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D.V. Alekseevsky · S. Marchiafava

A twistor construction of Kähler submanifolds of a quaternionic Kähler manifold

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Abstract. A class of minimal almost complex submanifolds of a Riemannian manifold $\tilde{M}^{4n}$ with a parallel quaternionic structure Q, in particular of a 4-dimensional oriented Riemannian manifold, is studied. A notion of Kähler submanifold is defined. Any Kähler submanifold is pluriminimal. In the case of a quaternionic Kähler manifold $\tilde{M}^{4n}$ of non zero scalar curvature, in particular, when $\tilde{M}^4$ is an Einstein, non Ricci-flat, anti-self-dual 4-manifold, we give a twistor construction of Kähler submanifolds M^{2n} of maximal possible dimension $2n$. More precisely, we prove that any such Kähler submanifold M^{2n} of $\tilde{M}^{4n}$ is the projection of a holomorphic Legendrian submanifold $L^{2n} \subset \mathcal{Z}$ of the twistor space $(\mathcal{Z}, \mathcal{H})$ of $\tilde{M}^{4n}$, considered as a complex contact manifold with the natural holomorphic contact structure $\mathcal{H} \subset T\mathcal{Z}$. Any Legendrian submanifold of the twistor space $\mathcal{Z}$ is defined by a generating holomorphic function. This is a natural generalization of Bryant's construction of superminimal surfaces in $S^4 = \mathbb{H}P^1$.

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1. Introduction

1.1. In 1982 R. Bryant [8] gave an explicit construction of superminimal conformal immersions of compact Riemann surfaces into the 4-sphere S^4 which led to several developments. The key idea of the construction is to use the twistor fibration $\pi : \mathbb{C}P^3 \to S^4$ which is a Riemannian submersion with fiber $S^2 \equiv \mathbb{C}P^1$, and whose horizontal distribution $\mathcal{H}$ is a holomorphic contact structure. An oriented surface $M^2 \subset S^4$ has a natural lift $L = J(M^2)$ into $\mathbb{C}P^3$ defined by the complex structure of M^2. The surface M^2 is superminimal if and only if its lift L is a holomorphic

D.V. Alekseevsky: Department of Mathematics, The Hull University, Cottingham Road, HU6 7RX, UK, e-mail: d.v.alekseevsky@hull.ac.uk

S. Marchiafava: Dipartimento di Matematica, Università di Roma “La Sapienza”, P.le A. Moro 2, 00185 Roma, Italy, e-mail: marchiaf@mat.uniroma1.it

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Legendrian submanifold of $\mathbb{C}P^3$, that is a horizontal holomorphic 2-dimensional submanifold. This reduces a description of superminimal surfaces $M^2 \subset S^4$ to a description of holomorphic Legendrian submanifolds L of $\mathbb{C}P^3$. R. Bryant constructed the holomorphic local coordinates u, p, q such that the contact distribution $\mathcal{H}$ is the kernel of the 1-form $\theta = du - pdq$. In terms of these coordinates a Legendrian submanifold has the form $L = L_f = \{u = f(q), p = \partial f/\partial q\}$ where $u = f(q)$ is a holomorphic function. Then the projection $M^2 = \pi(L_f)$ is a superminimal surface of S^4.

Bryant's construction was studied by several authors and generalized to immersions of surfaces into another manifolds for which the twistor construction still applies. Superminimal immersions of surfaces into 4-dimensional Riemannian manifolds were considered for example in [16]–[19], [22], [28], [12], [33], [29], [23]. In particular, H.B. Lawson constructed the superminimal immersions of surfaces into the complex projective plane $\mathbb{C}P^2$, which is a quaternionic Kähler manifold as well as the 4-sphere $S^4 \equiv \mathbb{H}P^1$. Superminimal immersions of surfaces into higher dimensional manifolds, and particularly into quaternionic Kähler manifolds were studied in [12], [9], [11], [23].

