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Regularity of almost minimizers of quasi-convex variational integrals with subquadratic growth

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Abstract. We prove a small excess regularity theorem for almost minimizers of a quasi-convex variational integral of subquadratic growth. The proof is direct, and it yields an optimal modulus of continuity for the derivative of the almost minimizer. The result is new for general almost minimizers, and in the case of absolute minimizers it considerably simplifies the existing proof.

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Key words. quasi-convexity – almost minimizers – partial regularity – subquadratic growth

1. Introduction

In this paper we study the regularity properties of almost minimizers of variational integrals

$$\mathcal{F}(u) = \int_U f(Du)\,dx\,, \tag{1}$$

where U is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and $u : U \to \mathbb{R}^N$ is a map defined on U taking values in $\mathbb{R}^N$, $N \geq 1$. Here

$$Du = \left(\frac{\partial u^i}{\partial x_\alpha}\right)_{1\leq\alpha\leq n,\ 1\leq i\leq N} : U \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$$

is the derivative of the function u and $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ denotes the space of all $n \times N$ matrices. The function $f\colon \mathrm{Hom}(R^n, \mathbb{R}^N) \to \mathbb{R}$ is a C^2-integrand of subquadratic p-growth, i.e. for some $L > 0$ and $p \in (1, 2)$ there holds:

$$|f(A)| \leq L(1 + |A|^p), \tag{2}$$

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for any $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. Further we assume that f is strictly quasi-convex, meaning that there holds

$$\int_U \big(f(A + D\varphi) - f(A)\big)\, dx \geq \lambda \int_U \big(1 + |A|^2 + |D\varphi|^2\big)^{\frac{p-2}{2}} |D\varphi|^2\, dx, \qquad (3)$$

for some fixed constant $\lambda > 0$, for any $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ and $\varphi \in C_0^1(U, \mathbb{R}^N)$. The condition of quasi-convexity was introduced by Morrey in [28]; for the notion of strict quasi-convexity in the case $p \geq 2$ we refer to the paper of Evans [19]. In [10] and [11] this condition was generalized to the case $1 < p < 2$ (i.e. the case of integrands with subquadratic growth) in the form given in (3).

For every $u \in W^{1,p}(U, \mathbb{R}^N)$ the variational integral $\mathcal{F}(u)$ is defined. We say that u is an $(\mathcal{F}, \omega)$-*minimizing function* (or an $(\mathcal{F}, \omega)$-*minimizer*) if there holds

$$\begin{aligned} &\int_{B_\rho(x_0)} f(Du)\, dx \\ &\leq \int_{B_\rho(x_0)} f(Du + D\varphi)\, dx + \omega(\rho) \int_{B_\rho(x_0)} \big(1 + |D\varphi|^p + |Du|^p\big)\, dx, \end{aligned} \qquad (4)$$

for any ball $B_\rho(x_0) \subset\subset U$ and any $\varphi \in W^{1,p}(U, \mathbb{R}^N)$ with $\mathrm{spt}\,\varphi \subset B_\rho(x_0)$. Here the modulus of almost minimality $\omega\colon [0, \infty) \to [0, \infty)$ is a non-decreasing bounded function with $\lim_{\rho \downarrow 0} \omega(\rho) = \omega(0) = 0$ satisfying the Dini condition

$$\Omega(r) = \int_0^r \frac{\sqrt{\omega(\rho)}}{\rho}\, d\rho < \infty, \qquad (5)$$

for any $r > 0$. In particular, note that the case $\omega \equiv 0$ corresponds to the case of $\mathcal{F}$-minimizers.

For examples, and motivation for studying almost minimizers (which are also immediately seen to be applicable to the subquadratic case) we refer the reader to the papers [15] and [5]. In particular we note here the example of Sverak given in [31] of a quasi-convex integrand with subquadratic growth which is not convex, nor even polyconvex.

In the fundamental paper [19] Evans considered strictly quasi-convex integrands f in the superquadratic case (i.e. $p \geq 2$). He proved that if f is of class C^2 and satisfies the growth assumption

$$|D^2 f(A)| \leq L(1 + |A|^{p-2}), \qquad (6)$$

for any $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ (note that this is stronger than (2)), then any $\mathcal{F}$-minimizing function $u \in W^{1,p}(U, \mathbb{R}^N)$ is of class $C^{1,\alpha}$ on some open subset $U_0 \subset U$ whose complement has n-dimensional Lebesgue measure zero. In [1] this result was generalized to integrands f of p-growth with $p \geq 2$ satisfying the weaker assumption (2); in that paper the authors were also able to handle the case of integrands f depending also on x and u, i.e. the case of quasi-convex variational integrals of the form $\mathcal{F}(u) = \int_U f(x, u(x), Du(x))\, dx$.

In [11] the result of Evans (under the weaker growth assumption (2)) was extended to the case of quasi-convex variational integrals with an integrand of subquadratic growth; cf. the earlier paper [10] for a treatment of the particular case $\frac{2n}{n+2} < p < 2$.

The proofs of the regularity results for $\mathcal{F}$-minimizing functions u of the above-mentioned papers are based on a blow-up technique originally developed by De Giorgi [13] and Almgren [3], [4] in the setting of geometric measure theory, and first adapted to the setting of partial-regularity theory for elliptic systems by Giusti–Miranda [24]. The key step is to establish a certain *excess-decay estimate* for the *excess function* Φ. In the case $p \geq 2$ this function is given by

$$\Phi(x_0, \rho) = \left(\fint_{B_\rho(x_0)} \left(|Du - (Du)_{x_0,\rho}|^2 + |Du - (Du)_{x_0,\rho}|^p \right) dx \right)^{1/2},$$

whereas in the case $1 < p < 2$ one uses

$$\Phi(x_0, \rho) = \left(\fint_{B_\rho(x_0)} |V(Du) - V((Du)_{x_0,\rho})|^2 \, dx \right)^{1/2}, \tag{7}$$

where

$$V(A) = (1 + |A|^2)^{\frac{p-2}{4}} A \quad \text{for } A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N).$$

It is shown that if $\Phi(x_0, \rho)$ is small enough on a ball $B_\rho(x_0) \subset\subset U$, then for some fixed $\vartheta \in (0, 1)$ one has the excess improvement $\Phi(x_0, \vartheta\rho) \leq c\vartheta\Phi(x_0, \rho)$. Iteration of this result yields the excess-decay estimate which implies the regularity result. In each of the above-mentioned papers the excess improvement is established by an indirect argument.

Partial regularity for $(\mathcal{F}, \omega)$-minimizing functions was first established in [15] in the case of quasi-convex variational integrands of quadratic growth and later in [17] for the case of p-growth with $p \geq 2$ under the stronger growth assumption (6). Both papers use the method of $\mathcal{A}$-harmonic approximation, which in particular enables the authors to prove the excess-improvement estimate directly. We give a brief discussion of this technique in the following, but remark here that the technique has its origins in Simon's proof of the regularity theorem of Allard ([2]), see [30, Section 23], and cf. [6]. The technique has been successfully applied in the framework of geometric measure theory (see [18]), and to obtain partial-regularity results for general elliptic systems (see [16], [14], [25], [26], [27]).

We consider a bilinear form on $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ which is (strongly) elliptic in the sense of Legendre–Hadamard, i.e. for all $\eta \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^N$ there holds:

$$\mathcal{A}(\eta \otimes \xi, \eta \otimes \xi) \geq \kappa |\eta|^2 |\xi|^2,$$

for some positive constant κ. The method of $\mathcal{A}$-harmonic approximation is based on the fact that one is able to obtain a good approximation of functions $g \in W^{1,2}(B, \mathbb{R}^N)$, which are approximately $\mathcal{A}$-harmonic in a certain sense by $\mathcal{A}$-harmonic functions $h \in W^{1,2}(B, \mathbb{R}^N)$, in both the L^2-topology and in the weak topology of $W^{1,2}$. Here $h \in W^{1,2}(B, \mathbb{R}^N)$ is called $\mathcal{A}$-harmonic on B if there holds

$$\int_B \mathcal{A}(Dh, D\varphi) \, dx = 0 \quad \text{for any } \varphi \in C_0^1(B, \mathbb{R}^N).$$

In the situations considered in the above-mentioned papers [15] and [17] the required approximate $\mathcal{A}$-harmonicity of $(\mathcal{F}, \omega)$-minimizing functions $u \in W^{1,p}(U, \mathbb{R}^N)$ is a simple consequence of the minimizing property. A comparison argument involving u and the $\mathcal{A}$-harmonic approximation h implies an estimate (in some sense an L^2-excess-improvement), which when combined with Caccioppoli's inequality yields the desired excess-improvement estimate.

In the present article we extend this approach to the case of quasi-convex variational integrands of subquadratic growth fullfilling the weaker growth assumption (2). The main result then reads as follows:

Theorem. *Suppose f is a strictly quasi-convex function of class C^2 satisfying the subquadratic growth condition* (2). *Suppose further that $u \in W^{1,p}(U, \mathbb{R}^N)$ is $(\mathcal{F}, \omega)$-minimizing where the modulus $\omega\colon [0, \infty) \to [0, \infty)$ determining the almost minimality property is a non-decreasing bounded function such that $\lim_{\rho \downarrow 0} \omega(\rho) = \omega(0) = 0$, $\rho \mapsto \rho^{-2\beta}\omega(\rho)$ is non-increasing for some $\beta \in (0, 1)$, and which satisfies the Dini condition* (5). *Then there exists an open subset $U_0 \subset U$ of full Lebesgue measure such that u is of class $C^1_{\mathrm{loc}}(U_0, \mathbb{R}^N)$. Moreover, in a neighbourhood of any point $x_0 \in U_0$ the derivative Du of u has modulus of continuity given by $\rho \mapsto \rho^{\alpha} + \Omega(\rho)$ for any $\alpha \in (\beta, 1)$.*

The common feature in all the cases discussed above is the fact that one is in a situation where the considered maps $g\colon B \to \mathbb{R}^N$ admit a bound $\int_B |Dg|^2\,dx \le 1$ for the L^2-norm of the gradient. In the subquadratic case we are dealing with functions belonging to the Sobolev space $W^{1,p}$ with $1 < p < 2$, and hence one only has a bound for $\int_B |Dg|^p\,dx$, and this causes several problems. The main problem is to establish a suitable version of the $\mathcal{A}$-harmonic approximation lemma. For $p \ge 2$ it is straightforward to adapt the standard $\mathcal{A}$-harmonic approximation lemma by utilizing L^2-theory combined with the (standard) Sobolev inequality. For $p < 2$ we obviously do not have access to L^2-theory for functions in $W^{1,p}$, and so we have to generalize the approximation lemma directly. The formulation and the proof of the L^p-version (see Section 7) are much more involved, and the proof requires the following Sobolev–Poincaré-type inequality:

$$\left(\fint_{B_\rho(x_0)} \left| V\left(\frac{u - u_{x_0,\rho}}{\rho} \right) \right|^{\frac{2n}{n-p}} dx \right)^{\frac{n-p}{2n}} \le c(n, N, p) \left(\fint_{B_\rho(x_0)} |V(Du)|^2\,dx \right)^{\frac{1}{2}}.$$

We prove this inequality in Section 4. A weaker version of this Sobolev–Poincaré-type inequality has been proved in [11]. There the authors need to increase the radius of the ball on the right-hand side; we note that such a weaker version is not sufficient to establish our desired approximation lemma.

