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## Global and pullback attractors of set-valued skew product flows

*Dedicated to R. Conti on the occasion of his 80th birthday*

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**Abstract.** We investigate the asymptotic behaviour of a general set-valued skew product flow (SVSPF), that is, a set-valued cocycle mapping (coming from a non-autonomous differential equation or inclusion) driven by another, autonomous, system. Absorptivity conditions which ensure the existence of several types of attractors for such set-valued systems are established. The topological properties of and relations between these attractors, in forward and pullback senses and their strong and weak versions are analyzed. Several illustrative examples are also provided.

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**Key words.** set-valued skew product flow – pullback attraction – weak invariance

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### 1. Introduction

Set-valued analysis and attainability set functions are used to handle problems arising from differential equations without uniqueness, differential inclusions, or problems arising in control theory, viability theory, finance and economics amongst others, and have been widely studied in recent decades (see, e.g., [3, 21, 20, 25, 14, 15]). The investigation of the asymptotic behaviour of these phenomena, global, strong or weak stability and attraction properties, needs the concept of pullback attractors when non-autonomous equations are considered, in particular stochastic and random ones which are intrinsically non-autonomous (cf. [11, 10, 22, 16] and [17, 5] for the weak pullback case).

However, the non-autonomous case can also be viewed within the framework of skew-product flows, which allows us to transform the problem into an autonomous one (cf. [23, 6, 7, 9, 18, 8]) and to apply the classical theory for autonomous systems in a different, extended, phase space. On the other hand, if such an attractor exists, then we can recover the dynamics in the original phase space  $\mathbb{R}^d$  and its

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asymptotic properties by using suitable projections, which is important as this original phase space is often the one that is of interest or meaningful in modelling.

The paper is organized as follows: In Section 2, we recall the concepts of skew-product flows coming from a cocycle set-valued mapping with a related driving system. Conditions for existence of strong and weak attractors for an autonomous set-valued semidynamical system in an abstract metric space and its properties are analyzed in Section 3. Then, in Section 4 we apply these results to the case of a skew-product flow as introduced above, distinguishing between strong and weak cases. Here, the sectorial components in the original phase space  $\mathbb{R}^d$  of the attractor in the extended phase space (under several suitable conditions) will be analyzed, namely, for strong and weak asymptotic concepts as well as for their connections with the underlying non-autonomous semi-flow. The formalism used here allows us to obtain additional results which supplement those in earlier papers ([5, 17]).

We also illustrate our theory with some examples and, for the sake of clarity, give most of the proofs at the end of the paper.

## 2. Set-valued skew product flows

Hereafter, we will use the following notation:  $\mathcal{P}(X)$  and  $\mathcal{K}(X)$  for the set of nonempty and nonempty compact subsets of a given space  $X$ , respectively;  $H^*$  for the Hausdorff semi-distance,  $H^*(A, B) = \sup_{a \in A} \text{dist}(a, B)$ , and  $H$  for the Hausdorff distance,  $H(A, B) = \max(H^*(A, B), H^*(B, A))$ .

To establish an appropriate framework for our analysis, we consider given an autonomous driving system,  $\theta : \mathbb{R} \times P \rightarrow P$ , where  $P$  is a metric space, i.e. a group of homeomorphisms under composition on  $P$  with the properties

- i)  $\theta_0 p = p$  for all  $p \in P$ ,
- ii)  $\theta_{t+s} p = \theta_t \theta_s p$  for all  $s, t \in \mathbb{R}$ ,
- iii) the mapping  $(t, p) \mapsto \theta_t p$  is continuous.

For instance, one can think of an ordinary differential system in  $P = \mathbb{R}^l$  with an autonomous globally Lipschitz and dissipative vectorfield  $g$ , i.e.  $p' = g(p)$ .

A set-valued skew product flow (SVSPF for short) consists of an autonomous driving system  $\theta$  on a metric space  $P$  and a set-valued cocycle mapping (attainability set mapping)  $\Phi : \mathbb{R}^+ \times P \times \mathbb{R}^d \rightarrow \mathcal{K}(\mathbb{R}^d)$  satisfying the following properties:

1. Compactness  $\Phi(t, p, x)$  is a nonempty compact subset of  $\mathbb{R}^d$  for all  $t \geq 0$ ,  $p \in P$ ,  $x \in \mathbb{R}^d$ ;
2. Initial condition

$$\Phi(0, p, x) = \{x\}$$

for all  $p \in P$  and  $x \in \mathbb{R}^d$ ;

3. Cocycle property

$$\Phi(t + s, p, x) = \Phi(t, \theta_s p, \Phi(s, p, x))$$

for all  $t \geq 0, p \in P, x \in \mathbb{R}^d$ ;

4. Continuity in time

$$\lim_{s \rightarrow t} H(\Phi(s, p, x), \Phi(t, p, x)) = 0$$

for all  $s, t \geq 0$  and all  $p \in P$  and  $x \in \mathbb{R}^d$ ;

5. Upper semi continuity in parameter and initial conditions

$$\lim_{q \rightarrow p, y \rightarrow x} H^*(\Phi(t, q, y), \Phi(t, p, x)) = 0$$

uniformly in  $t \in [T_0, T_1]$  for any  $0 \leq T_0 < T_1 < \infty$  for all  $p \in P$  and  $x \in \mathbb{R}^d$ .

*Remark 1.* Assumptions 4 and 5 imply that  $\Phi$  is globally upper semi continuous (u.s.c.), i.e. if  $(t_n, p_n, x_n) \rightarrow (t, p, x)$  as  $n \rightarrow \infty$ , then  $H^*(\Phi(t_n, p_n, x_n), \Phi(t, p, x)) \rightarrow 0$ . Indeed,

$$\begin{aligned} H^*(\Phi(t_n, p_n, x_n), \Phi(t, p, x)) &\leq H^*(\Phi(t_n, p_n, x_n), \Phi(t_n, p, x)) \\ &\quad + H^*(\Phi(t_n, p, x), \Phi(t, p, x)) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

since the first term in the right hand side goes to zero by the u.s.c. in the second and third variables uniformly in time (property 5), and the second term goes to zero by the continuity (hence u.s.c.) of  $\Phi$  on its first variable (property 4).

A trajectory of a set-valued cocycle  $\Phi$  is a single-valued mapping  $\phi_p : [0, T] \rightarrow \mathbb{R}^d$  which, for the indicated  $p \in P$ , satisfies

$$\phi_p(t) \in \Phi(t - s, \theta_s p, \phi_p(s)) \quad \text{for all } 0 \leq s \leq t \leq T. \tag{1}$$

A trajectory  $\phi$  is called an entire trajectory if it is defined on all of  $\mathbb{R}$  and satisfies (1) for all  $s \leq t$ . (If necessary, to emphasize a particular  $p \in P$ , we will use the notation  $p$ -trajectory for the above definition).

Now let us denote

$$\mathcal{T}_{p,x}([0, T]) = \{\phi_p, \text{ trajectory such that } \phi_p(0) = x\}.$$

Then, we can establish the following generalization of a theorem of Barbashin in a similar way as in [3, 21, 14]:

**Theorem 2.** *The following properties holds:*

1.  $\mathcal{T}_{p,x}([0, T]) \neq \emptyset$  (there exist trajectories for all  $p, x$ , and  $T > 0$ )
2.  $\mathcal{T}_{p,x}([0, T]) \subseteq C([0, T]; \mathbb{R}^d)$  (continuity)
3.  $\mathcal{T}_{p,x}([0, T])$  is a compact subset of  $C([0, T]; \mathbb{R}^d)$
4.  $\mathcal{T}_{p_n, x_n}([0, T]) \rightarrow \mathcal{T}_{p,x}([0, T])$  (in  $H^*$  on  $C([0, T]; \mathbb{R}^d)$ ) as  $p_n \rightarrow p, x_n \rightarrow x$ .

*Remark 3.* It is worth noticing that Theorem 2 also holds true if we consider a Banach space  $X$  instead of  $\mathbb{R}^d$  (see the proof in Section 7).

We now consider a general autonomous set-valued semidynamical system (SVSDS for short) as in Szegö and Treccani [25], that is, a set-valued mapping  $\Pi : \mathbb{R}^+ \times Y \rightarrow \mathcal{P}(Y)$  satisfying suitable properties, and where  $Y$  is a connected metric space. To avoid unnecessary repetitions, such properties are the stated below for the special case in which the set  $Y$  is the extended phase space  $P \times \mathbb{R}^d$ .

A particular case of an autonomous set-valued semidynamical system is that generated by a set-valued skew product flow, namely with  $\Pi(t, p, x) = (\theta_t p, \Phi(t, p, x))$  and satisfying the following properties:

1.  $\Pi(t, p, x)$  is nonempty and compact;
2.  $\Pi(0, p, x) = \{(p, x)\}$ ;
3. The semigroup property:

$$\Pi(t + s, p, x) = \Pi(t, \Pi(s, p, x)) ;$$

4.  $t \mapsto \Pi(t, p, x)$  is continuous in the Hausdorff metric, for all  $p, x$ , i.e.

$$H_{P \times \mathbb{R}^d}(\Pi(s, p, x), \Pi(t, p, x)) \rightarrow 0, \text{ as } s \rightarrow t;$$

5.  $(p, x) \mapsto \Pi(t, p, x)$  is upper semicontinuous in the  $H^*$  sense, i.e.

$$H_{P \times \mathbb{R}^d}^*(\Pi(t, q, y), \Pi(t, p, x)) \rightarrow 0, \text{ as } q \rightarrow p, y \rightarrow x,$$

uniformly in compact intervals  $t \in [T_1, T_2]$ .

A trajectory for an SVSDS is a single-valued mapping  $\pi : [0, T] \rightarrow Y$  with  $\pi(t) \in \Pi(t - s, \pi(s))$  for all  $0 \leq s \leq t \leq T$ . Analogously, a trajectory (or  $p$ -trajectory) for a SVSPF is a single-valued mapping  $\pi_p : [0, T] \rightarrow P \times \mathbb{R}^d$  with  $\pi_p(t) \in \Pi(t - s, \pi_p(s))$  for all  $0 \leq s \leq t \leq T$  and the first component of  $\pi_p(0)$  equals to  $p$ .

