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The Lewy–Stampacchia inequality for the obstacle problem with quadratic growth in the gradient

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Abstract. In this paper we prove that at least one solution of the obstacle problem for a linear elliptic operator perturbed by a nonlinearity having quadratic growth in the gradient satisfies the Lewy–Stampacchia inequality.

Résumé. Nous démontrons dans cet article qu'au moins une solution du problème de l'obstacle pour un opérateur elliptique linéaire perturbé par une non linéarité à croissance quadratique par rapport au gradient vérifie l'inégalité de Lewy–Stampacchia.

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1. Introduction

The Lewy–Stampacchia inequality was first proved by Lewy and Stampacchia [5] in the case of the obstacle problem for the Laplace operator. It was then extended by many authors to the case of linear and nonlinear elliptic operators and became a powerful tool for proving existence and regularity results, giving rise to numerous papers, some references of which can be found e.g. in the book of Troianiello [8] and in our papers [6,7].

In this paper we consider the case where the linear elliptic operator $-\mathrm{div} A(x)Du + \lambda u$ acting from $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$ is perturbed by a nonlinear term $H(x, u, Du)$ which has at most a quadratic growth with respect to Du, and we prove the Lewy–Stampacchia inequality for at least one solution of the obstacle problem

$$(1.1)\qquad \begin{cases} u \in K(\psi) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega A(x)Du(Dv - Du)\,dx + \lambda \int_\Omega u(v-u)\,dx \\ \displaystyle - \int_\Omega H(x,u,Du)(v-u)\,dx - \int_\Omega f(v-u))\,dx \geq 0, \\ \forall v \in K(\psi) \cap L^\infty(\Omega), \end{cases}$$

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where

$$K(\psi) = \left\{ v \in H_0^1(\Omega) : \; v \geq \psi \text{ a.e. in } \Omega \right\}.$$

The present paper can be considered in some sense as a continuation of our study of the Lewy–Stampacchia inequality: In [6] we proved the Lewy–Stampacchia inequality for the obstacle problem for linear and monotone operators by using the penalization method. We then extended this method in [7] to the bilateral problem, and we proved there that every solution of the problem (and not only one solution as in [6]) satisfies the Lewy–Stampacchia inequality. In the present paper we now face the problem where a nonlinear term in $|Du|^2$ perturbs the linear operator.

2. Assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$ (no smoothness is assumed on $\partial\Omega$). Let A be a bounded and coercive matrix, i.e.

$$A \in (L^\infty(\Omega))^{N \times N}, \quad A(x) \geq \alpha I \quad \text{a.e. } x \in \Omega, \tag{2.1}$$

for some $\alpha > 0$. Let f, λ and ψ be such that

$$f \in L^\infty(\Omega), \tag{2.2}$$

$$\lambda \in \mathbb{R}, \quad \lambda > 0, \tag{2.3}$$

$$\psi \in H^1(\Omega), \quad \psi^+ \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{2.4}$$

The latest hypothesis implies that $K(\psi) \cap L^\infty(\Omega)$ is non-empty, since it contains ψ^+. Finally, let $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ be a Carathéodory function which satisfies

$$|H(x, s, \xi)| \leq C_0 + C_1 |\xi|^2. \tag{2.5}$$

We also assume that

$$\begin{aligned} &|H(x, s, \xi) - H(x, \psi(x), D\psi(x))| \\ &\qquad \leq C_2 |s - \psi(x)| + C_3 |\xi - D\psi(x)|^2 + C_4 |\xi - D\psi(x)| \end{aligned} \tag{2.6}$$

for some non-negative constants C_2, C_3, C_4.

Hypothesis (2.6) seems to be rather strong; observe nevertheless that it is satisfied when $H(x, s, \xi)$ has a quadratic growth in ξ and is locally Lipschitz continuous in s and ξ, namely satisfies

$$\begin{cases} |H(x, s, \xi) - H(x, s, \eta)| \leq C(1 + |\xi| + |\eta|)|\xi - \eta|, \\ |H(x, s, \xi) - H(x, t, \xi)| \leq C(1 + |\xi|^2)|s - t|, \end{cases} \tag{2.7}$$

and when

$$\psi \in W^{1,\infty}(\Omega); \tag{2.8}$$

Indeed in such a case we have

$$\begin{cases} |H(x,s,\xi) - H(x,\psi(x),D\psi(x))| \\ \le |H(x,s,\xi) - H(x,s,D\psi(x))| + |H(x,s,D\psi(x)) - H(x,\psi(x),D\psi(x))| \\ \le C(1+|\xi|+|D\psi(x)|)|\xi - D\psi(x)| + C(1+|D\psi(x)|^2)|s-\psi(x)| \\ \le C(1+|\xi - D\psi(x)|+2|D\psi(x)|)|\xi - D\psi(x)| + C(1+|D\psi(x)|^2)|s-\psi(x)|, \end{cases}$$

and therefore (2.6) is satisfied when $\psi \in W^{1,\infty}(\Omega)$.

We define the operator $B : H^1(\Omega) \to H^{-1}(\Omega) + L^1(\Omega)$ by

$$B(v) = -\operatorname{div}(A(x)Dv) + \lambda v - H(x,v,Dv) - f, \qquad \forall v \in H^1(\Omega). \tag{2.9}$$

In addition to the previous hypotheses, we assume that

$$B(\psi) \in \mathcal{M}(\Omega), \quad \text{with } B(\psi)^+ \in L^\infty(\Omega), \tag{2.10}$$

i.e. that $B(\psi)$ is a Radon measure whose positive part belongs to $L^\infty(\Omega)$. Since $B(\psi)$ belongs to $H^{-1}(\Omega) + L^1(\Omega)$, hypothesis (2.10) of course implies that $B(\psi)^- \in H^{-1}(\Omega) + L^1(\Omega)$.