Having proven the $\mathcal{A}$-harmonic approximation lemma, the other steps in the above-sketched strategy are fairly simply adaptable to the current setting. Firstly, we prove a Caccioppoli-type inequality for $(\mathcal{F}, \omega)$-minimizing functions u by suitably modifying the arguments from the case $p \ge 2$ (see Lemma 3 in Section 5). Secondly we show that $(\mathcal{F}, \omega)$-minimizing functions are approximately $\mathcal{A}$-harmonic, where $\mathcal{A}$ results from "freezing the coefficients". Finally we compare u with the

$\mathcal{A}$-harmonic approximation to obtain – via our Caccioppoli-type inequality – the desired excess improvement. We note that all these steps have to be performed by using the excess function Φ from (7) and its L^2-counterpart

$$\left(\fint_{B_\rho(x_0)} \left| V\left(\frac{u - \xi - (Du)_{x_0,\rho}(x - x_0)}{\rho} \right) \right|^2 dx \right)^{1/2},$$

where $\xi \in \mathbb{R}^N$.

2. Notation and statement of the result

Throughout the paper we consider functionals of the form

$$\mathcal{F}(u) = \int_U f(Du)\, dx, \tag{8}$$

where U is a domain in $\mathbb{R}^n$ and $f\colon \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \mathbb{R}$ satisfies the following structure conditions:

(H1) the function f is of class C^2 and there exists $p \in (1, 2)$ such that for some constant $L \geq 0$ and all $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ we have

$$|f(A)| \leq L(1 + |A|^p)\,;$$

(H2) the function f is *(strictly) quasi-convex*, i.e. there exists a constant $\lambda > 0$ such that

$$\int_{B_\rho(x_0)} \big(f(A + D\varphi) - f(A)\big) dx \geq \lambda \int_{B_\rho(x_0)} \big(1 + |A|^2 + |D\varphi|^2\big)^{\frac{p-2}{2}} |D\varphi|^2 dx,$$

for any $B_\rho(x_0) \subset\subset U$, $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ and $\varphi \in C_0^1\big(B_\rho(x_0), \mathbb{R}^N\big)$.

Note that **(H1)** implies in particular that $\mathcal{F}$ is defined on $W^{1,p}(U, \mathbb{R}^N)$.

We next remark that hypothesis **(H2)** implies that, for any function $\varphi \in C_0^1\big(B_\rho(x_0), \mathbb{R}^N\big)$, the non-negative function

$$I(t) = \int_\Omega \left(f(A + t\,D\varphi) - f(A) - \lambda\big(1 + |A|^2 + |t\,D\varphi|^2\big)^{\frac{p-2}{2}} |t\,D\varphi|^2 \right) dx$$

attains its absolute minimum at $t = 0$; hence $I''(0) \geq 0$, which implies

$$\int_\Omega D^2 f(A)(D\varphi, D\varphi)\, dx \geq 2\lambda\, (1 + |A|^2)^{\frac{p-2}{2}} \int_\Omega |D\varphi|^2\, dx.$$

This integral condition is equivalent to the *strong Legendre–Hadamard condition* (see [29, 4.4.1, 4.4.3] or [20, 5.1.10]), i.e. for all $\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^N$ we have

$$D^2 f(A)(\eta \otimes \xi, \eta \otimes \xi) \geq 2\lambda(1 + |A|^2)^{\frac{p-2}{2}} |\eta|^2 |\xi|^2\,. \tag{9}$$

Let us remark that we do not assume an explicit growth condition for the second derivatives of f. For our purposes it is sufficient that, for any given constant $M > 0$, there exists a constant $K_M < \infty$ such that there holds

$$\sup_{\substack{|A| \le M \\ A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)}} |D^2 f(A)| \le K_M. \tag{10}$$

Since f is quasi-convex and satisfies the growth condition from **(H1)** it is well known (see [11, Remark 3.1]) that there exist a constant c of the form $c = c(n, N, p) \cdot L$ such that the first derivatives Df of f satisfy the growth condition

$$|Df(A)| \le c\,(1 + |A|^{p-1}). \tag{11}$$

Finally, we note that hypothesis **(H1)** implies the existence of a modulus of continuity of $D^2 f$ on compact subsets of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. To be more precise: corresponding to $M > 0$ there exists a monotone non-decreasing, concave function $\nu_M\colon [0, \infty) \to [0, \infty)$ continuous at 0 such that $\nu_M(0) = 0$ and

$$|D^2 f(A) - D^2 f(B)| \le \nu_M(|A - B|), \tag{12}$$

for any $A, B \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ with $|A| \le M + 1$, $|B| \le M + 1$. (We work with $M + 1$ instead of M due to the fact that we apply (12) to matrices A, B with $|A| \le M$ and $|A - B| \le 1$.)

Now we are in the position to define precisely the notion of an almost minimizer.

Definition 1. *Consider a functional $\mathcal{F}$ as in* (8) *defined on the Sobolev space $W^{1,p}(U, \mathbb{R}^N)$, $1 < p < 2$, U a domain in $\mathbb{R}^n$, and a function $\omega\colon [0, \infty) \to [0, \infty)$. A function $u \in W^{1,p}(U, \mathbb{R}^N)$ is called $(\mathcal{F}, \omega)$-minimizing at x_0 in U if, for all $B_\rho(x_0) \subset\subset U$, there holds*

$$\mathcal{F}(u; B_\rho(x_0)) \le \mathcal{F}(u + \varphi; B_\rho(x_0)) + \omega(\rho) \int_{B_\rho(x_0)} \big(1 + |Du|^p + |D\varphi|^p\big) dx, \tag{13}$$

for all $\varphi \in C_0^1\big(B_\rho(x_0), \mathbb{R}^N\big)$. Here $\mathcal{F}(v; B_\rho(x_0)) = \int_{B_\rho(x_0)} f(Dv)\, dx$. A function $u \in W^{1,p}(U, \mathbb{R}^N)$ is called an $(\mathcal{F}, \omega)$-minimizer if u is $(\mathcal{F}, \omega)$-minimizing at every point x_0 in U.

We impose the following conditions on the function ω which characterizes the almost minimality:

- **(F1)** $\omega(\rho)$ is a non-decreasing function of ρ with $\lim_{\rho \downarrow 0} \omega(\rho) = \omega(0) = 0$ and $\omega(\rho) \le 1$ for all ρ;
- **(F2)** $\omega(\rho)/\rho^{2\beta}$ is a non-increasing function of $\rho > 0$ for some $\beta \in (0, 1)$;
- **(F3)** $\Omega(r) = \displaystyle\int_0^r \frac{\sqrt{\omega(\rho)}}{\rho}\, d\rho$ is finite for some $r > 0$.

The main result of our paper is the following:

Theorem 1. *Consider* $p \in (1, 2)$*, a domain* U *in* $\mathbb{R}^n$ *with* $n \geq 2$*, a function* ω *satisfying* **(F1)–(F3)** *and a function* $f : \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \mathbb{R}$ *which satisfies* **(H1)** *and* **(H2)**. *Further let* $\mathcal{F}$ *be the functional on* $W^{1,p}(U, \mathbb{R}^N)$ *given by* $\mathcal{F}(u) = \int_U f(Du)\, dx$ *and* $u \in W^{1,p}(U, \mathbb{R}^N)$ *be* $(\mathcal{F}, \omega)$*-minimizing on* U*. Then there exists a relatively closed subset* $\mathrm{Sing}\, u$ *of* U *such that*

$$u \in C^1(U \setminus \mathrm{Sing}\, u\, , \mathbb{R}^N)\, .$$

Furthermore $\mathrm{Sing}\, u \subset \Sigma_1 \cup \Sigma_2$*, where*

$$\Sigma_1 = \left\{ x_0 \in U : \liminf_{\rho \downarrow 0} \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^p dx > 0 \right\}\, , \ \textit{and}$$

$$\Sigma_2 = \left\{ x_0 \in U : \limsup_{\rho \downarrow 0} |(Du)_{x_0,\rho}| = \infty \right\}\, ;$$

in particular $\mathcal{L}^n(\mathrm{Sing}\, u) = 0$.

In addition, in a neighbourhood of any $x_0 \in U \setminus \mathrm{Sing}\, u$ *and for any* $\alpha \in (\beta, 1)$ *the derivative* Du *of* u *has modulus of continuity given by* $\rho \mapsto \rho^\alpha + \Omega(\rho)$. □

We close the section with a remark on notation. The various constants c appearing in the proofs and statements will depend on a number of structural parameters, and these dependencies will be indicated in the text or by self-explanatory notation without giving the explicit form of the constant in every case. Moreover, the constants c may change from line to line. We also assume, unless otherwise stated, that all constants appearing satisfy $c \geq 1$.

3. Preliminaries

Throughout the paper we shall use the functions $V{=}V_p : \mathbb{R}^k \to \mathbb{R}^k$ and $W{=}W_p : \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$V(\xi) = \frac{\xi}{(1 + |\xi|^2)^{\frac{2-p}{4}}}\, , \qquad W(\xi) = \frac{\xi}{\sqrt{1 + |\xi|^{2-p}}}, \tag{14}$$

for each $\xi \in \mathbb{R}^k$ and for any $p > 1$. From the elementary inequality $\|x\|_{\frac{2}{2-p}} \leq \|x\|_1 \leq 2^{1-\frac{2-p}{2}} \|x\|_{\frac{2}{2-p}}$ applied to the vector $x = (1, |\xi|^{2-p}) \in \mathbb{R}^2$ we deduce that

$$(1 + |\xi|^2)^{\frac{2-p}{2}} \leq 1 + |\xi|^{2-p} \leq 2^{p/2}(1 + |\xi|^2)^{\frac{2-p}{2}},$$

which immediately yields

$$|W(\xi)| \leq |V(\xi)| \leq c(p)\, |W(\xi)|. \tag{15}$$

The purpose of introducing W is the fact that – in contrast to $|V|^{2/p}$ – the function $|W|^{2/p}$ is a convex function on $\mathbb{R}^k$. This can easily be shown as follows: firstly a direct computation yields that $t \mapsto W^{2/p}(t) = t^{2/p}(1 + t^{2-p})^{-1/p}$ is convex and monotone increasing on $[0, \infty)$ with $W^{2/p}(0) = 0$; secondly we have

$$\left|W\big(\tfrac{\xi+\eta}{2}\big)\right|^{2/p} = W\big(\tfrac{|\xi+\eta|}{2}\big)^{2/p} \le W\big(\tfrac{|\xi|+|\eta|}{2}\big)^{2/p} \le \frac{W(|\xi|)^{2/p} + W(|\eta|)^{2/p}}{2} = \frac{|W(\xi)|^{2/p} + |W(\eta)|^{2/p}}{2},$$

for any $\xi, \eta \in \mathbb{R}^k$.

We use a number of properties of $V = V_p$ which can be found in [11, Lemma 2.1].

Lemma 1. *Let* $p \in (1, 2)$ *and* $V, W : \mathbb{R}^k \to \mathbb{R}^k$ *be the functions defined in* (14). *Then for any* $\xi, \eta \in \mathbb{R}^k$ *and* $t > 0$ *there holds:*

(i) $\frac{1}{\sqrt{2}} \min(|\xi|, |\xi|^{\frac{p}{2}}) \le |V(\xi)| \le \min(|\xi|, |\xi|^{\frac{p}{2}})$;
(ii) $|V(t\xi)| \le \max(t, t^{\frac{p}{2}})\, |V(\xi)|$;
(iii) $|V(\xi + \eta)| \le c(p)\, (|V(\xi)| + |V(\eta)|)$;
(iv) $\frac{p}{2}|\xi - \eta| \le \frac{|V(\xi)-V(\eta)|}{(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{4}}} \le c(k, p)\, |\xi - \eta|$;
(v) $|V(\xi) - V(\eta)| \le c(k, p)\, |V(\xi - \eta)|$;
(vi) $|V(\xi - \eta)| \le c(p, M)\, |V(\xi) - V(\eta)|$ *for all* η *with* $|\eta| \le M$.

The inequalities (i) – (iii) *also hold if we replace* V *by* W.

