{t-k_2}^{t} e^{-\omega(2\pi-\tau)} d\tau\right| + \omega|k_3| \left|\int_{t-k_2}^{2\pi} e^{-\omega(2\pi-\tau)} d\tau\right|. \end{aligned} \tag{5.39}$$

It is apparent that

$$R(k_1, k_2, k_3, t, \omega) \to 0 \quad \text{as} \quad (k_1, k_2, k_3) \to (0, 0, 0)$$

uniformly for $(t, \omega) \in [\pi, 2\pi] \times I_N^\mu$, so that there exists a $k^* > 0$ such that

$$R(k_1, k_2, k_3, t) < \frac{\zeta}{2} \quad \text{if} \quad |k_i| \le k^*, \ i = 1, 2, 3, \quad (t, \omega) \in [\pi, 2\pi] \times I_N^\mu. \tag{5.40}$$

Now we choose $\gamma > 0$ small enough such that

$$\gamma = \min\left\{k^*, \frac{\zeta}{2}\right\} \tag{5.41}$$

$$\left\{z_1 = (x_1, y_1) \in \sigma_1^\gamma : \frac{\zeta}{2} \le x_1 \le x^* - \frac{\zeta}{2}\right\} \subset \Omega_1. \tag{5.42}$$

Then (5.36), (5.37), and (5.42) yield that

$$d_{2N} \le t_{2N}(x_0, \omega) + \gamma \le 2\pi - \frac{\zeta}{2}$$

and

$$d_{2N+1} \ge t_{2N+1} - \gamma \ge 2\pi + \frac{\zeta}{2}.$$

Hence one has $2\pi \in [d_{2N}, d_{2N+1}]$. Now for $t \in [d_{2N}, 2\pi]$ we have

$$\dot{x}(t) = \omega[\sin t - y(t)] = \omega[\sin t - x(t) - h(t)] \tag{5.43}$$

with $h(t) = y(t) - x(t)$. Equation (5.38) implies that

$$|h(t)| \leq \gamma, \quad t \in [d_{2N}, 2\pi]. \tag{5.44}$$

Let $t_{2N} = t_{2N}(x_0, \omega)$. Then, using (5.43) and (5.31), and upon a straightforward calculation, we obtain

$$\begin{aligned} x(2\pi) &= x(d_{2N})e^{-\omega(2\pi - d_{2N})} + \omega \int_{d_{2N}}^{2\pi} e^{-\omega(2\pi-\tau)}[\sin\tau - h(\tau)]d\tau \\ &= x^* e^{-\omega(2\pi - t_{2N})} + +\omega \int_{t_{2N}}^{2\pi} e^{-\omega(2\pi-\tau)} \sin\tau d\tau + R^* \qquad (5.45) \\ &= \tilde{P}(x_0, \omega) + R^*, \end{aligned}$$

where

$$\begin{aligned} R^* &= \left[[x^* - (x^* - x(d_{2N})])e^{-\omega(2\pi - t_{2N} + (t_{2N} - d_{2N}))} - x^* e^{-\omega(2\pi - t_{2N})}\right] \\ &\quad + \omega \int_{t_{2N} - (t_{2N} - d_{2N})}^{t_{2N}} e^{-\omega(2\pi - \tau)} \sin \tau d\tau \qquad (5.46) \\ &\quad - \omega \int_{t_{2N} - (t_{2N} - d_{2N})}^{2\pi} e^{-\omega(2\pi - \tau)} h(\tau) d\tau. \qquad (5.47) \end{aligned}$$

Using (5.39)–(5.42), (5.44), and (5.46) we therefore deduce that

$$|R^*| \leq R\big(x^* - x(d_{2N}), t_{2N} - d_{2N}, \gamma\big) \leq \frac{\zeta}{2}. \tag{5.48}$$