The second part of hypothesis (2.10) seems to be rather strong since it will be more natural to assume only that $B(\psi)^+ \in H^{-1}(\Omega) + L^1(\Omega)$. Nevertheless this assumption is in some sense natural in the framework of the Lewy–Stampacchia inequality (see the comment after the statement of the theorem in Section 3).

Let us recall that under hypotheses (2.1)–(2.5), problem (1.1) admits at least one solution. This existence result was proved in [2] and [3]. Actually not all the assumptions made here are necessary since, in order to prove this existence result, it is in particular sufficient to assume that $K(\psi) \cap L^\infty(\Omega)$ is non-empty in lieu of (2.4). The proofs of [2] and [3] are based on an approximation of the function H by bounded functions. We will give here a different proof of the same result, based on an approximation of the obstable problem by penalization, because this method is at the root of our proof of the Lewy–Stampacchia inequality.

Note also that, as proved in [1], Section 3.3.3, uniqueness holds true for the solution u of (1.1) when $H(x,s,\xi)$ is C^1 in s and ξ and satisfies the first assertion of (2.7) as well as the structure condition

$$\lambda - \frac{\partial H}{\partial s}(x,s,\xi) \ge \lambda_0 > 0. \tag{2.11}$$

All the above hypotheses hold true in the model case where $H(x,s,\xi) = h(x,s)|\xi|^2$, if $h(x,s)$ is a bounded Carathéodory function which is Lipschitz continuous in s (and which is non-increasing if one wants (2.11) to be satisfied), and if $\psi \in W^{1,\infty}(\Omega)$ with $-\operatorname{div}(A(x)D\psi) \in \mathcal{M}(\Omega)$ and $(-\operatorname{div}(A(x)D\psi))^+ \in L^\infty(\Omega)$.

3. The Lewy–Stampacchia inequality

In the present paper we prove that, under the above assumptions, the Lewy–Stampacchia inequality holds true for at least one solution of the obstacle problem (1.1), namely:

Theorem (Lewy–Stampacchia inequality). *Assume that hypotheses* (2.1)–(2.6) *and* (2.10) *hold true. Then, for at least one solution u of the obstacle problem* (1.1)*, the distribution $B(u)$ defined by* (2.9) *satisfies*

$$0 \leq B(u) \leq B(\psi)^+. \tag{3.1}$$

The Lewy-Stampacchia inequality (3.1) implies that the distribution $\mu = B(u)$ given by

$$-\mathrm{div}(A(x)Du) + \lambda u - H(x,u,Du) - f = \mu \tag{3.2}$$

satisfies

$$0 \leq \mu \leq B(\psi)^+,$$

and therefore that μ belongs to $L^\infty(\Omega)$, since in the present work we assume that $B(\psi)^+$ belongs to $L^\infty(\Omega)$ (hypothesis (2.10)). In order to study (3.2) considered as an equation in u for a given μ, and in particular the existence of a solution of this equation, a natural hypothesis is to assume that μ belongs to $L^\infty(\Omega)$, or at least to $L^{\frac{N}{2}+\delta}(\Omega)$ with $\delta > 0$ (see e.g. [2] and [4]). Since the assumption that $B(\psi)^+$ belongs to $L^\infty(\Omega)$ ensures that μ belongs to $L^\infty(\Omega)$, it can be considered as a reasonable hypothesis, even if a priori it would be more natural to assume only that $B(\psi)^+ \in H^{-1}(\Omega) + L^1(\Omega)$ in (2.10).

4. Proof of the existence of a solution

In this section we assume that hypotheses (2.1)–(2.5) hold true and prove by penalization that there exists a solution of the obstacle problem (1.1).

Under these hypotheses there exists at least one solution u_ε of the penalized problem

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ -\mathrm{div}(A(x)Du_\varepsilon) + \lambda u_\varepsilon - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^- = H(x,u_\varepsilon,Du_\varepsilon) + f \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{4.1}$$

The existence of a solution u_ε of (4.1) is easily obtained by using the proof used in [2] and [3]: One considers the case where in (4.1) the function H is replaced by a bounded function H_n (like e.g. $H_n = H/(1+n|H|)$). For such an approximation H_n, the Schauder fixed point theorem provides the existence of a solution $u_\varepsilon^n \in H_0^1(\Omega)$, which also belongs to $L^\infty(\Omega)$ by the weak maximum principle. It is then easy to prove, for ε fixed and n tending to infinity, an a priori estimate in $L^\infty(\Omega)$ and then an a priori estimate in $H_0^1(\Omega)$, similarly to the a priori estimates proved below. The strong $H_0^1(\Omega)$ convergence is then proved as below and provides the existence of a solution u_ε of (4.1).