Proof. The assertions for V are taken from [11, Lemma 2.1]. The analogous inequalities for W are easy to derive in the case of (i) and (ii); the estimate in (iii) follows from the corresponding estimate for V and (15). □

For later purposes we state the following two simple estimates which can easily be deduced from Lemma 1 (i) and (vi). For $\xi, \eta \in \mathbb{R}^k$ with $|\eta| \le M$ we have for $|\xi - \eta| \le 1$ the estimate

$$|\xi - \eta|^2 \le c(p, M)\, |V(\xi) - V(\eta)|^2, \tag{16}$$

while for $|\xi - \eta| > 1$ we have

$$|\xi - \eta|^p \le c(p, M)\, |V(\xi) - V(\eta)|^2 . \tag{17}$$

The next lemma is a more general version of [11, Lemma 2.7], which itself is an extension of [21, Lemma 3.1, Chap. V]. The proof in [21] can easily be adapted to the present situation by replacing the condition of homogenity by Lemma 1 (ii).

Lemma 2. *Let* $0 \le \vartheta < 1$, $a, b \ge 0$, $v \in L^p(B_\rho(x_0))$ *and* g *be a non-negative bounded function satisfying*

$$g(t) \le \vartheta g(s) + a \int_{B_\rho(x_0)} \left|V\left(\tfrac{v}{s-t}\right)\right|^2 dx + b,$$

for all $\rho/2 \le t < s \le \rho$. *Then there exists a constant* $c = c(\vartheta)$ *such that*

$$g(\rho/2) \le c(\vartheta) \left(a \int_{B_\rho(x_0)} |V(\tfrac{v}{\rho})|^2 dx + b\right).$$

□

4. A Sobolev–Poincaré-type inequality in the subquadratic case

In this section we consider functions in $u \in W^{1,p}\big(B_\rho(x_0), \mathbb{R}^N\big)$ in the subquadratic case $1 < p < 2$ and prove an inequality of Sobolev–Poincaré-type which is appropiate to our situation. An analogous Poincaré-type inequality can be found in [9, Proposition 7.6]. A weaker version of this Sobolev–Poincaré-type inequality was proved in [11, Theorem 2.4] (see the comments in Section 1).

Theorem 2 (Sobolev–Poincaré-type inequality). *Let* $p \in (1,2)$, $B_\rho(x_0) \subset \mathbb{R}^n$ *with* $n \geq 2$, *and set* $p^\sharp = \frac{2n}{n-p}$. *Moreover, let* V *and* W *be the functions defined in* (14). *Then the following inequality holds:*

$$\left(\fint_{B_\rho(x_0)} \left| W\left(\frac{u - u_{x_0,\rho}}{\rho}\right)\right|^{p^\sharp} dx\right)^{\frac{1}{p^\sharp}} \leq c_s \left(\fint_{B_\rho(x_0)} |W(Du)|^2 dx\right)^{\frac{1}{2}},$$

for any $u \in W^{1,p}\big(B_\rho(x_0), \mathbb{R}^N\big)$; *here the constant* c_s *depends only on* n, N, *and* p. *Furthermore, the analogous inequality holds if we replace* W *by* V.

Proof. Similarly to the proof of [23, Lemma 7.16] we have for $x \in B_\rho(x_0)$ (since $|x - y| \leq 2\rho$ for $x, y \in B_\rho(x_0)$):

$$\frac{|u(x) - u_{x_0,\rho}|}{2\rho} \leq \fint_{B_\rho(x_0)} \fint_0^{|x-y|} |Du(x + r\omega)\, \omega| \, dr dy\,,$$

where here $\omega = \frac{y-x}{|y-x|}$.

Since $t \mapsto W^{2/p}(t)$ is non-decreasing and convex (see the beginning of Section 3) we apply $W^{2/p}$ to both sides of the previous inequality and obtain by Jensen's inequality and recalling $W(0) = 0$:

$$\begin{aligned} W^{2/p}\left(\frac{|u - u_{x_0,\rho}|}{2\rho}\right) &\leq \fint_{B_\rho(x_0)} \fint_0^{|x-y|} W^{2/p}(|Du(x + r\omega)\, \omega|) dr dy \\ &\leq \frac{(2\rho)^{n-1}}{(n-1)\mathcal{L}^n(B_\rho(x_0))} \int_{\mathbb{R}^n} |x - y|^{1-n}\, \widetilde{W}^{2/p}(|Du(y)|) dy, \end{aligned}$$

where we define

$$\widetilde{W}(|Du(x)|) = \begin{cases} 0 & : x \notin B_\rho(x_0) \\ W(|Du(x)|) & : x \in B_\rho(x_0). \end{cases}$$

Then [32, Theorem 2.8.4] applied with $f = \widetilde{W}^{2/p}(|Du|)$, $\alpha = 1$ (note that $\widetilde{W}^{2/p}(|Du|) \in L^p(\mathbb{R}^n)$ since $W^{2/p}(|Du|) \in L^p(B_\rho(x_0))$ by Lemma 1 (i)) yields

$$\left(\int_{B_\rho(x_0)}\left|W\left(\frac{u-u_{x_0,\rho}}{2\rho}\right)\right|^{\frac{2n}{n-p}}dx\right)^{\frac{n-p}{np}}$$
$$=\left(\int_{B_\rho(x_0)}W^{2/p}\left(\frac{|u-u_{x_0,\rho}|}{2\rho}\right)^{\frac{np}{n-p}}dx\right)^{\frac{n-p}{np}}$$
$$\leq\frac{c}{\rho}\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|x-y|^{1-n}\widetilde{W}^{2/p}(|Du(y)|)dy\right)^{\frac{np}{n-p}}dx\right)^{\frac{n-p}{np}}$$
$$\leq\frac{c}{\rho}\left(\int_{\mathbb{R}^n}\left(\widetilde{W}^{2/p}(|Du(x)|)\right)^{p}dx\right)^{\frac{1}{p}}$$
$$=\frac{c}{\rho}\left(\int_{B_\rho(x_0)}|W(Du(x))|^2dx\right)^{\frac{1}{p}},$$

where the constant c depends on n, N, and p only. Since $\frac{1}{\rho}=\rho^{\frac{n-p}{p}}\rho^{-\frac{n}{p}}$ and $|W(\xi/2)|\geq\frac{1}{2}|W(\xi)|$ by Lemma 1 (*ii*), we obtain the assertion of the theorem, first for W, and then also for V by (15). □

5. A Caccioppoli inequality

The first step in proving a regularity theorem for $(\mathcal{F},\omega)$-minimizing functions is to establish a suitable Caccioppoli inequality. Typically, such an inequality is used to prove *higher integrability* via Gehring's lemma (in a version due to Giaquinta and Modica [21]). Since we do not use the blow-up technique for proving regularity we need only a very weak version of the Caccioppoli inequality; this means that the constant c_c in Caccioppoli's inequality for the function $B_\rho(x_0)\ni x\mapsto u(x)-\xi-A(x-x_0)$, where $\xi\in\mathbb{R}^N$ and $A\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$ is allowed to depend on a bound $M>0$ of the matrix A. Throughout this section we consider functionals $\mathcal{F}$ of the form $\mathcal{F}(u)=\int_U f(Du)\,dx$, and we assume that the integrand f satisfies the hypotheses **(H1)**, **(H2)**. Moreover, we assume that the function ω from the $(\mathcal{F},\omega)$-minimality condition satisfies hypothesis **(F1)**. For the convenience of the reader we recall from Section 2, (10) the definition of K_M: for $M>0$ there exists $K_M<\infty$ such that $|D^2f(A)|\leq K_M$, for any $A\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$ with $|A|\leq M$.

Lemma 3 (Caccioppoli inequality). *Let $M>0$. Then there exist constants $c_c=c_c(n,N,p,L,\lambda,M,K_{M+1})$ and $\rho_0=\rho_0(n,N,p,\lambda,M,\omega(\cdot))$ such that for every ball $B_\rho(x_0)\subset\subset U$ with $\rho\leq\rho_0$, $\xi\in\mathbb{R}^N$, $A\in\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N)$ with $|A|\leq M$ and every $u\in W^{1,p}(U,\mathbb{R}^N)$ which is $(\mathcal{F},\omega)$-minimizing in U, there holds:*

$$\fint_{B_{\rho/2}(x_0)}|V(Du-A)|^2\,dx\leq c_c\left[\fint_{B_\rho(x_0)}\left|V\left(\frac{u-\xi-A(x-x_0)}{\rho}\right)\right|^2dx+\omega(\rho)\right].$$

Proof. Although the proof is close to the proof of similar inequalities given in [19, Lemma 3.1, 5.1] in the case $p \geq 2$ (see also [15, Lemma 3.2], [17, Lemma 1]), and [1, proof of Lemma II.5], [22, Proposition 4.1], [10, proof of Lemma 2.7] and [11, proof of Lemma 2.8] in the case $1 < p < 2$, we give the arguments for the convenience of the reader.

Let $B_\rho(x_0) \subset\subset U$. We choose $\rho/2 \leq t < s \leq \rho$ and a standard cut-off function $\eta \in C_0^1(B_\rho(x_0), [0, 1])$ with $\eta \equiv 1$ on $B_t(x_0)$, $\eta \equiv 0$ outside $B_s(x_0)$, and which satisfies $|\nabla\eta| \leq \frac{2}{s-t}$. For $\xi \in \mathbb{R}^N$ and $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ we abbreviate

$$v = u - \xi - A(x - x_0), \qquad \varphi = \eta\, v, \qquad \text{and} \qquad \psi = (1 - \eta)\, v\,.$$

Then

$$D\varphi + D\psi = Dv = Du - A\,, \tag{18}$$

and further there holds

$$|D\varphi|^p \leq c(p)\, \left(|Dv|^p + \left|\tfrac{v}{s-t}\right|^p\right) \quad \text{and} \quad |D\psi|^p \leq c(p)\, \left(|Dv|^p + \left|\tfrac{v}{s-t}\right|^p\right). \tag{19}$$

Using hypothesis **(H2)** and (18) we obtain

$$\lambda \int_{B_s(x_0)} (1 + |A|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 dx \leq I + II + III\,, \tag{20}$$

where

$$\begin{aligned} I &= \int_{B_s(x_0)} [f(Du - D\psi) - f(Du)]\, dx\,, \\ II &= \int_{B_s(x_0)} [f(Du) - f(Du - D\varphi)]\, dx\,, \quad \text{and} \\ III &= \int_{B_s(x_0)} [f(A + D\psi) - f(A)]\, dx\,. \end{aligned}$$

