Annali di Matematica 185, S93–S107 (2006)

Digital Object Identifier (DOI) 10.1007/s10231-004-0138-0

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On the existence of unbounded noncontinuable solutions

Received: October 16, 2003; in final form: February 17, 2004
Published online: April 18, 2005 – © Springer-Verlag 2005

Abstract. In this paper sufficient (necessary) conditions are given under which a differential equation of the nth order has a noncontinuable solution $y : [T, \tau) \to R, \tau < \infty$ fulfilling $\lim_{t\to\tau_-} |y^{(j)}(t)| = \infty$, $j = 0, 1, \ldots, n-1$.

Mathematics Subject Classification (2000). 34C10, 34C15, 34D05

Key words. singular solutions – unbounded solutions

1. Introduction

Consider the nth order differential equation

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}), \tag{1}$$

where $n \geq 2$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, $f \in C^\circ(R_+ \times R^n)$.

Denote by $[[x]]$ the entire part of a number x.

A solution y defined on $[T, \tau) \subset R_+$ is called noncontinuable if $\tau < \infty$ and y cannot be defined for $t = \tau$. Note that in this case $\limsup_{t\to\tau_-} |y^{(n-1)}(t)| = \infty$ and sometimes y is called singular. A noncontinuable solution is called nonoscillatory if it is different from zero in a left neighbourhood of τ.

The first results for the nonexistence of noncontinuable solutions are given by Wintner, see [8] or [11]; other results are obtained, e.g., in [4,5]. In particular, noncontinuable solutions do not exist if $h \in C^\circ(R_+)$ and

$$|f(t, x_1, \ldots, x_n)| \leq h(t) \sum_{i=1}^{n} |x_i| \quad \text{on } R_+ \times R^n.$$

Our main goal is to investigate nonoscillatory noncontinuable solutions only. Results for the existence of such solutions of (1) and its special cases are obtained, e.g., in [1–3,5,6,9–11] under the assumptions $\alpha \in \{-1, 1\}$, $k \in \{1, 2, \ldots, n\}$, $|f(t, x_1, \ldots x_n)| \geq r(t)|x_k|^\lambda$, and

$$\alpha f(t, x_1, \ldots, x_n) x_k > 0 \quad \text{for large } |x_j|, j = 1, 2, \ldots, n \tag{2}$$

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where $r \in C^\circ(R_+)$ and $r > 0$ on R_+; noncontinuable solutions exist if $\alpha = 1$ and $\lambda > 1$ ([13], Th 11.3). If (2) holds with $\alpha = -1$, results for the existence of nonoscillatory noncontinuable solutions are obtained only under very restrictive assumptions posed on f, see [3].

In the last period, the problem of the existence of noncontinuable solutions with prescribed asymptotics on the right-hand side point τ of the definition interval is studied. More precisely, let $\tau \in (0, \infty)$. Then sufficient (necessary) conditions for the existence of a noncontinuable solution y fulfilling

$$\lim_{t\to\tau_-} y(t) = c_\circ \in R, \ \lim_{t\to\tau_-} y^{(i)}(t), i = 1, 2, \ldots, n-1 \quad \text{exist}$$

are given in [7] (case $n = 2$) and in [1,3] (case $n \ge 2$).

The results are enlarged in [2] to the case

$$\lim_{t\to\tau_-} |y(t)| = \infty, \ \lim_{t\to\tau_-} y^{(i)}(t), i = 1, 2, \ldots, n-1 \quad \text{exist;}$$

it is clear that in this case $\lim_{t\to\tau_-} y^{(i)}(t) \operatorname{sgn} y(t) = \infty, i = 1, 2 \ldots, n-1$. So the following problem was solved: to give sufficient (necessary) conditions assuring that (1) has a solution y satisfying the boundary-value conditions

$$\tau \in (0, \infty), \ \lim_{t\to\tau_-} y^{(i)}(t) \operatorname{sgn} y(t) = \infty, \ i = 0, 1, \ldots, n-1 \tag{3}$$

and that is defined in a left neighbourhood of τ. Note that some proofs in [2] are not correct, see Remark 4 below.

In the present paper we study problems (1) and (3); our approach is more broad than in [2].

2. Lemmas

We need some lemmas. The first one is a special case of [12, Cor. 1.1.].

Lemma 1. *Let* $[a, b] \subset R_+$, $\tilde f \in C^\circ([a, b] \times R^n)$ *and*

$$\tilde f(t, x_1, \ldots, x_n) \ge 0 \quad on \quad [a, b] \times R^n.$$

Then the problem $u^{(n)} = \tilde f(t, u, \ldots, u^{(n-1)})$, $u^{(i-1)}(a) = 0$ *for* $i = 1, \ldots, n-1$, $u^{(n-1)}(b) = 0$ *has at least one solution.*

Lemma 2. *Let k be an integer,* $\beta \in \{-1, 1\}$, $\tau \in (0, \infty)$ *and* $M \in (0, \infty)$ *be such that* $k \ge 2M$ *and*

$$\beta f(t, x_1, \ldots, x_n) \ge 0 \quad for \ t \in [0, \tau], \beta x_i \ge M, i = 1, 2, \ldots, n. \tag{4}$$

Then there exist $T^* \in [0, \tau)$ *such that* $\tau - T^* \le 1$ *and the boundary-value problem (1),* $y^{(i)}(T) = \beta M, i = 0, 1, \ldots, n-2, y^{(n-1)}(\tau) = \beta k$ *has at least one solution* y_k *for every* $T \in [T^*, \tau)$. *Moreover,* T^* *does not depend on k and* $|y_k^{(n-1)}(t)| > M$ *on* $[T, \tau)$.

Proof. We prove the statement for $\beta = 1$; for $\beta = -1$ the proof is similar. Let $T \in [0, \tau)$ be such that

$$\tau - T < M/B, \ \tau - T \leq 1, \tag{5}$$

where

$$\begin{aligned} B = \max\{ & f(t, x_1, \dots, x_n) : t \in [0, \tau], M \leq x_i \leq M(n+1) \\ & \text{for} \quad i = 1, 2, \dots, n-1, M \leq x_n \leq 2M\} + 1 \end{aligned}$$

and denote $J = [T, \tau]$.