We recall that a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ is an oriented Riemannian (Einstein) manifold $(\tilde{M}^{4n}, \tilde{g})$ together with a parallel (skew-symmetric) quaternionic structure $Q \subset \mathrm{End}TM$ which is locally generated by an almost hypercomplex basis (J_1, J_2, J_3). It has a twistor space $\mathcal{Z} = \{J \in Q \mid J^2 = -\mathrm{Id}\}$ which is endowed with a natural complex structure and projection $\pi : \mathcal{Z} \to \tilde{M}^{4n}$. If the scalar curvature is non zero, $\mathcal{Z}$ carries a natural (pseudo) Kähler metric, such that the projection π is a Riemannian submersion, and a holomorphic contact structure, defined by S. Salamon [37].

More recently, some generalization of Bryant's construction to superminimal higher dimensional submanifolds of some quaternionic Kähler manifolds were considered, see [29], [3], [24], [25].

The main aim of this paper is to show that a natural generalization of Bryant's construction provides a simple description of Kähler submanifolds of a quaternionic Kähler manifold $\tilde{M}^{4n}$ with non zero scalar curvature and, in particular, gives an explicit description of such submanifolds in Wolf spaces (that is compact symmetric quaternionic Kähler manifolds).

A Kähler submanifold (M^{2m}, J_1) of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ is defined as a submanifold M^{2m} together with a section J_1 of the twistor bundle $\mathcal{Z}_{|M}$ which preserves TM and is parallel along M [2]. Such a submanifold M^{2m} with the induced metric $g = \tilde{g}_{|TM}$ and the complex structure $J = J_{1|TM}$ is a Kähler manifold. If $\tilde{M}^{4n}, n > 1$, is a quaternionic Kähler manifold with non zero scalar curvature then Kähler submanifolds are precisely the totally complex submanifolds studied by many authors, see for example [20], [41], [40], [2]. It is known, [20], that they are minimal, and even pluriminimal [34] (or (1,1) geodesic [24]), and that their maximal dimension is $2n$.

The generalization of Bryant's construction works as follows. If the scalar curvature of the quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ is non zero (like in the case of 4-sphere) there is a correspondence between holomorphic, horizontal submanifolds N of the twistor space equipped with the holomorphic contact structure, and

Kähler submanifolds M^{2m} of $\tilde{M}^{4n}$: the natural lift $N = J_1(M)$ of a Kähler submanifold M^{2m} to the twistor space $\mathcal{Z}$ is a holomorphic, horizontal submanifold of $\mathcal{Z}$ and conversely the projection $M^{2m} = \pi(N)$ to $\tilde{M}^{4n}$ of a holomorphic horizontal submanifold $N \subset \mathcal{Z}$ is a Kähler submanifold. In fact, this correspondence was already indicated by M. Takeuchi [40], who stated it in terms of totally complex submanifolds. Like in the case of S^4, the explicit construction of Legendrian submanifolds $L \subset \mathcal{Z}$ (i.e. maximal holomorphic horizontal submanifolds) and hence maximal Kähler submanifolds $M^{2n} \subset \tilde{M}^{4n}$ reduces to the construction of Darboux coordinates u, p_k, q^k such that a (local) contact 1-form θ with $\mathcal{H} = \operatorname{Ker}\theta$ has the form $\theta = du - \sum_k p_k dq^k$. A direct generalization of the Bryant's formulas provides such Darboux coordinates for the twistor space $\mathcal{Z} = \mathbb{C}P^{2n+1}$ of the quaternionic projective space $\tilde{M}^{4n} = \mathbb{H}P^n$. This allows to construct Kähler submanifolds of $\mathbb{H}P^n$ hence, due to results by F. Burstall and P. Kobak (see below), of any Wolf space explicitly.

The above definition of Kähler submanifold makes sense, in fact, for any Riemannian manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ with a parallel (skew-symmetric) quaternionic structure Q, in particular for an oriented Riemannian 4-manifold $(\tilde{M}^4, \tilde{g}, Q)$ where $Q = \tilde{g}^{-1} \circ \Lambda^2_+$ and Λ^2_+ is the bundle of self-dual 2-forms on M^4. In this case it was proved by T. Friedrich [16], see also [33] Prop.1, that the Kähler submanifolds $M^2 \subset \tilde{M}^4$ are precisely the superminimal surfaces. Moreover, if $(\tilde{M}^4, \tilde{g})$ is an Einstein anti-self-dual manifold with non zero scalar curvature then the twistor space $\mathcal{Z}$ is a complex contact 3-manifold and the proposed construction provides all superminimal surfaces of $\tilde{M}^4$. In particular if $\tilde{M}^4 = S^4$ or $\mathbb{C}P^2$, it reduces to Bryant's and, respectively, Lawson's construction.