**Proposition 4.**  $\pi_p$  is a trajectory for the SVSPF if and only if there exists a trajectory  $\phi_p$  of  $\Phi$  such that

$$\pi_p(t) = (\theta_t p, \phi_p(t)) \quad \forall t \in [0, T].$$

*The result holds true for any trajectory defined in any interval of time, and also for entire trajectories.*

*Remark 5.* An analogous result to Theorem 2 holds for an SVSPF  $\Pi = (\theta, \Phi)$  (actually for any general SVSDS), replacing  $C([0, T]; \mathbb{R}^d)$  by  $C([0, T]; P \times X)$  in all the statements, actually, in any time interval not necessarily in the positive half line.

This is straightforward because the multi-valued mapping  $F : \{(t, t_0) : t \geq t_0\} \times P \times X \rightarrow \mathcal{P}(P \times X)$  defined by  $F(t, t_0, p, x) := \Pi(t - t_0, (p, x))$  satisfies the required conditions, since  $\theta$  and  $\Phi$  do (cf. [3, 21, 14, 5]).

### 3. Attractors of SVSDS

Now we consider a SVSDS  $\Pi : \mathbb{R}^+ \times Y \rightarrow \mathcal{P}(Y)$  and recall the basic concepts on attractors, which will be applied in the following section to our skew product formulation, denoting then the extended phase space  $P \times \mathbb{R}^d$  by  $Y$ .

For the sake of clarity, subscripts  $s$  and  $w$  on attractors will denote *strong* and *weak* concepts.

**Definition 6.** A strong global attractor for an SVSDS  $\Pi$  is a nonempty compact subset  $\mathcal{A}_s \subset Y$  satisfying

1. *strong invariance:*  $\Pi(t, \mathcal{A}_s) = \mathcal{A}_s$  for all  $t \geq 0$
2. *strong attraction:* for every nonempty bounded subset  $D$  of  $Y$ ,

$$\text{dist}_Y(\Pi(t, D), \mathcal{A}_s) \rightarrow 0, \text{ as } t \rightarrow \infty.$$

A weak global attractor for an SVSDS  $\Pi$  is a nonempty compact subset  $\mathcal{A}_w \subset Y$  satisfying

1. *weak invariance:*  $\forall y \in \mathcal{A}_w$  there exists an entire trajectory  $\pi : \mathbb{R} \rightarrow Y$  with  $\pi(0) = y$  and  $\pi(t) \in \mathcal{A}_w$  for all  $t \in \mathbb{R}$
2. *weak attraction:* for every nonempty bounded subset  $D$  of  $Y$  and  $y_n \in D$ , there exist trajectories  $\pi_n : \mathbb{R}^+ \rightarrow Y$  and numbers  $\tau_n \rightarrow \infty$  with  $\pi_n(0) = y_n$  and

$$\text{dist}_Y(\pi_n(\tau_n), \mathcal{A}_w) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

For completeness, we recall some results ensuring the existence of such attractors.

#### 3.1. Strong global attractor of SVSDS

The most simple case of an autonomous semi-flow with a compact absorbing set  $\mathcal{B}_s$  (i.e. for every bounded set  $D$  there exists  $T_D \geq 0$  such that  $\Pi(t, D) \subset \mathcal{B}_s$  for all  $t \geq T_D$ ) is well known. Assume that  $\mathcal{B}_s$  is a nonempty compact absorbing set in  $Y$ . Without loss of generality, we can assume that  $\mathcal{B}_s$  is  $\Pi$ -positively invariant (i.e.  $\Pi(t, \mathcal{B}_s) \subset \mathcal{B}_s$ , for all  $t \geq 0$ ). Define

$$\mathcal{A}_s = \bigcap_{t \geq 0} \Pi(t, \mathcal{B}_s).$$

Then,  $a \in \mathcal{A}_s$  iff  $a \in \mathcal{B}_s$  and there exist  $\tau_n \rightarrow \infty$ ,  $a_n \in \Pi(\tau_n, \mathcal{B}_s)$  with  $a_n \rightarrow a$  as  $n \rightarrow \infty$ .

**Proposition 7.** The set  $\mathcal{A}_s$  has the following properties:

1. It is nonempty and compact and attracts bounded sets.
2. It is  $\Pi$ -invariant, therefore it is a global strong attractor. Actually it is the maximal invariant compact set, and also the minimal closed set that attracts bounded sets.
3. If  $\Pi(t, x)$  is connected for all  $(t, x) \in \mathbb{R}^+ \times Y$ , then  $\mathcal{A}_s$  is also connected.

*Remark 8.* The existence of an absorbing set for the construction of the attractor and its properties can be relaxed to that of an attracting set (see [19, Th. 1]), which can be more appropriate in some other situations (e.g. hyperbolic systems).

### 3.2. Weak attractors of SVSDS

We consider now an SVSDS  $\Pi(t, x)$  and establish existence of weak attractors. To this end, we introduce the concept of a weak absorbing set  $\mathcal{B}_w$ , i.e., a nonempty, compact set which in addition is

- weakly positively invariant: for all  $b \in \mathcal{B}_w$  there exists at least one trajectory  $\pi$  with  $\pi(0) = b$  and  $\pi(t) \in \mathcal{B}_w$  for all  $t \geq 0$ .
- weakly absorbing: for all bounded subset  $D$  there exists  $T_D \geq 0$  such that for any  $d \in D$  there exists a trajectory  $\pi$  with  $\pi(0) = d$  and  $\pi(t) \in \mathcal{B}_w$  for all  $t \geq T_D$ .

**Theorem 9.** *Assume there exists a weak absorbing set  $\mathcal{B}_w$  for an SVSDS  $\Pi$ . Then, there exists the maximal weak attractor  $\mathcal{A}_w$  w.r.t.  $\mathcal{B}_w$ , which is defined as the set of points  $a \in \mathcal{B}_w$  such that there exist  $b_n \in \mathcal{B}_w$ ,  $\tau_n \rightarrow \infty$  and trajectories  $\pi_n : \mathbb{R}^+ \rightarrow \mathcal{B}_w$  with  $\pi_n(0) = b_n$  and  $\text{dist}(\pi_n(\tau_n), a) \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Remark 10.* As in [5, Lemma 13], we observe that an entire trajectory  $\pi : \mathbb{R} \rightarrow Y$  satisfies that

$$\pi(t) \in \mathcal{B}_w \quad \text{if and only if} \quad \pi(t) \in \mathcal{A}_w.$$

Therefore,  $\mathcal{A}_w$  is the set of points reached by entire trajectories contained in  $\mathcal{B}_w$ .

## 4. Attractors of SVSPF

Now specialize to  $\Pi = (\theta, \Phi)$  with our skew product structure, i.e. we consider now that  $Y = P \times \mathbb{R}^d$ . We again split our analysis into two cases concerning the strong and weak situations.

### 4.1. (Strong) global attractor of SVSPF

Suppose that there exists a compact positive invariant absorbing set  $\mathcal{B}_s \subset P \times \mathbb{R}^d$  for  $\Pi$ . Then, there exists a strong global attractor

$$\mathcal{A}_s = \bigcap_{t \geq 0} \Pi(t, \mathcal{B}_s).$$

Let  $P^* = \text{Pr}_P(\mathcal{A}_s)$  be the projection of  $\mathcal{A}_s$  onto the space  $P$  and consider the decomposed notation

$$\mathcal{A}_s = \bigcup_{p \in P^*} \{p\} \times A_s(p).$$

**Proposition 11.** *Under the previous assumptions the following properties hold:*

1.  $P^*$  is nonempty, compact and  $\theta_t P^* = P^*$ . In fact,  $P^*$  is the global attractor of the (single-valued) autonomous driving system  $\theta$  on  $P$ .
2.  $A_s(p^*)$  is nonempty and compact for each  $p^* \in P^*$ . It also satisfies the invariance property  $A_s(\theta_t p^*) = \Phi(t, p^*, A_s(p^*))$ .
3. The mapping  $P^* \ni p \mapsto A_s(p)$  is upper semi continuous.

Now consider the restriction  $\Pi^*$  of  $\Pi$  to  $P^* \times \mathbb{R}^d$ . Since  $\theta_t P^* = P^*$  for all  $t \in \mathbb{R}$ , it follows that  $\Pi^*$  is an SVSPF on  $P^* \times \mathbb{R}^d$ .

Consider  $\mathcal{B}_s^* = \mathcal{B}_s \cap (P^* \times \mathbb{R}^d)$ . Then,  $\mathcal{B}_s^*$  absorbs sets in  $P^* \times \mathbb{R}^d$  under  $\Pi^*$  ( $\equiv \Pi$ ). Also  $\mathcal{B}_s^*$  is nonempty, compact and  $\Pi^*$ -positively invariant, so  $\Pi^*$  has a maximal global attractor

$$\mathcal{A}_s^* = \bigcup_{p \in P^*} \{p\} \times A_s^*(p),$$

in  $P^* \times \mathbb{R}^d$ . Then, we have the following result:

**Proposition 12.** *The strong global attractors of  $\Pi$  and  $\Pi^*$  coincide:  $\mathcal{A}_s^* \equiv \mathcal{A}_s$ .*

#### 4.2. Weak attractors of SVSPF

Suppose that an SVSPF  $\Pi$  has a weak attractor relative to the compact weak absorbing set  $\mathcal{B}_w \subset P \times \mathbb{R}^d$ , which is also given by

$$\mathcal{A}_w = \bigcup_{p \in P^*} \{p\} \times A_w(p),$$

(again we denote  $P^* = \text{Pr}_P \mathcal{A}_w$ ) as described above for general set-valued semi dynamical systems. Then, we have

**Proposition 13.** *The following properties hold:*

1.  $P^*$  is the global attractor for the single-valued driving system  $\theta$  on  $P$ .
2.  $A_w(p)$  is a nonempty compact set for each  $p \in P^*$ .
3. The map  $P^* \ni p \mapsto A_w(p) \in \mathcal{K}(\mathbb{R}^d)$  is u.s.c.
4.  $\mathcal{A}_w$  is weakly invariant, i.e. if  $(p, a) \in \mathcal{A}_w$ , there exists an entire trajectory  $\pi = (\theta, \varphi_p)$  such that  $\varphi_p(0) = a$  and  $\pi(t) = (\theta_t p, \varphi_p(t)) \in \mathcal{A}_w$  for all  $t \in \mathbb{R}$ , i.e.  $\varphi_p(t) \in A_w(\theta_t p)$ .