Hence (5.45) and (5.48) yield that

$$\frac{\zeta}{2} \leq x(2\pi) \leq x^* - \frac{\zeta}{2}.$$

Together with (5.42) and the fact of $z(2\pi, z_0) \in \Sigma_1^\gamma$ we conclude that

$$z(2\pi, z_0) \subset \left\{ z_1 = (x_1, y_1) \in \sigma_1^\gamma : \frac{\zeta}{2} \leq x_1 \leq x^* - \frac{\zeta}{2} \right\} \subset \Omega_1$$

all $z_0 \in \Omega_1$, $\omega \in I_N^\mu$, and $\rho \in (0, \rho_1]$. Moreover, it is clear that the solution $z(t, z_0)$ completes exactly N cycles for a 2π time period. □

Final remarks

1. We notice that, unlike the region Ω_2, we have cut a small corner near the origin in the region Ω_1. The reason we do so is that there is a discontinuity of solutions of (5.1) on the initial value at $(0, 0)$ as $\rho \to 0$. That is, the solution starting at $(0, 0)$ at $t = 0$ will stay in the $y \geq 0$ half plane for at least π time before it goes to the lower half plane. However, if a solution starts at a point $(0, y_0)$ with $y_0 > 0$ at time $t = 0$, then, if ρ is sufficiently small, the solution will jump

to Ω_2 in a short time. Hence this type of solution cannot be approximated by a solution of the reduced system.

2. Recall that (see (2.3)) $\omega = \frac{1}{K_1 \nu}$. Thus we have

$$\nu_N = \frac{1}{K_1 \omega_N}, \quad \nu_N^* = \frac{1}{K_1 \omega_N^*}$$

and

$$\nu_{N+1}^* < \nu_N < \nu_N^*, \quad N = 1, 2, \dots.$$

Moreover, ω_N and ω_N^* depend on x^* and β. It can be shown that ω_N and ω_N^* increase as x^* and β increase. A visible reason is that an increase in x^* and β will slow the oscillation. Hence, in order to complete N cycles in a 2π period we must increase the value of ω. Note that, by (2.3) and (2.4),

$$x^* = \frac{K_2 i_0}{a K_1}, \quad \beta = \frac{(K_2 - K_1) i_0}{a K_1}.$$

This means that ω_N and ω_N^* increase when the amplitude a decreases. That is, ν_N and ν_N^* decrease as a decreases. Figure 13 gives the diagram about the relation between ν_N, ν_N^*, and a.

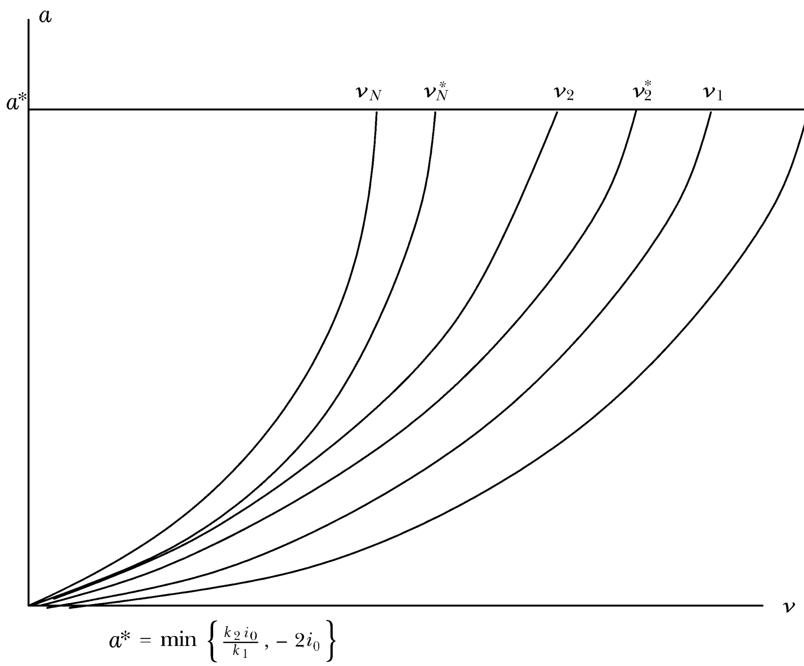


Fig. 13.

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