Annali di Matematica 183, 375–386 (2004)

Digital Object Identifier (DOI) 10.1007/s10231-003-0093-1

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On the Hilbert scheme of a Prym variety

Dedicated to the memory of Fabio Bardelli

Received: July 8, 2002
Published online: February 11, 2004 – © Springer-Verlag 2004

Introduction

We work over the field of complex numbers. In this paper we consider the Prym map $\mathcal{P} : \mathcal{R}_g \to \mathcal{A}_{g-1}$ from the moduli space of unramified double covers of projective irreducible and non-singular curves of genus $g \geq 6$ to the moduli space of principally polarized Abelian varieties of dimension $g - 1$. If $\pi : \tilde{C} \to C$ is such a double cover with C non-hyperelliptic, we consider the natural embedding $\tilde{C} \subset P$ (defined up to translation) of $\tilde{C}$ into the Prym variety P of π and we study the local structure of the Hilbert scheme $Hilb^P$ of P at the point $[\tilde{C}]$ (here and through the paper we adopt the notation $[-]$ for the point of a moduli space or of a Hilbert scheme which parametrizes the object $-$). We show that this structure is related to the local geometry of the Prym map, or more precisely with the validity of the infinitesimal version of Torelli's theorem for Pryms at $[\pi]$ (see Section 3 for the definitions).

The results we prove are the following.

Proposition. *If the infinitesimal Torelli theorem for Pryms holds at* $[\pi]$ *then* $Hilb^P$ *is non-singular of dimension* $g - 1$ *at* $[\tilde{C}]$ *(i.e.* $\tilde{C}$ *is unobstructed) and the only deformations of* $\tilde{C}$ *in* P *are translations.*

It is known that if the Clifford index of C is at least 3 then the condition of the proposition is satisfied. Therefore we have, in particular:

Corollary. *If* $Cliff(C) \geq 3$ *then* $Hilb^P$ *is non-singular of dimension* $g - 1$ *at* $[\tilde{C}]$ *(i.e.* $\tilde{C}$ *is unobstructed) and the only deformations of* $\tilde{C}$ *in* P *are translations.*

On the other side we have the following result:

Theorem. *Assume that the following conditions are satisfied:*

(a) the infinitesimal Torelli theorem for Pryms fails at $[\pi]$*;*
(b) $[\pi]$ *is an isolated point of the fibre* $\mathcal{P}^{-1}(\mathcal{P}([\pi]))$.

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Then $\tilde{C}$ is obstructed. Moreover the only local deformations of $\tilde{C}$ in P are translations and the only irreducible component of $Hilb^P$ containing $[\tilde{C}]$ is everywhere non-reduced.

Conversely, if $\tilde{C}$ is obstructed then the infinitesimal Torelli theorem for Pryms fails at $[\pi]$.

Using these results we give some examples in which $\tilde{C} \subset P$ is obstructed and some in which we have unobstructedness but the infinitesimal Torelli theorem for Pryms fails. The examples we construct are obtained from double covers belonging to $\mathcal{R}_6$ and to $\mathcal{R}_7$. For their construction we make use of a result proved in Section 4 which is a slight extension of a theorem of Recillas (see [12]).

The paper is divided into 5 sections. In Section 1 we discuss the Hilbert scheme of curves Abel–Jacobi embedded in their Jacobian. We prove that such curves are obstructed precisely when they are hyperelliptic of genus $g \geq 3$. This case is not relevant for what follows but it is worth keeping in mind the analogies between the two cases. In Section 2 we consider our problem and we study the conditions for the unobstructedness of $[\tilde{C}]$. We use the well-known cohomological description of certain tangent spaces and of maps between them. In Section 3 we relate these results with the infinitesimal Torelli theorem for Pryms and we prove our main result. In Section 4 we give a proof of the extension of Recillas' theorem. The final Section 5 contains the examples.

