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A stochastic model for fronts in porous media

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Abstract. We analyze a stochastic model for the motion of fronts in two-phase fluids and derive upscaled equations for the capillary pressure. This extends results of [11], where the same law for the capillary pressure was derived under an assumption on typical explosion patterns. With the work at hand we remove that assumption and show that in the stochastic case the upscaled equations hold almost surely.

Key words. homogenization – two-phase flow – surface tension – capillary pressure – porous media

1. Introduction

Two-phase flows in porous media are a major research field in engineering sciences due to their practical importance, applied, e.g., to water and air in soil or to oil and water in rock. In these applications one is mainly interested in the macroscopic behavior of the flow, i.e., one tries to find averaged equations on a scale that is large compared to the pore size. More recently, starting in the 1980s, the development of homogenization techniques has opened the field to mathematical analysis. At least for one-phase flows one was able to derive equations on the macroscale by averaging the equations of the microscale [9].

In this work we present such a homogenization procedure for a two-phase flow in a stochastic medium, extending results of [11]. We start from microscopic equations on the pore level that use the variables of pressure, velocity, and the position of the front; the equations couple velocity and pressure with Darcy's law and impose Laplace's law for the pressure at the boundary. We perform a homogenization and find equations that average over many pores but still resolve the free boundary between the two phases.

It remains to prescribe a geometry, i.e., the distribution of material and void. Unfortunately, general stochastic geometries seem to be inaccessible to today's analytical methods. We therefore introduce equations that mimic a filter geometry of elongated pores. The principal features of flows in such a filter are contained in our model, and we refer the reader to [10] for an application to the original free boundary problem. Concerning other approaches to the justification of two-phase equations we mention [1,2,7,8].

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We assume that the fluid (phase A) occupies the domain $\Omega := (-1,1)\times(-1,0)$ with periodicity conditions on the lateral boundaries. The upper boundary $\Gamma := (-1,1)\times\{0\}$ consists of two parts; with $\gamma \in (0,1/2)$ we set

$$\Gamma_1^\varepsilon := \varepsilon\cdot(\mathbb{Z}+(-\gamma,\gamma))\cap\Gamma,\quad \Gamma_2^\varepsilon := \varepsilon\cdot(\mathbb{Z}+(\gamma,1-\gamma))\cap\Gamma.$$

The small parameter ε is a measure of the pore size in the medium; we will always have $\varepsilon = 1/N$ with $N\in\mathbb{N}$. In this model, Γ_2^ε is the part of the free boundary where the fluid is in contact with the material, while along Γ_1^ε we have an interface between fluid A and fluid B. The free boundary is modeled with a piecewise constant height function:

$$h^\varepsilon : \Gamma_1^\varepsilon \to \mathbb{R}.$$

We use the $L^2(\Gamma)$-orthogonal projection Q_ε onto functions that are piecewise constant on Γ_1^ε.

$$\partial_t h^\varepsilon(x_1,t) = -\varepsilon Q_\varepsilon \partial_2 p^\varepsilon(x_1,0,t) \qquad \forall(x_1,0)\in\Gamma_1^\varepsilon,\forall t, \tag{1.1}$$

$$p^\varepsilon(x_1,0,t) = \mathcal{P}_0\left(\frac{x_1}{\varepsilon},\frac{h^\varepsilon(x_1,t)}{\varepsilon}\right) \qquad \forall(x_1,0)\in\Gamma_1^\varepsilon,\forall t, \tag{1.2}$$

$$\partial_2 p^\varepsilon(x_1,0,t) = 0 \qquad \forall(x_1,0)\in\Gamma_2^\varepsilon,\forall t, \tag{1.3}$$

$$-\Delta_x p^\varepsilon = 0 \qquad \text{in } \Omega\times(0,T). \tag{1.4}$$

As a driving mechanism we prescribe the inflow on the lower boundary $\Gamma_0 := (-1,1)\times\{-1\}$,

$$-\partial_2 p^\varepsilon(x_1,-1,t) = V_0(x_1) \qquad \forall(x_1,-1)\in\Gamma_0,\forall t. \tag{1.5}$$

We impose $\|V_0\|_{L^\infty}\le 1$, and we will always assume $p^\varepsilon > 0$ on Γ_0. The system is closed by prescribing initial values; we assume $0\le p^\varepsilon(t=0)\le p_1$.

The interesting feature of the above system is the phenomenon of *explosions*. Assume that the height in cell k of Γ_1^ε exceeds a critical value such that now $\partial_s\mathcal{P}_0$ is negative and the pressure in cell k is lower than in the vicinity. Then a fast process sets in due to an instabilty of the system: an increase in h results in a decrease in p, the pressure gradient creates a fast flow from neighboring cells toward cell k, the height is increased further, and the process speeds up. The height h is increased until $\mathcal{P}_0(k,h)$ coincides again with the average pressure of the neighboring cells. For small ε this process is fast, hence the name "explosion".

With Theorem 4.6 of [11] we found upscaled equations for system (1.1)–(1.5) in the deterministic case under a condition on the explosion patterns. The condition was, loosely speaking, that there is no point $(x,t)\in\Gamma\times(0,T)$ such that in the sequence of solutions $O(1/\varepsilon)$ explosions happen in an arbitrarily small neighborhood of (x,t). In a stochastic setting we can show that this happens only with probability 0 and that the upscaled system will be satisfied almost surely. This is the result of Theorem 1.1.

It remains to choose a stochastic law for the functions $\mathcal{P}_0$ that model the geometry. To keep the notation as simple as possible, we use the piecewise linear laws

$$\mathcal{P}_0(k,s) = p_{max}(k)\ p_0(s), \qquad p_0(s) = s \bmod 1. \tag{1.6}$$

The maximal pressures $p_{max}(k)$ are random variables. For a fixed interval $[p_1, p_2] \subset (0,\infty)$ and any $N = 1/\varepsilon$ we assume that the numbers $\omega_N(k) = p_{max}(k)$ are independent and uniformly distributed in the interval $[p_1, p_2]$. Note that the distributions of ω_{N_1} and ω_{N_2} are independent for $N_1 \neq N_2$. This is in contrast to many stochastic homogenization procedures where a single random vector ω induces the ε-dependent laws via scaling as in [4–6, 13]. If $N = 1/\varepsilon$ is fixed, we often suppress the index of ω_N. We study the pressure law

$$p^\varepsilon(x_1, 0, t) = \omega\left(\left[\frac{x_1}{\varepsilon}\right]\right) \cdot p_0\left(\frac{h^\varepsilon(x_1,t)}{\varepsilon}\right) \qquad \text{on } \Gamma_1^\varepsilon, \tag{1.7}$$

where $[q]$ denotes the closest integer to q. With (1.7) we restricted our study to functions $\mathcal{P}_0$ that are piecewise linear in s.

Our main result is the following.

Theorem 1.1. *With probability* 1 *there holds: every weak limit p^0 of a sequence of solutions p^ε to the above system has a representative, again denoted by p^0, that satisfies the upscaled equations as follows. $p^0(.,t)$ is harmonic, periodic across lateral boundaries, and satisfies the inflow condition* (1.5). *Moreover,*

$$0 \le p^0(x,t) \le p_1 \qquad \forall x \in \Gamma, \forall t. \tag{1.8}$$

Every point $(x,t) \in \Gamma \times (0,T)$ with $p^0(x,t) < p_1$ has a neighborhood in $\Gamma \times (0,T)$ on which

$$\partial_t \Theta(p^0) = -\partial_2 p^0 \tag{1.9}$$

holds in the sense of distributions. Everywhere on $\Gamma \times (0,T)$ holds the corresponding inequality

$$\partial_t \Theta(p^0) \le -\partial_2 p^0. \tag{1.10}$$

The function Θ is calculated with an expected value by

$$\Theta'(\rho) = 2\gamma \left\langle \frac{1}{\mathcal{P}_0'} \right\rangle. \tag{1.11}$$

Remarks concerning the upscaled equations. In the case of the piecewise linear law $\mathcal{P}_0$ considered here, the function Θ is linear and (1.11) simplifies to the k-average $\Theta(\rho) = 2\gamma\rho \langle 1/p_{max}(k)\rangle$. We wrote a more general expression in the theorem, one that remains valid in the nonlinear case under additional assumptions on the sequence p^ε (see [11]).