Moreover,  $\mathcal{A}_w$  is the largest weak invariant set in the absorbing set  $\mathcal{B}_w$ .

If we restrict  $\Pi$  to  $\Pi^*$  on  $P^* \times \mathbb{R}^d$ , we have that  $\Pi(t, (p, x)) = \Pi^*(t, (p, x))$  for all  $t \geq 0$  and  $(p, x) \in P^* \times \mathbb{R}^d$  since  $P^*$  is  $\theta$ -invariant.

Define

$$\mathcal{B}_w^* = \bigcup_{p \in P^*} \{p\} \times B_w(p) \subset \mathcal{B}_w = \bigcup_{p \in P} \{p\} \times B_w(p).$$

Note that  $A_w(p) \subset B_w(p)$  for all  $p \in P^*$ . So,  $B_w(p)$  is nonempty and compact. We also have that  $\mathcal{B}_w^*$  is weakly positive invariant since  $P^*$  is  $\theta$ -invariant and  $\mathcal{B}_w$  is weakly positive  $\Pi$ -invariant.

Then, we obtain a maximal weak attractor for  $\Pi^*$ ,  $\mathcal{A}_w^*$ , with respect to  $\mathcal{B}_w^*$  and we will use the notation

$$\mathcal{A}^* = \bigcup_{p \in P^*} \{p\} \times A_w^*(p).$$

*Remark 14.* Note that  $a \in A_w^*(p)$  if and only if there exist sequences  $t_n \rightarrow \infty$ ,  $(p_n, b_n) \in \mathcal{B}_w^*$ , trajectories  $\pi_n = (\theta, \varphi_{p_n})$  with  $\varphi_{p_n}(0) = b_n$  and  $\theta_{t_n} p_n \rightarrow p$  and  $\varphi_{p_n}(t_n) \rightarrow a$ .

Observe that  $\mathcal{A}_w^*$  consists of entire  $\Pi^*$ -trajectories, and, since it is weak invariant,  $A_w^*(p)$  is nonempty and compact for all  $p \in P^*$ .

$\mathcal{A}_w \subset \mathcal{B}_w^* \subset \mathcal{B}_w$  but  $\mathcal{A}_w$  is the maximal  $\Pi$ -weak invariant family contained in  $\mathcal{B}_w$  and  $\mathcal{A}_w^*$  is  $\Pi^*$ -weak invariant (and so  $\Pi$ -weak invariant), therefore one has that

$$\mathcal{A}_w^* \subset \mathcal{A}_w.$$

Indeed, as  $\Pi$  and  $\Pi^*$  coincide in  $\mathcal{B}_w^*$ , entire  $\Pi$ -trajectories in  $\mathcal{A}_w$  are  $\Pi^*$ -trajectories and conversely, by the same argument of maximality, we conclude that

$$\mathcal{A}_w = \mathcal{A}_w^*.$$

Thus, we can restrict ourselves to the dynamics on  $\mathcal{B}_w^*$  and we will study the relations between weak and strong skew-product attractors and their sections with respect to pullback weak and strong attractors in the following section.

### 5. Pullback structure of SVSPF attractors

Once again we split our analysis into two cases: strong and weak attractors.

#### 5.1. The strong case

As before,  $\mathcal{B}_s$  denotes a  $\Pi$ -positive invariant compact absorbing set and  $\mathcal{B}_s^* = \mathcal{B}_s \cap (P^* \times \mathbb{R}^d)$  is a  $\Pi^*$ -positive invariant compact absorbing set (we keep using the notation  $\mathcal{B}_s = \bigcup_{p \in P} \{p\} \times B_s(p)$ , so  $B_s(p) \neq \emptyset$  for  $p \in P^*$ ). This implies  $\Phi$ -positive invariance for the sections  $B_s(p)$  in  $P^*$ :

$$\Phi(t, p, B_s(p)) \subset B_s(\theta_t p) \quad \forall t \geq 0, \forall p \in P^*.$$

Indeed,  $\Pi^*(t, \mathcal{B}_s^*) \subset \mathcal{B}_s^*$  implies that

$$\Pi(t, (p, B_s(p))) = (\theta_t p, \Phi(t, p, B_s(p))) \subset \mathcal{B}_s^* = \bigcup_{q \in P^*} (q, B_s(q))$$

what necessarily means our claim if  $p \in P^*$ .



**Lemma 15.** *Define*

$$\hat{A}_s(p) = \bigcap_{t \geq 0} \Phi(t, \theta_{-t}p, B_s(\theta_{-t}p)) \quad \text{for } p \in P^*.$$

Then,  $\hat{A}_s(p)$  is nonempty and compact for all  $p \in P^*$ .

*Remark 16.* If  $B_s(p)$  and  $\Phi(t, p, x)$  are connected for all  $t \geq 0$ ,  $p \in P^*$  and  $x \in \mathbb{R}^d$ , then  $\hat{A}_s(p)$  is also connected, since it is the intersection of a nested family of nonempty, compact, connected sets.

**Proposition 17.** *The following identities hold:*

$$\hat{A}_s(p) = A_s^*(p) = A_s(p) \quad \forall p \in P^*. \quad (2)$$

Moreover, we also have the following result relating the sections with the notion of pullback attractor (see e.g. Kloeden and Schmalfuß [18, Def. 4.2]).

**Proposition 18.**  $\{\hat{A}_s(p)\}_{p \in P^*}$  is the pullback attractor for  $\Phi$  on  $P^* \times \mathbb{R}^d$ .

*Remark 19.*

- (i) Notice that Propositions 17 and 18 imply that the  $p$ -components of the strong global  $\Pi$ -attractor  $\mathcal{A}_s$ , which attracts in the forward and pullback senses, are the strong pullback attractor for  $\Phi$  when the dynamics are restricted to  $P^* \times \mathbb{R}^d$ .
- (ii) Here we started with a global attractor for the skew-product flow  $\Pi$  and have obtained a pullback attractor for  $\Phi$ . The converse is not true in general, i.e. if  $\{A_s(p), p \in P^*\}$  is a strong pullback attractor for  $\Phi$ , then  $\mathcal{A} = \bigcup_{p \in P^*} \{p\} \times A_s(p)$  may not be a global attractor for  $\Pi$  (see [8] for a counterexample in the single-valued case).
- (iii) In general, pullback attractors for set-valued flows are only negatively invariant, and strict invariance needs additional assumptions (the easiest is a lower semi continuous property for the flow). Here, the strict invariance holds since the driving system has a global attractor.

Now, bearing (2) in mind, we deduce an improvement of the u.s.c. result in Section 4.1 when the approximation in the parameter space comes from the driving system.

**Corollary 20.** *For every  $p \in P^*$ , the map  $t \mapsto A_s(\theta_t p)$  is continuous.*

*Proof.* It follows from the fact that  $\mathcal{A}_s^* \equiv \mathcal{A}_s$  is strongly  $\Pi$ -invariant (and thus  $\Phi(t, p, A_s(p)) = A_s(\theta_t p)$ ), and the continuity of  $\Phi$  on its first variable.  $\square$

5.2. *The weak case*

Assume that  $\mathcal{B}_w = \bigcup_{p \in P} \{p\} \times B_w(p)$  is a weak absorbing family for the skew-product flow and denote  $\mathcal{B}_w^* = \mathcal{B}_w \cap (P^* \times \mathbb{R}^d)$ . Then, analogously to the definitions of weak attractor for  $\Pi$  and an ordinary non-autonomous system [5], we will obtain a weak pullback attractor for  $\Phi$ , that is, a weakly invariant compact family  $\{A_w(p)\}_{p \in P^*}$  such that for every bounded set  $D \subset \mathbb{R}^d$  and  $x_n \in D$ , there exist sequences  $t_n \rightarrow \infty$  and  $\theta_{-t_n}p$ -trajectories  $\phi_n$  with  $\text{dist}(\phi_n(t_n, \theta_{-t_n}p, x_n), A_w(p)) \rightarrow 0$  as  $n \rightarrow \infty$ .

In order to do that, firstly observe that the family of sets  $\{B_w(p)\}_{p \in P}$  “is” (see Remark 22 below) a weak pullback absorbing family for  $\Phi$ , similarly to the concept of [5], which leads us to consider the following definition:

$$\hat{A}_w(p) = \left\{ a \in \mathbb{R}^d \mid \begin{array}{l} \exists t_n \rightarrow \infty, b_n \in B_w(\theta_{-t_n}p), p\text{-trajectories} \\ \varphi_n : [-t_n, 0] \rightarrow \mathbb{R}^d \text{ with } \varphi_n(-t_n) = b_n \text{ and } \varphi_n(0) \rightarrow a \end{array} \right\}.$$

We will now prove that  $\hat{A}_w(p) \subset A_w^*(p)$ .

Let  $a \in \hat{A}_w(p)$ , then there exist sequences  $t_n \rightarrow \infty, b_n \in B_w(\theta_{-t_n}p)$ , and  $p$ -trajectories  $\hat{\varphi}_n : [-t_n, 0] \rightarrow \mathbb{R}^d$  such that  $\hat{\varphi}_n(-t_n) = b_n$  and  $\hat{\varphi}_n(0) \rightarrow a$ .