Consider an auxiliary problem $k \in \{1, 2, \dots, \}, k \geq 2M$,

$$u^{(n)} = \tilde{f}(t, u, \dots, u^{(n-1)}), \ \ u^{(j)}(T) = 0, \ j = 0, 1, \dots, n-2; \ u^{(n-1)}(\tau) = 0, \tag{6}$$

where

$$\begin{aligned} \tilde{f}(t, u, \dots, u^{(n-1)}) &= f\big(t, \Phi(u + P(t)), \dots, \Phi(u^{(n-1)} + P^{(n-1)}(t))\big) \\ &\quad \times g\big(u^{(n-1)} + P^{(n-1)}(t)\big), \\ \Phi(v) &= \begin{cases} v & \text{for} \quad v \geq M \\ M & \text{for} \quad v < M, \end{cases} \\ P(t) &= M \sum_{j=0}^{n-2} \frac{(t-T)^j}{j!} + \frac{k}{(n-1)!}(t-T)^{n-1}, \\ g(v) &= \begin{cases} 1 & \text{for} \quad v \geq M \\ 2v/M - 1 & \text{for} \quad M/2 \leq v < M \\ 0 & \text{for} \quad v < M/2. \end{cases} \end{aligned}$$

From this and according to (4)$\tilde{f}\big(t, u, \dots, u^{(n-1)}\big) \geq 0$ on $[T, \tau) \times R^n$. If we put $[a, b] = [T, \tau]$ all assumptions of Lemma 1 are fulfilled and hence (6) has a solution $\bar{u}$.

As $P^{(j)}(T) = M$ for $j = 0, 1, \dots, n-2$ and $P^{(n-1)}(\tau) = k$, the transformation $y = u + P$ transforms problem (6) into

$$\begin{aligned} y^{(n)} &= f\big(t, \Phi(y), \dots, \Phi(y^{(n-1)})\big) g(y^{(n-1)}), \\ y^{(j)}(T) &= M, \ j = 0, 1, \dots, n-2, \\ y^{(n-1)}(\tau) &= k \end{aligned} \tag{7}$$

and hence $y = \bar{u} + P$ is a solution of (7). Moreover, (4), (7), and the definition of g yield $y^{(n)}(t) \geq 0$ on J and we prove that $y^{(n-1)}(t) \geq 0$ on J. Assume there exists $t_0 \in [T, \tau)$ such that $y^{(n-1)}(t_0) < 0$. Then, in view of $y^{(n-1)}(\tau) = k > 0$, because $y^{(n-1)}$ is increasing on J, there exists $t_1 \in (t_1, \tau)$ such that $y^{(n-1)}(t) < 0$ on $[t_0, t_1)$ and $y^{(n-1)}(t_1) = 0$. From this and from the definition of g we have $y^{(n)}(t) \equiv 0$ on $[t_0, t_1]$, which contradicts $y^{(n-1)}(t_0) < 0$ and $y^{(n-1)}(t_1) = 0$. From this

$$y^{(n)}(t) \geq 0, \ y^{(n-1)}(t) \geq 0 \quad \text{is nondecreasing on} \quad J, \tag{8}$$

and hence

$$M \le y^{(j)}(t) \quad \text{are nondecreasing on } J \text{ for } i = 0, 1, \ldots, n-2. \tag{9}$$

We estimate $y^{(n-1)}$ from below and prove that

$$M < y^{(n-1)}(t), t \in J. \tag{10}$$

Suppose that (10) is not valid. As $k \ge 2M$ and as (8) holds, there exist T_1 and T_2 such that

$$T \le T_1 < T_2 \le \tau, \; y^{(n-1)}(T_1) = M, \; y^{(n-1)}(T_2) = 2M.$$

From this, using (5), (7), and (8) we have

$$\begin{aligned} y^{(i)}(t) &= \sum_{j=0}^{n-i-2} \frac{M(t-T)^j}{j!} + \int_T^t \frac{(t-\sigma)^{n-i-2}}{(n-i-2)!} y^{(n-1)}(\sigma) d\sigma \\ &\le M(n-1) + 2M = M(n+1), i = 0, 1, \ldots, n-2, t \in J, \end{aligned}$$

which, together with (9) and (10), yields

$$\begin{aligned} &M \le y^{(i)}(t) \le M(n-1), i = 0, 1, \ldots, n-2, t \in J, \\ &M \le y^{(n-1)}(t) \le 2M, t \in [T_1, T_2]. \end{aligned} \tag{11}$$

Hence

$$\begin{aligned} M &= y^{(n-1)}(T_2) - y^{(n-1)}(T_1) = \int_{T_1}^{T_2} y^{(n)}(t) dt \\ &= \int_{T_1}^{T_2} f\big(t, \Phi(y(t)), \ldots, \Phi(y^{(n-1)}(t))\big) g(y^{(n-1)}(t))\, dt \\ &= \int_{T_1}^{T_2} f(t, y(t), \ldots, y^{(n-1)}(t)) dt \le B(T_2 - T_1) \le B(\tau - T). \end{aligned}$$

This contradicts the first inequality in (5) and so (10) holds. From this and from (11)

$$\Phi(y^{(i)}(t)) \equiv y^{(i)}(t), \; i = 0, 1, \ldots, n-1, \; g(y^{(n-1)}(t)) \equiv 1,$$

and y is a solution of problem (1), $y^{(i)}(T) = M, i = 0, 1, \ldots, n-1$, $y^{(n-1)}(\tau) = k$, too.

Note that, according to (5), T does not depend on k. Moreover, if (5) holds for $T = T^*$, then (5) is valid for $T \in [T^*, \tau)$, too. □

Remark 1. It follows from (5) that for a fixed number M and for sufficiently small positive τ we have $T^* = 0$.