1. The case of curves in their Jacobians

Consider a projective non-singular irreducible curve C of genus $g \geq 2$, let $JC := Pic^0(C)$ be the Jacobian variety of C, and let $j : C \to JC$ be an Abel–Jacobi map. We want to study the local structure of the Hilbert scheme $Hilb^{JC}$ of JC at $[j(C)]$ (the point parametrizing $j(C)$). Since j is an embedding we will identify C with $j(C)$. We have an exact sequence of locally free sheaves on C:

$$(1) \qquad 0 \to T_C \to T_{JC|C} \to N_C \to 0.$$

We have a canonical isomorphism $T_{JC|C} \cong H^1(\mathcal{O}_C) \otimes \mathcal{O}_C$ and therefore the cohomology sequence of (1) is:

(2)

$$0 \to H^1(\mathcal{O}_C) \to H^0(N_C) \xrightarrow{\delta} H^1(T_C) \xrightarrow{\sigma} H^1(\mathcal{O}_C) \otimes H^1(\mathcal{O}_C) \to H^1(N_C).$$

The family of translations of C in JC is parametrized by JC itself, and the map $H^1(\mathcal{O}_C) \to H^0(N_C)$ in (2) is precisely the characteristic map of this family at the point 0. Therefore we have the following Lemma, whose proof is obvious:

Lemma 1.1. *The following conditions are equivalent:*

(a) $Hilb^{JC}$ is non-singular of dimension g at $[C]$;
(b) $\delta = 0$;
(c) σ is injective.

If these conditions are satisfied then the only local deformations of C in JC are translations.

Using this lemma we can prove the following:

Theorem 1.2. *Suppose that C has genus $g \geq 3$:*

(a) if C is non-hyperelliptic then $Hilb^{JC}$ is non-singular of dimension g at $[C]$;
(b) if C is hyperelliptic then the connected component of $Hilb^{JC}$ containing $[C]$ is irreducible of dimension g and everywhere non-reduced with Zariski tangent space of dimension $2g-2$.

In both cases the only deformations of C in JC are translations.

Proof. The transpose of σ is the multiplication map:

$$\sigma^{\vee} : H^0(\omega_C) \otimes H^0(\omega_C) \to H^0\big(\omega_C^{\otimes 2}\big)$$

(see [5, Lemma 3]). This map, by Noether's theorem, is surjective if C is non-hyperelliptic and has co-rank $g-2$ if C is hyperelliptic (see [1]). Therefore, in view of Lemma 1.1, part (a) follows.

Now assume that C is hyperelliptic and that $\bar{C} \subset JC$ is a closed subscheme such that $[\bar{C}]$ belongs to the connected component of $Hilb^{JC}$ containing $[C]$. By the criterion of Matsusaka–Ran (see [7]) $\bar{C} = C_1 \cup \cdots \cup C_r$ is a reduced curve of compact type, and JC and $JC_1 \times \cdots \times JC_r$ are isomorphic as ppav's. Then it follows that $r = 1$ and $\bar{C}$ is irreducible and non-singular because C is. Now we apply Torelli's theorem to conclude that $\bar{C}$ is a translate of C. It follows that the connected component of $Hilb^{JC}$ containing $[C]$ is irreducible of dimension g and parametrizes the translates of C. On the other hand by (2) we have $h^0(N_C) = 2g-2 > g$. The conclusion follows. □

Theorem 1.2 can be interpreted in terms of the Torelli morphism,

$$\tau : M_g \to A_g,$$

from the moduli stack of projective non-singular curves of genus g to the moduli stack of principally polarized Abelian varieties of dimension g. The surjectivity of $\sigma^{\vee}$ is equivalent to that of the multiplication map

$$S^2 H^0(\omega_C) \to H^0\big(\omega_C^{\otimes 2}\big),$$

which is the codifferential of τ at $[C]$. Hence the surjectivity of this map is equivalent to the infinitesimal Torelli theorem for C (see [11]). Therefore Theorem 1.2 implies the following:

Corollary 1.3. *C is unobstructed in JC if and only if the infinitesimal Torelli theorem holds for C.*

Remarks. (i) The proof of Theorem 1.2(a) has already appeared in [5], but the argument does not appear to be complete. A proof is also given in [3] using the semiregularity map, but it is more complicated; moreover, the semiregularity map does not seem to be able to detect what happens in case (b).

(ii) In the case $g = 2$ we have that C is unobstructed in JC because the semiregularity map $H^1(N_C) \to H^2(\mathcal{O}_{JC})$ is injective since $H^1(\mathcal{O}_{JC}(C)) = 0$ by the ampleness of C in JC.