In the lift-off condition (1.10) one actually has some freedom. For a C^1-function p^0 it is only necessary to express that for $p^0(x,0,t) = p_1$ and $-\partial_2 p^0(x,0,t)$ negative, $\partial_t p^0(x,t)$ is also negative. This way the pressure enters a regime in which equality (1.9) can be applied, and the further evolution is thus determined.

The averaged equations of the theorem have a unique local solution (see [10]).

Methods in the proof. With the above choice of $\mathscr{P}_0$, the concept of an explosion simplifies. We say that $(\varepsilon[x/\varepsilon], t)$ is an explosion point if the pressure vanishes in the corresponding cell, $p^\varepsilon(x, t) = 0$. Since the pressure is nonnegative and does not vanish identically, there holds $\partial_t h^\varepsilon > 0$ in the point of an explosion, i.e., the pressure jumps down from $p_{max}(k)$ to 0. After this instant, in the vicinity of the explosion point, the pressure gradients are of order $O(1/\varepsilon)$ and lead to a fast motion toward a newly equilibrated pressure. The principal task in the proof of Theorem 1.1 is to investigate the distribution of explosion points.

The principal steps of the proof are listed below. Later on we will provide precise statements and verify them for constants $C_i, \kappa_i > 0$.

1. Many cells can sustain at most the pressure $p_1 + C_1\varepsilon^{\kappa_1}$.
2. The pressures $p^\varepsilon|_\Gamma$ are everywhere bounded by $p_1 + C_2\varepsilon^{\kappa_2}$.
3. Explosions cannot cluster, and the averaged equations hold.

Step 1 has a direct proof that is carried out in Lemma 1.3 and Corollary 1.4. *Step 2* is the most involved and the main purpose of Section 2. Proposition 2.1 provides the global estimate with $\kappa_2 \simeq 1/3$. *Step 3* is carried out in Section 3. The upscaled equations for p^0 follow as in [11].

The rescaled equations. We can scale all independent variables and the height function with the factor N. We write $x = (x_1, x_2)$ and t for the new independent variables, H for the new height function, and keep p for the pressure. The rescaled equations read

$$\partial_t H(x_1, t) = -Q_1 \partial_2 p(x_1, 0, t) \qquad (x_1, 0) \in \bar{\Gamma}_1, \tag{1.12}$$

$$p(x_1, 0, t) = \omega_N([x_1]) \cdot p_0(H(x_1, t)) \qquad (x_1, 0) \in \bar{\Gamma}_1, \tag{1.13}$$

$$\partial_2 p(x_1, 0, t) = 0 \qquad (x_1, 0) \in \bar{\Gamma}_2, \tag{1.14}$$

$$-\Delta_x p(., t) = 0 \qquad \text{in } \bar{\Omega}. \tag{1.15}$$

Here x varies among $|x_1| < N$, $-N < x_2 < 0$. We will most often work with these rescaled equations.

The time distance between explosions. Our goal is to show the unlikelihood of having many explosions in a small region. A first result in this direction is expressed in the following remark with a straightforward proof.

Remark 1.2. Assume that for all $\varepsilon = N^{-1} \le N_0^{-1}$ the estimate $p^\varepsilon \le p_1 + C\, N^{-\kappa}$ holds. Let the rescaled system have explosions in the spatial points $A, B \in \mathbb{Z}$ at times t_A and t_B. Then, for some $c > 0$ and all $N \ge N_0 > 0$, there holds

$$t_A = t_B \quad \text{or} \quad |t_A - t_B| \ge c \log(N).$$

One of our aims must now be to exclude the possibility of many simultaneous explosions.

Large gaps between outlets of low maximal pressure. To derive a priori estimates for p^ε, we have to use the fact that there are many outlets with low pressure. This statement is made precise in the remainder of this section.

In the following discussion we use the arbitrary numbers $a \in (0, 1)$ and $\alpha > 0$. We will later use the choices $a = 1/2$ and $a = 1/3$; α will always be small. With $[q] := \max\{k \in \mathbb{Z}, k \leq q\}$ we make the following definition.

Definition of a gap: Way say that ω_N has a gap if there are $[N^{a+\alpha}]$ consecutive integers i in $\{-N+1, \dots, N\}$ with $\omega_N(i) \geq p_1 + N^{-a}$.

Lemma 1.3 (The probability of gaps). *The probabilities $\mathbb{P}_N$ that ω_N has a gap satisfy*

$$\sum_N \mathbb{P}_N < \infty.$$

Proof. With $c = 1/(p_2 - p_1)$ we calculate for the probability of a gap at a given position

$$P_N = \left(1 - cN^{-a}\right)^{[N^{a+\alpha}]} \leq \left(\left(1 - cN^{-a}\right)^{N^a}\right)^{(N^\alpha - 1)}.$$

The asymptotic behavior for $N \to \infty$ is $P_N \sim \exp(-cN^\alpha)$, an exponential decay. Therefore, the probability for a gap at any position can be estimated by $\mathbb{P}_N \leq 2N \cdot P_N \sim 2N \exp(-cN^\alpha)$ for large N. This is still summable. □

We can now conclude with the Borel–Cantelli lemma for the sequence of geometries.

Corollary 1.4. *With probability* 1 *we find $N_0 \in \mathbb{N}$ such that the sequence of geometries $(\omega_N)_{N \in \mathbb{N}}$ satisfies: all the geometries ω_N, $N \geq N_0$ are without a gap.*

2. Estimates for the solution sequence

In this section we derive an upper bound of the following form for the solution sequence p^ε: there exist $C_2, \kappa_2 > 0$ such that with probability 1 there is an N_0 such that

$$p^\varepsilon|_\Gamma \leq p_1 + C_2 \varepsilon^{\kappa_2} \quad \forall \varepsilon = \frac{1}{N} \leq \frac{1}{N_0}. \tag{2.1}$$

We will use the upper bound in two ways. The first is the immediate implication that with probability 1 the limit pressure $p^0|_\Gamma$ is bounded by p_1, as was stated in (1.8). The second use is in Section 3, where we conclude from the upper bound for p^ε lower bounds for spatial and temporal distances between explosions. We refer the reader to [3] for a related problem.

The proofs are based on the fact that with probability 1 the solution sequence has no gaps (Corollary 1.4). We can therefore assume that for every $O(N^{1/3+\alpha})$ cells, we find one cell with maximal pressure below $p_1 + N^{-1/3}$. We often suppress the superscript ε of p^ε in the calculations.

Proposition 2.1 (Global upper bound). *For arbitrary $\kappa < \frac{1}{3}$ there exist $C, N_0 > 0$ such that with probability* 1 *all pressure functions p^ε, $\varepsilon = N^{-1} \le N_0^{-1}$ satisfy the L^∞ estimate*

$$\|p^\varepsilon|_\Gamma\|_{L^\infty} \le p_1 + C\varepsilon^\kappa. \tag{2.2}$$

The proof of this proposition is done in four steps, which are formulated in subsequent lemmas. In the first step we analyze the following Eqs. (2.3)–(2.8). They are a stationary version of (1.12)–(1.15). Another simplification is introduced. We assume that there is only one "active outlet", positioned at $x = 0$, denoted by $\Gamma_D = (-\gamma, \gamma) \times \{0\} \subset \Gamma$. The subscript D shall remind us that we have a Dirichlet condition on this part of the boundary.

$$Q_1 \partial_2 p(x_1, 0) = 0 \qquad \text{on } \bar\Gamma_1 \setminus \Gamma_D, \tag{2.3}$$

$$p(x_1, 0) = p_1 + \delta_p \qquad \text{on } \Gamma_D, \tag{2.4}$$

$$p(x_1, 0) = Q_1 p(x_1, 0) \qquad \text{on } \bar\Gamma_1, \tag{2.5}$$

$$\partial_2 p(x_1, 0) = 0 \qquad \text{on } \bar\Gamma_2, \tag{2.6}$$

$$-\partial_2 p(x_1, 0) = N^{-1} \qquad \text{on } \bar\Gamma_0, \tag{2.7}$$

$$-\Delta_y p(x_1, x_2) = 0 \qquad \text{in } \bar\Omega. \tag{2.8}$$

Equation (2.6) corresponds to the bound $\|V_0\|_\infty \le 1$. Our first result concerns the above equations with a further simplification: we study the periodic case. Let the domain be $\Omega_{per} = (-L, L) \times (-N, 0)$ with periodicity conditions identifying the boundary $\{x_1 = -L\}$ with $\{x_1 = L\}$.