Lemma 3. *Let* $[a,b)\subset R_+$, $\tilde f\in C^\circ([a,b)\times R^n)$, $\Phi\in C^\circ[a,b)$ *be a nondecreasing function and let for* $m\in\{1,2,\dots\}$ *fixed the equation* $u^{(n)}=\tilde f(t,u,\dots,u^{(n-1)})$ *have a solution* u_m*, defined on* $[a,b)$*, such that*

$$\sum_{i=0}^{n-1}\left|u_m^{(i)}(t)\right|\le\Phi(t)\quad for\quad t\in[a,b).$$

Then there exists a subsequence $\{u_{m_j}\}_{j=1}^\infty$ *of* $\{u_m\}_{m=1}^\infty$ *such that sequences* $\{u_{m_j}^{(i)}\}_{j=1}^\infty$, $i=0,1,\dots,n-1$ *converge locally uniformly on* $[a,b)$ *and* $u(t)=\lim_{j\to\infty}u_{m_j}(t)$ *for* $t\in[a,b)$ *is a solution of* $u^{(n)}=\tilde f(t,u,\dots,u^{(n-1)})$.

Proof. The assertion follows by applying the Arzelà–Ascoli Theorem, see e.g., [11] Lemma 10.2. □

Lemma 4 ([11], Lemma 11.2). *Let* $y\in C^{(n)}[a,b)$, $s\in\{0,1,\dots,n-2\}$, $\delta\in\left(0,\frac{1}{n-s}\right)$ *and*

$$y^{(i)}(t)y(t)>0\quad for\ \ i=0,1,\dots,n-1,\ \ y^{(n)}(t)y(t)\ge 0\quad on\ \ [a,b).$$

Then

$$\prod_{i=s}^{n-1}|y^{(i)}(t)|^{-\varepsilon}\ge\omega\int_t^b|y^{(n)}(\sigma)|^\delta|y^{(s)}(\sigma)|^\gamma\,d\sigma,\,t\in(a,b),$$

where

$$\varepsilon=\frac{2[1-(n-s)\delta]}{(n-s)(n-s-1)}>0,\ \ \gamma=\frac{(n-s+1)\delta-2}{n-s-1}$$

and

$$\omega=\varepsilon\prod_{i=1}^{n-s}[\delta+(n-s-i)\varepsilon]^{-\delta-(n-s-i)\varepsilon}.$$

Lemma 5. *Let* $\lambda\le 1+\frac{1}{n-1}$, $\tau<\infty$ *and* $M_1>0$. *Let* y *be a solution of (1) defined on* $[T,\tau)\subset R_+$ *such that* $\lim_{t\to\tau_-}|y^{(n-1)}(t)|=\infty$ *and*

$$|y^{(n)}(t)|\le M_1|y^{(n-1)}(t)|^\lambda,\ \ t\in[T,\tau).$$

Then $\lim_{t\to\tau_-}|y(t)|=\infty$.

Proof. Let $\lim_{t\to\tau_-}y^{(n-1)}(t)=\infty$. The case $\lim_{t\to\tau-}y^{(n-1)}(t)=-\infty$ can be studied similarly. Then y has a limit for $t\to\tau_-$ and suppose that $\lim_{t\to\tau-}y(t)=c<\infty$. Moreover, $\bar T\in[T,\tau)$ exists such that $y^{(n-1)}(t)>0$ on $[\bar T,\tau)$ and the assumptions of the lemma yield

$$y^{(n)}(t)\le M(y^{(n-1)}(t))^\lambda,\ \ t\in[\bar T,\tau). \tag{12}$$

Let $\lambda > 1$. Then the integration of (12) on $[t, \tau)$ yields

$$y^{(n-1)}(t) \geq [\lambda_1 M_1(\tau - t)]^{-\lambda_1},\ t \in [\bar{T}, \tau), \lambda_1 = 1/(\lambda - 1).$$

From this and from $n - 1 \leq \lambda_1$ we have

$$\begin{aligned} c = y(\tau-) &= \sum_{i=0}^{n-2} \frac{y^{(i)}(\bar{T})}{i!}(\tau - \bar{T})^i + \int_{\bar{T}}^{\tau} \frac{(\tau - s)^{n-2}}{(n-2)!}\, y^{(n-1)}(s) ds \\ &\geq -\sum_{i=0}^{n-2} \frac{|y^{(i)}(\bar{T})|}{i!}(\tau - \bar{T})^i + \frac{(\lambda_1 M_1)^{-\lambda_1}}{(n-2)!} \int_{\bar{T}}^{\tau} (\tau - s)^{n-2-\lambda_1}\, ds = \infty. \end{aligned}$$

The contradiction proves the statement of the lemma.

Let $\lambda \leq 1$. Then the integration of (12) yields

$$\left(y^{(n-1)}(\tau-)\right)^{1-\lambda} - y^{(n-1)}(\bar{T})^{1-\lambda} \leq M_1(\tau - \bar{T})(1 - \lambda) \quad \text{for} \quad \lambda < 1$$

and

$$\lim_{t \to \tau-} \log(y^{(n-1)}(t)/y^{(n-1)}(\bar{T})) \leq M_1(\tau - \bar{T}) \quad \text{for} \quad \lambda = 1;$$

the contradiction to $\lim_{t \to \tau_-} y^{(n-1)}(t) = \infty$ shows that this case is impossible. □

3. Main results

We begin our consideration with two nonexistence results.

Theorem 1. *Let τ and M be positive constants and $I \subset R_+$ be a left neighborhood of τ such that $\beta f(t, x_1, \ldots, x_n) \leq 0$ for $\beta x_i \geq M$, $i = 1, 2, \ldots, n$, $\beta \in \{-1, 1\}$, $t \in I$. Then there exists no solution y of (1) fulfilling (3).*

Proof. The theorem is a mild generalization of Theorem 1(i) in [2], and its proof is similar. □

The following theorem solves the case in which f has the opposite sign.