2. The Hilbert scheme of the Prym variety at $[\tilde{C}]$

Let now $\pi : \tilde{C} \to C$ be an unramified double cover of a projective non-singular irreducible curve C of genus $g \geq 3$, so that $\tilde{C}$ has genus $\tilde{g} = 2g - 1$. Let $\eta \in Pic^0(C)$ be the 2-division point corresponding to π. We have a canonical isogeny $J\tilde{C} \to JC \times P$, where P is the Prym variety of π. *Throughout this section we assume C to be non-hyperelliptic*. Under this hypothesis we have an embedding $\alpha : \tilde{C} \to P$ which is obtained as the composition

$$\tilde{C} \to J\tilde{C} \to JC \times P \to P$$

(see [7]). We will identify $\tilde{C}$ with $\alpha(\tilde{C})$. We want to study the Hilbert scheme $Hilb^P$ locally at the point $[\tilde{C}]$.

In analogy with the situation studied in Section 1, we consider the exact sequence of locally free sheaves on $\tilde{C}$,

$$0 \to T_{\tilde{C}} \to T_{P|\tilde{C}} \to N_{\tilde{C}} \to 0. \tag{3}$$

We have a canonical isomorphism $T_{P|\tilde{C}} \cong H^1(C, \eta) \otimes \mathcal{O}_{\tilde{C}}$ so that the cohomology sequence of (3) becomes

(4)

$$\begin{aligned} 0 \longrightarrow H^1(C,\eta) \longrightarrow\ & H^0(\tilde{C}, N_{\tilde{C}}) \xrightarrow{\ \delta\ } H^1(\tilde{C}, T_{\tilde{C}}) \\ & \xrightarrow{\ \sigma\ } H^1(C,\eta) \otimes H^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) \longrightarrow H^1(\tilde{C}, N_{\tilde{C}}). \end{aligned}$$

Along the same lines of Section 1 we can state the following:

Lemma 2.1. *The following conditions are equivalent:*

(a) $Hilb^P$ is non-singular of dimension $g-1$ at $[\tilde{C}]$;
(b) $\delta = 0$;
(c) σ is injective.

If these conditions are satisfied then the only local deformations of $\tilde{C}$ in P are translations.

In order to understand the conditions of Lemma 2.1 we must study the map σ, or equivalently, its transpose $\sigma^\vee$. We have canonical isomorphisms:

$$H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})^\vee \cong H^0(\tilde{C}, \omega_{\tilde{C}}) \cong H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes \eta)$$

and

$$H^1(\tilde{C}, T_{\tilde{C}})^\vee \cong H^0\big(\tilde{C}, \omega_{\tilde{C}}^{\otimes 2}\big) \cong H^0\big(C, \omega_C^{\otimes 2} \otimes \eta\big) \oplus H^0\big(C, \omega_C^{\otimes 2}\big)$$

corresponding to the decompositions into $+1$ and -1 eigenvalues under the action induced by the involution on $\tilde{C}$. Hence

$$\begin{aligned} \sigma^\vee : H^0(C, \omega_C \otimes \eta) \bigotimes \big[H^0(C, \omega_C) \oplus H^0(C, \omega_C \otimes \eta)\big] \\ \to H^0\big(C, \omega_C^{\otimes 2} \otimes \eta\big) \oplus H^0\big(C, \omega_C^{\otimes 2}\big), \end{aligned}$$

and it is induced by multiplication of sections [2, p. 382]. Therefore, after decomposing the domain of $\sigma^\vee$ as

$$H^0(C,\omega_C\otimes\eta)\bigotimes\big[H^0(C,\omega_C)\oplus H^0(C,\omega_C\otimes\eta)\big]$$
$$=\big[H^0(C,\omega_C\otimes\eta)\otimes H^0(C,\omega_C)\big]\bigoplus\big[H^0(C,\omega_C\otimes\eta)\otimes H^0(C,\omega_C\otimes\eta)\big],$$

we see that $\sigma^\vee=\mu\oplus\nu$, where

$$\mu : H^0(C,\omega_C\otimes\eta)\otimes H^0(C,\omega_C)\to H^0\big(C,\omega_C^{\otimes 2}\otimes\eta\big)$$

and

$$\nu : H^0(C,\omega_C\otimes\eta)\otimes H^0(C,\omega_C\otimes\eta)\to H^0\big(C,\omega_C^{\otimes 2}\big).$$