To estimate $I + III$ we note that

$$\begin{aligned} I + III &= \int_{B_s(x_0)} \int_0^1 [Df(A + \tau D\psi) - Df(Du - \tau D\psi)] d\tau D\psi\, dx \\ &= \int_{B_s(x_0)} \int_0^1 (Df(A + \tau D\psi) - Df(A))\, d\tau D\psi\, dx \\ &\quad + \int_{B_s(x_0)} \int_0^1 (Df(A) - Df(A + Dv - \tau D\psi))\, d\tau D\psi\, dx \\ &= I' + III'\,, \end{aligned}$$

with the obvious labelling. On $B_s(x_0) \cap \{|D\psi| > 1\}$ we use (11) and the fact that $|A| \leq M$ to deduce the following estimate for the integrand in I':

$$|Df(A + \tau D\psi) - Df(A)||D\psi| \leq c\, |D\psi|^p\,,$$

where the constant c depends only on n, N, p, L, and M. On the other hand, on $B_s(x_0) \cap \{|D\psi| \le 1\}$ we use the bound for the second derivative $D^2 f$ of f, i.e. (10) with M replaced by $M+1$, to infer

$$|Df(A+\tau D\psi) - Df(A)||D\psi| \le K_{M+1}\,|D\psi|^2,$$

where $K_{M+1} = \sup_{|A|\le M+1} |D^2 f(A)|$. Using the last two estimates, we obtain

$$|I'| \le c \int_{B_s(x_0)\cap\{|D\psi|>1\}} |D\psi|^p dx + K_{M+1} \int_{B_s(x_0)\cap\{|D\psi|\le 1\}} |D\psi|^2 dx\,, \tag{21}$$

where $c = c(n, N, p, L, M)$. In view of Lemma 1 (i) we infer

$$|I'| \le c \int_{B_s(x_0)} |V(D\psi)|^2 dx\,,$$

where c depends on n, N, p, L, M, and K_{M+1} only. Using Lemma 1 (i), (ii) and (iii) we estimate $V(D\psi)$ as follows:

$$\begin{aligned}
|V(D\psi)| &= |V((1-\eta)Dv - \nabla\eta\otimes v)| \\
&\le c(p)\ [|V((1-\eta)Dv)| + |V(\nabla\eta\otimes v)|] \\
&\le c(p)\ \Big[|V(Dv)| + \min\Big\{\big|\tfrac{2v}{s-t}\big|\ ,\ \big|\tfrac{2v}{s-t}\big|^{\frac{p}{2}}\Big\}\Big] \\
&\le c(p)\ \Big[|V(Dv)| + \Big|V\left(\tfrac{v}{s-t}\right)\Big|\Big]\ .
\end{aligned}$$

In view of the previous estimate we conclude, noting that $D\psi \equiv 0$ on $B_t(x_0)$,

$$|I'| \le c \int_{B_s(x_0)\setminus B_t(x_0)} \Big[|V(Dv)|^2 + \Big|V\left(\tfrac{v}{s-t}\right)\Big|^2\Big]\,dx\,, \tag{22}$$

where c depends on the same quantities as before, i.e. on n, N, p, L, M, and K_{M+1}.

The second term III' is estimated similarly. Firstly, on the set $B_s(x_0) \cap \{|Dv| + |D\psi| > 1\}$ we use (11) to bound the integrand in III' from above by

$$|Df(A) - Df(A+Dv-\tau D\psi)||D\psi| \le c\ (|Dv| + |D\psi|)^{p-1}\,|D\psi|\,,$$

where $c = c(n, N, p, L, M)$. Secondly, on the complement $B_s(x_0) \cap \{|Dv| + |D\psi| \le 1\}$ we use (10) to bound the second derivatives $D^2 f$ (note that $|A+Dv-\tau D\psi| \le M+1$ in this case) and obtain:

$$|Df(A) - Df(A+Dv-\tau D\psi)||D\psi| \le K_{M+1}(|Dv| + |D\psi|)|D\psi|\,.$$

Using again Lemma 1 we deduce:

$$\begin{aligned}
|III'| &\le c \int_{B_s(x_0)\setminus B_t(x_0))\cap\{|Dv|+|D\psi|>1\}} [|Dv| + |D\psi|]^p\,dx \\
&\quad + K_{M+1} \int_{B_s(x_0)\setminus B_t(x_0))\cap\{|Dv|+|D\psi|\le 1\}} [|Dv| + |D\psi|]^2\,dx \\
&\le c \int_{B_s(x_0)\setminus B_t(x_0)} |V(|Dv| + |D\psi|)|^2 dx \\
&\le c \int_{B_s(x_0)\setminus B_t(x_0)} |V(Dv)|^2 + |V(D\psi)|^2 dx \\
&\le c \int_{B_s(x_0)\setminus B_t(x_0)} \Big(|V(Dv)|^2 + \Big|V\left(\tfrac{v}{s-t}\right)\Big|^2\Big)\,dx,
\end{aligned} \tag{23}$$

where the constant c depends on the same quantities as the constant in (22), namely on n, N, p, L, M, and K_{M+1}.

To estimate II we use the $(\mathcal{F}, \omega)$-minimality of u, i.e. (13), to conclude

$$\begin{aligned} |II| &\le \omega(s) \int_{B_s(x_0)} (1 + |Du|^p + |D\varphi|^p)\, dx \\ &\le \omega(s) \int_{B_s(x_0)} (1 + 2^{p-1} M^p + 2^{p-1} |Dv|^p + |D\varphi|^p)\, dx \\ &\le c(p, M)\, \alpha_n \omega(s)\, s^n + c(p)\, \omega(s) \int_{B_s(x_0)} (|Dv| + |D\varphi|)^p\, dx. \end{aligned}$$

To estimate the second term of the right-hand side of this inequality we argue similiarly as above, i.e. we split the domain of integration into two parts $B_s(x_0) \cap \{|Dv| + |D\varphi| \le 1\}$ and $B_s(x_0) \cap \{|Dv| + |D\varphi| > 1\}$. On the set where $|Dv| + |D\varphi| \le 1$ we trivially find

$$\omega(s) \int_{B_s(x_0) \cap \{|Dv|+|D\varphi| \le 1\}} (|Dv| + |D\varphi|)^p\, dx \le \alpha_n \omega(s)\, s^n,$$

whereas on $B_s(x_0) \cap \{|Dv| + |D\varphi| > 1\}$ we argue similarly to (23), using (19) to estimate $|D\varphi|^p$, and obtain:

$$\begin{aligned} &\omega(s) \int_{B_s(x_0) \cap \{|Dv|+|D\varphi| > 1\}} (|Dv| + |D\varphi|)^p\, dx \\ &\le c(p)\, \omega(s) \int_{B_s(x_0)} \left[|V(Dv)|^2 + \left| V\left(\tfrac{v}{s-t}\right)\right|^2 \right] dx. \end{aligned}$$

Thus we have established:

$$|II| \le c(p, M)\, \alpha_n \omega(s)\, s^n + c(p)\ \ \omega(s) \int_{B_s(x_0)} \left[|V(Dv)|^2 + \left| V\left(\tfrac{v}{s-t}\right)\right|^2 \right] dx. \quad (24)$$

Finally, on $B_t(x_0)$ we use Lemma 1 (iv) and (vi) to bound the integrand of the left-hand side of (20) from below:

$$\begin{aligned} \lambda(1 + |A|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 &= \lambda(1 + |A|^2 + |Dv|^2)^{\frac{p-2}{2}} |Dv|^2 \\ &\ge c(p)\, \lambda\, (1 + |A|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - A|^2 \\ &\ge c(n, N, p, \lambda)\, |V(Du) - V(A)|^2 \\ &\ge c(n, N, p, \lambda, M)\ \ |V(Dv)|^2 . \end{aligned}$$

Using this in (20) together with the estimates (22), (23), and (24) we finally arrive at:

$$\begin{aligned} c \int_{B_t(x_0)} |V(Dv)|^2 dx \le \widetilde{c} \left(\int_{B_s(x_0) \setminus B_t(x_0)} \left(|V(Dv)|^2 + \left| V\left(\tfrac{v}{s-t}\right)\right|^2 \right) dx + \omega(s)\, s^n \right) \\ + \widehat{c}\, \omega(s) \int_{B_s(x_0)} \left(|V(Dv)|^2 + \left| V\left(\tfrac{v}{s-t}\right)\right|^2 \right) dx, \end{aligned}$$

where the constants $c, \widetilde{c}$, and $\widehat{c}$ have the following dependencies: $c = c(n, N, p, \lambda, M)$, $\widetilde{c} = \widetilde{c}(n, N, p, L, M, K_{M+1})$, $\widehat{c} = \widehat{c}(p)$. Now, we choose $\rho_0 = \rho_0(n, N, p, \lambda, M)$ such that $\widehat{c}\,\omega(\rho_0) \le \frac{c}{2}$; this choice of ρ_0 is possible in view of our hypothesis **(F1)** on ω. "Filling the hole" yields, with $\vartheta = \frac{2\widetilde{c}}{c+2\widetilde{c}} < 1$:

$$\int_{B_t(x_0)} |V(Dv)|^2 dx \le \vartheta \int_{B_s(x_0)} |V(Dv)|^2 dx + c\left[\int_{B_\rho(x_0)} \left|V\left(\tfrac{v}{s-t}\right)\right|^2 dx + \omega(s)\, s^n\right],$$

for a constant $c = c(n, N, p, L, \lambda, M, K_{M+1})$. The proof is now completed by applying Lemma 2. □

6. Approximate harmonicity

For a ball $B_\rho(x_0) \subset\subset U$, a function $u \in W^{1,p}\big(B_\rho(x_0), \mathbb{R}^N\big)$, a vector $\xi \in \mathbb{R}^N$, and a linear function $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ we define

$$\Phi(x_0, \rho, A) = \left(⨍_{B_\rho(x_0)} |V(Du) - V(A)|^2\, dx\right)^{\frac{1}{2}}. \tag{25}$$

Lemma 4 (Approximate A-harmonicity). *For any $M > 0$ there exists a constant $c_e = c_e(n, N, p, L, M, K_{M+2})$ such that for every $u \in W^{1,p}(U, \mathbb{R}^N)$ that is $(\mathcal{F}, \omega)$-minimizing in U, every ball $B_\rho(x_0) \subset\subset U$ and every $A \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ with $|A| \le M$ we have:*

$$\left|⨍_{B_\rho(x_0)} D^2 f(A)(Du - A, D\varphi) dx\right| \le c_e\left(\sqrt{\nu_M(\Phi)}\Phi + \Phi^2) + \sqrt{\omega(\rho)}\right) \sup_{B_\rho(x_0)} |D\varphi|,$$

for all $\varphi \in C_0^1(B_\rho(x_0), \mathbb{R}^N)$. Here we have abbreviated $\Phi(x_0, \rho, A)$ by Φ.

Proof. We write B_ρ for $B_\rho(x_0)$. We assume initially that $|D\varphi| \le 1$. For $0 < s \le 1$ we use the $(\mathcal{F}, \omega)$-minimality of u to deduce (see [15, inequality (4.4)])

$$\begin{aligned}
&\int_{B_\rho} D^2 f(A)(Du - A, D\varphi)\, dx \\
&\ge \frac{1}{s}\Bigg[\int_{B_\rho}\int_0^s \big(Df(Du) - Df(Du + \tau D\varphi)\big) D\varphi\, d\tau\, dx \\
&\qquad + s\int_{B_\rho}\int_0^1 \big(D^2 f(A) - D^2 f(A + \tau(Du - A))\big)(Du - A, D\varphi)\, d\tau\, dx \\
&\qquad - \omega(\rho)\int_{B_\rho} (1 + |Du|^p + s^p) dx\Bigg] \\
&= \int_{B_\rho} (I + II + III)\, dx,
\end{aligned}$$

with the obvious labelling. To estimate I, II and III we distinguish the cases $|Du - A| \leq 1$, $|Du - A| > 1$. In the case that $|Du - A| \leq 1$ we use (16) and obtain

$$|Du - A|^2 \leq c\,|V(Du) - V(A)|^2\,, \tag{26}$$

while in the case $|Du - A| > 1$ we infer from (17)

$$|Du - A|^p \leq c\,|V(Du) - V(A)|^2\,; \tag{27}$$

note that the constants in (26) and (27) depend only on p and M.

Estimate for *I*: Since $|D^2 f(B)| \leq K_{M+2}$ on $\{B \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N) : |B| \leq M + 2\}$, on the set $B_\rho \cap \{|Du - A| \leq 1\}$ we have the bound:

$$|I| = \frac{1}{s}\left|\int_0^s \int_0^1 D^2 f(Du + \sigma\tau D\varphi)(\tau D\varphi, D\varphi)\, d\sigma\, d\tau\right| \leq \frac{s}{2} K_{M+2}.$$

In the case $B_\rho \cap \{|Du - A| > 1\}$ we use (11) and (27) to deduce:

$$|I| \leq c\,|Du - A|^p \leq c\,|V(Du) - V(A)|^2\,,$$

where $c = c(n, N, p, L, M)$.