Lemma 2.2 (Stationary, periodic estimate). *For $q < \infty$ arbitrary there exists $C > 0$ such that every solution p of* (2.3)–(2.8) *on the periodic domain Ω_{per} satisfies*

$$\|p|_\Gamma\|_{L^\infty(\Gamma)} \le p_1 + \delta_p + C\left(\frac{L}{N}\right)^{1/2} L^{1/q}. \tag{2.9}$$

The constant C is independent of N and L.

Proof. The proof is based on L^2-type estimates. Let C_p be the L^∞ norm of p (bounded for fixed N, L). We multiply $-\Delta p = 0$ by $p - (p_1 + \delta_p)$ and perform an integration by parts using

$$\int_{\bar\Gamma \setminus \Gamma_D} p \cdot \partial_2 p = \int_{\bar\Gamma \setminus \Gamma_D} Q_1 p \cdot \partial_2 p = \int_{\bar\Gamma \setminus \Gamma_D} p \cdot Q_1 \partial_2 p = 0.$$

We find the H^1 estimate

$$\int_\Omega |\nabla p|^2 \le \int_{\bar\Gamma_0} (-\partial_2 p) \cdot (p - p_1 - \delta_p) \le C_p \frac{2L}{N}.$$

In particular, we have a bound for the Dirichlet integral over the subdomain $\Omega_L := (-L, L) \times (-L, 0)$. We introduce $x = (x_1, x_2) = L \cdot \tilde x = (L\tilde x_1, L\tilde x_2)$ and the function $\tilde{p}$ defined on the standard domain $\tilde{\Omega} = (-1, 1) \times (-1, 0)$ by $\tilde{p}(\tilde{x}) := p(x)$. We find

$$\|\nabla \tilde{p}\|^2_{L^2(\tilde{\Omega})} = \int_{\Omega_L} |\nabla p|^2 \leq C_p \frac{2L}{N}.$$

We can use a trace theorem on the (N- and L-independent) domain $\tilde{\Omega}$ and find with the average $a = \frac{1}{2} \int_{\tilde{\Omega}} \tilde{p}$

$$\|(\tilde{p} - a)|_{\tilde{\Gamma}}\|_{L^q(\tilde{\Gamma})} \leq C(q) \left(C_p \frac{2L}{N} \right)^{1/2}.$$

We now consider that outlet $Y = (x_i - \gamma, x_i + \gamma) \times \{0\}$, where $|p - a|$ takes its maximal value. With the rescaled outlet $\tilde{Y} = L^{-1} Y$ we can write

$$\begin{aligned} \|(p - a)|_{\bar{\Gamma}}\|_{L^\infty} &= \|(\tilde{p} - a)|_{\tilde{\Gamma}}\|_{L^\infty} = |\tilde{p} - a|(\tilde{Y}) = \\ &= \left(\frac{1}{|\tilde{Y}|} \int_{\tilde{Y}} |\tilde{p} - a|^q \right)^{1/q} \\ &\leq \left(\frac{L}{2\gamma} \right)^{1/q} \left(\int_{\tilde{\Gamma}} |\tilde{p} - a|^q \right)^{1/q} \leq C'(q) L^{1/q} \left(C_p \frac{L}{N} \right)^{1/2}. \end{aligned}$$

This is a bound on oscillations along $\bar{\Gamma}$. Assume that C_p is bounded independently of N, L. Then, since in one outlet we have the prescribed value $p_1 + \delta_p$ for p, we found the result.

Concerning C_p we can calculate, comparing p with the function $(x_1, x_2) \to \|p|_{\bar{\Gamma}}\|_\infty + |x_2| N^{-1}$, and using the last inequality

$$C_p \leq \|p|_{\bar{\Gamma}}\|_\infty + N N^{-1} \leq p_1 + \delta_p + 2C'(q) L^{1/q} \left(C_p \frac{L}{N} \right)^{1/2} + 1.$$

This inequality is of the form $C_p \leq c_1 + c_2 \sqrt{C_p}$ and yields a uniform bound for C_p. Note that we can assume $L^{1+2/q} \leq N$, since otherwise the result follows trivially. □

The next two Lemmas 2.3 and 2.4 confirm that estimate (2.9) remains valid in the general situation in which the domain is not periodic.

Lemma 2.3 (Monotonicity of the periodic solution). *The stationary periodic solution p of equations* (2.3)–(2.8) *satisfies the monotonicity*

$$\partial_1 p(x_1, x_2) \geq 0 \qquad \forall x_1 \in (0, L), x_2 \in (-N, 0).$$

Proof. The proof is based on geometric considerations with similarities to the analysis of two-dimensional ordinary differential equations.

Assume that in the boundary point $(z_0, 0)$ we have $\partial_1 p(z_0, 0) < 0$. We consider the function $q : \xi \mapsto p(\xi, 0)$ on the interval $[z_0, L]$ with minimum in a point $z_1 \in (z_0, L]$. In the case $z_1 \in \mathrm{cl}(\bar{\Gamma}_1)$ we find a cell $(j - \gamma, j + \gamma) \subset [z_0, L]$, $j \in \mathbb{Z}$, and by $Q_1 \partial_2 p = 0$ we can assume $\partial_2 p(z_1, 0) \geq 0$. In this case, but also in the case $z_1 \in \bar{\Gamma}_2$, we know by the Hopf lemma that z_1 cannot be a local minimum of p.

We can now follow the streamline $s \to \gamma(s)$ of the velocity field $v = -\nabla p$ toward lower pressure, starting from $(z_1, 0)$. By streamline we mean the following: In regular points $\gamma(s)$ we consider $\gamma'(s) = -\lambda \nabla p$ with positive λ. If $\gamma(s)$ happens to be on the stable manifold of a saddle point, we let γ parametrize first the stable and then an unstable manifold. To make a choice, we decide to "turn right", that is, we rotate $\gamma'(s)$ clockwise on the saddle. Continuing γ we necessarily reach the boundary, since otherwise we found an interior minimum of the harmonic function p. Since p is decreasing along the streamline and z_1 is a minimum of q on $[z_0, L]$, the streamline cannot end on $[z_0, L]$. By the boundary condition it also cannot end on Γ_0. We denote the endpoint of the streamline by $z_2 \in [0, z_0)$. An illustration is given in Fig. 1.

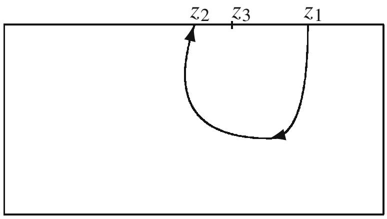

Fig. 1. An impossible streamline of p

We finally find a point $z_3 \in [z_2, z_1]$ where $q|_{[z_2,z_1]}$ has a maximum. The construction implies $z_3 \in (z_2, z_1)$. The pressure function on the streamline is below $p(z_1, 0)$; therefore z_3 is a local maximum of the function p in the domain enclosed by the streamline and Γ. Then $\partial_2 p \geq 0$ in the cell containing z_3, not vanishing identically, a contradiction to (2.3). We proved that there is no point $z_0 > 0$ with $\partial_1 p(z_0, 0) < 0$.

We have seen that p is a harmonic function with monotonically increasing Dirichlet data on $\bar{\Gamma} \cap \{x_1 > 0\}$ and Neumann conditions on $\bar{\Gamma}_0$. Note that we can consider $u := p - \frac{x_2}{N}$ to have homogeneous Neumann data. Such a function is necessarily monotone increasing in x_1 in all points (x_1, x_2). To see this it suffices to consider the symmetric extension of u across $\{x_2 = -N\}$ and to use the maximum principle for $\partial_1 u$. □

Assume that p is a solution of (2.3), (2.5)–(2.8) on the original domain $\bar{\Omega} = (-N, N) \times (-N, 0)$, and let Γ_D be the periodic sequence of cells with period $2L$,

$$\Gamma_D := \bar{\Gamma} \cap \bigcup_{k \in \mathbb{Z}} (2k \cdot L - \gamma, 2k \cdot L + \gamma).$$

Assume that on Γ_D there holds $p \leq p_1 + \delta_p$ [replacing (2.5)]. Then we can compare p with the function

$$\hat{p}(x_1, x_2) = p_L(x_1 - 2kL, x_2) \quad \text{for } x_1 \in [(2k-1)L, (2k+1)L],$$

where p_L is the solution of Lemma 2.2. There holds $p \leq \hat{p}$, and the estimates for p_L remain valid for p.