The following lemma is well known (see [2, Prop. 7.7]):

Lemma 2.2. *The sheaf $\omega_C\otimes\eta$ is not very ample if and only if there exist points $x,y,z,t\in C$ such that $\eta\cong\mathcal{O}_C(x+y-z-t)$. If these conditions are satisfied then the map ν is not surjective.*

Proposition 2.3. *(i) In each of the following cases the map μ is surjective:*

(a) C is not bi-elliptic;
(b) ν is surjective;

(ii) if C is not bi-elliptic then $\operatorname{cork}(\sigma^\vee)=\operatorname{cork}(\nu)$. *In particular if C is not bi-elliptic the surjectivity of $\sigma^\vee$ is equivalent to the surjectivity of ν.*

Proof. (i) Note first that in both cases (a) and (b) the linear series $|\omega_C\otimes\eta|$ is base point free and is not composed with an involution: in fact, in case (a) since C is not hyperelliptic, $|\omega_C\otimes\eta|$ is base point free; moreover if it were composed with an involution then, since $\deg(\omega_C\otimes\eta)=2g-2$, the morphism

$$\phi_\eta : C\to\mathbf{P}^{g-2}$$

would be of degree 2 onto a curve of degree $g-1$, which has genus ≤ 1, a contradiction. In the case (b) the assertion is true by Lemma 2.2.

Let $\underline{b}:=P_1+\cdots+P_{g-3}\in C^{(g-3)}$ be general. Consider the exact sequence on C

$$0\to\omega_C\otimes\eta(-\underline{b})\to\omega_C\otimes\eta\to\mathcal{T}\to 0,$$

where $\mathcal{T}$ is a torsion sheaf supported on $\underline{b}$. Multiplying firstly by $H^0(\omega_C)$ and taking cohomology, and secondly by ω_C and taking cohomology, we obtain the following commutative diagram with exact rows, where the vertical maps are given by multiplication:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\omega_C\otimes\eta(-\underline{b}))\otimes H^0(\omega_C) & \longrightarrow & H^0(\omega_C\otimes\eta)\otimes H^0(\omega_C) & \longrightarrow & H^0(\mathcal{T})\otimes H^0(\omega_C) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\mu_{\underline{b}}} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\bar\mu} & & \\
0 & \longrightarrow & H^0\big(\omega_C^{\otimes 2}\otimes\eta(-\underline{b})\big) & \longrightarrow & H^0\big(\omega_C^{\otimes 2}\otimes\eta\big) & \longrightarrow & H^0(\mathcal{T}\otimes\omega_C) & \longrightarrow & 0.
\end{array}$$

Since $|\omega_C \otimes \eta|$ is not composed with an involution and $\underline{b}$ is generic, $\omega_C \otimes \eta(-\underline{b})$ is base point free, and by the base point free pencil trick we find

$$\ker(\mu_{\underline{b}}) = H^0(\underline{b} \otimes \eta) = 0,$$

hence

$$\mathrm{rk}(\mu_{\underline{b}}) = 2g = h^0\big(\omega_C^{\otimes 2} \otimes \eta(-\underline{b})\big),$$

i.e. $\mu_{\underline{b}}$ is surjective. On the other hand $\bar{\mu}$ is surjective because ω_C is globally generated. The conclusion follows from the above diagram.

(ii) follows immediately from part (i) and from the relation between the maps σ, μ, ν. □

Collecting all we have said so far we can state the following:

Corollary 2.4. *If ν is surjective then $Hilb^P$ is non-singular of dimension $g-1$ at $[\tilde{C}]$ (i.e. $\tilde{C}$ is unobstructed) and the only local deformations of $\tilde{C}$ in P are translations.*

As an application we can prove the following:

Corollary 2.5. *If Cliff$(C) \geq 3$ then $Hilb^P$ is non-singular of dimension $g-1$ at $[\tilde{C}]$ (i.e. $\tilde{C}$ is unobstructed) and the only local deformations of $\tilde{C}$ in P are translations.*

Proof. It follows easily from a result of [6] (see e.g. [8]) that if Cliff$(C) \geq 3$ then the map ν is surjective. Therefore the corollary follows from Corollary 2.4. □