Lemma 4.1. *Every solution u_ε of the penalized problem* (4.1) *satisfies*

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le \sup\{(C_0 + \|f\|_{L^\infty(\Omega)})/\lambda,\ \|\psi^+\|_{L^\infty(\Omega)}\}. \tag{4.2}$$

Proof of Lemma 4.1. Let us first give a proof of (4.2) based on the strong maximum principle, which is licit when everything (namely Ω, f, A, H, ψ, u) is sufficiently smooth (say C^2). In this case, let x_0 be a point where $u_\varepsilon(x)$ achieves its maximum. If $x_0 \in \partial\Omega$, then

$$\sup_{x\in\Omega} u_\varepsilon(x) = u_\varepsilon(x_0) = 0 \le \sup\{(C_0 + \|f\|_{L^\infty(\Omega)})/\lambda,\ \|\psi^+\|_{L^\infty(\Omega)}\}.$$

If $x_0 \in \Omega$, we have, at the point x_0,

$$\begin{aligned} -(\operatorname{div}(A(x)Du_\varepsilon))(x_0) + \lambda u_\varepsilon(x_0) - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-(x_0) \\ = H(x_0, u_\varepsilon(x_0), Du_\varepsilon(x_0)) + f(x_0). \end{aligned}$$

Since u_ε achieves its maximum at x_0, we have $Du_\varepsilon(x_0) = 0$ and $-\operatorname{div}(A(x_0)Du_\varepsilon(x_0)) \ge 0$, which, using the growth condition (2.5), implies

$$\lambda u_\varepsilon(x_0) - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-(x_0) \le C_0 + f(x_0) \le C_0 + \|f\|_{L^\infty(\Omega)}.$$

If $u_\varepsilon(x_0) \ge \psi(x_0)$, then $\lambda u_\varepsilon(x_0) \le C_0 + \|f\|_{L^\infty(\Omega)}$. On the other hand, if $u_\varepsilon(x_0) \le \psi(x_0)$, then $u_\varepsilon(x_0) \le \|\psi^+\|_{L^\infty(\Omega)}$. In all cases we have proved that

$$\sup_{x\in\Omega} u_\varepsilon(x) = u_\varepsilon(x_0) \le \sup\left\{(C_0 + \|f\|_{L^\infty(\Omega)})/\lambda,\ \|\psi^+\|_{L^\infty(\Omega)}\right\}. \tag{4.3}$$

Similarly, at a point x_0 where u_ε achieves its minimum, one proves that

$$\inf_{x\in\Omega} u_\varepsilon(x) = u_\varepsilon(x_0) \ge -(C_0 + \|f\|_{L^\infty(\Omega)})/\lambda), \tag{4.4}$$

which completes the proof of (4.2).

In the general case where we cannot use the above proof based on the strong maximum principle, we follow the proof given in [2] and [3]. We define

$$\varphi(s) = s\exp(Ms^2), \tag{4.5}$$

and we use in (4.1) the test function $\varphi(w_\varepsilon^+)$ with

$$w_\varepsilon = u_\varepsilon - m, \quad m = \sup\left\{(C_0 + \|f\|_{L^\infty(\Omega)})/\lambda,\ \|\psi^+\|_{L^\infty(\Omega)}\right\}, \quad M = \frac{C_1^2}{4\alpha^2}$$

in order to prove (4.3), and the test function $\varphi(-w_\varepsilon^-)$ with

$$w_\varepsilon = u_\varepsilon + m, \quad m = (C_0 + \|f\|_{L^\infty(\Omega)})/\lambda, \quad M = \frac{C_1^2}{4\alpha^2}$$

in order to prove (4.4). The computations to be done in these proofs are very similar to the computations we will perform in the proof of Lemma 4.2 below.

Lemma 4.2. *Every solution u_ε of the penalized problem* (4.1) *is bounded in* $H_0^1(\Omega)$ *independently of* ε.

Proof of Lemma 4.2. We use in (4.1) the test function $\varphi(u_\varepsilon - v^\star)$, with φ given by (4.5), $v^\star \in K(\psi) \cap L^\infty(\Omega)$ (for example $v^\star = \psi^+$), and M a constant to be specified later. We obtain

$$(4.6)\qquad \begin{cases} \displaystyle\int_\Omega A(x)Du_\varepsilon(Du_\varepsilon - Dv^\star)\varphi'(u_\varepsilon - v^\star)\,dx \\ \displaystyle +\lambda\int_\Omega (u_\varepsilon - v^\star)\varphi(u_\varepsilon - v^\star)\,dx - \frac{1}{\varepsilon}\int_\Omega (u_\varepsilon - \psi)^-\varphi(u_\varepsilon - v^\star)\,dx \\ \displaystyle = \int_\Omega H(x, u_\varepsilon, Du_\varepsilon)\varphi(u_\varepsilon - v^\star)\,dx + \int_\Omega (f - \lambda v^\star)\varphi(u_\varepsilon - v^\star)\,dx. \end{cases}$$

Using the coercivity (2.1), the fact that φ is increasing, the fact that

$$-(u_\varepsilon - \psi)^-\varphi(u_\varepsilon - v^\star) \geq 0 \quad \text{a.e. in } \Omega$$

(which follows from φ increasing and $v^\star \geq \psi$), and the growth condition (2.5), we obtain

$$(4.7)\qquad \begin{cases} \displaystyle\alpha\int_\Omega |Du_\varepsilon|^2\varphi'(u_\varepsilon - v^\star)\,dx \\ \displaystyle\leq \int_\Omega \big(C_0 + C_1|Du_\varepsilon|^2 + |f| + \lambda|v^\star|\big)|\varphi(u_\varepsilon - v^\star)|\,dx \\ \displaystyle + \int_\Omega A(x)Du_\varepsilon Dv^\star\varphi'(u_\varepsilon - v^\star)\,dx \\ \displaystyle\leq \int_\Omega (C_0 + |f| + \lambda|v^\star|)|\varphi(u_\varepsilon - v^\star)|\,dx + C_1\int_\Omega |Du_\varepsilon|^2|\varphi(u_\varepsilon - v^\star)|\,dx \\ \displaystyle + \int_\Omega \|A\|_{L^\infty(\Omega)}|Du_\varepsilon||Dv^\star||\varphi'(u_\varepsilon - v^\star)|\,dx. \end{cases}$$