Denote  $p_n = \theta_{-t_n}p$  and  $\varphi_n : [0, t_n] \rightarrow \mathbb{R}^d$  given by

$$t \mapsto \varphi_n(t) = \hat{\varphi}_n(t - t_n).$$

It is obvious that  $\pi_n = (\theta, \varphi_n)$  are  $\Pi^*$ -trajectories with  $\pi_n(0) = (p_n, b_n)$  and

$$\pi_n(t_n) = (\theta_{t_n}p_n, \varphi_n(t_n)) = (p, \hat{\varphi}_n(0)) \rightarrow (p, a).$$

Remark 14 implies that  $\hat{A}_w(p) \subset A_w^*(p)$ .

As for the other inclusion, we have:  $\mathcal{A}_w^*$  is weakly invariant, which means that for all  $(p, a) \in \mathcal{A}_w^*$ , there exists an entire trajectory  $\pi$  of  $\Pi^*$  with  $\pi(t) \in \mathcal{A}_w^*$  for all  $t \in \mathbb{R}$ , i.e.  $\pi(t) = (\theta_t p, \varphi_p(t))$  with  $\varphi_p$  an entire trajectory of  $\Phi$  such that  $\varphi_p(0) = a \in A_w(p)$  and  $\varphi_p(t) \in A_w(\theta_t p)$  for all  $t \in \mathbb{R}$ .

Consider any sequence  $t_n \rightarrow \infty$  and define  $p_n = \theta_{-t_n}p$ . Take  $b_n = \varphi_p(-t_n) \in A_w(\theta_{-t_n}p) \subset B_w(\theta_{-t_n}p)$ . Then, trivially  $\varphi_n(t) \equiv \varphi_p(t)$  for all  $n$  and  $t$ , jointly with the chosen values  $b_n, t_n$  and  $p_n$ , gives that  $a \in \hat{A}_w(p)$  and therefore  $A_w^*(p) \subset \hat{A}_w(p)$ .

Therefore, we have proved the following result:

**Proposition 21.** *Under the above assumptions,  $\hat{A}_w(p) = A_w^*(p) = A_w(p)$  for all  $p \in P^*$ , i.e. the maximal weak attractor of  $\Pi$  in  $\mathcal{B}_w$ , resp.  $\Pi^*$  in  $\mathcal{B}_w^*$  give the maximal weak pullback attractor of  $\Phi$  with respect to the absorbing family  $\{B_w(p), p \in P^*\}$ .*

*Remark 22.* By Proposition 13 and the continuity of  $\theta$ , the set-valued mapping

$$t \mapsto \hat{A}_w(\theta_t p) = A_w(\theta_t p)$$

is (only) upper semi continuous (in comparison with [5, Prop. 11]).

### 6. Examples

Our first example related to the strong structure of a skew product flow is given by the differential inclusion

$$x'(t) \in G(t, x(t)) + \beta p(t, x(t)), \tag{3}$$

where the multi-valued mapping  $G$  has good properties (for instance, it has closed and convex values in  $\mathcal{P}(\mathbb{R}^d)$  and is Lipschitz continuous) and satisfies:

1. almost periodic dependence on its first variable,
2. a dissipativity condition: for all  $y \in G(t, x)$  it holds  $(x, y) \leq -\alpha_0|x|^2 + \alpha_1$ , with  $\alpha_i > 0, i = 1, 2$ ,
3. and  $p$  is single-valued, almost periodic on its first variable, and satisfies

$$\begin{aligned} |p(t, x)| &\leq \gamma_1(t)|x| + \gamma_2(t), \\ |p(t, x) - p(t, y)| &\leq \gamma_3(t)|x - y| \end{aligned}$$

with  $\gamma_i (i = 1, 2, 3)$  continuous, positive functions and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \gamma_1(s) ds = C_f < \infty. \tag{4}$$

Under these assumptions, it is easy to deduce the existence of an attractor for the skew-product flow generated by the above problem if  $\beta$  is positive and small enough. The parameter space  $P$  is the product of the hulls of the mappings  $G$  and  $p$  in appropriate function spaces.

The almost periodic time dependence is not essential. Consider the driving system

$$p'(t) = g(p(t)), \tag{5}$$

with  $g : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$g(y) = \begin{cases} -y - 1 & \text{if } y \leq -1, \\ y^2 - 1 & \text{if } y \in (-1, 1), \\ 1 - y & \text{if } y \geq 1. \end{cases}$$

Then, it is easy to check that the non-autonomous problem

$$x'(t) \in [-1, -1/2]x(t) + p(t)$$

generates a set-valued cocycle mapping, and jointly with (5), a skew-product flow with a non-trivial global attractor in the strong sense.

Other examples, arising from differential equations without uniqueness, partial differential equations with periodic forcing terms, delay and functional differential models among others can be found in [9, 6–8].

A simple example to illustrate the weak case can be provided by using an almost periodic or even periodic function of time. Suppose we have the differential

inclusion

$$x'(t) \in x(t)[- \sin^2 t, 0], \tag{6}$$

which clearly generates a set-valued cocycle mapping. Since for every initial value  $x_0$ , the constant function  $\phi(t) = x_0$  is a solution of the problem, there does not exist a strong attractor for the skew product flow. However, the set  $\{0\}$  is a forward attractor for the non-autonomous system (and  $[0, \pi] \times \{0\}$  is a global attractor for the skew-product flow, where  $P = \mathbb{R}$  is equipped with the usual shift modulo  $\pi$ , which is considered as the driving operator). Indeed,

$$\frac{x'}{x} = - \sin^2 t = -1/2(1 - \cos(2t))$$

generates the solutions for the initial data  $x_0$

$$x(t) = x_0 e^{-t/2+1/4 \sin(2t)}.$$

Finally, in [13] we can find some interesting examples on practical problems arising in controllability theory for linear differential systems written in the form:

$$x'(t) = A(t)x + Bu, \quad u \in U,$$

where  $A$  is an almost periodic operator and  $B$  a linear operator acting on a control set  $U$ . (Here  $P$  is the closure of the set  $\{A(t) : t \in \mathbb{R}\}$ ).

## 7. Proofs

### 7.1. Proof of Theorem 2: Barbashin's Theorem for set-valued cocycle maps

First, we prove a Barbashin Theorem which will be needed below. We follow the ideas in [14] to construct a trajectory. We proceed in several steps.

#### Attainability functions

Because of the cocycle property, for general points  $x_0, x_1$ , times  $0 \leq t \leq t_1$ , and  $p \in P$ , we have

$$\Phi(t_1, p, x_0) = \Phi(t, \theta_{t_1-t} p, \Phi(t_1 - t, p, x_0)).$$

Then,  $x_1 \in \Phi(t_1, p, x_0)$  if and only if

$$\exists x \in \Phi(t_1 - t, p, x_0) \quad \text{such that} \quad x_1 \in \Phi(t, \theta_{t_1-t} p, x).$$

This suggests the following definition of a restricted backwards attainability function:

$$x \in G(x_0, p; x_1, t_1; t) \iff x_1 \in \Phi(t, \theta_{t_1-t} p, x). \tag{7}$$

Then, one has the following result:

**Lemma 23.** *If  $x_1 \in \Phi(t_1, p, x_0)$ , then  $G(x_0, p; x_1, t_1; t)$  is nonempty and closed, and  $G(x_0, p; x_1, t_1; 0) = \{x_1\}$ .*

The mapping  $t \in [0, t_1] \rightarrow A(t) \in \mathcal{P}(\mathbb{R}^N)$  defined by

$$A(t) = \Phi(t_1 - t, p, x_0) \cap G(x_0, p; x_1, t_1; t) \tag{8}$$

has nonempty compact images and is continuous in  $t$  with respect to the Hausdorff metric.

*Proof.* Owing to the cocycle property, it is obvious that  $G(x_0, p; x_1, t_1; t)$  is nonempty. To prove that it is closed, we proceed as in [21]. Suppose  $y_i \in G(x_0, p; x_1, t_1; t)$  with  $y_i \rightarrow y$  as  $i \rightarrow \infty$ , then  $x_1 \in \Phi(t, \theta_{t_1-t}p, y)$ . Indeed,

$$\begin{aligned} & d(x_1, \Phi(t, \theta_{t_1-t}p, y)) \\ & \leq d(x_1, \Phi(t, \theta_{t_1-t}p, y_i)) + H^*(\Phi(t, \theta_{t_1-t}p, y_i), \Phi(t, \theta_{t_1-t}p, y)). \end{aligned}$$

The first term on the right hand side is zero, and the second one converges to zero due to the u.s.c. of  $\Phi$ , so  $d(x_1, \Phi(t, \theta_{t_1-t}p, y)) = 0$  and  $x_1 \in \Phi(t, \theta_{t_1-t}p, y)$ .

The statement of  $G(x_0, p; x_1, t_1; 0) = \{x_1\}$  is trivial.

It is clear that  $A(t)$  is nonempty (see the construction of  $G$  and (7)), and also compact since it is the intersection of a compact set and a closed one. We now need to prove the continuity of  $t \mapsto A(t)$ , i.e. we have to check that  $H(A(s), A(s_0)) \rightarrow 0$  as  $s \rightarrow s_0$ .