Theorem 2. *Let τ, M, and M_1 be positive constants and $I \subset R_+$ a left neighborhood of τ. Suppose that one of the following assumptions holds.*

(i) Let $\alpha_j \in R$ for $j = 1, 2, \ldots, n-1$, $\bar{\alpha} = \max(0, \alpha_1, \ldots, \alpha_{n-1})$ be such that $\bar{\alpha} + \lambda \leq 1$ and

$$0 \leq \beta f(t, x_1, \ldots, x_n) \leq M_1 \sum_{i=1}^{n-1} |x_i|^{\alpha_i} |x_n|^{\lambda} \tag{13}$$

for $t \in I$, $\beta x_j \geq M$, $j = 1, 2, \ldots, n$ and $\beta \in \{-1, 1\}$;

(ii) Let $\alpha \in R$, $s \in \{0, 1, \ldots, n-2\}$ *and* $\lambda \geq 0$ *be such that*

$$M_1|x_{s+1}|^{\alpha}|x_n|^{\lambda} \leq \beta f(t, x_1, \ldots, x_n) \tag{14}$$

for $t \in I$, $\beta x_j \geq M$, $j = 1, 2, \ldots, n$ *and* $\beta \in \{-1, 1\}$. *Further, let either*

$$\min(\alpha, 0) + \lambda > 1 + \frac{1}{n-1} \tag{15}$$

or

$$\lambda > 1 \qquad \text{and} \quad s > \frac{1}{\alpha + \lambda - 1} \tag{16}$$

or

$$\alpha + \lambda > 1 + \frac{1}{n-1} \quad \text{and} \quad s > \frac{2n + (n-1)(\alpha + \lambda - 1)}{3(\alpha + \lambda - 1) + 2}. \tag{17}$$

Then there exists no solution y of (1) such that (3) holds.

Proof. Let, contrarily, $y : [\tau_1, \tau) \to R$ be a solution of (1) such that $[\tau_1, \tau) \subset I$ and

$$\lim_{t \to \tau_-} y^{(i)}(t) = \infty, \ i = 0, 1, \ldots, n-1. \tag{18}$$

The opposite case $\lim_{t\to\tau_-} y^{(i)}(t) = -\infty$ for $i = 0, 1, \ldots, n-1$ can be studied similarly.

Suppose without loss of generality that $M \geq 1$. Then there exists $\bar{T} \in [\tau_1, \tau)$ such that

$$y^{(i)}(t) \geq M \geq 1 \quad \text{on } [\bar{T}, \tau), i = 0, 1, \ldots, n-1 \tag{19}$$

and

$$\tau - \bar{T} \leq \frac{1}{2}, \ (\tau - \bar{T})(1 - \alpha - \lambda)M_1(n-1) < 1, \tag{20}$$

where $\alpha = 1 - \lambda$ in case (i). Note that, due to (13) and (14), we have $y^{(n)}(t) > 0$ on $[\bar{T}, \tau)$, and hence (19) yields

$$y^{(i)}, i = 0, 1, \ldots, n-1 \quad \text{are increasing on} \quad [\bar{T}, \tau). \tag{21}$$

From this, from (20), and from Taylor series theorem we have

$$y^{(i)}(t) = y^{(i)}(\bar{T}) + \int_{\bar{T}}^{t} y^{(i+1)}(\sigma)d\sigma \leq K + \frac{1}{2}y^{(i+1)}(t),$$

$$t \in [\bar{T}, \tau), i = 0, 1, \ldots, n-2 \quad \text{where} \quad K = \sum_{j=0}^{n-2} y^{(j)}(\bar{T});$$

hence, by virtue of (18), there exists $T \in [\bar{T}, \tau)$ such that

$$y^{(i)}(t) \leq y^{(i+1)}(t), \ t \in J = [T, \tau), i = 0, 1, \ldots, n-2. \tag{22}$$

Suppose that (i) is valid. At first, consider the case $\bar{\alpha} + \lambda < 1$.

If $\alpha_i \le 0, i \in \{1, \dots, n-1\}$, then (19) and $\bar{\alpha} \ge 0$ yield

$$\left(y^{(i-1)}(t)\right)^{\alpha_i} \le 1 \le \left(y^{(n-1)}(t)\right)^{\bar{\alpha}}, t \in J;$$

similarly in the case $\alpha_i > 0, i \in \{1, \dots, n-1\}$ we obtain from (19) and (22)

$$1 \le \left(y^{(i-1)}(t)\right)^{\alpha_i} \le \left(y^{(n-1)}(t)\right)^{\alpha_i} \le \left(y^{(n-1)}(t)\right)^{\bar{\alpha}}, t \in J.$$

Hence

$$\sum_{i=1}^{n-1} \left(y^{(i-1)}(t)\right)^{\alpha_i} \le (n-1)\left(y^{(n-1)}(t)\right)^{\bar{\alpha}}, t \in J$$

and (1), (13), and (19) yield

$$y^{(n)}(t) \le M_1 \sum_{i=1}^{n-1} \left(y^{(i-1)}(t)\right)^{\alpha_i} \left(y^{(n-1)}(t)\right)^{\lambda} \le \bar{M}_1 \left(y^{(n-1)}(t)\right)^{\bar{\alpha}+\lambda},$$

where $\bar{M} = (n-1)M_1$. From this the integration yields

$$\left(y^{(n-1)}(t)\right)^{1-\bar{\alpha}-\lambda} - \left(y^{(n-1)}(T)\right)^{1-\bar{\alpha}-\lambda} \le (1-\lambda-\bar{\alpha})\bar{M}_1(\tau - T), \ t \in J; \qquad (23)$$

this inequality contradicts $\bar{\alpha} + \lambda < 1$ and $\lim_{t \to \tau_-} y^{(n-1)}(t) = \infty$. If $\bar{\alpha} + \lambda = 1$, the proof is similar. It is sufficient to replace (23) with

$$\log \frac{y^{(n-1)}(t)}{y^{(n-1)}(T)} \le \bar{M}_1(\tau - T), t \in J.$$

Suppose that (ii) is valid. Then (1), (14), and (19) yield

$$y^{(n)}(t) \ge M_1 \left(y^{(s)}(t)\right)^{\alpha} \left(y^{(n-1)}(t)\right)^{\lambda}, t \in J. \qquad (24)$$