Choosing M such that

$$\alpha\varphi'(s) - C_1|\varphi(s)| = \exp(Ms^2)(\alpha + 2\alpha Ms^2 - C_1|s|) \geq \exp(Ms^2)\frac{\alpha}{2} \geq \frac{\alpha}{2} \quad \forall s \in \mathbb{R},$$

which holds true whenever $M \geq C_1^2/4\alpha^2$, and using $f \in L^\infty(\Omega)$, $v^\star \in L^\infty(\Omega) \cap H^1(\Omega)$, $\varphi' \geq 1$, and Lemma 4.1, we deduce from (4.7) that u_ε is bounded in $H_0^1(\Omega)$.

Lemma 4.3. *There exists a subsequence, still denoted by* ε, *and a function* $u \in K(\psi) \cap L^\infty(\Omega)$ *such that* u_ε *tends to* u *strongly in* $H_0^1(\Omega)$ *and a.e. in* Ω.

Proof of Lemma 4.3. Using Lemmas 4.1 and 4.2 and the Rellich compactness theorem, there exist a subsequence (still denoted by ε) and a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$(4.8)\qquad u_\varepsilon \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \quad \text{weakly* in } L^\infty(\Omega) \ \text{ and a.e. in } \Omega.$$

Taking $(u_\varepsilon - v^\star)$, with $v^\star \in K(\psi) \cap L^\infty(\Omega)$, as test function in (4.1), we obtain, since all the other terms are bounded, that

$$\left| \frac{1}{\varepsilon} \int_\Omega -(u_\varepsilon - \psi)^- (u_\varepsilon - v^\star)\, dx \right| \le C,$$

which implies, writing $u_\varepsilon - v^\star = (u_\varepsilon - \psi) + (\psi - v^\star)$, that

$$\frac{1}{\varepsilon} \int_\Omega |(u_\varepsilon - \psi)^-|^2\, dx \le C$$

since $v^\star \ge \psi$. In view of (4.8) this implies $u \ge \psi$, and therefore we have

$$u \in K(\psi) \cap L^\infty(\Omega). \tag{4.9}$$

Since $u \in K(\psi) \cap L^\infty(\Omega)$, we can take $v^\star = u$ in (4.6). Using the facts that

$$(u_\varepsilon - u)\varphi(u_\varepsilon - u) \ge 0 \quad \text{and} \quad -(u_\varepsilon - \psi)^- \varphi(u_\varepsilon - u) \ge 0$$

and the growth condition (2.5), we obtain

$$\begin{cases} \displaystyle\int_\Omega A(x)(Du_\varepsilon - Du)(Du_\varepsilon - Du)\varphi'(u_\varepsilon - u)\, dx \\ \displaystyle\le \int_\Omega (C_0 + C_1|Du_\varepsilon|^2 + |f| + \lambda|u|)|\varphi(u_\varepsilon - u)|\, dx \\ \displaystyle- \int_\Omega A(x)Du(Du_\varepsilon - Du)\varphi'(u_\varepsilon - u)\, dx. \end{cases}$$

We write $|Du_\varepsilon|^2 \le 2|Du_\varepsilon - Du|^2 + 2|Du|^2$, use the coercivity (2.1), and finally choose M such that

$$\alpha\varphi'(s) - 2C_1|\varphi(s)| = \exp(Ms^2)(\alpha + 2\alpha M s^2 - 2C_1|s|) \ge \exp(Ms^2)\frac{\alpha}{2} \ge \frac{\alpha}{2} \quad \forall s \in \mathbb{R},$$

which holds true whenever $M \ge C_1^2/\alpha^2$. We obtain

$$\begin{cases} \displaystyle\frac{\alpha}{2} \int_\Omega |Du_\varepsilon - Du|^2\, dx \\ \displaystyle\le \int_\Omega (C_0 + 2C_1|Du|^2 + |f| + \lambda|u|)|\varphi(u_\varepsilon - u)|\, dx \\ \displaystyle- \int_\Omega A(x)Du(Du_\varepsilon - Du)\varphi'(u_\varepsilon - u)\, dx, \end{cases}$$

from which we deduce Lemma 4.3 in view of (4.8).

Taking $(v - u_\varepsilon)$, with $v \in K(\psi) \cap L^\infty(\Omega)$, as a test function in (4.1), it is then easy to pass to the limit in each term and to obtain that u is a solution of the obstacle problem (1.1).

5. Proof of the Lewy–Stampacchia inequality

To simplify the notation, we set in this section

$$h = B(\psi) = -\mathrm{div}(A(x)D\psi) + \lambda\psi - H(x, \psi, D\psi) - f, \tag{5.1}$$

where the operator B is defined by (2.9).