In the stochastic case we do not have a periodic distribution of the cells with low pressure, and the above reasoning has to be refined. The next lemma states that the result remains valid if the interhole distances are bounded. Here the monotonicity result of Lemma 2.3 is used.

Lemma 2.4 (Stationary, nonperiodic estimate). *We consider Eqs.* (2.3), (2.5)–(2.8) *on the periodic domain* $\bar{\Omega} = (-N, N) \times (-N, 0)$ *together with the condition* $p \le p_1 + \delta_p$ *on* Γ_D *with*

$$\Gamma_D = \bigcup_{j \in J} Y_j \qquad Y_j := (j - \gamma, j + \gamma) \times \{0\}.$$

The index set $J \subset \{-N+1, \ldots, N\}$ *is arbitrary, but the distances* d_j^- *and* $d_j^+ = d_{j+1}^-$ *to the left and right neighboring outlet in* J *satisfy* $d_j^\pm \le 2L$.

Then every solution p *satisfies*

$$\|p|_\Gamma\|_{L^\infty(\Gamma)} \le p_1 + \delta_p + C \left(\frac{L}{N}\right)^{1/2} L^{1/q}. \tag{2.10}$$

Proof. The idea is again to use the solution p_L of the periodic Eqs. (2.3)–(2.8) in order to construct a comparison solution for p. With the (j-dependent) distances $L^- = \frac{1}{2} d_j^-$ and $L^+ = \frac{1}{2} d_j^+$ we can consider the function p_L restricted to the domain $(-L^-, L^+) \times (-N, 0)$. We glue together these functions and find $\hat{p}$ on Ω as above. Then $\hat{p}$ is continuous and satisfies the upper and lower boundary conditions, since p_L satisfies them as well.

Along the gluing lines $\{(x_1, x_2) : x_1 = j + L^+, j \in J, -N < x_2 < 0\}$ we cannot expect the Laplace equation to be satisfied since $\hat{p}$ is not C^1 on these lines. Nevertheless, Lemma 2.3 assures that $\partial_1 p(d_j/2 - 0) \ge 0$ and, by symmetry, $\partial_1 p(-d_j/2 + 0) \le 0$. We conclude that the singular part of $\partial_1^2 \hat{p}$ is nonpositive and therefore $\Delta \hat{p} \le 0$ in the distributional sense. We can use the comparison principle and find $p \le \hat{p}$ and therefore estimate (2.10). □

We have now completed the stationary estimate and can turn to the instationary one.

Proof of Proposition 2.1. We first show that the stationary solution $\hat{p}$ is an upper bound for instationary solutions. We set

$$\Gamma_D := \left\{(k - \gamma, k + \gamma) | k \in \{-N+1, \ldots, N\}, \omega(k) < p_1 + \delta_p\right\}.$$

With this choice the pressure p has the bound $p_1 + \delta_p$ on Γ_D. Note that we neither claim that the cells of Γ_D show explosions nor that other cells do not show explosions.

For this choice of Γ_D we consider the stationary solution $\hat{p}$ of (2.3)–(2.8) that was studied so far. We claim that $\hat{p}$ is an upper bound for the (rescaled) instationary solution $p(t)$. We will use once more the fact that no "negative explosions" can happen; in a point on Γ with $p = Q_1 p = 0$ there holds $\partial_t H = -Q_1 \partial_2 p > 0$.

The difference $u := p - \hat{p}$ is harmonic and initially negative. It satisfies $-\partial_2 u \le 0$ on $\bar{\Gamma}_0$, is periodic across the lateral boundaries, and satisfies $u = Q_1 u$ on $\bar{\Gamma}_1$. In points of explosions the function u jumps down, and else there holds

$$\partial_t u = \partial_t p = -\omega([.]) p_0'(H(.)) \cdot Q_1 \partial_2 p = -\omega([.]) Q_1 \partial_2 u \quad \text{on } \bar{\Gamma} \setminus \Gamma_D. \tag{2.11}$$

Furthermore, $u = p - \hat{p} < 0$ on Γ_D. These inequalities together imply that u satisfies a parabolic maximum principle and remains negative for all times. Indeed, let us consider the first time instance t such that $\max\{u(x,t)|x \in [-N,N] \times [-N,0]\} = 0$, and let x be a point with $u(x,t) = 0$. By the maximum principle for harmonic functions, x can be chosen on the boundary. By the Hopf lemma, the Neumann derivative in a maximum on the boundary must be positive; therefore, $x \notin \bar{\Gamma}_0 \cup \bar{\Gamma}_2$; it is immediate that $x \notin \Gamma_D$. It remains to consider $x \in \bar{\Gamma}_1 \setminus \Gamma_D$. The calculation (2.11) together with the Hopf lemma shows $\partial_t u(x,t) < 0$, a contradiction to the choice of t. Note that we used here $p_0' > 0$. We conclude that the estimate for $\hat{p}$ of Lemma 2.4 provides a bound for p.

In order to conclude the proof we set $\alpha = 1/3 - \kappa$. For every N we are given a geometry $\omega_N : [-N+1, \dots, N] \to [p_1, p_2]$. We choose a level $p_1 + \delta_p$ by setting $a = 1/3$ and thus $\delta_p := N^{-1/3}$. With probability 1 the sequence ω_N has no gaps of length $L := N^{1/3+\alpha}$ from some N_0 on by Corollary 1.4. We now choose $q > 4$ such that $\frac{1}{3q} \le \frac{\alpha}{4}$. Evaluating (2.10) yields

$$\begin{aligned}\|p|_{\bar{\Gamma}}\|_{L^\infty} &\le p_1 + \delta_p + C(q)\left(\frac{L}{N}\right)^{1/2} L^{1/q} \\ &\le p_1 + N^{-1/3} + C(q) N^{-1/3+\alpha/2} N^{1/(3q)+\alpha/q} \\ &\le p_1 + C N^{-1/3+\alpha}.\end{aligned}$$

This proves the proposition. □

3. Upscaled equations

In this section we conclude the proof of Theorem 1.1. The key in the proof is the derivation of additional regularity properties (besides L^∞) of the limiting pressure p^0. Precisely, we introduce

$$p_\delta^\varepsilon(x,t) := \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} p^\varepsilon(\xi, 0, t)\, d\xi,$$

and the same for p^0. We will show that almost surely there is a representative of p^0 such that for all $\delta > 0$, $|x| < 1$, the function $p_\delta^0(x, .)$ is continuous.

The proof of the theorem is divided into two parts. In 3.1 we calculate an upper bound for the number of explosions in a low-pressure region; using Proposition 3.3, the additional regularity of p^0 can be shown and the upscaled equations can be derived as in [11]. Since we treat the linear case here, the derivation of the upscaled equations can actually be simplified, and we provide it in 3.2.

3.1. The number of explosions in low-pressure regions

In order to estimate the number of explosions we must (a) find a lower bound for the time distance between two explosions in one cell and (b) find an upper bound for the number of cells that exhibit explosions. It is in the proof of (a) that we exploit the piecewise linearity of the law $\mathcal{P}_0$.