3. Hilb$^{\mathbf{P}}$ and the infinitesimal Torelli theorem for Pryms

We keep the notations of Section 2. Consider the Prym morphism

$$\mathcal{P} : \mathcal{R}_g \to \mathcal{A}_{g-1},$$

which goes from the coarse moduli scheme of étale double covers of curves of genus g to the coarse moduli scheme of ppav of dimension $g-1$, $g \geq 6$. These schemes have singularities due to the presence of automorphisms of the objects they classify. Therefore if we want to study the infinitesimal properties of $\mathcal{P}$ it is more natural to consider the corresponding moduli stacks R_g, A_{g-1}. The Prym construction defines a morphism of stacks

$$\mathrm{Pr} : R_g \to A_{g-1}.$$

Then the map ν considered in Section 2 coincides with the co-differential of Pr at $[\pi]$ (see [2, Proposition 7.5], which implies this statement modulo obvious modifications). Therefore the surjectivity of ν is equivalent to Pr being an immersion at $[\pi]$ (see [2, 7.6]). In this case we say that the infinitesimal Torelli theorem for Pryms holds at $[\pi]$, according to the terminology most commonly used nowadays. In view of Corollary 2.4 we can therefore state the following:

Proposition 3.1. *If the infinitesimal Torelli theorem for Pryms holds at* $[\pi]$ *then* $Hilb^P$ *is non-singular of dimension* $g-1$ *at* $[\tilde{C}]$ *(i.e.* $\tilde{C}$ *is unobstructed) and the only local deformations of* $\tilde{C}$ *in* P *are translations.*

In the case $\mathrm{Cliff}(C) \leq 2$ the infinitesimal Torelli theorem for Pryms in general fails, i.e. in general ν is not surjective. Our next goal is to relate the obstructedness of $\tilde{C}$ in P to the failure of the infinitesimal Torelli theorem for Pryms. The main result of this section is the following:

Theorem 3.2. *Assume that the following conditions are satisfied:*

(a) the infinitesimal Torelli theorem for Pryms fails at $[\pi]$*;*
(b) $[\pi]$ *is an isolated point of the fibre* $\mathcal{P}^{-1}(\mathcal{P}([\pi]))$.

Then $\tilde{C}$ *is obstructed. Moreover the only local deformations of* $\tilde{C}$ *in* P *are translations; in particular the only irreducible component of* $Hilb^P$ *containing* $[\tilde{C}]$ *is everywhere non-reduced of dimension* $g-1$.

Conversely, if $\tilde{C}$ *is obstructed then the infinitesimal Torelli theorem fails at* $[\pi]$.

Proof. By (a) the map δ in the exact sequence (4) is non-zero. Assume by contradiction that $[\tilde{C}]$ is unobstructed. Then we can find a non-singular curve $S \subset Hilb^P$ passing through $[\tilde{C}]$ such that $\delta(T_{S,[\tilde{C}]}) \neq 0$. This condition implies that the functorial morphism $S \to \mathcal{M}_{\tilde{g}}$ defined by the family of curves $\mathcal{C} \to S$ (which can be assumed to be smooth) is not constant and therefore this family does not consist of curves all isomorphic to $\tilde{C}$. But this is impossible: in fact for each curve $\tilde{C}'$ in the family we have

$$\tilde{C}' \equiv_{\mathrm{num}} \tilde{C} \equiv_{\mathrm{num}} \frac{2}{(g-2)!}\Xi^{g-2},$$

so that by a theorem of Welters (see [13]) there is an étale double cover $\pi' : \tilde{C}' \to C'$, (P, Ξ) is the Prym variety of π' and $\tilde{C}'$ is Prym embedded. But this contradicts condition (b) if $\tilde{C}'$ is not isomorphic to $\tilde{C}$ because $[\pi'] \in \mathcal{P}^{-1}(\mathcal{P}([\pi]))$.

This analysis also shows that locally the only deformations of $\tilde{C}$ in P are translations; and since $\delta \neq 0$ the Zariski tangent space of $Hilb^P$ at $[\tilde{C}]$ has dimension larger than $g-1$. This also proves the last assertion.

The converse is a special case of Proposition 3.1. □

The theorem does not say anything in the case when $[\pi]$ is a non-isolated point of the fibre $\mathcal{P}^{-1}(\mathcal{P}([\pi]))$. We will see in Section 5 that in this case there are examples where $[\tilde{C}]$ is unobstructed.