Estimate for *II*: On $B_\rho \cap \{|Du - A| \leq 1\}$ we have $|A + \tau(Du - A)| \leq M + 1$. We then use (12), (10), (26) to obtain:

$$\begin{aligned} |II| &\leq \sqrt{2\,K_{M+1}}\,\sqrt{\nu_M(|Du - A|)}\,|Du - A| \\ &\leq c\,\sqrt{\nu_M(|V(Du) - V(A)|)}\,|V(Du) - V(A)|\,, \end{aligned}$$

where the constant c depends only on p, M, and K_{M+1}. On the set $B_\rho \cap \{|Du - A| > 1\}$ we use (11), (10) and (27) to infer:

$$\begin{aligned} |II| &= \left|D^2 f(A)(Du - A, D\varphi) + Df(A)D\varphi - Df(Du)D\varphi\right| \\ &\leq c\,|Du - A|^p \leq c\,|V(Du) - V(A)|^2\,, \end{aligned}$$

where $c = c(n, N, p, L, M, K_M)$.

Estimate for *III*: Here we see on $B_\rho \cap \{|Du - A| \leq 1\}$:

$$|III| \leq c\,\frac{\omega(\rho)}{s}\,,$$

while on $B_\rho \cap \{|Du - A| > 1\}$ there holds, using (26):

$$|III| \leq c\,\frac{\omega(\rho)}{s}|Du - A|^p \leq c\,\frac{\omega(\rho)}{s}|V(Du) - V(A)|^2\,.$$

In each case the constant depends only on p and M.

Combining the estimates for I, II, and III, applying Hölder's and Jensen's inequalities and recalling the fact that $\omega(\rho) \leq 1$ (see **(F1)**), and choosing $s = \sqrt{\omega(\rho)}$ we finally arrive at:

$$\fint_{B_\rho} D^2 f(A)(Du - A, D\varphi)\, dx \geq -\,c_e\,\left(\sqrt{\nu_M(\Phi)}\Phi + \Phi^2 + \sqrt{\omega(\rho)}\right),$$

where the constant c_e depends only on n, N, p, L, M, K_{M+2}.

The corresponding estimate for $\fint_{B_\rho} D^2 f(A)(Du - A, D\varphi)\,dx$ from above can be shown completely analogously. This proves the lemma for $|D\varphi| \le 1$, and rescaling yields the general result. □

7. $\mathcal{A}$-harmonic approximation

In this section we present the $\mathcal{A}$-harmonic approximation lemma, and refer the reader to the introduction for further comments and references concerning the origins of this kind of lemma.

The first result we need is a simple consequence of the a priori estimates for solutions of linear elliptic sytems of second order with constant coefficients (see [11, Proposition 2.10] for a similar result). Note that for the proof of the $\mathcal{A}$-harmonic approximation lemma we require only the result concerning smoothness, but the a priori estimate (29) will be needed in Section 8.

Lemma 5. *Let $h \in W^{1,1}\big(B_\rho(x_0), \mathbb{R}^N\big)$ be such that*

$$\int_{B_\rho(x_0)} \mathcal{A}(Dh, D\varphi)\,dx = 0, \tag{28}$$

for any $\varphi \in C_0^1\big(B_\rho(x_0), \mathbb{R}^N\big)$, where $\mathcal{A} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ is elliptic in the sense of Legendre–Hadamard with ellipticity constant κ and upper bound K. Then $h \in C^\infty\big(B_\rho(x_0), \mathbb{R}^N\big)$ and

$$\rho \sup_{B_{\rho/2}(x_0)} |D^2 h| + \sup_{B_{\rho/2}(x_0)} |Dh| \le c_a \fint_{B_\rho(x_0)} |Dh|\,dx, \tag{29}$$

where the constant c_a depends only on n, N, κ and K.

Proof. Let $v \in W^{1,2}\big(B_\rho(x_0), \mathbb{R}^N\big)$ be a weak solution of (28). From [11, proof of Proposition 2.10] we infer that

$$\sup_{B_{\rho/2}} |v| \le c \fint_{B_\rho} |v|\,dx\,,$$

where the constant c depends only on n, N, κ and K.

For $0 < r < \rho$ set $\tau = r/\rho$. Then for any $\varepsilon > 0$ we can find $x_* \in B_r$ such that (noting that $B_{\frac{(1-\tau)\rho}{2}}(x_*) \subset B_\rho$):

$$\begin{aligned} \sup_{B_r} |v| = \sup_{B_{\tau\rho}} |v| &\le \varepsilon + |v(x_*)| \le \varepsilon + \sup_{B_{\frac{(1-\tau)\rho}{4}}(x_*)} |v| \le \varepsilon + c \fint_{B_{\frac{(1-\tau)\rho}{2}}(x_*)} |v|\,dx \\ &\le \varepsilon + c\,\frac{2^n}{(1-\tau)^n} \fint_{B_\rho} |v|\,dx = \varepsilon + 2^n\,c\left(\tfrac{\rho}{\rho - r}\right)^n \fint_{B_\rho} |v|\,dx\,. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, considering $\varepsilon \searrow 0$ yields, for Dv and $D^2 v$:

$$\sup_{B_r} |D^2 v| \le c\left(\tfrac{\rho}{\rho - r}\right)^n \fint_{B_\rho} |D^2 v|\,dx, \quad \sup_{B_r} |Dv| \le c\left(\tfrac{\rho}{\rho - r}\right)^n \fint_{B_\rho} |Dv|\,dx, \tag{30}$$

where the constant c depends only on n, N, κ and K. Combining (30) with Caccioppoli's inequality for solutions of linear elliptic systems with constant coefficients we derive:

$$\begin{aligned}\sup_{B_{\rho/2}} |D^2 v| &\leq c \fint_{B_{\frac{3}{5}\rho}} |D^2 v|\, dx \leq c \left(\fint_{B_{\frac{3}{5}\rho}} |D^2 v|^2\, dx \right)^{\frac{1}{2}} \\ &\leq \frac{c}{\rho} \left(\fint_{B_{\frac{4}{5}\rho}} |Dv|^2\, dx \right)^{\frac{1}{2}} \leq \frac{c}{\rho} \left(\fint_{B_{\frac{4}{5}\rho}} |Dv|\, dx \right)^{\frac{1}{2}} \left(\sup_{B_{\frac{4}{5}\rho}} |Dv| \right)^{\frac{1}{2}} \\ &\leq \frac{c}{\rho} \fint_{B_\rho} |Dv|\, dx,\end{aligned}$$

where the constant c has the same dependencies as above. This estimate together with (30) yields (29) for $v \in W^{1,2}$, and a standard approximation argument (see [11, proof of Proposition 2.10, Step 2]) yields the desired a priori estimate for general v in $W^{1,1}$. □

The next result, the $\mathcal{A}$-harmonic approximation lemma, is, as discussed in the introduction, the key ingredient in proving our regularity result.

Lemma 6. *Let κ, K be positive constants. Then for any $\varepsilon > 0$ there exist $\delta = \delta(n, N, \kappa, K, \varepsilon) \in (0, 1]$ with the following property: for any bilinear form $\mathcal{A}$ on* $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ *which is elliptic in the sense of Legendre–Hadamard with ellipticity constant κ and upper bound K, for any $v \in W^{1,p}\big(B_\rho(x_0), \mathbb{R}^N\big)$ satisfying*

$$\fint_{B_\rho(x_0)} |W(Dv)|^2\, dx \leq \gamma^2 \leq 1 \quad \textit{and}$$

$$\fint_{B_\rho(x_0)} \mathcal{A}(Dv, D\varphi)\, dx \leq \gamma\, \delta \sup_{B_\rho(x_0)} |D\varphi| \quad \textit{for all } \varphi \in C_0^1\big(B_\rho(x_0), \mathbb{R}^N\big),$$

there exists an $\mathcal{A}$-harmonic function h satisfying

$$\fint_{B_\rho(x_0)} |W(Dh)|^2\, dx \leq 1 \quad \textit{and} \quad \fint_{B_\rho(x_0)} \left|W\big(\tfrac{v-\gamma h}{\rho}\big)\right|^2 dx \leq \gamma^2 \varepsilon\,.$$

Here a function h is termed $\mathcal{A}$-harmonic if it satisfies $\int_{B_\rho(x_0)} \mathcal{A}(Dh, D\varphi)\, dx = 0$, for all $\varphi \in C_0^1\big(B_\rho, \mathbb{R}^N\big)$.

Proof. We assume that $x_0 = 0$ and $\rho = 1$. The general case follows by rescaling v via $x \mapsto \rho \cdot v\,((x - x_0)/\rho)$.

If the conclusion of the lemma was false, we could find $\varepsilon > 0$ and sequences $\{\mathcal{A}_k\}$ of bilinear forms with uniform ellipticity bound κ and uniform upper bound K, $\{g_k\}$ with $g_k \in W^{1,p}(B, \mathbb{R}^N)$ and $\{\gamma_k\}$ with $\gamma_k \in (0, 1]$ such that

$$\fint_B |W(Dg_k)|^2\, dx \leq \gamma_k^2 \leq 1 \quad \text{and} \quad \left| \fint_B \mathcal{A}_k(Dg_k, D\varphi) dx \right| \leq \gamma_k \frac{1}{k} \sup_B |D\varphi|, \tag{31}$$

for all $\varphi \in C_0^1(B, \mathbb{R}^N)$, but

$$\fint_B |W(g_k - \gamma_k h)|^2 dx > \gamma_k^2\, \varepsilon \quad \text{for all } h \in \mathcal{H}_k, \tag{32}$$

where here

$$\mathcal{H}_k = \left\{ f \in W^{1,p}(B, \mathbb{R}^N) : f \text{ is } \mathcal{A}_k\text{-harmonic on } B \text{ and} \fint_B |W(Df)|^2 dx \leq 1 \right\}.$$

Without loss of generality we can assume that $\fint_B g_k\, dx = 0$; otherwise we replace g_k by $g_k - \fint_B g_k\, dx$. Furthermore, passing to a subsequence we can assume that $\{\gamma_k\}$ is monotone.