We start with the case  $H^*(A(s), A(s_0)) \rightarrow 0$ . Suppose it is not so, i.e. there exist a constant  $\varepsilon > 0$  and a sequence  $s_i \rightarrow s_0$ , such that  $H^*(A(s_i), A(s_0)) \geq \varepsilon$ . As  $A(s_i)$  is compact, the maximum is achieved at some point, say  $z_i$ :

$$H^*(A(s_i), A(s_0)) = d(z_i, A(s_0)) \geq \varepsilon.$$

Since  $z_i \in A(s_i) \subset \Phi(t_1 - s_i, p, x_0)$ , they belong to a compact set (by the continuity of  $\Phi$  and its compact values), so there exists a convergent subsequence  $z_{i'} \rightarrow z_0 \in \Phi(t_1 - s_0, p, x_0)$ . On the other hand, as  $z_i \in G(x_0, p; x_1, t_1; s_i)$ , i.e.  $x_1 \in \Phi(s_i, \theta_{t_1-s_i}p, z_i)$ , because of the u.s.c. of  $\Phi$  on its three variables (see Remark 1) and as  $(s_i, \theta_{t_1-s_i}p, z_i) \rightarrow (s_0, \theta_{t_1-s_0}p, z_0)$ , we have that  $x_1 \in \Phi(s_0, \theta_{t_1-s_0}p, z_0)$ , i.e.  $z_0 \in G(x_0, p; x_1, t_1; s_0)$  and therefore  $z_0 \in A(s_0)$ , which leads to a contradiction, since then  $H^*(A(s_i), A(s_0)) = d(z_i, A(s_0)) \leq d(z_i, z_0) \rightarrow 0$ .

We now prove that  $H^*(A(s_0), A(s)) \rightarrow 0$  as  $s \rightarrow s_0$ . Arguing by contradiction, assume there exist a constant  $\varepsilon > 0$  and a sequence  $\{s_i\}_{i \geq 1}$  with  $s_i \rightarrow s_0$  as  $i \rightarrow \infty$ , such that  $H^*(A(s_0), A(s_i)) \geq \varepsilon$ . Consider  $z_i^0 \in A(s_0)$  such that  $d(z_i^0, A(s_i)) = H^*(A(s_0), A(s_i))$ . This is possible since  $A(s_0)$  is compact, and indeed, w.l.o.g. we can assume that  $z_i^0 \rightarrow z_0 \in A(s_0)$ . To finish, now it is enough to prove that there exists  $z_i \in A(s_i)$  such that  $z_i \rightarrow z_0$ . We split into two cases (the general case is a combination of these):

$s_i \leq s_0$  Consider

$$z_0 \in A(s_0) = \Phi(t_1 - s_0, p, x_0) \cap G(x_0, p; x_1, t_1; s_0),$$

thus  $x_1 \in \Phi(s_0, \theta_{t_1-s_0}p, z_0)$ . On the one hand,

$$\Phi(s_0 - s_i, \theta_{t_1-s_0}p, z_0) \cap G(z_0, \theta_{t_1-s_0}p; x_1, s_0; s_i) \neq \emptyset$$

by the same reasons as  $A(t)$ . On the other hand, one can easily see that

$$G(x_0, p; x_1, t_1; s_i) = G(z_0, \theta_{t_1-s_0} p; x_1, s_0; s_i),$$

and

$$\Phi(s_0 - s_i, \theta_{t_1-s_0} p, z_0) \subset \Phi(t_1 - s_i, p, x_0).$$

Therefore,

$$\emptyset \neq \Phi(s_0 - s_i, \theta_{t_1-s_0} p, z_0) \cap G(z_0, \theta_{t_1-s_0} p; x_1, s_0; s_i) \subset A(s_i).$$

Pick

$$z_i \in \Phi(s_0 - s_i, \theta_{t_1-s_0} p, z_0) \cap G(z_0, \theta_{t_1-s_0} p; x_1, s_0; s_i).$$

By the continuity of  $\Phi$  we have that  $z_i \rightarrow z_0$ .

$s_i \geq s_0$  The same argument shows that

$$\emptyset \neq \Phi(t_1 - s_i, p, x_0) \cap G(x_0, p; z_0, t_1 - s_0; s_i - s_0),$$

and we also have the following inclusion

$$G(x_0, p; z_0, t_1 - s_0; s_i - s_0) \subset G(x_0, p; x_1, t_1; s_i)$$

by means of the cocycle property of  $\Phi$ . Thus,

$$\begin{aligned} \emptyset \neq \Phi(t_1 - s_i, p, x_0) \cap G(x_0, p; z_0, t_1 - s_0; s_i - s_0) \\ \subset \Phi(t_1 - s_i, p, x_0) \cap G(x_0, p; x_1, t_1; s_i) = A(s_i). \end{aligned}$$

By choosing

$$z_i \in \Phi(t_1 - s_i, p, x_0) \cap G(x_0, p; z_0, t_1 - s_0; s_i - s_0),$$

we have that  $z_i \in \Phi([0, t_1 - s_0], p, x_0)$  for all  $i$ , which is compact. Therefore, there exists a converging subsequence  $z_{i'} \rightarrow \xi$ . On the other hand, as long as  $z_0 \in \Phi(s_i - s_0, \theta_{t_1-s_0} p, z_i)$ , particularizing to the case of the subsequence  $z_{i'}$  commented above, the u.s.c. of  $\Phi$  implies  $z_0 \in \Phi(0, \theta_{t_1-s_0} p, \xi) = \{\xi\}$ , so  $z_0 = \xi$  (observe that therefore the whole sequence converges), and the proof is complete. □

**Corollary 24.** *Every  $\Phi$ -trajectory is continuous.*

*Proof.* Let  $\phi_p : [0, T] \rightarrow \mathbb{R}^d$  be a  $p$ -trajectory, i.e.  $\phi_p(t) \in \Phi(t - s, \theta_s p, \phi_p(s))$  for all  $0 \leq s \leq t \leq T$ . We consider  $t_a$  fixed, and  $t \rightarrow t_a$ . Separate the proof in two cases:

$t > t_a$  Then,

$$\phi(t) \in \Phi(t - t_a, \theta_{t_a} p, \phi_p(t_a)) \rightarrow \Phi(0, \theta_{t_a} p, \phi_p(t_a)) = \{\phi_p(t_a)\}.$$

$t < t_a$  By easy computations, we see that

$$\phi(t) \in \Phi(t - t_0, \theta_{t_0} p, \phi_p(t_0)) \cap G(\phi(t_0), \theta_{t_0} p, \phi(t_a), t_a - t_0, t_a - t).$$

Observe that this set has the form described in the definition (8) for  $A(\tilde{t})$ , but with  $\tilde{x}_0 = \phi_p(t_0)$ ,  $\tilde{p} = \theta_{t_0} p$ ,  $\tilde{x}_1 = \phi_p(t_a)$ ,  $\tilde{t}_1 = t_a - t_0$ ,  $\tilde{t} = t_a - t$ . Thus, now we have that  $A(\tilde{t}) \rightarrow A(0)$  as  $\tilde{t} = t_a - t \rightarrow 0$ , but  $A(0) = \{\phi_p(t_a)\}$ , which gives us the desired result. □

*Proof of Theorem 2.* We prove Theorem 2 following the ideas in [14, 21].

In order to do that, firstly we show how to construct trajectories.

Consider  $a \leq b$  and  $(\theta_a p, x) \in P \times \mathbb{R}^d$ . We will construct a trajectory  $\phi_p$  with  $\phi_p(a) = x$  passing through any other (given) point at time  $b$ , say  $\phi_p(b)$ , which necessarily belongs to  $\Phi(b - a, \theta_a p, \phi_p(a))$ . A first step is to choose the mid time point  $(a + b)/2$  of the interval  $(a, b)$  and an image for it. By the previous considerations, this image must be in

$$\Phi((b - a)/2, \theta_a p, \phi_p(a)) \cap G(\phi_a(a), \theta_a p, \phi_p(b), b - a, (b - a)/2),$$

which is a nonempty compact set. We iterate this process in the intervals  $[a, (a + b)/2]$  and  $[(a + b)/2, b]$ . Repeating this process, we can obtain a dyadic sequence of points which satisfies the relation for being a trajectory. The problem of completing from this set to the whole interval is solved by density: let  $t$  be a non dyadic point of  $[a, b]$ , and  $t'$  and  $t''$  any dyadic points such that  $t' < t < t''$ , then we choose

$$\begin{aligned} \phi_p(t) &\in K(t) \\ &= \bigcap_{t' < t < t''} \Phi(t - t', \theta_{t'} p, \phi_p(t')) \cap G(\phi_p(t'), \theta_{t'} p, \phi_p(t''), t'' - t', t'' - t). \end{aligned}$$

Now, we need to check that each of these sets is nonempty (which can be seen again as in (8)), and that this construction is consistent, i.e. the finite intersection property holds. The arguments are the same as in [14].

Indeed, for any  $s_1 < s_2 < t < s_3 < s_4$ , we have that the following sets are well defined and the inclusions hold:

$$\Phi(\phi_p(s_2), \theta_{s_2} p, t - s_2) \subset \Phi(\phi_p(s_1), \theta_{s_1} p, t - s_1)$$

and

$$\begin{aligned} G(\phi_p(s_1), \theta_{s_1} p; \phi_p(s_3), s_3 - s_1; s_3 - t) &= G(\phi_p(s_2), \theta_{s_2} p; \phi_p(s_3), s_3 - s_2; s_3 - t) \\ &\subset G(\phi_p(s_1), \theta_{s_1} p; \phi_p(s_4), s_4 - s_1; s_4 - t) \\ &= G(\phi_p(s_2), \theta_{s_2} p; \phi_p(s_4), s_4 - s_2; s_4 - t). \end{aligned}$$

Thus, we have proved the existence of at least one trajectory. We recall that every trajectory has been proved to be continuous.

Now we prove that  $\mathcal{T}_{p,x}([0, T])$  is compact in  $C([0, T]; \mathbb{R}^d)$ . Let be  $\{\phi_n\} \subset \mathcal{T}_{p,x}([0, T])$ . As  $\Phi(T, p, x)$  is compact, there exists a subsequence  $\{\phi_{n_1}(T)\} \subset \{\phi_n(T)\}$  converging to a point denoted  $\phi(T)$ . By the same reason, there exists another subsequence  $\{\phi_{n_2}(T/2)\} \subset \{\phi_{n_1}(T/2)\}$  converging to a point  $\phi(T/2)$  and we iterate this procedure. By a diagonal argument, we obtain a subsequence relabeled again with index  $m$ , converging in all the dyadic numbers of  $[0, T]$ :  $\phi_m(pT/2^q) \rightarrow \phi(pT/2^q)$ .