Let (15) hold. Put $\sigma = \min(\alpha, 0)$. If $\alpha \le 0$, then $\sigma = \alpha$ and, (19) and (22) yield

$$\left(y^{(s)}(t)\right)^{\alpha} \ge \left(y^{(n-1)}(t)\right)^{\alpha} = \left(y^{(n-1)}(t)\right)^{\sigma}, \ t \in J;$$

similarly, if $\alpha > 0$, then $\sigma = 0$ and

$$\left(y^{(s)}(t)\right)^{\alpha} \ge 1 = \left(y^{(n-1)}(t)\right)^{\sigma}, \ t \in J.$$

From this and from (15), (19), and (24) we obtain

$$y^{(n)}(t) \ge M_1 \left(y^{(n-1)}(t)\right)^{\sigma+\lambda} \ge M_1 \left(y^{(n-1)}(t)\right)^{1+\frac{1}{n-1}+\varepsilon}, t \in J,$$

where $\varepsilon = \frac{1}{2}[\sigma + \lambda - 1 - \frac{1}{n-1}] > 0$. Hence, the integration on $[t, \tau)$ and (18) yield $y^{(n-1)}(t) \le \left[M_1 \lambda_1^{-1}(\tau - t)\right]^{-\lambda_1}, t \in J, \lambda_1 = \left(\frac{1}{n-1} + \varepsilon\right)^{-1}$. From this and from

Taylor series theorem

$$\infty = y(\tau-) = \sum_{i=0}^{n-2} \frac{y^{(i)}(T)}{i!}(\tau - T)^i + \int_T^\tau \frac{(\tau-\sigma)^{n-2}}{(n-2)!} y^{(n-1)}(\sigma)d\sigma$$
$$\leq \sum_{i=0}^{n-2} \frac{y^{(i)}(T)}{i!}(\tau - T)^i + \frac{1}{(n-2)!}\left(\frac{\lambda_1}{M_1}\right)^{\lambda_1} \int_T^\tau (\tau-\sigma)^{n-2-\lambda_1} d\sigma < \infty$$

as $n - 2 - \lambda_1 > -1$. The contradiction proves the statement in this case.

Let (16) hold. Then the second inequality in (16) yields $\alpha + \lambda > 1 + \frac{1}{n-1}$ and, taking into account the argument used when (15) holds, it is enough to suppose $\alpha > 0$. The integration of (24) on $[t, \tau)$, (18), and (21) yields

$$\left(y^{(n-1)}(t)\right)^{1-\lambda} \geq (\lambda - 1)M_1 \int_t^\tau \left(y^{(s)}(\sigma)\right)^\alpha d\sigma$$
$$\geq (\lambda - 1)M_1(\tau - t)\left(y^{(s)}(t)\right)^\alpha, \ t \in J;$$

hence, using (22), we have

$$\left(y^{(s)}(t)\right)^{\alpha+\lambda-1} \leq \left(y^{(s)}(t)\right)^\alpha \left(y^{(n-1)}(t)\right)^{\lambda-1} \leq \frac{1}{(\lambda-1)M_1(\tau - t)}$$

or $y^{(s)}(t) \leq \frac{M_2}{(\tau-t)^{\lambda_2}}$, $t \in J$ where $\lambda_2 = \frac{1}{\lambda+\alpha-1}$ and $M_2 = \left((\lambda-1)M_1\right)^{-\lambda_2}$. Then

$$\infty = y(\tau-) = \sum_{i=0}^{s-1} \frac{y^{(i)}(T)}{i!}(\tau - T)^i + \int_T^\tau \frac{(\tau-\sigma)^{s-1}}{(s-1)!} y^{(s)}(\sigma)d\sigma$$
$$\leq \sum_{i=0}^{s-1} \frac{y^{(i)}(T)}{i!}(\tau - T)^i + \frac{M_2}{(s-1)!} \int_T^\tau (\tau-\sigma)^{s-1-\lambda_2} d\sigma < \infty \qquad (25)$$

as $s - 1 - \lambda_2 > -1$. The contradiction proves the statement in this case.

Let (17) hold. With respect to the proved conclusion for (15), it is enough to suppose that $\alpha > 0$. Put $\delta = 2/[(\alpha + \lambda)(n - s - 1) + n - s + 1]$. As $\alpha + \lambda > 1$, then $\delta \in (0, \frac{1}{n-s})$ and Lemma 4 can be applied to

$$\varepsilon = \frac{2(\lambda + \alpha - 1)}{(n-s)[(\alpha+\lambda)(n-s-1)+n-s+1]}, \gamma = \frac{(n-s+1)\delta - 2}{n-s-1} < 0;$$

note that $(\lambda + \alpha)\delta + \gamma = 0$.

Hence Lemma 4 (with $[a, b) = [T, \tau)$), (14), and (22) yield

$$(y^{(s)}(t))^{n-s} \leq y^{(s)}(t) \dots y^{(n-1)}(t) \leq \left[\omega \int_t^\tau (y^{(n)}(\sigma))^\delta (y^{(s)}(\sigma))^\gamma d\sigma\right]^{-\frac{1}{\varepsilon}}$$
$$\leq \left(\omega M_1^\delta \int_t^\tau (y^{(s)}(\sigma))^{\alpha\delta+\gamma} \ (y^{(n-1)}(\sigma))^{\lambda\delta} d\sigma\right)^{-\frac{1}{\varepsilon}} \qquad (26)$$
$$\leq \left(\omega M_1^\delta \int_t^\tau (y^{(s)}(\sigma))^{(\alpha+\lambda)\delta+\gamma} d\sigma\right)^{-\frac{1}{\varepsilon}} = M_3(\tau - t)^{-\frac{1}{\varepsilon}}, \ t \in J,$$

where $M_3 = (\omega M_1^{\delta})^{-1/\varepsilon}$. Let $\lambda_2 = \frac{1}{\varepsilon(n-s)}$. Then (17) yields

$$\begin{aligned} s - \lambda_2 &= s - \frac{(\alpha+\lambda)(n-s-1) + n - s + 1}{2(\lambda+\alpha-1)} \\ &= \frac{s\big[3(\lambda+\alpha-1)+2\big] - (\alpha+\lambda)(n-1) - n - 1}{2(\lambda+\alpha-1)} \\ &> \frac{2n + (n-1)(\alpha+\lambda-1) - (\alpha+\lambda)(n-1) - n - 1}{2(\lambda+\alpha-1)} = 0. \end{aligned}$$

From this and from (26) we have that (25) holds and the contradiction gives the assertion. □

Remark 2. (i) A special case of Theorem 2(ii) with (15) and $\alpha = 0$ is published in [2] Th. 1. (ii).
(ii) Note that if (16) holds, then $\alpha + \lambda > 1 + \frac{1}{n-1}$ and $s \neq 0$.