The only known examples of compact quaternionic Kähler manifolds with positive scalar curvature are the Wolf spaces, that is the symmetric compact quaternionic Kähler manifolds. Let $\tilde{M}_1$, $\tilde{M}_2$ be two Wolf spaces of the same dimension. Their twistor spaces $\mathcal{Z}_1$, $\mathcal{Z}_2$ are homogeneous complex contact manifolds. Generalizing the results by Bryant and Lawson in dimension 4, F. Burstall proved that there exists a birational correspondence $\varphi : \mathcal{Z}_1 \to \mathcal{Z}_2$ which preserves the contact distributions and hence transforms Legendrian submanifolds into Legendrian submanifolds [9]. This establishes a birational correspondence between maximal Kähler submanifolds of M_1 and M_2. P. Kobak [25] described this correspondance φ explicitly. In particular, if $M_1 = \mathbb{H}P^n$ we can construct a Kähler submanifold of the Wolf space M_2 as the projection on M_2 of the Legendrian submanifolds $\varphi(L_f)$ where L_f is a Legendrian submanifold of $\mathbb{C}P^{2n+1}$ defined by a holomorphic function f.

In fact, we consider a more general class of almost complex submanifolds (M, J_1) of a manifold $(\tilde{M}, \tilde{g}, Q)$ with a parallel quaternionic structure that is a submanifold M together with a section J_1, $J_1^2 = -\mathrm{id}$, of the twistor bundle $\pi : \mathcal{Z}|_M \to M$ which preserves TM. It is well known that any almost complex submanifold of a Kähler manifold is complex and minimal. In the case of a Riemannian manifold $(\tilde{M}^{4n}, \tilde{g})$ with a parallel quaternionic structure Q, in particular for a quaternionic Kähler manifold, the situation is different. The almost complex structure $J = J_1|_M$ of an almost complex submanifold may be not integrable, or it may be integrable, but not parallel, such that $(M, \tilde{g}|M, J)$ is a Hermitian, but not a Kähler manifold. The section J_1 defines a natural lift $N = J_1(M^{2m})$ of an almost complex submanifold into the twistor space $\mathcal{Z}$. Imposing some additional condition on J_1 which implies integrability of J_1, we define a notion of supercomplex submanifold (M^{2m}, J_1). In fact, in Section 4 we prove that the lift $J_1(M^{2m})$ of an almost complex submanifold (M^{2m}, J_1) of a quaternionic Kähler manifold is a complex submanifold of the twistor space if and only if it is supercomplex. The result holds also for surfaces of an oriented 4-dimensional Riemannian manifold. In this case the notion of supercomplex submanifold reduces to Th. Friedrich's notion [18] of t-holomorphic surface in an oriented Riemannian 4-manifold. A result from [2] implies that for $n > 1$ an almost complex submanifold (M^{2m}, J_1) is supercomplex if and only if the complex structure $J = J_1|_{TM^{2m}}$ is integrable. Remark that almost complex structures J_1 on an open submanifold M^{4n} of a quaternionic Kähler manifold $\tilde{M}^{4n}$ and their integrability were studied in [5].

In Section 3, we give necessary and sufficient conditions for an almost complex submanifold (M^{2m}, J_1) of a Riemannian manifold $\tilde{M}^{4n}$ with a parallel quaternionic structure Q to be minimal. A general formula for the mean curvature vector of (M^{2m}, J_1) is given in terms of Lee form of the (local) extension of J_1 to $\tilde{M}^{4n}$. Under the hypothesis that the quaternionic Kähler manifold has non zero scalar curvature, a characterization of the supercomplex submanifolds (M, J_1) whose immersion is pluriminimal is given, proving that they are the Kähler submanifolds or the quaternionic (hence totally geodesic) submanifolds.