By Lemma 1 (i) we see

$$\begin{aligned}\fint_B |Dg_k|^p\, dx &= \frac{1}{\mathcal{L}^n(B)} \left(\int_{B\cap\{|Dg_k|\leq 1\}} |Dg_k|^p dx + \int_{B\cap\{|Dg_k|>1\}} |Dg_k|^p dx \right)\\ &\leq c(p) \left(\fint_B |W(Dg_k)|^p dx + \fint_B |W(Dg_k)|^2 dx \right)\\ &\leq c(p) \left(\gamma_k^p + \gamma_k^2\right).\end{aligned}$$

Hence, passing to a further subsequence (also labelled with k) we obtain the existence of $g \in W^{1,p}(B, \mathbb{R}^N)$ and $\mathcal{A} \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ such that there holds:

$$\begin{cases} \frac{g_k}{\gamma_k} \to g \ \text{ weakly in } W^{1,p}(B, \mathbb{R}^N),\\ \frac{g_k}{\gamma_k} \to g \ \text{ strongly in } L^p(B, \mathbb{R}^N),\\ \frac{g_k}{\gamma_k} \to g \ \text{ a.e. on } B,\\ \mathcal{A}_k \to \mathcal{A} \ \text{ in } \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N). \end{cases} \tag{33}$$

Using the lower semicontinuity of $u \mapsto \int_B |W(Du)|^2 dx$ with respect to weak convergence in $W^{1,p}(B, \mathbb{R}^N)$ (which follows from the convexity of $\xi \mapsto |W(\xi)|^2$, which in turn follows from the convexity of $\xi \mapsto |W(\xi)|^{2/p}$, and Lemma 1 (i); see e.g. [12, Chapter 4, Theorem 2.3]) we obtain:

$$\fint_B |W(Dg)|^2 dx \leq \liminf_{k\to\infty} \fint_B \left|W\left(\tfrac{Dg_k}{\gamma_k}\right)\right|^2 dx \leq \liminf_{k\to\infty} \gamma_k^{-2} \fint_B |W(Dg_k)|^2 dx \leq 1, \tag{34}$$

where in the second-last inequality we have used Lemma 1 (ii). For $\varphi \in C_0^1(B, \mathbb{R}^N)$ we have

$$\begin{aligned}&\int_B \mathcal{A}(Dg, D\varphi)\, dx\\ &= \int_B \mathcal{A}\left(Dg - \tfrac{Dg_k}{\gamma_k}, D\varphi\right) dx + \int_B (\mathcal{A} - \mathcal{A}_k)\left(\tfrac{Dg_k}{\gamma_k}, D\varphi\right) dx + \int_B \mathcal{A}_k\left(\tfrac{Dg_k}{\gamma_k}, D\varphi\right) dx.\end{aligned}$$

Passing to the limit $k \to \infty$ we see that the first term on the right-hand side converges to 0 due to the weak convergence of $\frac{g_k}{\gamma_k}$ to g (see (33)); the same holds for the second term in view of the uniform bound of $\frac{Dg_k}{\gamma_k}$ in L^p and the convergence of the $\mathcal{A}_k$'s (see (33)); and the third term vanishes in the limit $k \to \infty$ via (31). This shows that the weak limit g is $\mathcal{A}$-harmonic on B, i.e.

$$\fint_B \mathcal{A}(Dg, D\varphi)\,dx = 0 \quad \text{for all } \varphi \in C_0^1(B, \mathbb{R}^N). \tag{35}$$

Therefore, by Lemma 5, we see that g is smooth on the open ball B.

For fixed $\frac{1}{2} \le \rho < 1$ (to be specified later) let $v_k \in W^{1,2}(B_\rho, \mathbb{R}^N)$ denote the unique solution of the Dirichlet problem

$$\fint_{B_\rho} \mathcal{A}_k(Dv_k, D\varphi)dx = 0 \quad \text{for all } \varphi \in C_0^1\big(B_\rho, \mathbb{R}^N\big), \quad v_k = g \text{ on } \partial B_\rho. \tag{36}$$

Using the strong Legendre–Hadamard condition with uniform ellipticity bound κ for each $\mathcal{A}_k$, the $\mathcal{A}_k$-harmonicity of v_k on B_ρ, the $\mathcal{A}$-harmonicity of g, the smoothness of g on B_ρ, and Hölder's inequality we see that

$$\begin{aligned}
\kappa \fint_{B_\rho} |Dv_k - Dg|^2 dx &\le \fint_{B_\rho} \mathcal{A}_k(Dv_k - Dg, Dv_k - Dg)\,dx \\
&= -\fint_{B_\rho} \mathcal{A}_k(Dg, Dv_k - Dg)\,dx \\
&= \fint_{B_\rho} (\mathcal{A} - \mathcal{A}_k)(Dg, Dv_k - Dg)\,dx \\
&\le |\mathcal{A} - \mathcal{A}_k| \sup_{B_\rho} |Dg| \left(\fint_{B_\rho} |Dv_k - Dg|^2\,dx \right)^{1/2}.
\end{aligned}$$

Therefore we conclude:

$$\lim_{k\to\infty} \fint_{B_\rho} |Dv_k - Dg|^2 dx = 0\,. \tag{37}$$

Since $v_k - g \in W_0^{1,2}\big(B_\rho, \mathbb{R}^N\big)$ this implies, via Poincaré's inequality:

$$v_k \to g \;\text{ strongly in } W^{1,2}\big(B_\rho, \mathbb{R}^N\big). \tag{38}$$

We now define, for $y \in B$:

$$\begin{aligned}
\widetilde{v}_k(y) &= v_k(\rho y)\,, \\
\widetilde{g}(y) &= g(\rho y)\,, \\
b_k &= \max\left\{1, \left[\fint_B |W(D\tilde{v}_k)|^2 dx\right]^{1/p}\right\}, \quad \text{and} \\
\widehat{v}_k &= \widetilde{v}_k / b_k\,.
\end{aligned}$$

Note in particular that there holds $\widehat{v}_k \in \mathcal{H}_k$.

We now show that we can choose $\rho \in (1/2, 1)$ sufficiently close to 1 such that

$$\fint_B |W(g_k - \gamma_k \widehat{v}_k)|^2 dx < \tfrac{1}{2}\,\varepsilon\,\gamma_k^2\,, \tag{39}$$

for k sufficiently large. This would contradict (32) and would prove the assertion of the lemma.

Writing $p^\sharp$ for the W-Sobolev–Poincaré conjugate from Theorem 2, i.e. $p^\sharp = \frac{2n}{n-p}$, we note that there holds $p^\sharp > 2$. Hence we can write $\frac{1}{2}$ as a convex combination of 1 and $\frac{1}{p^\sharp}$, i.e. there exists $t \in (0, 1)$ such that

$$1 = 2t + \frac{2(1-t)}{p^\sharp}.$$

Therefore, we can estimate, via Hölder's inequality:

$$\begin{aligned} &⨍_B |W(g_k - \gamma_k \widehat{v}_k)|^2 dx \\ &= ⨍_B |W(g_k - \gamma_k \widehat{v}_k)|^{2t} |W(g_k - \gamma_k \widehat{v}_k)|^{2(1-t)} dx \\ &\le \left(⨍_B |W(g_k - \gamma_k \widehat{v}_k)| dx \right)^{2t} \left(⨍_B |W(g_k - \gamma_k \widehat{v}_k)|^{p^\sharp} dx \right)^{\frac{2(1-t)}{p^\sharp}}. \end{aligned} \tag{40}$$

In the following we estimate both factors on the right-hand side of (40) seperately. Using Lemma 1 (i) we see that

$$\begin{aligned} \left(⨍_B |W(g_k - \gamma_k \widehat{v}_k)|\, dx \right)^{2t} &\le \gamma_k^{2t} \left(⨍_B \left| \widehat{v}_k - \tfrac{g_k}{\gamma_k} \right| dx \right)^{2t} \\ &\le \gamma_k^{2t} \left(⨍_B |\widehat{v}_k - \widetilde{v}_k|\, dx + ⨍_B \left| \widetilde{v}_k - \tfrac{g_k}{\gamma_k} \right| dx \right)^{2t} \\ &= \gamma_k^{2t} [I_k + II_k]^{2t}, \end{aligned} \tag{41}$$

with the obvious definition for I_k and II_k.

To estimate I_k in (41) we note that Lemma 1 (ii) and (38), i.e. the strong convergence of v_k to g in $W^{1,2}(B_\rho, \mathbb{R}^N)$, imply

$$\int_B |W(D\widetilde{v}_k)|^2 dx \le \rho^{p-n} \int_{B_\rho} |W(Dv_k)|^2 dx \to \rho^{p-n} \int_{B_\rho} |W(Dg)|^2 dx,$$

as $k \to \infty$. Hence, in view of (34) we find:

$$\limsup_{k\to\infty} ⨍_B |W(D\widetilde{v}_k)|^2 dx \le \rho^{p-n} \quad \text{and hence} \quad \limsup_{k\to\infty}\ b_k \le \rho^{\frac{p-n}{p}}.$$

Noting that $⨍_B g\, dx = 0$, $b_k \ge 1$, and applying Poincaré's inequality to g on B we deduce, in view of (38),

$$\begin{aligned} I_k = \tfrac{b_k - 1}{b_k} ⨍_B |\widetilde{v}_k|\, dx &\le (b_k - 1) ⨍_B |\widetilde{v}_k|\, dx = (b_k - 1) ⨍_{B_\rho} |v_k|\, dx \\ &\le (b_k - 1) ⨍_{B_\rho} |v_k - g|\, dx + c\,(b_k - 1) ⨍_B |Dg|\, dx \\ &\le o(1) + c\,(b_k - 1) ⨍_B |Dg|\, dx, \end{aligned}$$

where $c = c(n, N)$.

Distinguishing the cases $|Dg| \leq 1$ and $|Dg| > 1$ and using Lemma 1 (i) as well as (34) we find

$$\fint_B |Dg|\, dx \leq 2\sqrt{2} \left(\fint_B |W(Dg)|^2\, dx \right)^{\frac{1}{2}} \leq 2\sqrt{2}\,, \tag{42}$$

which, combined with the second-last inequality, implies

$$\limsup_{k\to\infty} I_k \leq c \left(\rho^{\frac{p-n}{p}} - 1 \right), \tag{43}$$

where the constant c depends only on n and N.

The second term II_k on the right-hand side of (41) is estimated as follows:

$$\begin{aligned} II_k &\leq \fint_B |\widetilde{v}_k - \widetilde{g}|\ dx + \fint_B |\widetilde{g} - g|\, dx + \fint_B \left|g - \tfrac{g_k}{\gamma_k}\right| dx \\ &= o(1) + \fint_B |\widetilde{g} - g|\ dx\,. \end{aligned}$$

Here we have used (33) twice (i.e. the strong L^2-convergence $\frac{g_k}{\gamma_k} \to g$ as $k \to \infty$). Elementary calculus yields:

$$\begin{aligned} \int_B |\widetilde{g} - g|\, dx &= \int_B |g(\rho x) - g(x)|\, dx \\ &= \int_B \left| \int_0^1 Dg(tx + (1-t)\rho x)\,(1-\rho)x\, dt \right| dx \\ &\leq (1-\rho) \int_0^1 (t + (1-t)\rho)^{1-n} \int_{B_{t+(1-t)\rho}} |Dg(y)| dy\, dt \\ &\leq (1-\rho)\rho^{1-n} \int_B |Dg(y)| dy\,. \end{aligned}$$

Hence (42) implies:

$$\limsup_{k\to\infty} II_k \leq c(n)\,(1-\rho)\,.$$

Combining the estimates for I_k and II_k, we arrive at:

$$\limsup_{k\to\infty} [I_k + II_k] \leq c(n, N) \left(\rho^{\frac{n-p}{p}} - \rho \right). \tag{44}$$