As  $\{\phi_n\}$  are trajectories, then  $\phi_n(t) \in \Phi(t - s, \theta_s p, \phi_n(s))$  for all  $0 \leq s \leq t \leq T$ , in particular for dyadic numbers, whence  $\phi(t) \in \Phi(t - s, \theta_s p, \phi(s))$  by the u.s.c. of  $\Phi$ . To extend it to the whole interval to obtain a trajectory, we proceed as before.

Let us prove that  $\phi_n \rightarrow \phi$  in  $C([0, T]; \mathbb{R}^d)$ . The pointwise convergence follows easily. Indeed, for any  $t$ , let us write

$$\phi_n(t) - \phi(t) = \phi_n(t) - \phi_n(t_D) + \phi_n(t_D) - \phi(t_D) + \phi(t_D) - \phi(t),$$

with  $t_D$  a dyadic number close enough to  $t$  such that  $|\phi(t_D) - \phi(t)| \leq \varepsilon/3$ , and with  $H(\Phi(t, p, x), \Phi(t_D, p, x)) \leq \varepsilon/3$ . Then we can choose  $n(t_D)$  such that for all  $n \geq n(t_D)$ , one has  $|\phi_n(t_D) - \phi(t_D)| \leq \varepsilon/3$ . However, for the uniform convergence one needs to be more careful. We follow the proof in [21, Th. 6.2]. By a contradiction argument, if it does not hold, there exist a constant  $\varepsilon > 0$ , sequences  $t_n$ , with  $t_n \rightarrow t \in [0, T]$ , and  $\phi_n$  such that

$$|\phi_n(t_n) - \phi(t)| > \varepsilon. \tag{9}$$

Consider a dyadic number  $\tau \in \begin{cases} (t, T] & \text{if } t < T \\ \{T\} & \text{if } t = T. \end{cases}$

As  $\{\phi_n(t_n)\} \subset \Phi([0, T], p, x)$ , which is compact, there exists a convergent subsequence (we do not relabel it)  $\phi_n(t_n) \rightarrow z$ . Then, there exists  $n_\tau \in \mathbb{N}$  such that for all  $n \geq n_\tau$ , we have that  $t_n < \tau$ . Since  $\phi_n(\tau) \in \Phi(\tau - t_n, \theta_{t_n} p, \phi_n(t_n))$  and  $\tau$  is dyadic, we have  $\phi_n(\tau) \rightarrow \phi(\tau)$  and so, the global u.s.c. of  $\Phi$  implies that

$$\phi(\tau) \in \Phi(\tau - t, \theta_t p, z).$$

Using now the continuity of  $\phi$  and the density of dyadic numbers we have:

$$\phi(t) = \lim_{\tau \rightarrow t} \phi(\tau) \in \limsup_{\tau \rightarrow t} \Phi(\tau - t, \theta_t p, z) = \Phi(0, \theta_t p, z) = \{z\},$$

which contradicts (9).

Finally, we prove the upper semicontinuity result claimed in (4):

$$H^*(\mathcal{T}_{p_n, x_n}([0, T]), \mathcal{T}_{p, x}([0, T])) \rightarrow 0 \quad \text{if } (p_n, x_n) \rightarrow (p, x).$$

According to the last section, it is equivalent to prove  $\varepsilon$ -u.s.c. We proceed again by a contradiction argument. Suppose there exist a positive constant  $\varepsilon > 0$ , a sequence of pairs  $(p_n, x_n)$  converging to  $(p, x)$  in  $P \times \mathbb{R}^d$ , and trajectories  $\phi_n \in \mathcal{T}_{p_n, x_n}([0, T])$  such that  $\phi_n \notin B_{C([0, T]; \mathbb{R}^d)}(\mathcal{T}_{p, x}([0, T]), \varepsilon)$ . We will prove that for a subsequence  $\phi_{n'}$ , it is satisfied that  $\phi_{n'} \rightarrow \phi \in \mathcal{T}_{p, x}([0, T])$ , which will give us the contradiction.

First, we have that  $\phi_n(0) = x_n \rightarrow x$ . As  $\phi_n(T) \in \Phi(T, p_n, x_n)$ , by the compact values and u.s.c. of  $\Phi$ , there exists a subsequence (which we do not relabel) converging to an element  $\phi(T)$  in  $\Phi(T, p, x)$ . The same argument can be applied to this subsequence at time  $T/2$ , and, taking a diagonal subsequence, for a countable set of numbers (the dyadic in  $[0, T]$ ), defining a set of points  $\phi(kT/2^m)$ . Of course, they satisfy the trajectory property, as far as  $\phi_n$  are:

$$\phi_n(t) \in \Phi(t - s, \theta_s p_n, \phi_n(s)),$$

for dyadic  $0 \leq s \leq t \leq T$ . By the u.s.c. of  $\Phi$  we have

$$\phi(t) \in \Phi(t - s, \theta_s p, \phi(s)).$$

The extension to the whole interval is done as above, preserving the trajectory property. To finish, the uniform convergence of  $\phi_n$  to  $\phi$  is deduced as in the previous case.



7.2. Autonomous strong attractors of SVSDS. Proof of Proposition 7

Clearly,  $\mathcal{A}_s$  is nonempty since it is the intersection of a nested family of compact sets. Moreover, it is compact too.

Since any bounded set is absorbed by  $\mathcal{B}_s$ , it is enough to see that  $\mathcal{B}_s$  is attracted by  $\mathcal{A}_s$ . If not, there exist  $\varepsilon > 0$  and a sequence  $x_n \in \Pi(t_n, \mathcal{B}_s)$ , with  $t_n \rightarrow \infty$ , such that  $\text{dist}(x_n, \mathcal{A}_s) \geq \varepsilon > 0$ . But  $x_n \in \mathcal{B}_s$  for all  $n \geq n(\mathcal{B}_s)$  by the absorbing property of  $\mathcal{B}_s$ , and as it is compact, there is a subsequence converging (we do not relabel) to  $x \in \mathcal{A}_s$ , which is a contradiction.

We check  $\Pi$ -invariance in two steps: First, we prove that  $\Pi(t, \mathcal{A}_s) \subset \mathcal{A}_s$  as in the single-valued case. Indeed,

$$\begin{aligned} \Pi(t, \mathcal{A}_s) &= \Pi\left(t, \bigcap_{r \geq 0} \Pi(r, \mathcal{B}_s)\right) \subset \bigcap_{r \geq 0} \Pi(t, \Pi(r, \mathcal{B}_s)) \\ &= \bigcap_{r \geq 0} \Pi(t+r, \mathcal{B}_s) = \bigcap_{r \geq t} \Pi(r, \mathcal{B}_s) = \bigcap_{r \geq 0} \Pi(r, \mathcal{B}_s) = \mathcal{A}_s \end{aligned}$$

where we have used the semigroup property of  $\Pi$  and the positive invariance of  $\mathcal{B}_s$ .

For the converse,  $\mathcal{A}_s \subset \Pi(t, \mathcal{A}_s)$ , pick  $a \in \mathcal{A}_s$ . Then, there exist sequences  $\tau_n \rightarrow \infty$ , and  $a_n \in \Pi(\tau_n, \mathcal{B}_s)$  with  $a_n \rightarrow a$ . Consider  $t > 0$  and  $n(t)$  such that for all  $n \geq n(t)$ ,  $\tau_n - t \geq 0$ . Then, for all  $n \geq n(t)$ :

$$a_n \in \Pi(\tau_n, \mathcal{B}_s) = \Pi(t, \Pi(\tau_n - t, \mathcal{B}_s))$$

and thus there exists a sequence  $a'_n \in \Pi(\tau_n - t, \mathcal{B}_s)$  with  $a_n \in \Pi(t, a'_n)$ . Since  $\mathcal{B}_s$  is compact and positively invariant, we deduce from

$$a'_n \in \Pi(\tau_n - t, \mathcal{B}_s) \subset \mathcal{B}_s$$

the existence of a convergent subsequence  $a'_{n_j} \rightarrow a' \in \mathcal{B}_s$  if  $j \rightarrow \infty$ . Of course,  $\tau_{n_j} - t \rightarrow \infty$  and therefore,  $a' \in \mathcal{A}_s$ . By the upper semicontinuity we have that

$$H^*(\Pi(t, a'_{n_j}), \Pi(t, a')) \rightarrow 0 \quad j \rightarrow \infty,$$

and from  $a_{n_j} \rightarrow a$  and  $a_{n_j} \in \Pi(t, a'_{n_j})$  we obtain  $a \in \Pi(t, a') \subset \Pi(t, \mathcal{A}_s)$  as desired.

Since a compact set  $K$  that is  $\Pi$ -invariant satisfies by the attraction property,  $H^*(\Pi(t, K), \mathcal{A}_s) \rightarrow 0$ , and we have that  $H^*(K, \mathcal{A}_s) = H^*(\Pi(t, K), \mathcal{A}_s)$ , then  $H^*(K, \mathcal{A}_s) = 0$  and so  $K \subset \mathcal{A}_s$ .

On the other hand, for a closed set  $B$  attracting bounded sets, we have that  $H^*(\mathcal{A}_s, B) = H^*(\Pi(t, \mathcal{A}_s), B) \rightarrow 0$  and therefore  $H^*(\mathcal{A}_s, B) = 0$  and  $\mathcal{A}_s \subset B$ .