Lemma 3.1. *For every $\kappa < 1/3$, with probability 1 there exist $c_0, \varepsilon_0 > 0$ such that in every ε-system with $\varepsilon < \varepsilon_0$ there holds: the minimal time span Δt between two explosions in the same outlet is bounded from below by*

$$\Delta t \geq c_0 N^{\kappa - 1}.$$

Proof. We consider rescaled variables and use the comparison principle. The subsequent explosion happens later if additional explosions take place, and it happens also later if the initial values are lower. We can therefore assume without loss of generality that the solution in the point of the first explosion is $p(0) = 0$, $p(k) = p_1 + CN^{-\kappa} \forall k \in \mathbb{Z}, k \neq 0$, where we used the global a priori bound of Proposition 2.1. We can furthermore assume $-\partial_2 p = N^{-1}$ on $\bar{\Gamma}_0$. We decompose the pressure function as

$$p(x, 0) = p^+(x) + p^-(x)$$

into the "positive" part $p^+(0) = p_1$, $p^+(k) = p_1 + CN^{-\kappa} \forall k \neq 0$ satisfying $-\partial_2 p = N^{-1}$ on $\bar{\Gamma}_0$, and the "negative" part $p^-(0) = -p_1$, $p^-(k) = 0 \forall k \neq 0$ and a homogeneous Neumann condition on $\bar{\Gamma}_0$. By linearity the solution p can be found as the sum of the solutions to initial values p^+ and p^- (we denote the time dependent solution again by $p^\pm$). Inequality (1.8) of Lemma A.2 yields for the evolution of the negative part an estimate $p^-(0, t) \leq -p_1 ct^{-1}$. The evolution of the positive part satisfies by the maximum principle $p^+(0, t) \leq p_1 + CN^{-\kappa} + C_1 t N^{-1}$ for all $t > 0$. We conclude that in the moment of the second explosion

$$p_1 \leq p(0, t) = p^+(0, t) + p^-(0, t) \leq p_1 + CN^{-\kappa} + C_1 t N^{-1} - \frac{p_1 c}{t}.$$

We conclude $t \geq cN^\kappa$, and rescaling by a factor N yields the result. $\square$

In what follows we have to find upper bounds for the number of explosions. We will always calculate the expected value of cells where explosions are possible and then conclude with the help of the following theorem of Bernstein, quoted from [12].

Theorem 3.2 (Bernstein). *For fixed $n \in \mathbb{N}$, let $X_i, 1 \leq i \leq n$ be independent random variables with $|X_i| \leq c$ and $Var(X_i) > 0$ for all i. Set $S_n := \sum_{i=1}^n X_i$ and $\tau^2 := \sum_{i=1}^n Var(X_i)$. Then for every $t > 0$ there holds*

$$\mathbb{P}(S_n - ES_n \geq t) \leq \exp\left(-\frac{t^2}{2\tau^2 + \frac{2}{3}tc}\right).$$

Here E denotes the expected value.

We will use this result as follows. The random variable $X_i = 1$ indicates that an explosion is possible at position i, otherwise $X_i = 0$. Assume that in a region of width $n = N^a$ we have $\mathbb{P}(X_i = 1) = N^{-b}$ with $b < a$, and we therefore expect $ES_n = N^{a-b}$ explosions. We calculate for large N the probability of $2N^{a-b}$ explosions using $\tau^2 \le n \cdot N^{-b} = N^{a-b}$ and $t := N^{a-b}$:

$$\mathbb{P}\left(S_n \ge 2N^{a-b}\right) \le \mathbb{P}\left(S_n - ES_n \ge t\right) \le \exp\left(-\frac{t^2}{2\tau^2 + \frac{2}{3}tc}\right)$$
$$\le \exp\left(-\frac{N^{2(a-b)}}{2N^{a-b} + \frac{2}{3}N^{a-b}}\right) \le \exp\left(-\frac{3}{8}N^{a-b}\right).$$

This is summable in N, and also expressions of the form $N^q\mathbb{P}(S_n \ge 2N^{a-b})$ remain summable over $N \in \mathbb{N}$. With the Borel–Cantelli lemma we conclude that with probability 1, along a sequence $N = N_0, N_0 + 1, \ldots$, in no subset of length n does a large deviation from the expected value occur.

Proposition 3.3. *There exists $\beta > 0$ such that with probability 1 the following property holds. If (x, t) is a point with $p^\varepsilon_\delta(x, t) \le p_1 - \Delta\rho$ along a sequence $\varepsilon \to 0$ with $\delta > 0$ small enough compared to $\Delta\rho$, then there exist $\Delta t, c, \varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$*

(i) All explosions in $(x - \delta, x + \delta) \times (t, t + \Delta t)$ happen in the time interval $(t, t + c\varepsilon^\kappa)$,
(ii) There are at most $O(\varepsilon^{-1+\beta})$ explosions in the region of (i).

Proof. **(i)** The idea of proof is, loosely speaking, the following. The low pressure average creates in every outlet an $O(1)$ flow into the domain, and the pressure decreases at a finite rate. After time $O(\varepsilon^\kappa)$ the maxima are below p_1 and no further explosions can happen.

In order to make the argument precise, we consider the three intervals $S_k = (x - k\cdot\delta, x + k\cdot\delta)$ with $k = 1, 2, 3$. Note that the averages $p^\varepsilon_{(k\delta)}(x, t)$ are also below $p_1 - \Delta\rho/4$ for small ε. To find an upper bound for the pressure p^ε, we construct a comparison solution $\bar{p}^\varepsilon$ which at time t coincides with p^ε in the spatial interval S_2 and is set to $\bar{p}^\varepsilon = p_1 + C\varepsilon^\kappa$ outside S_2. We extend the linear pressure laws such that $\bar{p}^\varepsilon$ has no explosions. By linearity of the laws we can now decompose the initial values of $\bar{p}^\varepsilon$. We set $\bar{p}_B(t) = 0$ in S_2 and $\bar{p}_B(t) = 2C\varepsilon^\kappa$ outside S_2 and define $\bar{p}_A(t) = \bar{p}^\varepsilon(t) - \bar{p}_B(t)$. We first study the time evolution of $\bar{p}_B$. On the interval S_1 we find by Remark A.2 of [11] a bound of the form

$$\bar{p}_B(\xi, \tau) \le C(\delta)\Delta t\ \varepsilon^\kappa \qquad \forall \xi \in S_1, \tau \in (t, t + \Delta t).$$

In particular, choosing a small Δt, we achieve $\bar{p}_B \le \frac{1}{2}C\varepsilon^\kappa$ in S_1.

The values of $\bar{p}_A$ must now be analyzed. Note that $\bar{p}_A(t)$ satisfies $\bar{p}_A(t) = p_1 - C\varepsilon^\kappa$ outside S_2. We again use Remark A.2 of [11] to conclude that $\bar{p}_A(\tau) \le p_1 - \frac{1}{2}C\varepsilon^\kappa$ outside S_3 for Δt small compared to δ. We now consider maxima of $\bar{p}_A$ with values above $p_1 - \frac{1}{2}C\varepsilon^\kappa$ (they are all in S_3). We can calculate $\partial_2\bar{p}_A \ge c_H > 0$ with the help of the quantitative Hopf Lemma 3.1 of [11]. Therefore, in these maxima $\partial_t \bar{p}_A \leq -C\partial_2 \bar{p}_A \leq -Cc_H$, and after the time $O(\varepsilon^\kappa)$ the values of all maxima are below $p_1 - \frac{1}{2}C\varepsilon^\kappa$. Then all values of $\bar{p}^\varepsilon = \bar{p}_A + \bar{p}_B$ are below p_1 in S_1 and no further explosions are possible.

(ii) For the second assertion we use the rescaled pressure function p to count the number of explosions in a domain $(\bar{n}_0 - \delta N, \bar{n}_0 + \delta N) \times (\bar{t}_0, \bar{t}_0 + cN^{1-\kappa})$. For small σ we can use the same number $\kappa = 1/3 - \sigma$ for the a priori estimate of p^ε and the time distance of Lemma 3.1. Without loss of generality we set $\bar{t}_0 = 0, \bar{n}_0 = 0$. Assume that we have $N^{1-\beta}$ explosions. We fix $\tau \in (\kappa, 1-\kappa)$ and divide the spatial interval into $O(N^\tau)$ segments of length $L = O(N^{1-\tau})$ and find a segment S with $O(N^{1-\beta-\tau})$ explosions. Let t_0 be the time instance of the last explosion in S and note $t_0 \leq cN^{1-\kappa}$ by (i). We now divide the time interval into two parts, $(0, t_0 - N^{1-\tau})$ and $(t_0 - N^{1-\tau}, t_0)$.

We claim that less than half of the explosions happen in the second time interval. The expected number of active outlets is calculated as the total number of cells in the segment multiplied by $N^{-\kappa}$, the fraction of active outlets. We therefore expect to have $O(N^{1-\tau-\kappa})$ active outlets in S. The number of explosions of each outlet is bounded by the total time divided by $\Delta t = O(N^\kappa)$ of Lemma 3.1, $N^{1-\tau-\kappa}$. The total number of explosions in S in the second time interval can be bounded with probability 1 by any number that is large compared to $N^{2-2\tau-2\kappa}$. By assumption we have $O(N^{1-\beta-\tau})$ explosions in $(0, t_0)$; the claim is proved for $2-2\tau-2\kappa < 1-\beta-\tau$, or $1 - 2\kappa + \beta < \tau$.