We rewrite the penalized problem (4.1) as

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ -\mathrm{div}(A(x)(Du_\varepsilon - D\psi)) + \lambda(u_\varepsilon - \psi) - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^- \\ = H(x, u_\varepsilon, Du_\varepsilon) - H(x, \psi, D\psi) - h^+ + h^- \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{5.2}$$

Lemma 5.1. *Under the hypotheses of the theorem, for every solution u_ε of the penalized problem* (4.1)*, we have, when $\varepsilon \le 1/(2C_2)$,*

$$\left\| \frac{1}{\varepsilon}(u_\varepsilon - \psi)^- \right\|_{L^\infty(\Omega)} \le 2\|h^+\|_{L^\infty(\Omega)}, \tag{5.3}$$

where the constant C_2 appears in hypothesis (2.6).

Proof of Lemma 5.1. As in the proof of Lemma 4.1, let us first give a proof based on the strong maximum principle in the case where everything is sufficiently smooth. In this case, let x_0 be a point where $-\frac{1}{\varepsilon}(u_\varepsilon(x) - \psi(x))$ achieves its maximum (since $\frac{1}{\varepsilon}(u_\varepsilon - \psi)^- = \sup\{-\frac{1}{\varepsilon}(u_\varepsilon - \psi), 0\}$, it will not be necessary to consider here the point where $-\frac{1}{\varepsilon}(u_\varepsilon(x) - \psi(x))$ achieves its minimum). If $x_0 \in \partial\Omega$, Lemma 5.1 is proved since $u_\varepsilon = 0$ on $\partial\Omega$ and $\psi \le 0$ on $\partial\Omega$ implies $-\frac{1}{\varepsilon}(u_\varepsilon - \psi)^-(x_0) = 0$. If $x_0 \in \Omega$, we have at the point x_0

$$\begin{cases} (-\mathrm{div}(A(x)(Du_\varepsilon - D\psi)))(x_0) + \lambda(u_\varepsilon - \psi)(x_0) - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-(x_0) \\ = H(x_0, u_\varepsilon(x_0), Du_\varepsilon(x_0)) - H(x_0, \psi(x_0), D\psi(x_0)) - h^+(x_0) + h^-(x_0). \end{cases}$$

Since $u_\varepsilon - \psi$ achieves its minimum at x_0, we have

$$D(u_\varepsilon - \psi)(x_0) = 0 \quad \text{and} \quad -\mathrm{div}(A(x_0)D(u_\varepsilon - \psi)(x_0)) \le 0.$$

On the other hand using hypothesis (2.6) we have

$$\begin{cases} H(x_0, u_\varepsilon(x_0), Du_\varepsilon(x_0)) - H(x_0, \psi(x_0), D\psi(x_0)) \\ = H(x_0, u_\varepsilon(x_0), D\psi(x_0)) - H(x_0, \psi(x_0), D\psi(x_0)) \ge -C_2|u_\varepsilon(x_0) - \psi(x_0)|, \end{cases}$$

and therefore, since $h^-(x_0) \ge 0$, we have

$$\lambda(u_\varepsilon - \psi)(x_0) - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-(x_0) \ge -C_2|(u_\varepsilon - \psi)(x_0)| - h^+(x_0).$$

Assuming that $(u_\varepsilon - \psi)(x_0) \le 0$ (if not we have $\frac{1}{\varepsilon}(u_\varepsilon - \psi)^-(x_0) = 0$ and Lemma 5.1 is proved), this implies that we have

$$\left(\lambda + \frac{1}{\varepsilon} - C_2\right)(u_\varepsilon - \psi)^-(x_0) \le h^+(x_0).$$

Therefore, since $\lambda > 0$, we have, for $\varepsilon \leq 1/(2C_2)$,

$$\sup_{x\in\Omega} \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-(x) = \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-(x_0) \leq 2h^+(x_0) \leq 2\|h^+\|_{L^\infty(\Omega)},$$

which proves Lemma 5.1.

In the general case where we cannot use the above proof based on the strong maximum principle, we use in (5.2) the test function $\varphi(-w_\varepsilon^-)$, with φ given by (4.5) and

$$\text{(5.4)} \qquad \begin{cases} w_\varepsilon = -\dfrac{1}{\varepsilon}(u_\varepsilon - \psi)^- + m, \quad m = 2\|h^+\|_{L^\infty(\Omega)} + \delta, \\ M = M_\varepsilon = \varepsilon^2 \dfrac{\left(C_3 + C_4^2/(2\delta)\right)^2}{8\alpha^2}, \end{cases}$$

where $\delta > 0$ is given.

Since w_ε^- belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, the use of $\varphi(-w_\varepsilon^-)$ as test function in (5.2) is licit. Using hypothesis (2.6) and the Young inequality, we obtain, denoting by $\langle .,. \rangle$ the duality pairing between an element of $H^{-1}(\Omega) + L^1(\Omega)$ and an element of $H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$\text{(5.5)} \quad \begin{cases} \displaystyle\int_\Omega A(x)D(u_\varepsilon - \psi)D(-w_\varepsilon^-)\varphi'(-w_\varepsilon^-)\,dx + \lambda \int_\Omega (u_\varepsilon - \psi)\varphi(-w_\varepsilon^-)\,dx \\ \displaystyle + \int_\Omega -\frac{1}{2\varepsilon}(u_\varepsilon - \psi)^-\varphi(-w_\varepsilon^-)\,dx + \int_\Omega \left(-\frac{1}{2\varepsilon}(u_\varepsilon - \psi)^- + \frac{m}{2}\right)\varphi(-w_\varepsilon^-)\,dx \\ \displaystyle \leq C_2 \int_\Omega |u_\varepsilon - \psi||\varphi(-w_\varepsilon^-)|\,dx \\ \displaystyle + \left(C_3 + \frac{C_4^2}{2\delta}\right)\int_\Omega |Du_\varepsilon - D\psi|^2|\varphi(-w_\varepsilon^-)|\,dx + \frac{\delta}{2}\int_\Omega |\varphi(-w_\varepsilon^-)|\,dx \\ \displaystyle - \int_\Omega h^+\varphi(-w_\varepsilon^-)\,dx + \langle h^-, \varphi(-w_\varepsilon^-)\rangle + \frac{m}{2}\int_\Omega \varphi(-w_\varepsilon^-)\,dx, \end{cases}$$