4. Further considerations

A theorem of Recillas [12] says that if $\pi : \tilde{C} \to C$ is a double cover with C trigonal (but not hyperelliptic) of genus g then $\mathcal{P}([\pi]) = [JX]$ with X a 4-gonal curve, and the pair (X, g^1_4) is uniquely determined. A consequence of this result and of a theorem of Mumford [9], which gives a list of the Prym varieties which are Jacobians, is that

$$\mathcal{P}^{-1}([JX]) = W^1_4(X) \tag{5}$$

set-theoretically if $g-1 \geq 6$. This says in particular that for $g \geq 11$ the Prym map is 1-1 on $\mathcal{R}_{g,T}$ (= the locus of étale double covers of trigonal curves): this follows from the fact that $W^1_4(X)$ consists of at most one point if $g(X) \geq 10$. If $g-1=5$ then (5) is not true but we have a strict inclusion $\supset$ (see Section 4). The following proposition gives some further information which will suffice for some applications.

Proposition 4.1. *Assume that X is a non-singular irreducible curve of genus $g-1 \geq 5$, non-hyperelliptic nor trigonal, and such that every g^1_4 on X has no divisors of the form $2P+2Q$ or $4P$ and let $\pi : \tilde{C} \to C$ be an unramified double cover, with C trigonal of genus g, such that $\mathcal{P}([\pi]) = [JX]$. Then there is a canonical isomorphism between the kernel of the differential of* Pr *at $[\pi]$ and the Zariski tangent space of $W^1_4(X)$ at the line bundle L corresponding to π.*

Proof. Since X is not trigonal we may view $W^1_4(X)$ as parametrizing 4-1 morphisms of X into $\mathbf{P}^1$. Let $\varphi : X \to \mathbf{P}^1$ be the 4-1 cover defined by L.

Let $B = Spec(\mathbf{C}[\epsilon])$ and consider a family of deformations of φ parametrized by B:

$$\begin{array}{ccc} X \times B & \longrightarrow & \mathbf{P}^1 \times B \\ & \searrow \quad \swarrow & \\ & B. & \end{array}$$

To this family we can associate a family of deformations of π just extending Recillas's construction, as follows. Consider the second relative symmetric product over $\mathbf{P}^1 \times B$:

$$\tilde{\mathcal{C}} := S^{(2)}_{\mathbf{P}^1 \times B}(X \times B),$$

which comes endowed with an induced family of morphisms of degree 6

$$\begin{array}{cccc} f^{(2)}: & \tilde{\mathcal{C}} & \longrightarrow & \mathbf{P}^1 \times B \\ & & \searrow \quad \swarrow & \\ & & B. & \end{array}$$

On $\tilde{\mathcal{C}}$ there is a natural involution ι commuting with the projection to B. Letting $\mathcal{C} = \tilde{\mathcal{C}}/\iota$, we obtain a family parametrized by B:

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{\Pi} & \mathcal{C} \\ & \searrow \quad \swarrow & \\ & B & \end{array} \tag{6}$$

such that $f^{(2)}$ factors through Π. Therefore (6) is a first-order deformation of π; moreover (6) is contained in $\mathcal{P}^{-1}([JX])$ by construction and therefore it is an element of $\ker(d\mathrm{Pr}_{[\pi]})$.

Conversely, assume a family (6) given, and assume that (6) is contained in $\ker(d\mathrm{Pr}_{[\pi]})$. Then we also have a family of triple covers of $\mathbf{P}^1$:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathbf{P}^1 \times B \\ & \searrow \quad \swarrow & \\ & B. & \end{array}$$

Correspondingly we have an inclusion $\mathbf{P}^1 \times B \subset S_B^{(3)}(\mathcal{C})$, and an étale morphism of degree 8

$$\Pi^{(3)} : S_B^{(3)}(\tilde{\mathcal{C}}) \to S_B^{(3)}(\mathcal{C})$$

induced by Π. Let $\mathcal{D} := \Pi^{(3)-1}(\mathbf{P}^1 \times B)$. The involution ι on $\tilde{\mathcal{C}}$ induces an involution on $S_B^{(3)}(\tilde{\mathcal{C}})$ which commutes with $\Pi^{(3)}$ and induces an involution on $\mathcal{D}$. We obtain a commutative diagram:

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathbf{P}^1 \times B \\ \downarrow & \nearrow & \downarrow \\ \mathcal{D}/\iota & \longrightarrow & B, \end{array} \tag{7}$$

where the diagonal morphism defines a family of deformations of X with an assigned g_4^1 on the family. The assumption that (6) is contained in $\ker(d\mathrm{Pr}_{[\pi]})$ means that the family of ppav obtained as Pryms of the family (6) is the family of Jacobians of $\mathcal{D}/\iota \to B$ and that it is trivial. By the infinitesimal Torelli theorem for Jacobians $\mathcal{D}/\iota \to B$ is the trivial family as well. Therefore diagram (7) gives us a family of deformations of φ. □

5. Examples

5.1. Let $\bar{X} \subset \mathbf{P}^2$ be an irreducible sextic having 4 distinct nodes $N_1, \ldots, N_4$, and let X be the normalization of $\bar{X}$, which has genus 6. If no three among $N_1, \ldots, N_4$ are on a line then $W_4^1(X)$ consists of five non-singular points, the g_4^1's cut by the four pencils of lines through each of the nodes and by the pencil of conics containing $N_1, \ldots, N_4$. From Proposition 4.1 it then follows that the infinitesimal Prym-Torelli theorem holds at the five double covers $\pi : \tilde{C} \to C$ of trigonal curves of genus 7 such that $\mathcal{P}([\pi]) = [JX]$.

Assume now that $\bar{X}$ has three of the nodes, say N_1, N_2, N_3, on a line. Then the g_4^1 defined by N_4 and that defined by the pencil of conics are identified to a unique element L of $W_4^1(X)$ with a 1-dimensional Zariski tangent space. Applying Proposition 4.1 to this case we see that the infinitesimal Prym-Torelli theorem fails at the double cover π corresponding to (X, L) in the fibre $\mathcal{P}^{-1}([JX])$. Moreover, since equality (5) implies that $\mathcal{P}^{-1}([JX])$ is finite, from Theorem 3.2 we deduce that $\tilde{C}$ is obstructed in JX and that the only component of $Hilb^P$ containing $[\tilde{C}]$ is everywhere non-reduced and consists of translates of $[\tilde{C}]$.

A count of parameters shows that in this way we get a 14-dimensional locus where the infinitesimal Prym-Torelli theorem fails inside the 18-dimensional space $\mathcal{R}_7$.

5.2. Let's consider the Prym map $\mathcal{P} : \mathcal{R}_6 \to \mathcal{A}_5$. This case has been extensively studied in [4] and offers a wide variety of examples, but it is not yet completely understood from the point of view of the infinitesimal Prym-Torelli theorem. Recall that both domain and co-domain are irreducible of dimension 15. Some loci where the infinitesimal Prym-Torelli theorem fails are the following.

5.2.1. Consider a non-singular curve $C \subset \mathbf{P}^4$ obtained as a general hyperplane section of a Reye congruence in $\mathbf{P}^5$, i.e. of an Enriques surface S of degree 10 contained in a non-singular quadric. Then C is a curve of genus 6, embedded with a Prym canonical linear series $|\omega \otimes \eta|$; since C is contained in a quadric it follows that the map ν is not surjective, and therefore the infinitesimal Prym-Torelli theorem fails at the double cover $\pi : \tilde{C} \to C$ associated to η.

Naranjo–Verra proved that the fibre $\mathcal{P}^{-1}(\mathcal{P}([\pi]))$ is discrete [10]. Therefore from Theorem 3.2 it follows that $Hilb^P$ is obstructed at $\tilde{C}$.

Note that a count of parameters shows that the locus of double covers π constructed in this way has dimension $14 = 9 + 5$ (9 for the moduli of Enriques surfaces and 5 for the hyperplane sections), i.e. it is a divisor in $\mathcal{R}_6$. In particular it follows that a general such curve C is not trigonal since trigonal curves depend on 13 parameters.