Let us turn our attention to existence results.

Theorem 3. *Let* $0 \le \bar{\tau} < \tau < \infty, 0 \le \lambda \le \lambda_1, \beta \in \{-1, 1\}, M > 0, M_1 > 0, M_2 > 0, \alpha \in R, s \in \{0, 1, \dots, n-2\}, \alpha_j \in R$ *for* $j = 1, 2, \dots, n-1$ *and* $\bar{\alpha} = \max(0, \alpha_1, \dots, \alpha_{n-1})$ *be such that*

$$1 < \alpha + \lambda, \ \bar{\alpha} + \lambda_1 \le 1 + \frac{1}{n-1}$$

and

$$M_1 |x_{s+1}|^{\alpha} |x_n|^{\lambda} \le \beta f(t, x_1, \dots, x_n) \le M_2 \sum_{i=1}^{n-1} |x_i|^{\alpha_i} |x_n|^{\lambda_1}$$
$$\text{for} \quad t \in [\bar{\tau}, \tau], \beta x_i \ge M, i = 1, 2, \dots, n. \tag{27}$$

Then there exists a solution y of (1), defined in a left neighborhood of τ, *such that (3) holds.*

Proof. We prove the statement for $\beta = 1$; for $\beta = -1$ the proof is similar. Suppose, without loss of generality, that $M \ge 1$. As according to (27) all assumptions of Lemma 2 are fulfilled, there exists $T \in [\bar{\tau}, \tau)$ such that

$$\tau - T \le 1 \tag{28}$$

and for $k \in \{k_0, k_0 + 1, \dots\}, k_0 \ge 2M$ the boundary-value problem

$$\begin{aligned} y^{(n)} &= f(t, y, \dots, y^{(n-1)}), t \in [T, \tau], \\ y^{(i)}(T) &= M, i = 0, 1, \dots, n-2, \ y^{(n-1)}(\tau) = k \end{aligned} \tag{29}$$

has a solution y_k; at the same time, T does not depend on k. Let $J = [T, \tau]$. Lemma 2 implies

$$y_k^{(i)} \ge M \quad \text{and} \quad y_k^{(i)} \quad \text{are increasing on} \quad J \quad \text{for} \quad i = 0, 1, \dots, n-1. \tag{30}$$

Now, we estimate the derivatives of y_k independently on k. By virtue of (28)–(30)

$$y_k^{(i)}(t) = M + \int_T^t y_k^{(i+1)}(\sigma)d\sigma \le M + y_k^{(i+1)}(t)(t-T) \le 2y_k^{(i+1)}(t),$$

$$y_k^{(i)}(t) \le 2^{n-1} y_k^{(n-1)}(t) \tag{31}$$

$$t \in J, i = 0, 1, \dots, n-2,\ k = k_0, k_0+1, \dots .$$

We apply Lemma 4 to $\delta = 2/[(n-s-1)(\alpha+\lambda)+n-s+1]$ and $[a,b] = J$. Due to $\alpha + \lambda > 1$, we have $\delta \in (0, \frac{1}{n-s})$. If γ, ω, and

$$\varepsilon = \frac{2(\alpha+\lambda-1)}{(n-s)\big[(n-s-1)(\alpha+\lambda)+n-s+1\big]} > 0$$

are given by Lemma 4, then $(\alpha+\lambda)\delta + \gamma = 0$ and

$$\prod_{i=s}^{n-1} y_k^{(i)}(t) \le \left[\omega \int_t^\tau \big(y_k^{(n)}(\sigma)\big)^\delta \big(y_k^{(s)}(\sigma)\big)^\gamma d\sigma\right]^{-1/\varepsilon}. \tag{32}$$

As (30) yields $M^{n-s-1} y_k^{(n-1)}(t) \le \prod_{i=s}^{n-1} y_k^{(i)}(t)$, $t \in J$ and as (31) yields $y_k^{(n-1)}(t) \ge 2^{1-n} y_k^{(s)}(t)$, $t \in J$, it follows from $\varepsilon > 0$, $\lambda\delta \ge 0$, (1), (32), and from the first inequality in (27) that

$$\begin{aligned} M^{n-s-1} y_k^{(n-1)}(t) &\le \left[\omega \int_t^\tau M_1^\delta \big(y_k^{(s)}(\sigma)\big)^{\alpha\delta+\gamma} \big(y_k^{(n-1)}(\sigma)\big)^{\lambda\delta} d\sigma\right]^{-1/\varepsilon} \\ &\le \left[\omega M_1^\delta \int_t^\tau 2^{(1-n)\lambda\delta} \big(y_k^{(s)}(\sigma)\big)^{(\alpha+\lambda)\delta+\gamma} d\sigma\right]^{-1/\varepsilon} \\ &= \left[2^{(1-n)\lambda\delta} \omega M_1^\delta \int_t^\tau d\sigma\right]^{-1/\varepsilon} = C(\tau-t)^{-\frac{1}{\varepsilon}}, t \in J, \end{aligned}$$

where $C = (\omega M_1^\delta 2^{(1-n)\lambda\delta})^{-1/\varepsilon}$. Hence, from this and from (30) and (31)

$$M \le y_k^{(i)}(t) \le 2^{n-1} M^{1+s-n} C(\tau-t)^{-\frac{1}{\varepsilon}}, t \in J, i = 0, 1, \dots, n-1,$$

and Lemma 3 yields the existence of a subsequence of $\{y_k\}$ (we denote it by $\{y_k\}$ for simplicity) that converges locally uniformly (together with the derivatives up to the order $n-1$) to a solution y of (1) on J. Note that (30) yields $y^{(i)}$, $i = 0, 1, \dots, n-1$ are increasing on J.