or in other terms

$$I_\varepsilon + II_\varepsilon + III_\varepsilon + \frac{1}{2}\int_\Omega w_\varepsilon \varphi(-w_\varepsilon^-)\,dx \leq IV_\varepsilon + V_\varepsilon + VI_\varepsilon + VII_\varepsilon + VIII_\varepsilon + IX_\varepsilon.$$

Since $h^- \geq 0$ and $\varphi(-w_\varepsilon^-) \leq 0$, we have

$$VIII_\varepsilon = \langle h^-, \varphi(-w_\varepsilon^-)\rangle \leq 0.$$

Define

$$F_\varepsilon = \{x \in \Omega : w_\varepsilon(x) < 0\}.$$

Then $\varphi(-w_\varepsilon^-) = 0$ on $\Omega \backslash F_\varepsilon$, and since $m > 0$, we necessarily have $-\frac{1}{\varepsilon}(u_\varepsilon - \psi)^- < 0$ on F_ε, i.e. $u_\varepsilon - \psi < 0$ on F_ε. Therefore

$$II_\varepsilon = \lambda \int_\Omega (u_\varepsilon - \psi)\varphi(-w_\varepsilon^-)\,dx \geq 0,$$

while

$$IV_\varepsilon - III_\varepsilon = \int_\Omega \left(C_2 - \frac{1}{2\varepsilon}\right)(u_\varepsilon - \psi)^- |\varphi(-w_\varepsilon^-)|\, dx \le 0$$

whenever $\varepsilon \le 1/(2C_2)$. Similarly

$$VI_\varepsilon + VII_\varepsilon + IX_\varepsilon = \int_\Omega \left(\frac{\delta}{2} + h^+ - \frac{m}{2}\right)|\varphi(-w_\varepsilon^-)|dx \le 0$$

in view of the definition of $m = 2\|h^+\|_{L^\infty(\Omega)} + \delta$. Finally, since

$$D(-w_\varepsilon^-) = Dw_\varepsilon = D\left(-\frac{1}{\varepsilon}(u_\varepsilon - \psi)^-\right) = \frac{1}{\varepsilon}D(u_\varepsilon - \psi) \quad \text{on } F_\varepsilon,$$

while $D(-w_\varepsilon^-) = 0$ on $\Omega\backslash F_\varepsilon$, we have

$$\begin{cases} I_\varepsilon - V_\varepsilon = \displaystyle\int_{F_\varepsilon} A(x)D(u_\varepsilon - \psi)D(u_\varepsilon - \psi)\frac{1}{\varepsilon}\,\varphi'(-w_\varepsilon^-)\, dx \\ \quad -\left(C_3 + \dfrac{C_4^2}{2\delta}\right)\displaystyle\int_{F_\varepsilon} |D(u_\varepsilon - \psi)|^2|\varphi(-w_\varepsilon^-)|\, dx \\ \ge \displaystyle\int_{F_\varepsilon} |D(u_\varepsilon - \psi)|^2\left(\frac{\alpha}{\varepsilon}\,\varphi'(-w_\varepsilon^-) - \left(C_3 + \frac{C_4^2}{2\delta}\right)|\varphi(-w_\varepsilon^-)|\right) dx \ge 0, \end{cases}$$

since in view of the definition of $M = M_\varepsilon = \varepsilon^2(C_3 + C_4^2/(2\delta))^2/(8\alpha^2)$, we have

$$\frac{\alpha}{\varepsilon}\,\varphi'(s) - \left(C_3 + \frac{C_4^2}{2\delta}\right)|\varphi(s)| = \exp(Ms^2)\left(\frac{\alpha}{\varepsilon} + 2\frac{\alpha}{\varepsilon}Ms^2 - \left(C_3 + \frac{C_4^2}{2\delta}\right)|s|\right) \ge 0$$

for every $s \in \mathbb{R}$.

This proves that when $\varepsilon \le 1/(2C_2)$, we have $\int_\Omega w_\varepsilon\varphi(-w_\varepsilon^-)\, dx \le 0$, i.e. $w_\varepsilon \ge 0$ or $\frac{1}{\varepsilon}(u_\varepsilon - \psi)^- \le m = 2\|h^+\|_{L^\infty(\Omega)} + \delta$. Since this result holds true for every $\delta > 0$, this proves (5.3) and Lemma 5.1.

We will now assume for a while that, in addition to hypothesis (2.10) assumed in the theorem, we have

$$\text{(5.6)} \qquad \begin{cases} h = h^\oplus - h^\ominus, \quad \text{with} \quad h^\oplus \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad h^\oplus \ge 0, \\ \qquad\qquad\qquad\qquad\quad h^\ominus \in H^{-1}(\Omega) + L^1(\Omega), \quad h^\ominus \ge 0. \end{cases}$$

In a further step, we will remove this hypothesis. Note that hypothesis (5.6) implies (2.10) (but that the converse is not true, since $h^\oplus$ is assumed to belong to $H_0^1(\Omega)$ in (5.6)), and that $h^\oplus$ and $h^\ominus$ are not supposed to coincide with h^+ and h^-, although this can be the case as well.