5.2.2. Consider an irreducible sextic $\bar{C} \subset \mathbf{P}^2$ with four nodes such that two of its bi-tangents meet in one of the nodes, say N. Then the normalization C has genus 6 and the g^1_4 defined by the pencil of lines through N has two divisors of the form $2P + 2Q$. It follows that there is a 2-division point $\eta \in Pic(C)$ such that $\omega \otimes \eta$ is not very ample and the map ν is not surjective (use Lemma 2.2). Therefore the infinitesimal Prym-Torelli theorem fails at the double cover $\pi : \tilde{C} \to C$ associated to η. The locus in $\mathcal{R}_6$ defined by this family of examples is disjoint from the previous one because there the line bundles $\omega \otimes \eta$ were very ample. It is not clear to us what the dimensions of the fibres $\mathcal{P}^{-1}(\mathcal{P}([\pi]))$ are and therefore whether $[\tilde{C}]$ is obstructed in this case.

5.2.3. Another locus where the infinitesimal Prym-Torelli fails is $\mathcal{R}_{6,T}$, the locus of double covers of trigonal curves. It has dimension 13, and the restriction of $\mathcal{P}$ to $\mathcal{R}_{6,T}$ has general fibre of dimension 1, as it follows from Recillas's theorem recalling that $W^1_4(X)$ for a curve X of genus 5 has dimension 1.

What is interesting here is that $W^1_4(X) = \Theta_{sing}$, the singular locus of the theta divisor of JX: for a general X this is a non-singular curve of genus 11 which has an involution ι with quotient a non-singular plane quintic C. The double cover $\pi : W^1_4(X) \to C$ is associated with a 2-division point η such that $\mathcal{O}(1) \otimes \eta$ is an even theta-characteristic (see [4] for details). Moreover $\mathcal{P}([\pi]) = [JX]$ again, by [9]. Therefore we see that for a general X of genus 5 we have

$$\mathcal{P}^{-1}([JX]) = W^1_4(X) \cup \{[\pi]\}.$$

In particular the fibre of $\mathcal{P}$ is not equi-dimensional. Moreover ν is surjective at $[\pi]$ (see [4, part II, Section 5]) and therefore the curve $W^1_4(X) = \Theta_{sing}$ is unobstructed in JX. Note that this gives an example of a double cover π of a curve of Clifford index 1 (namely a non-singular plane quintic) at which the infinitesimal Prym–Torelli theorem holds.

Note also that, since $\mathrm{cork}(\nu)$ is 1-dimensional if $\pi : \tilde{C} \to C$ is a double cover of a general trigonal curve of genus 6, we have that $\mathrm{cork}(\sigma)$ is 1-dimensional as well, by Proposition 2.3 (clearly a general trigonal C is not bi-elliptic). Therefore δ has rank 1 and

$$h^0(\tilde{C}, N_{\tilde{C}}) = g.$$

With some extra effort one can easily show that in this case $\tilde{C}$ is unobstructed in JX. In fact consider a small (1-dimensional) neighborhood A of $[\pi]$ in the fibre $\mathcal{P}^{-1}([JX])$ and let

$$\begin{array}{ccc} \tilde{\mathfrak{C}} & \subset & JX \times A \\ \downarrow & & \\ A & & \end{array} \qquad (8)$$

be the corresponding 1-parameter family of deformations of $\tilde{C}$ in JX. Since this family has varying moduli, in the exact sequence (4) we have $0 \neq \delta(v) \in H^1(\tilde{C}, T_{\tilde{C}})$ if $v \neq 0$ is a tangent vector to A at the point a_0 parametrizing $\tilde{C}$, and $\delta(v)$ generates $\mathrm{Im}(\delta)$. Now consider a small neighborhood B of 0 in JX and build a new family:

$$\begin{array}{ccc} \tilde{\mathfrak{C}}' & \subset & JX \times A \times B \\ \downarrow & & \\ A \times B, & & \end{array}$$

whose fibre over (a, b) is the curve $t_b^*(\tilde{\mathfrak{C}}_a)$, i.e. the translate by b of the fibre $\tilde{\mathfrak{C}}_a$ of the family (8). It is clear that the characteristic map

$$T_{A\times B,(a_0,0)} \to H^0(\tilde{C}, N_{\tilde{C}})$$

is an isomorphism, proving that $\tilde{C}$ is unobstructed. Note that $h^0(\tilde{C}, N_{\tilde{C}}) > g - 1$ in this case, and $\tilde{C}$ has non-trivial moduli.

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