We now study the effect of the explosions in the first time interval. By Lemma A.2 each explosion reduces the values of $p(t_0)$ on S by at least $c_0 t_0^{-1} \geq \frac{c_0}{c} N^{\kappa-1}$. The total (pointwise) loss of pressure by explosions has the asymptotics

$$\Delta\rho = O(N^{\kappa-1}) \cdot O(N^{1-\beta-\tau}) = O(N^{\kappa-\beta-\tau}).$$

For $\tau < 2\kappa - \beta$ this is large compared to $CN^{-\kappa}$. The pressure $p(t_0)$ is therefore everywhere below p_1 and the explosion at t_0 yields a contradiction. The proof used that κ can be chosen close to $1/3$ and $\beta > 0$ small in order to make it possible to find $\tau \in (\kappa, 1-\kappa)$ with $1 - 2\kappa + \beta < \tau < 2\kappa - \beta$. □

3.2. Derivation of the upscaled equations

The derivation of the upscaled equations in [11] was based on Corollary 3.2 therein, stating that no explosions can happen in a region of low average pressure. The corollary cannot be used here since in the stochastic case the inequality $p^\varepsilon \leq p_1$ does not hold in general. One can easily construct examples in which the local pressure average is small, but nevertheless explosions can happen. We must use here the weaker statement of Proposition 3.3 to derive the upscaled equations. We start with a result on additional regularity; as the representative of p^0 we use its maximal function.

Proposition 3.4. *Almost surely the limiting pressure has the following property. Let $\delta_0 > 0$ be arbitrarily small and $(x, \bar{t})$ a point with limiting pressure not* *maximal,*

$$\limsup_{\delta \to 0} p_\delta^0(x, \bar{t}) = \rho < p_1. \tag{3.1}$$

Then there exist $\rho_0 < p_1$, $0 < \delta < \delta_0$, $t_1 < \bar{t} < t_2$, *and* $\varepsilon_0 > 0$, *such that*

$$p_\delta^\varepsilon(x, t) \le \rho_0 \quad \forall t \in (t_1, t_2), \varepsilon < \varepsilon_0, \tag{3.2}$$

where ρ_0 *can be chosen independently of* δ_0. *The limiting pressure satisfies*

$$p_\delta^0(x, t) \le \rho_0 \quad \forall t \in (t_1, t_2). \tag{3.3}$$

Proof. We first note that (3.3) is an immediate consequence of (3.2), since spatio-temporal averages of p^ε converge to the corresponding averages of p^0. It remains to show (3.2). We sketch the proof and refer to Proposition 3.6 of [11] for more details.

We choose $\Delta\rho > 0$ small compared to $p_1 - \rho$, and then $\delta > 0$ small with $|p_\delta^0(x, \bar{t}) - \rho|$ small; the precise smallness conditions are given below. In order to find t_1 we argue by contradiction and assume that for a subsequence $\varepsilon \to 0$ and a sequence $t_1^\varepsilon \nearrow \bar{t}$ there holds

$$p_\delta^\varepsilon\big(x, t_1^\varepsilon\big) \ge p_1 - \Delta\rho. \tag{3.4}$$

By continuity of p^ε we can actually assume equality in (3.4) by choosing the largest t_1^ε in the vicinity of $\bar{t}$ satisfying the inequality. From (3.1) we conclude that for another sequence $t^\varepsilon \searrow \bar{t}$

$$p_\delta^\varepsilon(x, t^\varepsilon) \le \rho + \Delta\rho.$$

Exploiting that the pressure in t_1^ε is large, we first verify that redistribution of mass between the cells is a small effect. We consider the cell midpoints $x_k \in \varepsilon\mathbb{Z} \cap (x - \delta, x + \delta)$ and calculate for the positive parts in the change of effective height $\bar{h}^\varepsilon = h^\varepsilon \bmod \varepsilon$,

$$\begin{aligned}\frac{1}{2\delta} \sum_k \left(\bar{h}^\varepsilon\big(x_k, t^\varepsilon\big) - \bar{h}^\varepsilon\big(x_k, t_1^\varepsilon\big)\right)_+ &\le \frac{1}{2\delta} \sum_k \varepsilon \left(\frac{p_1 + C\varepsilon^\kappa}{p_{max}(k)} - \frac{p^\varepsilon\big(x_k, t_1^\varepsilon\big)}{p_{max}(k)} \right) \\ &\le \frac{1}{p_1} \left(p_1 - p_\delta^\varepsilon\big(x, t_1^\varepsilon\big)\right) + o(1) \le \frac{1}{p_1} \Delta\rho + o(1)\end{aligned}$$

for $\varepsilon \to 0$. We can now conclude that the average height is decreased between t_1^ε and t^ε. We calculate along the upper boundary

$$\begin{aligned}\rho - p_1 + 2\Delta\rho &\ge \frac{1}{2\delta} \int_{B_\delta(x)} p^\varepsilon(\xi, \tau)\, d\xi \Big|_{t_1^\varepsilon}^{t^\varepsilon} \\ &\ge \frac{1}{2\delta 2\gamma} \int_{B_\delta(x) \cap \Gamma_1^\varepsilon} p^\varepsilon(\xi, \tau)\, d\xi \Big|_{t_1^\varepsilon}^{t^\varepsilon} + o(1)\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2\delta}\sum_k \left(\bar{h}^\varepsilon\left(x_k, t^\varepsilon\right) - \bar{h}^\varepsilon\left(x_k, t_1^\varepsilon\right)\right)_+ \cdot \inf\{p_{max}(.)\} \\
&\qquad + \frac{1}{2\delta}\sum_k \left(\bar{h}^\varepsilon\left(x_k, t^\varepsilon\right) - \bar{h}^\varepsilon\left(x_k, t_1^\varepsilon\right)\right)_- \cdot \sup\{p_{max}(.)\} + o(1) \\
&\geq (\Delta\rho/p_1)(p_1 - p_2) + o(1) + \frac{p_2}{2\delta}\sum_k \left(\bar{h}^\varepsilon\left(x_k, t^\varepsilon\right) - \bar{h}^\varepsilon\left(x_k, t_1^\varepsilon\right)\right).
\end{aligned}$$

For small ε and small $\Delta\rho$ we conclude for the change in average height

$$\frac{p_2}{2\delta}\sum_k \left(\bar{h}^\varepsilon\left(x_k, t^\varepsilon\right) - \bar{h}^\varepsilon\left(x_k, t_1^\varepsilon\right)\right) \leq \rho - p_1 + C\Delta\rho.$$

Under the assumption $\Delta\rho < (p_1 - \rho)/2C$ we conclude an order $O(\varepsilon)$ loss in average effective height in the ε-system.

We now exploit conservation of mass in the system, expressed through $\operatorname{div} v^\varepsilon = 0$ for $v^\varepsilon = -\nabla p^\varepsilon$ and (1.1) in the integrated form. With $R = (x-\delta, x+\delta) \times (-1, 0)$ and $\Sigma = (x-\delta, x+\delta) \times \{0\}$,

$$\partial_t \int_{x-\delta}^{x+\delta} \frac{h^\varepsilon}{\varepsilon} = \int_\Sigma v^\varepsilon \cdot e_2 = -\int_{\partial R\setminus\Sigma} v^\varepsilon \cdot n.$$

The total flow of v^ε through $\partial R \setminus \Sigma$ is at most $O(1)$, since p^ε is bounded. Therefore, the loss of mass through these boundaries in the considered timespan is of order $O(|t^\varepsilon - t_1^\varepsilon|)$. We conclude that this flow cannot reduce the height by order $O(\varepsilon)$. The loss in effective height must therefore be induced explosions; $O(1/\varepsilon)$ explosions must have taken place in the ε-system. This yields a contradiction with Proposition 3.3 (ii). The construction of t_2 follows the same pattern. □

Proof of Theorem 1.1. The representative p^0 of the limiting pressure is clearly harmonic, periodic across lateral boundaries, and satisfies the inflow condition on the lower boundary. Inequality (1.8) follows from the upper bound on the pressure in (2.2). Furthermore, by Proposition 3.4, for every point $(\bar{x}, \bar{t})$ with $p^0(\bar{x}, \bar{t}) < p_1$ we find $\delta_1 > 0$ small and $t_1 < \bar{t} < t_2$ with $p^0_{\delta_1}(\bar{x}, t) < p_1$ for $t \in [t_1, t_2]$. With Proposition 3.3 (i) we find a neighborhood $U \ni (\bar{x}, \bar{t})$ such that all ε-systems have no explosions in U. It remains to verify (1.9) and (1.10) in U (or in a smaller neighborhood).