For the last statement, we adapt the proof given in Gobbino and Sardella [12, Th. 3.1]; see also [9]. By contradiction, if  $\mathcal{A}_f$  is not connected, then  $\mathcal{A}_f = A_1 \cup A_2$  with  $A_i$  nonempty disjoint compact sets ( $i = 1, 2$ ). Take  $\varepsilon > 0$  such that

$B(A_1, \varepsilon) \cap B(A_2, \varepsilon) = \emptyset$ , and define for  $i = 1, 2$

$$Y_i = \{y \in Y : \Pi(t, y) \in B(A_i, \varepsilon) \text{ for } t \text{ large enough}\}.$$

As  $\mathcal{A}_f$  is a global attractor and taking into account the assumptions on connectedness for  $\Pi$ , these sets are well defined, and they are nonempty, disjoint, contain the respective set  $A_i$ , and complete the whole space  $Y$ . We will see they are open sets, which will finish the proof. Indeed, given  $y \in Y_i$  and a neighborhood  $B$ , it is attracted by  $\mathcal{A}_f$ , call  $T(B, \varepsilon) \geq 0$  the absorption time for  $B$  to  $B(\mathcal{A}_f, \varepsilon)$ , i.e.  $\Pi(t, B) \subset B(\mathcal{A}_f, \varepsilon)$  for all  $t \geq T(B, \varepsilon)$ . In particular, as  $\Pi(t, y) \in B(Y_i, \varepsilon)$ , a neighborhood  $U \subset B$  of  $y$  satisfies by u.s.c. that  $\Pi(t, U) \subset B(A_i, \varepsilon)$ , and so  $U \subset Y_i$ , which means  $Y_i$  is open as desired.

### 7.3. Autonomous weak attractors of SVSDS. Proof of Theorem 9

**Nonempty and compact:** consider any sequences  $\tau_n \rightarrow \infty$  and  $b_n \in \mathcal{B}_w$ . By the weak positive invariance of  $\mathcal{B}_w$ , there exist trajectories  $\pi_n$  with  $\pi_n(0) = b_n$  and  $\pi_n(t) \in \mathcal{B}_w$  for all  $t \geq 0$ . In particular,  $a_n = \pi_n(\tau_n) \in \mathcal{B}_w$ , and by the compactness of  $\mathcal{B}_w$  we can extract a subsequence  $a_{n_j}$  converging to an element  $a$  in  $\mathcal{B}_w$  as  $j \rightarrow \infty$ . Taking  $\{\tau_{n_j}, b_{n_j}, a_{n_j}\}_j$  as the original sequences, we have that  $a \in \mathcal{A}_w$  which is therefore nonempty.

To show that  $\mathcal{A}_w$  is compact, we only need to see that it is closed since is contained in the compact set  $\mathcal{B}_w$ . Suppose  $a_k \in \mathcal{A}_w$  and  $a_k \rightarrow a$  as  $k \rightarrow \infty$ . Then, there exist sequences  $\tau_{k,n} \rightarrow \infty$  as  $n \rightarrow \infty$  and trajectories  $\pi_{k,n}$  with  $\pi_{k,n}(t) \in \mathcal{B}_w$  for all  $t \geq 0$  and  $\pi_{k,n}(\tau_{k,n}) \rightarrow a_k$  as  $n \rightarrow \infty$ . Pick  $n_k$  so that

$$|\pi_{k,n_k}(\tau_{k,n_k}) - a_k| \leq 1/k \quad \text{and} \quad t_{k+1,n_{k+1}} \geq t_{k,n_k} + 1 \quad \forall k \in \mathbb{Z}^+.$$

Then

$$\begin{aligned} |\pi_{k,n_k}(\tau_{k,n_k}) - a| &\leq |\pi_{k,n_k}(\tau_{k,n_k}) - a_k| + |a_k - a| \\ &\leq 1/k + |a_k - a| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Taking  $\{\pi_{k,n_k}, \tau_{k,n_k}\}_k$  as the original sequences, we have again that  $a \in \mathcal{A}_w$ , which is closed as desired, and hence compact.

**Weak positive invariance:** Take  $a \in \mathcal{A}_w$ , then there exist a sequence  $\tau_n \rightarrow \infty$  and trajectories  $\pi_n$  with  $\pi_n(t) \in \mathcal{B}_w$  for all  $t \geq 0$ , such that  $\pi_n(\tau_n) \rightarrow a$  as  $n \rightarrow \infty$ . If we denote  $v_n(t) := \pi_n(\tau_n + t)$ , it is obvious that  $v_n$  is a trajectory and  $v_n(t) \in \mathcal{B}_w$  for all  $t \geq 0$  and  $v_n(0) \rightarrow a \in \mathcal{A}_w$ . Applying Barbashin's Theorem 2 on an interval, say  $[0, T]$ , we obtain a convergent subsequence  $v_{n_j}(t) \rightarrow v(t)$  uniformly for  $t \in [0, T]$ . Naturally,  $v : [0, T] \rightarrow \mathcal{B}_w$  is a trajectory and  $v(0) = a$ . Moreover,  $v(t) \in \mathcal{A}_w$  since  $\pi_{n_j}(\tau_{n_j}) \in \mathcal{B}_w$ ,  $\pi_{n_j}(\tau_{n_j} + t) \rightarrow v(t)$  and  $\tau_{n_j} + t \rightarrow \infty$ . A diagonal argument shows again that we can obtain a trajectory defined on all of  $\mathbb{R}^+$ .

**Weak negative invariance:** The same idea can be used backwards in time. For any  $T > 0$ , consider  $n_T$  such that  $\tau_n - T \geq 0$  for all  $n \geq n_T$ , and write

$$v_n : [-T, 0] \rightarrow \mathcal{B}_w : s \mapsto v_n(s) := \pi_n(\tau_n + s).$$

Barbashin’s theorem can be applied successively on intervals  $[-T, 0]$ ,  $[-2T, -T]$ ,  $\dots$  and by a diagonal argument we obtain the existence of a trajectory  $\bar{v} : \mathbb{R}^- \rightarrow \mathcal{B}_w$ , which indeed takes values in  $\mathcal{A}_w$  as before, with  $\bar{v}(0) = a$ . The concatenation of  $v$  and  $\bar{v}$  gives us the invariance of  $\mathcal{A}_w$  as desired.

**Weak attraction:** Let  $D$  be a bounded subset of  $\mathbb{R}^d$ . Since  $\mathcal{B}_w$  is weakly absorbing, there exists a time  $T_D > 0$  such that for each  $d_n \in D$  there exists a trajectory  $\pi_n$  with  $\pi_n(0) = d_n$  and  $\pi_n(t) \in \mathcal{B}_w$  for all  $t \geq T_D$ .

By the weak positive invariance of  $\mathcal{B}_w$ , we can consider trajectories  $\tilde{\pi}_n : \mathbb{R}^+ \rightarrow \mathcal{B}_w$  with  $\tilde{\pi}_n(0) = \pi_n(T_D)$ . Since  $\mathcal{B}_w$  is compact, for any sequence  $\tau_{n,k} \rightarrow \infty$  as  $k \rightarrow \infty$ , there exist subsequences  $\tilde{\pi}_n(\tau_{n,k'}) \rightarrow a_n$  as  $k' \rightarrow \infty$  for some  $a_n \in \mathcal{B}_w$  (for each  $n$ ). By definition of  $\mathcal{A}$ , we have that  $a_n \in \mathcal{A}_w$ . Define

$$\pi_n^*(t) := \begin{cases} \pi_n(t) & 0 \leq t \leq T_D, \\ \tilde{\pi}_n(t - T_D) & t \geq T_D. \end{cases}$$

Then,  $\pi_n^*(0) = d_n$  and  $\pi_n^*(\tau_{n,k'} + T_D) \rightarrow a_n$  as  $k' \rightarrow \infty$ . Pick  $k'_n$  such that  $\tau_{n,k'_n} < \tau_{n+1,k'_{n+1}}$  and  $\text{dist}(\pi_n^*(\tau_{n,k'_n} + T_D), \mathcal{A}_w) \leq 1/n$ . Therefore we have obtained for the trajectories  $\pi_n^*$  which start at  $d_n$  that

$$\text{dist}(\pi_n^*(\tau_{n,k'_n} + T_D), \mathcal{A}_w) \rightarrow 0.$$

Thus, we have weak attraction.

The maximality statement w.r.t.  $\mathcal{B}_w$  comes from its definition.

#### 7.4. Attractors for SVSPF and their restrictions

7.4.1. *Proof of Proposition 11.* The first assertion is obvious.

For the second one, as  $P^*$  is the projection onto  $P$  of the attractor  $\mathcal{A}_s$ , for every  $p^* \in P^*$ ,  $A_s(p^*)$  is nonempty. Compactness follows from that of  $\mathcal{A}_s$  and the continuity of the projection of  $P^* \times \mathbb{R}^d$  onto  $\mathbb{R}^d$ . The  $\Phi$ -invariance of  $A_s(p)$  follows trivially from the  $\Pi$ -invariance of  $\mathcal{A}_s$ .

We now prove the third claim. Since  $A_s(p)$  is compact, it is equivalent to prove  $\varepsilon$ -u.s.c. (cf. [1]). Suppose not, then there exists a constant  $\varepsilon > 0$  and  $p_n \rightarrow p$  (elements of  $P^*$ ) such that  $A_s(p_n) \not\subset B(A_s(p), \varepsilon)$ , i.e. there exists a sequence  $x_n \in A_s(p_n)$  with  $x_n \notin B(A_s(p), \varepsilon)$ . By the sectorial definition of  $A_s(p)$ , for each  $(p_n, x_n)$ , there exist sequences  $t_m^n \rightarrow \infty$  as  $m \rightarrow \infty$  and  $y_m^n \in \Phi(t_m^n, p_m^n, b_m^n)$ , with  $(p_m^n, b_m^n) \subset \mathcal{B}_s$ , such that  $(\theta_{t_m^n}^n p_m^n, y_m^n) \rightarrow (p_n, x_n)$  as  $m \rightarrow \infty$ . Pick  $m(n)$  strictly increasing such that  $t_{m(n)}^n$  is also strictly increasing. From  $(\theta_{t_{m(n)}^n}^n p_{m(n)}^n, y_{m(n)}^n)$  we can extract a subsequence converging to a pair  $(p, y) \in \{p\} \times A_s(p)$ , as they belong to  $\mathcal{B}_s$  asymptotically. But this means that a subsequence of  $x_n$  approximates  $y \in A_s(p)$ , which is a contradiction.