Lemma 5.2. *In addition to the hypotheses of the theorem, assume that hypothesis* (5.6) *holds true. Then, for every solution u_ε of the penalized problem* (4.1)*, the function z_ε defined by*

$$\text{(5.7)} \qquad z_\varepsilon = h^\oplus - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-$$

satisfies

$$z_\varepsilon^- \to 0 \quad \textit{strongly in } L^2(\Omega). \tag{5.8}$$

Proof of Lemma 5.2. Since $h^\oplus$ is assumed to belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$ by hypothesis (5.6), the function z_ε belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and $-z_\varepsilon^-$ can be used as test function in (5.2). Using hypothesis (2.6), we obtain

$$\begin{cases} \int_\Omega A(x)D(u_\varepsilon - \psi)D(-z_\varepsilon^-)\,dx + \lambda \int_\Omega (u_\varepsilon - \psi)(-z_\varepsilon^-)\,dx \\ + \int_\Omega \left(h^\oplus - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^-\right)(-z_\varepsilon^-)\,dx \\ \le C_2 \int_\Omega |u_\varepsilon - \psi| z_\varepsilon^-\,dx + C_3 \int_\Omega |Du_\varepsilon - D\psi|^2 z_\varepsilon^-\,dx \\ + C_4 \int_\Omega |Du_\varepsilon - D\psi| z_\varepsilon^-\,dx + \langle h^\ominus, -z_\varepsilon^-\rangle, \end{cases}$$

or in other terms

$$I_\varepsilon + II_\varepsilon + \int_\Omega |z_\varepsilon^-|^2\,dx \le III_\varepsilon + IV_\varepsilon + V_\varepsilon + VI_\varepsilon.$$

Since $h^\ominus \ge 0$, we have

$$VI_\varepsilon = \langle h^\ominus, -z_\varepsilon^-\rangle \le 0.$$

Define

$$E_\varepsilon = \{x \in \Omega : z_\varepsilon(x) < 0\}.$$

Then $z_\varepsilon^- = 0$ on $\Omega \backslash E_\varepsilon$, and since $h^\oplus \ge 0$, we necessarily have $-\frac{1}{\varepsilon}(u_\varepsilon - \psi)^- < 0$ on E_ε, i.e. $u_\varepsilon - \psi < 0$ on E_ε. Therefore

$$II_\varepsilon = \lambda \int_\Omega (u_\varepsilon - \psi)(-z_\varepsilon^-)\,dx \ge 0,$$

while

$$III_\varepsilon = C_2 \int_{E_\varepsilon} (u_\varepsilon - \psi)^- z_\varepsilon^-\,dx = C_2\varepsilon \int_{E_\varepsilon} \left(h^\oplus - z_\varepsilon\right) z_\varepsilon^-\,dx = C_2\varepsilon \int_{E_\varepsilon} \left(h^\oplus + z_\varepsilon^-\right) z_\varepsilon^-\,dx.$$

On the other hand,

$$D(-z_\varepsilon^-) = Dz_\varepsilon = Dh^\oplus - \frac{1}{\varepsilon}D(u_\varepsilon - \psi)^- = Dh^\oplus + \frac{1}{\varepsilon}D(u_\varepsilon - \psi) \quad \text{on } E_\varepsilon,$$

while $D(-z_\varepsilon^-) = 0$ on $\Omega \backslash E_\varepsilon$. Therefore we have

$$I_\varepsilon = \int_{E_\varepsilon} A(x)D(u_\varepsilon - \psi)Dz_\varepsilon dx = \varepsilon \int_{E_\varepsilon} A(x)D\left(z_\varepsilon - h^\oplus\right)Dz_\varepsilon dx.$$

Since we have $Du_\varepsilon - D\psi = \varepsilon(Dz_\varepsilon - Dh^\oplus)$ on E_ε, we have proved that

$$(5.9)\qquad \begin{cases} \varepsilon \displaystyle\int_{E_\varepsilon} A(x)\big(Dz_\varepsilon - Dh^\oplus\big)D\big(z_\varepsilon - h^\oplus\big)\,dx + \int_\Omega |z_\varepsilon^-|^2\,dx \\ \leq -\varepsilon \displaystyle\int_{E_\varepsilon} A(x)D\big(z_\varepsilon - h^\oplus\big)Dh^\oplus dx + C_2\varepsilon \int_{E_\varepsilon} \big(h^\oplus + z_\varepsilon^-\big)z_\varepsilon^-\,dx \\ +C_3\varepsilon^2 \displaystyle\int_{E_\varepsilon} \big|Dz_\varepsilon - Dh^\oplus\big|^2 z_\varepsilon^-\,dx + C_4\varepsilon \int_{E_\varepsilon} \big|Dz_\varepsilon - Dh^\oplus\big| z_\varepsilon^-\,dx. \end{cases}$$

Since $h^+ \leq h^\oplus$, the $L^\infty(\Omega)$ bound (5.3) of Lemma 5.1 implies that for $\varepsilon \leq 1/(2C_2)$, we have

$$\|z_\varepsilon\|_{L^\infty(\Omega)} \leq \|h^\oplus\|_{L^\infty(\Omega)} + \left\|\frac{1}{\varepsilon}(u_\varepsilon - \psi)^-\right\|_{L^\infty(\Omega)} \leq 3\|h^\oplus\|_{L^\infty(\Omega)} = C^\star.$$