In order to estimate the second factor of the right-hand side of (40) we use Lemma 1 (iii), the Sobolev–Poincaré-type estimate from Theorem 2, (31), Jensen's and Hölder's inequality to deduce, writing $(\widehat{v}_k)_B$ for $⨍_B \widehat{v}_k\, dx$,

$$\begin{aligned}
&\left(⨍_B |W(g_k - \gamma_k \widehat{v}_k)|^{p^\sharp} dx\right)^{\frac{1}{p^\sharp}} \\
&\le c\left[\left(⨍_B |W(g_k - \gamma_k \widehat{v}_k + \gamma_k (\widehat{v}_k)_B)|^{p^\sharp} dx\right)^{\frac{1}{p^\sharp}} + |W(\gamma_k (\widehat{v}_k)_B)|\right] \\
&\le c\left[\left(⨍_B |W(D(g_k - \gamma_k \widehat{v}_k))|^2 dx\right)^{\frac{1}{2}} + |W(\gamma_k (\widehat{v}_k)_B)|\right] \\
&\le c\left[\gamma_k + \left(⨍_B |W(\gamma_k D\widehat{v}_k)|^2 dx\right)^{\frac{1}{2}} + \left(⨍_B |W(\gamma_k \widehat{v}_k)|^2 dx\right)^{\frac{1}{2}}\right], \qquad (45)
\end{aligned}$$

with a constant $c = c(n, N, p)$. Using Lemma 1 (i) and (ii) (note $b_k \ge 1$, $\rho < 1$) we obtain for the second term of the right-hand side of the previous inequality:

$$\begin{aligned}
&\left(⨍_B |W(\gamma_k D\widehat{v}_k)|^2 dx\right)^{\frac{1}{2}} \\
&\le c\left[\left(⨍_B \left|W\left(\gamma_k \left(D\widehat{v}_k - \tfrac{D\widetilde{g}}{b_k}\right)\right)\right|^2 dx\right)^{\frac{1}{2}} + \left(⨍_B \left|W\left(\gamma_k \tfrac{D\widetilde{g}}{b_k}\right)\right|^2 dx\right)^{\frac{1}{2}}\right] \\
&\le \frac{c}{b_k^{p/2}}\left[\gamma_k \rho \left(⨍_{B_\rho} |Dv_k - Dg|^2 dx\right)^{\frac{1}{2}} + \rho^{\frac{p-n}{2}} \left(⨍_B |W(\gamma_k Dg)|^2 dx\right)^{\frac{1}{2}}\right], \quad (46)
\end{aligned}$$

with a constant $c = c(p)$. For the second term in (46) we now show the bound

$$⨍_B |W(\gamma_k Dg)|^2 dx \le 2\gamma_k^2. \qquad (47)$$

We first consider the case that γ_k is non-increasing. Then $\frac{\gamma_k}{\gamma_\ell} \ge 1$ for $\ell > k$. Using the fact that $\gamma_k \frac{Dg_\ell}{\gamma_\ell}$ converges weakly in $L^p(B, \mathbb{R}^N)$ to $\gamma_k Dg$, together with the lower semicontinuity of the functional $w \mapsto \int_B |W(Dw)|^2 dx$ with respect to weak convergence in $W^{1,p}(B, \mathbb{R}^N)$ (see the derivation of (34)), Lemma 1 (ii) and (31) we find:

$$⨍_B |W(\gamma_k Dg)|^2 dx \le \liminf_{\ell\to\infty} ⨍_B \left|W\left(\gamma_k \tfrac{Dg_\ell}{\gamma_\ell}\right)\right|^2 dx \le \liminf_{\ell\to\infty} \gamma_k^2 \frac{⨍_B |W(Dg_\ell)|^2 dx}{\gamma_\ell^2} \le \gamma_k^2 .$$

In the case that γ_k is non-decreasing, i.e. $\frac{\gamma_k}{\gamma_\ell} \le 1$ for $\ell > k$, we note that we have $\lim_{\ell\to\infty} \gamma_\ell = \gamma \le 1$. Estimating the integral $⨍_B |W(\gamma_k Dg)|^2 dx$ similarly to above (note that in this case we use Lemma 1 (ii) with $0 < t \le 1$) we infer for k sufficiently large that there holds

$$⨍_B |W(\gamma_k Dg)|^2 dx \le \lim_{\ell\to\infty} \gamma_k^2 \frac{\gamma_\ell^{2-p}}{\gamma_k^{2-p}} = \gamma_k^2 \frac{\gamma^{2-p}}{\gamma_k^{2-p}} \le 2\gamma_k^2 .$$

Combining (46) and (47) we deduce:

$$\left(\fint_B |W(\gamma_k D\widehat{v}_k)|^2 dx\right)^{\frac{1}{2}} \le \frac{c\,\gamma_k}{b_k^{p/2}}\left[\rho\left(\fint_{B_\rho} |Dv_k - Dg|^2 dx\right)^{\frac{1}{2}} + \rho^{\frac{p-n}{2}}\right], \qquad (48)$$

with a constant $c = c(p)$.

The term $\left(\fint_B |W(\gamma_k \widehat{v}_k)|^2 dx\right)^{\frac{1}{2}}$ can be treated analogously; we only have to use the Poincaré-type inequality for W which follows from Theorem 2 (by Hölder's inequality we can replace $p^\sharp$ by 2). We then obtain

$$\left(\fint_B |W(\gamma_k \widehat{v}_k)|^2 dx\right)^{\frac{1}{2}} \le \frac{c\,\gamma_k}{b_k^{p/2}}\left[\rho\left(\fint_{B_\rho} |v_k - g|^2 dx\right)^{\frac{1}{2}} + \rho^{-\frac{n}{2}}\right],$$

where now the constant c depends on n, N and p.

Collecting terms we obtain from (45):

$$\limsup_{k\to\infty} \frac{1}{\gamma_k}\left(\fint_B |W(g_k - \gamma_k \widehat{v}_k)|^{p^\sharp} dx\right)^{\frac{1}{p^\sharp}} \le c(n, N, p)\,.$$

Combining this with (44), (40) and (41) we can finally conclude

$$\limsup_{k\to\infty} \frac{1}{\gamma_k^2} \fint_B |W(g_k - \gamma_k \widehat{v}_k)|^2 dx \le c\left(\rho^{\frac{n-p}{p}} - \rho\right)^{2t},$$

where c has the same dependencies as above. Therefore we can choose $\rho \in (1/2, 1)$ sufficiently close to 1 such that (39) holds, thus providing the desired contradiction to (32). This finishes the proof of the lemma. □

8. Proof of Theorem 1

Let $M > 0$ be fixed. We consider a point $x_0 \in U$ such that $|(Du)_{x_0,\rho}| \le M$. From (9) and (10) we conclude that $\mathcal{A} = D^2 f((Du)_{x_0,\rho})$ is elliptic in the sense of Legendre–Hadamard with ellipticity constant $\kappa = 2\lambda(1 + M^2)^{\frac{p-2}{2}}$ and upper bound $K = K_M = \sup_{|A|\le M} |D^2 f(A)|$. Therefore we are in a position to apply Lemma 4 and Lemma 6 with $A = (Du)_{x_0,\rho}$, κ and K.

For $\vartheta \in (0, 1/4]$ to be specified later we set $\varepsilon = \vartheta^{n+4}$. With $\delta = \delta(n, N, \kappa, K, \vartheta) \in (0, 1]$ we denote the constant from Lemma 6 corresponding to the quantities n, N, κ, K and the particular choice of $\varepsilon = \vartheta^{n+4}$.

Throughout the rest of this section we use the abbreviation:

$$\Phi(\rho) = \Phi(x_0, \rho, (Du)_{x_0,\rho}),$$

where Φ is given in (25). We also define

$$\Gamma(\rho) = \sqrt{\Phi^2(\rho) + 4\frac{\omega(\rho)}{\delta^2}}, \quad w = u - (Du)_{x_0,\rho}(x - x_0) \quad \text{and} \quad \gamma = c_1 c_e \Gamma(\rho),$$

where c_1 stands for the constant $c(p, M)$ from Lemma 1 (vi). From Lemma 4 we have the constant $c_e = c_e(n, N, p, L, M, K_{M+2})$, from Lemma 5 the constant

$c_a = c_a(n, N, \kappa, K)$, and finally from Lemma 3 the constant $c_c = c_c(n, N, p, L, \lambda, M, K_{M+1})$ and the radius $\rho_0 = \rho_0(n, N, p, \lambda, M, \omega(\cdot))$.

We next establish an initial excess-improvement estimate, assuming that the excess $\Phi(\rho)$ is initially sufficiently small. The precise statement is:

Lemma 7 (Excess-improvement). *Assume that the following smallness conditions hold for some $\rho \in (0, \rho_0]$:*

$$2\sqrt{2}\, c_a \gamma \le 1\,, \quad \sqrt{\nu_M\,(\Phi(\rho))} + \Phi(\rho) \le \delta/2 \quad \textit{and} \quad c_e\, c_a\, \Phi(\rho) \le 1\,.$$

Then the following growth condition holds:

$$\Phi^2(\vartheta\rho) \le \widetilde{c}\,\vartheta^2 \Phi^2(\rho) + \widehat{c}\omega(\rho),$$

with constants $\widetilde{c} = \widetilde{c}(n, N, p, L, \lambda, M, K_{M+2})$ and $\widehat{c} = \widehat{c}(n, N, p, L, \lambda, M, K_{M+2}, \vartheta)$.

Proof. From the definition of w we find, using (15) and Lemma 1 (vi):

$$\fint_{B_\rho(x_0)} |W(Dw)|^2\, dx \le \fint_{B_\rho(x_0)} |V(Dw)|^2\, dx \le c_1\, \Phi^2(\rho) \le \gamma^2.$$

We infer from Lemma 4 and the smallness condition $\sqrt{\nu_M\,(\Phi(\rho))} + \Phi(\rho) \le \delta/2$:

$$\begin{aligned}
&\left| \fint_{B_\rho(x_0)} D^2 f((Du)_{x_0,\rho})(Dw, D\varphi)\, dx \right| \\
&\le c_e \left[\sqrt{\nu_M\,(\Phi(\rho))}\Phi(\rho) + \Phi^2(\rho) + \sqrt{\omega(\rho)} \right] \sup_{B_\rho(x_0)} |D\varphi| \\
&= \gamma\ \frac{\sqrt{\nu_M\,(\Phi(\rho))}\Phi(\rho) + \Phi^2(\rho) + \sqrt{\omega(\rho)}}{c_1 \Gamma(\rho)} \sup_{B_\rho(x_0)} |D\varphi| \\
&\le \gamma\, \delta \sup_{B_\rho(x_0)} |D\varphi|.
\end{aligned}$$

Hence the hypotheses of Lemma 6 are fulfilled. Therefore we can find a function $h \in W^{1,p}\big(B_\rho(x_0), \mathbb{R}^N\big)$ which is $D^2 f((Du)_{x_0,\rho})$-harmonic such that

$$\fint_{B_\rho(x_0)} |W(Dh)|^2\, dx \le 1 \quad \text{and} \quad \fint_{B_\rho(x_0)} \left| W\big(\tfrac{w-\gamma h}{\rho}\big) \right|^2 dx \le \gamma^2\, \varepsilon = \gamma^2\, \vartheta^{n+4}. \tag{49}$$

With the help of Lemma 1 (iii) and (v) we deduce

$$\begin{aligned}
\Phi(\vartheta\rho) &= \fint_{B_{\vartheta\rho}(x_0)} |V(Du) - V((Du)_{x_0,\vartheta\rho})|^2\, dx \\
&\le c \fint_{B_{\vartheta\rho}(x_0)} |V(Du - (Du)_{x_0,\vartheta\rho})|^2\, dx \\
&\le c \fint_{B_{\vartheta\rho}(x_0)} |V(Du - (Du)_{x_0,\rho} - \gamma\, Dh(x_0))|^2\, dx \\
&\quad + c\, |V((Du)_{x_0,\vartheta\rho} - (Du)_{x_0,\rho} - \gamma\, Dh(x_0))|^2,
\end{aligned} \tag{50}$$

where the constant c depends only on n, N and p.