We prove that

$$\lim_{t \to \tau_-} y(t) = \infty.$$

Thus, suppose that (see (30))

$$M \le y(t) \le Q_1 < \infty \quad \text{on} \quad [T, \tau). \tag{33}$$

Then according to (30) and $M \geq 1$, inequalities $y^{(i)}(t) \geq 1, i = 0, 1, \ldots, n-1$ hold on J. Let $\alpha_i > 0$. Then (31) and $\bar{\alpha} > 0$ yield

$$\left(y_k^{(i-1)}(t)\right)^{\alpha_i} \leq \left(y_k^{(i-1)}(t)\right)^{\bar{\alpha}} \leq 2^{(n-1)\bar{\alpha}}\left(y_k^{(n-1)}(t)\right)^{\bar{\alpha}},\ t \in J.$$

Similarly, if $\alpha_i \leq 0$, (31) and $\bar{\alpha} \geq 0$ yield

$$\left(y_k^{(i-1)}(t)\right)^{\alpha_i} \leq 1 \leq \left(y_k^{(n-1)}(t)\right)^{\bar{\alpha}} \leq 2^{(n-1)\bar{\alpha}}\left(y_k^{(n-1)}(t)\right)^{\bar{\alpha}},\ t \in J.$$

Hence, (1) and (27) yield

$$\begin{aligned} y_k^{(n)}(t) &\leq M_2 \sum_{i=1}^{n-1} \left(y_k^{(i-1)}(t)\right)^{\alpha_i} y_k^{(n-1)}(t)^{\lambda_1} \leq M_3\left(y_k^{(n-1)}(t)\right)^{\bar{\alpha}+\lambda_1} \\ &\leq M_3\left(y_k^{(n-1)}(t)\right)^{1+\frac{1}{n-1}}, t \in J, k \geq k_0, \end{aligned} \tag{34}$$

where $M_3 = (n-1)2^{(n-1)\bar{\alpha}} M_2$.

We prove that $\lim_{t\to\tau_-} y^{(n-1)}(t) = \infty$. Hence, suppose, contrarily, that

$$\lim_{t\to\tau_-} y^{(n-1)}(t) = Q_2 < \infty.$$

Let $T_1 \in [T, \tau)$ be such that $\tau - T_1 \leq \frac{n-1}{4M_3}\left(\frac{1}{Q_2+1}\right)^{\frac{1}{n-1}}$. The uniform convergence of $\{y_k^{(n-1)}\}$ to $y^{(n-1)}$ on $[T, T_1]$ yields the existence of $\bar{k} \geq k_0$ such that $\bar{k} \geq 4^{n-1}(Q_2+1)$ and

$$\left|y_k^{(n-1)}(t) - y^{(n-1)}(t)\right| < 1 \quad \text{for} \quad k \geq \bar{k} \quad \text{and} \quad t \in [T, T_1].$$

Because $y^{(n-1)}$ is increasing, we obtain

$$y_k^{(n-1)}(T_1) \leq 1 + Q_2,\ k \geq \bar{k},$$

and the integration of (34) on $[T_1, \tau]$ yields

$$\begin{aligned} 1 + Q_2 \geq y_k^{(n-1)}(T_1) &\geq \left(k^{-\frac{1}{n-1}} + \frac{M_3}{n-1}(\tau - T_1)\right)^{1-n} \\ &\geq \left(\bar{k}^{-\frac{1}{n-1}} + \frac{1}{4}\left(\frac{1}{Q_2+1}\right)^{\frac{1}{n-1}}\right)^{1-n} \geq 2^{n-1}(Q_2+1). \end{aligned}$$

The contradiction proves that $\lim_{t\to\tau_-} y^{(n-1)}(t) = \infty$. Hence from this and from (34) it follows that the assumptions of Lemma 5 are fulfilled and, thus, (32) contradicts its conclusion. Hence y is unbounded on J and, as it is increasing on J, $\lim_{t\to\tau_-} y(t) = \infty$ and (3) holds. □

Remark 3. It follows from the proof of Theorem 3 and from Remark 1 that for sufficiently small positive τ, the solution y is defined on $[0, \tau)$.

We illustrate the situation on special cases of (1). Consider the differential equations

$$y^{(n)} = r(t)h(y)g(y^{(n-1)}), \tag{35}$$

$$y^{(n)} = r(t)h_1(y, \dots, y^{n-2}), \tag{36}$$

$$y^{(n)} = r(t)h(y^{(s)})g(y^{(n-1)}), s \in \{0, 1, \dots, n-2\}, \tag{37}$$

where $r \in C^\circ(R_+), h \in C^\circ(R), g \in C^\circ(R), h_1 \in C^\circ(R^{n-1}), M \in (0,\infty)$, $\tau \in (0,\infty), h(x)x > 0$ for $|x| \geq M, r(\tau) \neq 0, g(x) > 0$ for $|x| \geq M$, $\beta h_1(x_1, \dots, x_{n-2}) > 0$ for $\beta x_i \geq M, i = 1, \dots, n-2$, and $\beta \in \{-1, 1\}$.

Corollary 1. *Let* $\alpha_i \in R, i = 1, 2, \dots, n-1, \bar{\alpha} = \max(\alpha_1, \dots, \alpha_{n-1}), \alpha \in R$, $M > 0, M_1 > 0, M_2 > 0$, *and* $s \in \{0, 1, \dots, n-2\}$.

(i) *Let* $1 < \alpha \leq \bar{\alpha} \leq 1 + \frac{1}{n-1}, r(\tau) > 0,$

$$M_1|x_{s+1}|^\alpha \leq |h_1(x_1, \dots, x_{n-1})|, \ |x_i| \geq M, i = 0, 1, \dots, n-1, \tag{38}$$

and

$$|h_1(x_1, \dots, x_{n-1})| \leq M_2 \sum_{i=1}^{n-1} |x_i|^{\alpha_i}, \ |x_i| \geq M, i = 0, 1, \dots, n-1. \tag{39}$$

Then (36) has a solution y fulfilling (3).