Taking a further subsequence of $\varepsilon \to 0$ we can assume that, additionally to $p^\varepsilon \to p^0$ in $L^\infty_w(\Omega)$, also $p^\varepsilon|_\Gamma \to p^0|_\Gamma$ in $L^\infty_w(\Gamma)$. We exploit here that harmonic functions have a trace on the boundary.

We consider an arbitrary point $(x, t) \in U$ and $\delta > 0$. The calculations start from the differentiated version of the microscopic pressure law $p^\varepsilon = \mathcal{P}_0(\varepsilon^{-1}h^\varepsilon)$ in the points $x_k = \varepsilon k \in (x-\delta, x+\delta)$,

$$\partial_t p^\varepsilon(x_k, t) = \mathcal{P}_0'\big(k, \varepsilon^{-1}h^\varepsilon(x_k, t)\big) \cdot \varepsilon^{-1}\partial_t h^\varepsilon(x_k, t). \tag{3.5}$$

Dividing by $\mathcal{P}_0'(k, .) = p_{max}(k)$ and introducing $\Phi_k(\rho) = \rho/p_{max}(k)$ we find

$$\frac{d}{dt}\Phi_k\big(p^\varepsilon(x_k, t)\big) = \varepsilon^{-1}\partial_t h^\varepsilon(x_k, t) = \frac{1}{\varepsilon 2\gamma}\int_{x_k-\gamma\varepsilon}^{x_k+\gamma\varepsilon} \big(-\partial_2 p^\varepsilon(\xi, t)\big)\, d\xi. \tag{3.6}$$

Summing over the spatial points x_k and integrating in time yields

$$\begin{aligned}\frac{1}{\Delta t}\frac{\varepsilon}{2\delta}\sum_k\left[\Phi_k\big(p^\varepsilon(x_k,t+\Delta t)\big)-\Phi_k\big(p^\varepsilon(x_k,t)\big)\right]\\ =\frac{1}{\Delta t}\frac{1}{2\gamma}\int_t^{t+\Delta t}\big(-\partial_2 p^\varepsilon_\delta(x,\tau)\big)\,d\tau.\end{aligned}\tag{3.7}$$

We exploited that taking x_1 averages and application of ∂_2 commute. The right-hand side converges for $\varepsilon\to 0$ as a distribution,

$$\int_t^{t+\Delta t}\big(-\partial_2 p^\varepsilon_\delta\big)\,d\tau \to \int_t^{t+\Delta t}\big(-\partial_2 p^0_\delta\big)\,d\tau.$$

Here the integrands are interpreted as distributions: for $\varphi\in C^2(\Omega\cup\Gamma)$,

$$\int_\Gamma -\partial_2 p^0_\delta\cdot\varphi|_\Gamma := -\int_\Gamma \big(p^0_\delta\big)|_\Gamma\,(\partial_2\varphi)|_\Gamma+\int_\Omega p^0_\delta\,\Delta\varphi,$$

and the same for p^ε. In the convergence we use $p^\varepsilon|_\Gamma\to p^0|_\Gamma$ in L^∞_w.

We next consider the left-hand side of (3.7) and its limit as $\varepsilon\to 0$. We choose the function $\Theta(\rho)$ as the expected value $\Theta(\rho)=\langle 2\gamma\Phi_k(\rho)\rangle_k=2\gamma\rho\,\langle 1/p_{max}(k)\rangle_k$. Then, almost surely, averages of Φ_k coincide with Θ,

$$\Theta(\rho):=\lim_{\varepsilon\to 0}2\gamma\frac{\varepsilon}{2\delta}\sum_k\Phi_k(\rho).$$

We must now use the fact that p^ε has small oscillations in x. Since there are no exlosions in U, Lemma 4.2 of [11] assures that the functions $p^\varepsilon(.,t)$ and $p^\varepsilon(.,t+\Delta t)$ are uniformly continuous. Therefore, if we replace in (3.7) $p^\varepsilon(x_k,t)$ by $p^\varepsilon_\delta(x,t)$, we introduce an error term which vanishes for $\delta\to 0$. We use the fact that Φ_k are uniformly Lipschitz continuous.

Furthermore, in regions without explosions, the averages p^ε_δ are uniformly Lipschitz continuous; therefore $p^\varepsilon_\delta\to p^0_\delta$ pointwise and we can replace $p^\varepsilon_\delta(x,t)$ by $p^0_\delta(x,t)$. Taking $\varepsilon\to 0$ we find

$$\begin{aligned}\frac{1}{\Delta t}\left[\Theta\big(p^0_\delta(x,t+\Delta t)\big)-\Theta\big(p^0_\delta(x,t)\big)\right]+o(1)\\ =\frac{1}{\Delta t}\int_t^{t+\Delta t}\big(-\partial_2 p^0_\delta(x,\tau)\big)\,d\tau\end{aligned}\tag{3.8}$$

with $o(1)\to 0$ for $\delta\to 0$. Linearity of Θ allows to take the limits $\delta\to 0$ and $\Delta t\to 0$ in the sense of distributions. We arrive at the evolution law (1.9).

To show inequality (1.10) one essentially has to do the same calculations, starting with inequality $\le$ in (3.5). The technical difficulty is that, due to the presence of explosions, the functions p^ε are no longer uniformly continuous. We refer to [11] for the proof and note that the following weak lift-off condition can be easily read off from (3.7).

There is $c>0$ such that for all $(x,0,t)\in\bar\Omega\times[0,T]$ and $\Delta t,\delta>0$ holds

$$p^0_\delta(x,0,t+\Delta t)\le p_1+c\int_t^{t+\Delta t}\big(-\partial_2 p^0_\delta(x,\tau)\big)\,d\tau.\tag{3.9}$$

□

Appendix A. Decay rate for single explosions

Our aim is to prove Lemma A.2. To this end we study the equations on the unbounded domains $\Omega = \mathbb{R} \times \mathbb{R}_-$ with upper boundary $\Gamma \equiv \mathbb{R}$; the cell width is still $\gamma\varepsilon$. Here, we are not interested in the effect of explosions and simplify the equations to

$$\Delta u^\varepsilon(t) = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\partial_2 u^\varepsilon(t) = 0 \qquad \text{on } \Gamma_2^\varepsilon, \tag{1.2}$$

$$u^\varepsilon(t) = Q_\varepsilon u^\varepsilon(t) \qquad \text{on } \Gamma_1^\varepsilon, \tag{1.3}$$

$$\partial_t u^\varepsilon(t) = -a_0^\varepsilon([.]) Q_\varepsilon \partial_2 u^\varepsilon(t) \qquad \text{on } \Gamma_1^\varepsilon. \tag{1.4}$$

We always assume that u^ε vanishes at infinity for all t. The following lemma is an elementary homogenization result. The only purpose of the lemma is to show that not all mass of a bounded domain can be lost in an arbitrary short time.

Lemma A.1. *For $\varepsilon = N^{-1}$ let $a_0^\varepsilon : \mathbb{R} \to [a_1, a_2] \subset (0, \infty)$, constant on all intervals $\varepsilon(k - 1/2, k + 1/2)$, $k \in \mathbb{Z}$, be an arbitrary sequence of coefficient functions and u^ε a family of solutions of* (1.1)–(1.4) *with initial condition*

$$u^\varepsilon(x_1, 0, 0) = \begin{cases} \varepsilon^{-1} & x_1 \in (-\varepsilon\gamma, \varepsilon\gamma), \\ 0 & \textit{else.} \end{cases}$$

Then, for a subsequence $\varepsilon \to 0$, we find limits u^0 and $\bar{a}$, $u^\varepsilon|_\Gamma \to u^0|_\Gamma \in L^\infty(0, T; L^1(\Gamma))$ uniformly on compact subsets of $(0, \infty) \times \Gamma$ and $(a_0^\varepsilon)^{-1} \to (\bar{a})^{-1}$ in L_w^∞. u^0 solves in the sense of distributions

$$\Delta u^0(t) = 0 \qquad \textit{in } \Omega, \forall t > 0, \tag{1.5}$$

$$\partial_t u^0 = -\frac{1}{2\gamma} \bar{a} \cdot \partial_2 u^0 \qquad \textit{on } \Gamma. \tag{1.6}$$

Furthermore, for every $\delta > 0$ the expression

$$V^0(t) := \int_{-\delta}^{\delta} u^0(x_1, 0, t)\, dx_1 \tag{1.7}$$

on $[0, t_\delta]$ is uniformly bounded from below by some constant $c_\delta > 0$.