We next estimate the right-hand side of (50). Decomposing $B_{\vartheta\rho}(x_0)$ into the set with $|Du-(Du)_{x_0,\rho}-\gamma\, Dh(x_0)| \le 1$ and that with $|Du-(Du)_{x_0,\rho}-\gamma\, Dh(x_0)| > 1$, than using Lemma 1 (i) and Hölder's inequality we obtain:

$$\begin{aligned}|(Du)_{x_0,\vartheta\rho} - (Du)_{x_0,\rho} - \gamma\, Dh(x_0))| &\le \fint_{B_{\vartheta\rho}(x_0)} \big|Du - (Du)_{x_0,\rho} - \gamma\, Dh(x_0))\big|\, dx \\ &= \sqrt{2}\ \left(I^{1/2} + I^{1/p}\right),\end{aligned}$$

where we have abbreviated

$$I = \fint_{B_{\vartheta\rho}(x_0)} \big|V(Du - (Du)_{x_0,\rho} - \gamma\, Dh(x_0))\big|^2\, dx.$$

Now, since $|V(A)| = V(|A|)$ and $t \mapsto V(t)$ is monotone increasing, we deduce from (50), also using Lemma 1 (i) and (ii), that there holds:

$$\Phi(\vartheta\rho) \le c\, \left(I + V^2(I^{1/2} + I^{1/p})\right) \le c\,(I + I^{2/p}), \tag{51}$$

where c depends only on n, N and p. Therefore it remains for us to estimate the quantity I. By considering the cases $|Dh| \le 1$ and $|Dh| > 1$ seperately and keeping in mind (49) we have (cf. (42)):

$$\fint_{B_\rho(x_0)} |Dh|\, dx \le 2\sqrt{2}. \tag{52}$$

Using the assumption $|(Du)_{x_0,\rho}| \le M$ and Lemma 5, this shows:

$$\begin{aligned}|(Du)_{x_0,\rho}| + \gamma|Dh(x_0)| \le M + \gamma|Dh(x_0)| &\le M + \gamma\, c_a \fint_{B_\rho(x_0)} |Dh|\, dx \\ &\le M + 2\sqrt{2}\, c_a\gamma \le M + 1\,.\end{aligned}$$

Caccioppoli's inequality (i.e. Lemma 3 applied on $B_{\vartheta\rho}(x_0)$ with $\gamma\, h(x_0)$, respectively $(Du)_{x_0,\rho} + \gamma\, Dh(x_0)$, instead of ξ, respectively A; note that the constant c_c depends only on n, N, p, L, λ, $M+1$ and K_{M+2}) and Lemma 1 (iii) yield:

$$\begin{aligned}I &\le c_c\left[\fint_{B_{2\vartheta\rho}(x_0)} \left|V\left(\tfrac{w-\gamma\, h(x_0)-\gamma Dh(x_0)(x-x_0)}{2\vartheta\rho}\right)\right|^2 dx + \omega(2\vartheta\rho)\right] \\ &\le c\ \left[\fint_{B_{2\vartheta\rho}(x_0)} \left(\left|V\left(\tfrac{w-\gamma h}{2\vartheta\rho}\right)\right|^2 + \left|V\left(\gamma\ \tfrac{h-h(x_0)-Dh(x_0)(x-x_0)}{2\vartheta\rho}\right)\right|^2\right) dx + \omega(\rho)\right],\end{aligned}$$

where the constant c is given by $c(p)\cdot c_c$. To estimate the right-hand side we use (15), Lemma 1 (ii) (note that $\frac{1}{2\vartheta} \ge 1$), (49) and recall the choice $\varepsilon = \vartheta^{n+4}$ to infer:

$$\begin{aligned}\fint_{B_{2\vartheta\rho}(x_0)} \left|V\left(\tfrac{w-\gamma h}{2\vartheta\rho}\right)\right|^2 dx &\le c(p) \fint_{B_{2\vartheta\rho}(x_0)} \left|W\left(\tfrac{w-\gamma h}{2\vartheta\rho}\right)\right|^2 dx \\ &\le c(p)(2\vartheta)^{-n} \fint_{B_\rho(x_0)} \left|W\left(\tfrac{w-\gamma h}{2\vartheta\rho}\right)\right|^2 dx \\ &\le c(p)(2\vartheta)^{-n-2} \fint_{B_\rho(x_0)} \left|W\left(\tfrac{w-\gamma h}{\rho}\right)\right|^2 dx \\ &\le c(p)\, 2^{-n-2}\vartheta^{-n-2}\gamma^2\varepsilon = c(n,p)\vartheta^2\gamma^2.\end{aligned}$$

Using Lemma 1 (i), Taylor's theorem applied to h on $B_{2\vartheta\rho}(x_0)$, Lemma 5 and (52) we obtain:

$$\begin{aligned}
&\fint_{B_{2\vartheta\rho}(x_0)} \left|V\left(\gamma \tfrac{h-h(x_0)-Dh(x_0)(x-x_0)}{2\vartheta\rho}\right)\right|^2 dx \\
&\leq \gamma^2 \fint_{B_{2\vartheta\rho}(x_0)} \left|\tfrac{h-h(x_0)-Dh(x_0)(x-x_0)}{2\vartheta\rho}\right|^2 dx \\
&\leq \frac{\gamma^2}{4\vartheta^2\rho^2} \sup_{B_{2\vartheta\rho}(x_0)} |h(x)-h(x_0)-Dh(x_0)(x-x_0)|^2 \leq 8\, c_a^2 \vartheta^2 \gamma^2.
\end{aligned}$$

Combining these estimates and keeping in mind the dependencies of the constants as well as the definitions of γ and Γ we see:

$$I \leq \widetilde{c}\,\vartheta^2 \Phi^2(\rho) + \widehat{c}\omega(\rho),$$

where $\widetilde{c}$ depends on n, N, p, L, λ, M and K_{M+2} and $\widehat{c}$ depends on the same quantities and additionally on ϑ (the dependency from ϑ occurs due to the fact that δ depends on ϑ). Inserting this into (51) we easily find (recalling also that $\Phi(\rho) \leq 1$ and $\omega(\rho) \leq 1$):

$$\Phi^2(\vartheta\rho) \leq \widetilde{c}\,\vartheta^2 \Phi^2(\rho) + \widehat{c}\omega(\rho),$$

where the constants $\widetilde{c}$, $\widehat{c}$ have the same dependencies as above. This proves the asserted estimate. □

The assumptions of Lemma 7 are satisfied in points $x_0 \in U \setminus (\Sigma_1 \cup \Sigma_2)$, and the resultant excess improvement can be iterated by the use of the Dini condition **(F3)**, to yield an *excess-decay estimate* for $\Phi(r)$, i.e. we have

$$\Phi(x_0, r, (Du)_r) \leq \mathrm{const}\Big[\Big(\frac{r}{\rho}\Big)^\alpha \Phi(x_0, \rho, (Du)_\rho) + \sqrt{\omega(r)}\Big],$$

for all $0 < r \leq \rho$ and α with $\beta < \alpha < 1$ (where β is the constant from **(F2)**) (cf. [15, proof of Theorem 2.2, p. 683 ff.] and [11, proof of Theorem 3.2, p. 163]). The regularity result then follows from the fact that this excess-decay estimate implies:

$$\begin{aligned}
\fint_{B_r(x)} |V(Du) - (V(Du))_{x,r}|^2\, dx &\leq \fint_{B_r(x)} |V(Du) - (V(Du)_{x,r})|^2\, dx \\
&\leq \mathrm{const}\left[\Big(\frac{r}{\rho}\Big)^\alpha \Phi(x_0, \rho, (Du)_\rho) + \sqrt{\omega(r)}\right],
\end{aligned}$$

for any x in a neighbourhood of x_0. From this estimate we conclude (by Campanato's characterization of Hölder continuous functions, cf. [7], [8]) that $V(Du)$ has the modulus of continuity $\rho \mapsto \rho^\alpha + \Omega(\rho)$. By Lemma 1 (iv) this modulus of continuity carries over to Du.

References

1. Acerbi, E., Fusco, N.: A regularity theorem for minimizers of quasiconvex variational integrals. Arch. Ration. Mech. Anal. **99**, 261–281 (1987)
2. Allard, W.K.: On the first variation of a varifold. Ann. Math. (2) **95**, 225–254 (1972)
3. Almgren, F.J.: Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. Ann. Math. (2) **87**, 321–391 (1968)
4. Almgren, F.J.: Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. Mem. Am. Math. Soc. **165**, (1976)
5. Anzelotti, G.: On the $C^{1,\alpha}$-regularity of ω-minima of quadratic functionals. Boll. Unione Mat. Ital. C (6) **2**, 195–212 (1983)
6. Bombieri, E.: Regularity theory for almost minimal currents. Arch. Ration. Mech. Anal. **7**, 99–130 (1982)
7. Campanato, S.: Proprietà di una famiglia di spazi funzionali. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **18**, 137–160 (1964)
8. Campanato, S.: Equazioni ellitichi del II^e ordine e spazi $\mathcal{L}^{2,\lambda}$. Ann. Mat. Pura Appl., IV. Ser. **69**, 321–381 (1965)
9. Conti, S., Fonseca, I., Leoni, G.: A Γ–Convergence Result for the Two–Gradient Theory of Phase Transitions. Commun. Pure Appl. Math. **165**, 141–164 (1998)
10. Carozza, M., Passarelli di Napoli, A.: A regularity theorem for minimisers of quasiconvex integrals: the case $1 < p < 2^*$. Proc. R. Soc. Edinb., Sect. A, Math. **126**, 1181–1199 (1996)
11. Carozza, M., Fusco, N., Mingione, G.: Partial Regularity of Minimizers of Quasiconvex Integrals with Subquadratic Growth. Ann. Mat. Pura Appl., IV. Ser. **175**, 141–164 (1998)
12. Dacorogna, B.: Direct Methods in the Calculus of Variations. Berlin, Heidelberg, New York: Springer 1989
13. De Giorgi, E.: Frontiere orientate di misura minima. Seminaro Mat. Scuola Norm. Sup. Pisa, 1–56 (1961)
14. Duzaar, F., Gastel, A.: Nonlinear elliptic systems with Dini continuous coefficients. Arch. Math. **78**, 58–73 (2002)
15. Duzaar, F., Gastel, A., Grotowski, J.F.: Partial regularity for almost minimizers of quasiconvex functionals. SIAM J. Math. Anal. **32**, 665–687 (2000)
16. Duzaar, F., Grotowski, J.F.: Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation. Manuscr. Math. **103**, 267–298 (2000)
17. Duzaar, F., Kronz, M.: Regularity of ω–minimizers of quasi–convex variational integrals with polynomial growth. Differ. Geom. Appl. **17**, 139–152 (2002)
18. Duzaar, F., Steffen, K.: Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals. J. Reine Angew. Math. **546**, 73–138 (2002)
19. Evans, L.C.: Quasiconvexitity and Partial Regularity in the Calculus of Variations. Arch. Ration. Mech. Anal. **95**, 227–252 (1986)
20. Federer, H.: Geometric Measure Theory. Berlin, Heidelberg, New York: Springer 1969
21. Giaquinta, M.: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Princeton: Princeton University Press 1983
22. Giaquinta, M., Modica, G.: Partial regularity of minimizers of quasiconvex integrals. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **3**, 185–208 (1986)
23. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Berlin, Heidelberg, New York: Springer 1983
24. Giusti, E., Miranda, M.: Sulla regolarità delle soluzioni deboli di una classe di sistemi ellittici quasi-lineari. Arch. Ration. Mech. Anal. **31**, 173–184 (1968)
25. Grotowski, J.F.: Boundary regularity for nonlinear elliptic systems. Calc. Var. Partial Differ. Equ. **15**, 353–388 (2002)
26. Grotowski, J.F.: Boundary regularity for quasilinear elliptic systems. Commun. Partial Differ. Equations **27**, 2491–2512 (2002)

27. Kronz, M.: Partial Regularity Results for Minimizers of Quasiconvex Functionals of Higher Order. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **19**, 81–112 (2002)
28. Morrey, C.B.: Quasi-convexity and the lower semicontinuity of multiple integral. Pac. J. Math. **2**, 25–53 (1952)
29. Morrey, C.B.: Multiple integrals in the calculus of variations. Berlin, Heidelberg, New York: Springer 1966
30. Simon, L.: Lectures on Geometric Measure Theory. Canberra: Australian National University Press 1983
31. Sverak, V.: Quasiconvex functions with subquadratic growth. Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **433**, 723–725 (1991)
32. Ziemer, W.P.: Weakly differentiable functions. Berlin, Heidelberg, New York: Springer 1989