(ii) *If* $r(\tau) < 0$
or

$$r(\tau) > 0, \ \bar{\alpha} \leq 1 \quad \textit{and (39) holds}$$

or

$$r(\tau) > 0, \alpha > 1 + \frac{1}{n-1}, \ s > \frac{\alpha(n-1) + n + 1}{3\alpha - 1}, \quad \textit{and (38)}$$

holds, then (36) has no solution y fulfilling (3).

Proof. (i) It is a consequence of Theorem 3, with $\lambda = \lambda_1 = 0$ and $[\bar{\tau}, \tau] \subset R_+$ being such that $r > 0$ on $[\bar{\tau}, \tau]$.

(ii) It is an application of Theorem 1, Theorem 2(i), and Theorem 2(ii) with (17). □

A simple application of Corollary 1 yields the following result.

Corollary 2. *Consider the equation* $y^{(n)} = r(t)\,|\,y^{(s)}\,|^\alpha \operatorname{sgn} y^{(s)}$, *where* $s \in \{1, \dots, n-2\}$, $\tau \in (0,\infty), r(\tau) \neq 0$ *and* $s > \frac{\alpha(n-1)+n+1}{3\alpha-1}$. *Then this equation has a solution y fulfilling (3) if and only if* $\alpha \in (1, 1 + \frac{1}{n-1}]$ *and* $r(\tau) > 0$.

Corollary 3. *(i) Let* $0 \le \lambda \le \bar{\lambda}$, $M > 0$, $M_1 > 0$, *and* $0 \le \alpha \le \bar{\alpha}$ *be such that*

$$M_1\ |x|^{\alpha} \le |f(x)|,\ |x|^{\lambda} \le g(x),\ |x| \ge M, \tag{40}$$

$$|f(x)| \le |x|^{\bar{\alpha}}, g(x) \le |x|^{\bar{\lambda}},\ |x| \ge M. \tag{41}$$

If $1 < \alpha + \lambda \le \bar{\alpha} + \bar{\lambda} \le 1 + \frac{1}{n-1}$ *and* $r(\tau) > 0$, *then (37) has a solution y fulfilling (3).*

(ii) If $r(\tau) < 0$
or $r(\tau) > 0$, *(41) holds with* $\bar{\alpha} \in R$ *and* $\max(\bar{\alpha}, 0) + \lambda \le 1$
or $r(\tau) > 0, \alpha \le 0, \lambda > 1 + \frac{1}{n-1}$ *and (40) holds*
or $r(\tau) > 0, \alpha > 0, \lambda > 1, s > 0, \alpha + \lambda > 1 + \frac{1}{s}$ *and (40) holds*
or

$$r(\tau) > 0, \alpha + \lambda > 1 + \frac{1}{n-1}, s > \frac{2n + (n-1)(\alpha + \lambda - 1)}{3(\alpha + \lambda - 1) + 2} \tag{42}$$

and (40) holds, then (37) does not have solutions y fulfilling (3).

Proof. The argument is similar to the one in Corollary 1. □

Corollary 4. *Consider the equation* $y^{(n)} = r(t)|y^{(s)}|^{\alpha}|y^{(n-1)}|^{\lambda} \operatorname{sgn} y^{(s)}$ *with* $\tau \in (0, \infty)$, $r(\tau) \ne 0$, $s \in \{1, \dots, n-2\}$, $\alpha \ge 0$, $\lambda \ge 0$ *and* $s > \frac{2n+(n-1)(\alpha+\lambda-1)}{3(\alpha+\lambda-1)+2}$. *Then this equation has a solution y fulfilling (3) if and only if* $\alpha + \lambda \in (1, 1 + \frac{1}{n-1}]$ *and* $r(\tau) > 0$.

Corollary 5. *Let* $\lambda \ge 0$, $M > 0$, $M_1 > 0$ *be such that* $|x|^{\lambda} \le g(x) \le M_1|x|^{\lambda}$ *for* $|x| \ge M$ *and let* $|h|$ *be bounded from below and from above by positive constants for* $|x| \ge M$. *Then (35) has a singular solution fulfilling (3) if and only if* $1 < \lambda \le 1 + \frac{1}{n-1}$ *and* $r(\tau) > 0$.

Proof. It is a special case of Corollary 3 with $\lambda = \lambda_1$ and $\bar{\alpha} = \alpha = 0$. □

Remark 4. The results of Theorem 2 are valid also for $\alpha = 0$ and $\lambda = \bar{\lambda}$. This case was studied in Theorem 2 in [2]. But the proof of this theorem has a gap and the upper estimation in (27) is missing. So the precise formulation is given by our Theorem 3 and, similarly, the precise formulation of Corollary 2 in [2] is given by Corollary 5.; Corollary 1 in [2] is not valid.

Remark 5. Condition (17) is not valid for $s = 0$ and for $n = 2$. There exists $s \in \{1, \dots, n-2\}$ for which (17) is valid (or the two last inequalities in (42) are valid) if $\alpha + \lambda > 1 + \frac{4}{2n-5}$ and $n \ge 3$. If $1 + \frac{1}{n-1} < \alpha + \lambda \le 1 + \frac{4}{2n-5}$ and $n \ge 3$, then there exists no $s \in \{1, \dots, n-2\}$ for which (17) holds.

The problem of the existence of solution y of (1) fulfilling (3) is not solved fully. The following example shows that solutions for which (3) is valid may exist in the case $\alpha + \lambda > 1 + \frac{1}{n-1}$.

Example 1. The equation $y^{(n)} = 3^{\frac{3}{4}}(n - \frac{2}{3})|y'|^{\frac{3}{4}}|y^{(n-1)}| \operatorname{sgn} y'$, $n \ge 3$ has a solution $y(t) = \frac{1}{(\tau - t)^{\frac{1}{3}}}$ of the form (3). At the same time $\alpha + \lambda = \frac{7}{4} > 1 + \frac{1}{n-1}$.

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