7.4.2. *Proof of Proposition 12.* Obviously  $\mathcal{A}_s^* \subset \mathcal{A}_s$ , since  $\mathcal{A}_s^*$  attracts a smaller class of sets, in fact, just those from  $P^* \times \mathbb{R}^d$  rather than  $P \times \mathbb{R}^d$ .

Now  $\Pi^*(t, \mathcal{A}_s) \equiv \Pi(t, \mathcal{A}_s) \equiv \mathcal{A}_s, \forall t \geq 0$  since  $\mathcal{A}_s \subset P^* \times \mathbb{R}^d$  and  $\Pi^* \equiv \Pi$  on  $P^* \times \mathbb{R}^d$ . Since  $\mathcal{A}_s^*$  attracts nonempty bounded subsets of  $P^* \times \mathbb{R}^d$  including  $\mathcal{A}_s$ , one has that

$$H^*(\mathcal{A}_s, \mathcal{A}_s^*) = H^*(\Pi^*(t, \mathcal{A}_s), \mathcal{A}_s^*) \rightarrow 0, \text{ as } t \rightarrow \infty,$$

i.e.  $H^*(\mathcal{A}_s, \mathcal{A}_s^*) \equiv 0$ , thus  $\mathcal{A}_s \subset \mathcal{A}_s^*$  and therefore  $\mathcal{A}_s \equiv \mathcal{A}_s^*$ .

Alternatively, one can also argue in the following way:  $\mathcal{A}_s$  is a  $\Pi^*$ -invariant set and the global attractor  $\mathcal{A}_s^*$  of  $\Pi^*$  in  $P^* \times \mathbb{R}^d$  is the maximal compact  $\Pi^*$ -invariant subset of  $P^* \times \mathbb{R}^d$ , so  $\mathcal{A}_s \subset \mathcal{A}_s^*$ .

7.4.3. *Proof of Proposition 13.* The statement about  $P^*$  is clear even in this weak framework since  $\theta$  is single-valued and the weak  $\Pi$ -invariance easily implies  $\theta$ -invariance for  $P^*$ , and the weak attraction property of  $\mathcal{A}_w$  implies the strong attraction property for  $\theta$  in  $P^*$ .

That  $A_w(p)$  is nonempty is trivial, as mentioned before, since  $P^*$  is the projection of  $\mathcal{A}_w$  onto  $P$ . It is also compact, the proof is the same in Theorem 9, since for all  $a_k \in P \times \mathbb{R}^d$  we have that  $\Pr_P(a_k) = p$  and so the limit  $a$ .

Let us prove that  $P^* \ni p \mapsto A_w(p)$  is u.s.c. We want to see that if  $p' \rightarrow p$ , then  $H^*(A_w(p'), A_w(p)) \rightarrow 0$ . If not, there exist a constant  $\varepsilon > 0$  and a sequence  $p_n \rightarrow p$  with  $\varepsilon \leq H^*(A_w(p_n), A_w(p)) = \text{dist}(a_n, A_w(p))$ , where we have used that  $A_w(p_n)$  is compact. Therefore,

$$\varepsilon \leq \text{dist}(a_n, a) \quad \forall a \in A_w(p). \tag{10}$$

As  $A_w(p_n) \subset \Pr_{\mathbb{R}^d} \mathcal{A}_w$ , which is compact, from  $\{a_n\}$  we extract a convergent subsequence (which we do not relabel), and so  $(p_n, a_n) \rightarrow (p, a)$ . From the weak invariance of  $\mathcal{A}_w$ , there exists at least one entire trajectory  $\pi_n$  passing through each  $(p_n, a_n)$ . Barbashin's theorem (Th. 2, see Remark 5) provides a converging subsequence  $\{\pi_{n_1}\}_{n_1}$  on the interval  $[-1, 1]$  to a trajectory. Applying it again to this subsequence we obtain another one denoted by  $\{\pi_{n_2}\}_{n_2}$ , which is uniformly converging on  $[-2, 2]$ . A diagonal argument gives an entire trajectory  $\pi$  such that  $\pi(0) = (p, a)$ . By Remark 10, we have that  $(p, a) \in \mathcal{A}_w$ , so  $a \in A_w(p)$ , which contradicts (10).

The last statement is obvious.

### 7.5. Pullback structure of SVSPF attractors

7.5.1. *Proof of Lemma 15.* Thanks to the  $\Phi$ -positive invariance of  $B_s(p)$  for  $p \in P^*$ , and the cocycle property, we have that

$$\begin{aligned} \Phi(t+r, \theta_{-t-r} p, B_s(\theta_{-t-r} p)) &= \Phi(t, \theta_{-t} p, \Phi(r, \theta_{-t-r} p, B_s(\theta_{-t-r} p))) \\ &\subset \Phi(t, \theta_{-t} p, B_s(\theta_{-t} p)). \end{aligned}$$

Thus, these sets are nested, and by the u.s.c. of  $\Phi$  on its third variable and the compactness of  $B_s(p)$  for  $p \in P^*$ , they are compact. Thus,  $\hat{A}_s(p)$  is nonempty and compact.

7.5.2. *Proof of Proposition 17.* We only need to check the first identity since the second one has already been proved.

First, let us prove that  $\hat{A}_s(p) \subset A_s^*(p)$ :

$$\begin{aligned} \{p\} \times \hat{A}_s(p) &= \{p\} \times \bigcap_{t \geq 0} \Phi(t, \theta_{-t}p, B_s(\theta_{-t}p)) \\ &= \bigcap_{t \geq 0} \{p\} \times \Phi(t, \theta_{-t}p, B_s(\theta_{-t}p)) \\ &= \bigcap_{t \geq 0} \{\theta_t(\theta_{-t}p)\} \times \Phi(t, \theta_{-t}p, B_s(\theta_{-t}p)) \\ &= \bigcap_{t \geq 0} \Pi(t, (\theta_{-t}p, B_s(\theta_{-t}p))) \\ &\subset \bigcap_{t \geq 0} \Pi(t, \mathcal{B}_s^*) \equiv \mathcal{A}_s^*. \end{aligned}$$

Here we have used that  $P^*$  is  $\theta$ -invariant. Thus  $\{p\} \times \hat{A}_s(p) \subset \mathcal{A}_s^*$  what implies that  $\{p\} \times \hat{A}_s(p) \subset \{p\} \times A_s^*(p)$  and therefore  $\hat{A}_s(p) \subset A_s^*(p)$  as desired.

As for the converse, notice that  $\Pi^*(t, \mathcal{A}_s^*) = \mathcal{A}_s^*$  for all  $t \geq 0$ . So, in particular,

$$\Phi(t, p, A_s^*(p)) = A_s^*(\theta_t p) \quad \text{for all } t \geq 0, p \in P^*.$$

Moreover, we know that

$$\hat{A}_s(p) \subset A_s^*(p) \subset B_s(p).$$

Thus, setting  $\theta_{-t}p$  instead of  $p$ ,

$$A_s^*(p) = \Phi(t, p, A_s^*(\theta_{-t}p)) \subset \Phi(t, \theta_{-t}p, B_s(\theta_{-t}p))$$

for all  $t \geq 0$  and  $p \in P^*$ . Therefore, we finally obtain that

$$A_s^*(p) \subset \bigcap_{t \geq 0} \Phi(t, \theta_{-t}p, B_s(\theta_{-t}p)) = \hat{A}_s(p).$$

7.5.3. *Proof of Proposition 18.* Suppose not, then there exist a positive constant  $\varepsilon$ , a bounded set  $D$  and sequences  $t_n \rightarrow \infty$ ,  $y_n \in \Phi(t_n, \theta_{-t_n}p, d_n)$  with  $p \in P^*$  and  $d_n \in D$  such that

$$\text{dist}(y_n, \hat{A}_s(p)) > \varepsilon > 0.$$

There exists  $T_D(p)$  such that  $\Phi(t, \theta_{-t}p, D) \subset B_s(p)$  for all  $t \geq T_D(p)$ . So, as  $B_s(p)$  is compact, there exists a converging subsequence (denoted the same)  $y_n \rightarrow x \in B_s(p)$ . We will see that in fact  $x \in \hat{A}_s(p)$ , which will be a contradiction.

Consider any  $\tau > 0$  and take  $n(\tau)$  big enough such that  $t_n - \tau > 0$  and  $\Phi(t_n - \tau, \theta_{-t_n}p, d_n) \subset B_s(\theta_{-\tau}p)$  which is possible since  $t_n \rightarrow \infty$  and the family  $B_s(p)$  is  $\Phi$ -pullback absorbing.

Then, we have that

$$y_n \in \Phi(t_n, \theta_{-t_n} p, d_n) = \Phi(\tau, \theta_{-\tau} p, \Phi(t_n - \tau, \theta_{-t_n} p, d_n)),$$

which shows that  $x \in \hat{A}_s(p)$  as desired.

Now we show that this is the minimal pullback attractor, i.e. it coincides with the closure of the union of the omega-limit sets on each  $p$ -fiber:

$$\overline{\bigcup_{D \text{ bounded}} \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \Phi(t, \theta_{-t} p, D)}.$$

Indeed, this is a trivial consequence of the fact that for all  $p \in P^*$ ,  $B_s(p)$  is contained in a compact set  $K$ , namely,  $K = \text{Pr}_{\mathbb{R}^d} \mathcal{B}_s$ :

$$\Phi(t, \theta_{-t} p, B_s(\theta_{-t} p)) \subset \Phi(t, \theta_{-t} p, K)$$

and so

$$\hat{A}_s(p) \subset \overline{\bigcup_{D \text{ bounded}} \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \Phi(t, \theta_{-t} p, D)}.$$

The other inclusion is obvious, using the minimality of the omega-limit set for  $D$  at fiber  $p$ .

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