Using the coercivity (2.1) we deduce from (5.9) that

$$\begin{cases} \varepsilon\alpha \displaystyle\int_{E_\varepsilon} \big|Dz_\varepsilon - Dh^\oplus\big|^2 dx + \int_\Omega |z_\varepsilon^-|^2 dx \\ \leq \varepsilon\|A\|_{L^\infty(\Omega)} \displaystyle\int_\Omega \big|Dz_\varepsilon - Dh^\oplus\big||Dh^\oplus| dx + \varepsilon C_2 C^\star \int_{E_\varepsilon} \big(h^\oplus + C^*\big)dx \\ +\varepsilon^2 C_3 C^\star \displaystyle\int_{E_\varepsilon} \big|Dz_\varepsilon - Dh^\oplus\big|^2 dx + \varepsilon C_4 C^\star \int_{E_\varepsilon} \big|Dz_\varepsilon - Dh^\oplus\big| dx, \end{cases}$$

which using Young inequality twice implies that for ε sufficiently small (say $\varepsilon \leq \alpha/(2C_3C^\star)$), we have

$$\int_\Omega |z_\varepsilon^-|^2\,dx \leq \varepsilon C,$$

which proves Lemma 5.2.

5.1. A first step in the direction of the Lewy–Stampacchia inequality

From the definition (5.7) of z_ε, we deduce that

$$(5.10)\qquad 0 \leq \frac{1}{\varepsilon}(u_\varepsilon - \psi)^- = h^\oplus - z_\varepsilon \leq h^\oplus + z_\varepsilon^-.$$

But from the penalized equation (4.1) and the definition (2.9) of operator B we have

$$\frac{1}{\varepsilon}(u_\varepsilon - \psi)^- = B(u_\varepsilon),$$

and from Lemma 4.3 we deduce that

$$B(u_\varepsilon) \to B(u) \quad \text{strongly in } H^{-1}(\Omega) + L^1(\Omega),$$

where u is a solution of the obstacle problem (1.1). Therefore, by (5.10) and Lemma 5.2, we have proved that

$$(5.11)\qquad 0 \leq B(u) \leq h^\oplus$$

holds true when (5.6) is assumed to hold true.

This is a first step in the direction of the Lewy–Stampacchia inequality: When in addition to hypothesis (2.10), which asserts that $B(\psi)^+ \in L^\infty(\Omega)$, one assumes that $B(\psi)^+ \in L^\infty(\Omega) \cap H_0^1(\Omega)$, which implies that (5.6) holds true with $h^\oplus = h^+ = B(\psi)^+$ and $h^\ominus = h^- = B(\psi)^-$, inequality (5.11) is nothing but the Lewy–Stampacchia inequality (3.1).

5.2. End of the proof of the Lewy–Stampacchia inequality

We now prove that the Lewy–Stampacchia inequality holds true for one solution of (1.1) under the sole hypotheses of the theorem.

When only hypothesis (2.10) holds true, i.e. when

$$B(\psi) \in \mathcal{M}(\Omega), \quad \text{with } B(\psi)^+ \in L^\infty(\Omega),$$

we consider a sequence $h_n^\oplus$ such that

$$\begin{cases} h_n^\oplus \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad h_n^\oplus \geq 0, \quad \|h_n^\oplus\|_{L^\infty(\Omega)} \leq C, \\ h_n^\oplus \rightharpoonup B(\psi)^+ \quad \text{weakly* in } L^\infty(\Omega) \quad \text{and a.e. in } \Omega, \end{cases}$$

and we define

$$h_n^\ominus = B(\psi)^-, \quad h_n = h_n^\oplus - h_n^\ominus, \quad f_n = f + B(\psi)^+ - h_n^\oplus,$$

$$B_n(v) = -\mathrm{div}(A(x)Dv) + \lambda v - H(x, v, Dv) - f_n, \quad \forall v \in H^1(\Omega).$$

We now consider the obstacle problem $(1.1)_n$ where in (1.1) f is replaced by f_n. Observe that $f_n \in L^\infty(\Omega)$ and that

$$B_n(\psi) = B(\psi) + f - f_n = B(\psi) - B(\psi)^+ + h_n^\oplus = h_n^\oplus - h_n^\ominus.$$

Therefore (5.6) is now satisfied.[1] The result obtained above proves that there exists a solution u_n of $(1.1)_n$ for which (5.11) holds true, namely

$$0 \leq B_n(u_n) \leq h_n^\oplus. \tag{5.12}$$

Since $B_n(v) = B(v) + f - f_n$ and since f_n satisfies

$$\|f_n\|_{L^\infty(\Omega)} \leq C, \quad f_n \rightharpoonup f \quad \text{weakly* in } L^\infty(\Omega) \text{ and a.e. in } \Omega,$$

results very similar to those proved in Section 4 above imply that

$$u_n \to u \quad \text{strongly in } H_0^1(\Omega),$$

where u is a solution of (1.1). Passing to the limit in (5.12) we obtain

$$0 \leq B(u) \leq B(\psi)^+,$$

which is the Lewy–Stampacchia inequality.

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[1] We do not know whether $h_n^+ = h_n^\oplus$ or not; this is why we introduced the notation $h^\oplus$ in hypothesis (5.6).

Therefore, when the hypotheses of the theorem hold true, we have proved that there exists at least one solution of the obstacle problem (1.1) which satisfies the Lewy–Stampacchia inequality.

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