Proof. The family of functions u^ε is nonnegative, and a weighted integration of the boundary values yields a uniform bound for $u^\varepsilon(., 0, t) \in L^1(\Gamma)$. We therefore find a weak-$\star$ convergent subsequence $\varepsilon \to 0$ with $u^\varepsilon|_\Gamma \to u^0|_\Gamma$ in $L_w^\infty((0, T); \mathcal{M}(\Gamma))$. For the same subsequence we can assume $(a_0^\varepsilon)^{-1} \to (\bar{a})^{-1}$ in L_w^∞.

For fixed $t_1 > 0$ the functions $u^\varepsilon(., 0, t_1)$ are bounded, independent of $\varepsilon > 0$. One can see this with the help of the quantitative Hopf lemma and conservation of mass by showing that the maximum $y(t) = \max\{u^\varepsilon(x, t) | x \in \Gamma\}$ satisfies

$$\partial_t y(t) \le -c_0 y(t)^{3/2},$$

as long as $y(t) \geq C$. This implies that there is a bound for $u^0|_\Gamma \in L^\infty((t_1, T) \times \Gamma)$ for all $t_1 > 0$, that the energy of u^ε is finite, and that u^ε has $\partial_t u^\varepsilon$ and $\partial_2 u^\varepsilon$ bounded in L^2 (compare Proposition A.3 of [11]).

We now use the weak form of the equations: For every $\phi \in C_0^2(\Omega \cup \Gamma \times \mathbb{R}_+)$ with $\partial_2 \phi = 0$ on $\Gamma \times \mathbb{R}_+$ we find

$$\begin{aligned} 0 &= \int_0^\infty \int_\Omega u^\varepsilon \cdot \Delta\phi + \int_0^\infty \int_\Gamma \partial_2 u^\varepsilon \cdot \phi \\ &= \int_0^\infty \int_\Omega u^\varepsilon \cdot \Delta\phi - \int_0^\infty \int_{\Gamma_1^\varepsilon} \frac{1}{a_0^\varepsilon} \partial_t u^\varepsilon \cdot Q_\varepsilon \phi + \int_0^\infty \int_\Gamma \partial_2 u^\varepsilon \cdot (\phi - Q_\varepsilon \phi) \\ &= \int_0^\infty \int_\Omega u^\varepsilon \cdot \Delta\phi + \int_0^\infty \int_{\Gamma_1^\varepsilon} \frac{1}{a_0^\varepsilon} u^\varepsilon \cdot \partial_t Q_\varepsilon \phi + \int_0^\infty \int_\Gamma \partial_2 u^\varepsilon \cdot (\phi - Q_\varepsilon \phi). \end{aligned}$$

Taking the limit $\varepsilon \to 0$ we find by the regularity estimate for $\partial_2 u^\varepsilon|_\Gamma \in L^2_{loc}((0, T) \times \Gamma)$

$$0 = \int_0^\infty \int_\Omega u^0 \cdot \Delta\phi + 2\gamma \int_0^\infty \int_\Gamma \frac{1}{\bar{a}} u^0 \cdot \partial_t \phi.$$

This is the weak form of the equations.

In order to show (1.7), we use a symmetric cutoff function $\eta \in C^2(\Gamma, \mathbb{R})$ with $\eta(r) = 1$ for $0 \leq r < R/2$ and $\eta(r) = 0$ for $r > R$. With $\phi(x_1, x_2) = \eta(|x|)$ we calculate for the loss of an averaged mass

$$\begin{aligned} \partial_t V_\eta^\varepsilon(t) &:= \partial_t \int_{\Gamma_1^\varepsilon} \eta(x_1) \frac{1}{a_0^\varepsilon(x_1)} u^\varepsilon(x_1, 0, t)\, dx_1 \\ &= -\int_\Gamma \eta(x_1) Q_\varepsilon \partial_2 u^\varepsilon(x_1, 0, t)\, dx_1 \\ &= -\int_\Gamma \eta(x_1) \partial_2 u^\varepsilon(x_1, 0, t)\, dx_1 + \int_\Gamma [\eta - Q_\varepsilon \eta](x_1) \partial_2 u^\varepsilon(x_1, 0, t)\, dx_1 \\ &= \int_\Omega \Delta\phi(x) u^\varepsilon(x, t)\, dx + \int_\Gamma [\eta - Q_\varepsilon \eta](x_1) \partial_2 u^\varepsilon(x_1, 0, t)\, dx_1. \end{aligned}$$

This quantity is uniformly bounded: For the first integral this follows from the uniform bounds for $u^\varepsilon|_\Gamma \in L^1$. For the second integral this is a consequence of $\|\eta - Q_\varepsilon \eta\|_{L^\infty} \leq C\varepsilon$ and $\|\partial_2 u^\varepsilon(., 0, t)\|_{L^1} \leq C\varepsilon^{-1}$, and the latter follows from the linearity of the operator $L^1(\Gamma) \ni u^\varepsilon|_\Gamma \to \partial_2 u^\varepsilon|_\Gamma \in L^1(\Gamma)$ and a scaling argument. We conclude that the family $V_\eta^\varepsilon : [0, 1] \to \mathbb{R}$ is uniformly Lipschitz continuous. We can assume that $V_\eta^\varepsilon \to V_\eta^0$ in a Hölder space C^α. Because of $V_\eta^0(0) = V_\eta^\varepsilon(0) \geq \frac{2\gamma}{a_2}$ and

$$V_\eta^\varepsilon(t) \to 2\gamma \int_\Gamma \eta(x_1) \frac{1}{\bar{a}(x_1)} u^0(x_1, 0, t)\, dx_1 \qquad a.e.\ t \in (0, t_\delta),$$

we find the lower bound for averages of u^0. □

We now turn to the principal result of this appendix. It verifies that the decay of a single peak is asymptotically like $1/t$ for $t \to \infty$, and the rate is independent of the coefficients a_0. In the proof we use the fact that the estimates of Lemma A.1

are independent of ε; we can reinterpret long time spans as finite time spans in an ε system.

Lemma A.2. *Consider Eqs.* (1.1)–(1.4) *with* $\varepsilon = 1$ *and with initial values* $u(0) = 1$*,* $u(k) = 0 \forall k \neq 0$*. Then there exists* $c_0 > 0$ *with*

$$u(x,t) \geq \frac{c_0}{t} \quad \forall t > 1, x \in (-t,t). \tag{1.8}$$

Proof. Note that, in the case $a_0 \equiv c$ and without the projection, the unique solution is the first x_1 derivative of the solution of Remark A.1 of [11]. This immediately yields the decay.

We do a rescaling by the factor $\varepsilon = 1/t$. Then we have to show for $u^\varepsilon(\varepsilon x, \varepsilon t) = \frac{1}{\varepsilon}u(x,t)$ that $u^\varepsilon(\varepsilon x, 1) \geq c_0$ for all $\varepsilon x \in (-1,1)$. Assume the contrary. Then there exists $x_1 \in (-1,1)$ and a sequence $\varepsilon \to 0$ such that solutions u^ε of (1.1)–(1.4) satisfy

$$u^\varepsilon(x_1, 0, 1) \to 0 \quad \text{for } \varepsilon \to 0.$$

By Lemma A.1, for some u^0 solving (1.5) and (1.6), $u^\varepsilon \to u^0$ uniformly on $(t_1,t_2) \times \Gamma$ for all $0 < t_1 < t_2 < \infty$. The solution u^0 has nonvanishing initial values by (1.7). One easily verifies that solutions of these equations never vanish for finite values of t and x_1. We found a contradiction. □

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