Sections 4 and 5 contain the twistor description of supercomplex and, respectively, Kähler submanifolds of a quaternionic Kähler manifold with non zero scalar curvature. Some results concerning the global structure of Legendrian submanifolds of a quaternionic Kähler manifold are also stated. In Section 6, we construct a Darboux coordinated for the twistor space $\mathbb{C}P^{2n+1}$ of $\mathbb{H}P^n$, which gives an explicit local construction of Kähler submanifolds of $\mathbb{H}P^n$ in terms of holomorphic functions on $\mathbb{C}P^{2n+1}$.

2. Almost complex, totally complex and Kähler submanifolds of a Riemannian manifold with parallel skew-symmetric quaternionic structure

2.1. Recall that a **(skew-symmetric) almost quaternionic structure** Q on a Riemannian $4n$-manifold (M^{4n}, g) is a rank-3 subbundle Q of the bundle of endomorphisms locally generated by 3 anticommuting skew-symmetric almost complex structures $(J_1, J_2, J_3 = J_1 J_2)$. The triple (J_1, J_2, J_3) is called an **admissible** (local) **basis** of Q. An almost quaternionic structure Q is called a **quaternionic structure** if it admits a **quaternionic connection**, that is a torsionless connection ∇ which preserves Q:

$$\nabla_X J_\alpha = \omega_\gamma(X) J_\beta - \omega_\beta(X) J_\gamma, \qquad X \in TM \tag{1}$$

where α, β, γ is a cyclic permutation of $1, 2, 3$ and $\omega_\alpha, \alpha = 1, 2, 3$, are local 1-forms which depend on the choice of an admissible basis (J_α).

A quaternionic structure Q on a Riemannian manifold (M^{4n}, g) is called a **parallel (skew-symmetric) quaternionic structure** if the Levi–Civita connection is a quaternionic connection and Q consists of skew-symmetric endomorphisms.

Any oriented 4-dimensional Riemannian manifold has two parallel (skew-symmetric) quaternionic structures $Q = g^{-1} \circ \Lambda^2_+$ and $Q' = g^{-1} \circ \Lambda^2_-$ associated with the decomposition $\Lambda^2(TM) = \Lambda^2_+ \oplus \Lambda^2_-$ of 2-forms into self-dual and anti-self-dual part. For $n > 1$, a Riemannian manifold $(\tilde{M}^{4n}, \tilde{g})$ with a parallel (skew-symmetric) quaternionic structure Q is called a **quaternionic Kähler manifold**. It is an Einstein manifold and its curvature has a decomposition

$$R = \nu R_{\mathbb{H}P^n} + W$$

where $R_{\mathbb{H}P^n}$ is the curvature tensor of the quaternionic projective space $\mathbb{H}P^n$ with the standard metric, ν is a constant which is called the **reduced scalar curvature**, such that $K = 4n(n+2)\nu$ is the scalar curvature, and W is the **quaternionic Weyl tensor** which verifies the identity $[W(X, Y), Q] = 0$ and has all contractions equal to zero.

For $n = 1$, we call an oriented Riemannian 4-manifold, together with the parallel quaternionic structure $Q = g^{-1} \circ \Lambda^2_+$, a quaternionic Kähler manifold if its curvature tensor has such a decomposition. Since $\mathbb{H}P^1 = S^4$, the tensor $R_{\mathbb{H}P^1}$ is the curvature tensor of constant curvature equal 1 and W is the anti-selfdual part of the Weyl tensor. Hence *4-dimensional quaternionic Kähler manifolds are the same as Einstein anti-self-dual manifolds.*

2.2. We will denote by $(\tilde{M}^{4n}, \tilde{g}, Q)$ a Riemannian manifold with a parallel quaternionic structure Q.

Definition 2.1. *A submanifold M^{2m} of $(\tilde{M}^{4n}, \tilde{g}, Q)$ together with a section $J_1 = J_1^M \in \Gamma(Q_{|M})$ such that*

1) $J_1^2 = -Id$,
2) $J_1 TM = TM$

is called **almost complex**.

In this case $(M^{2m}, g = \tilde{g}_{|TM}, J = J_{1|TM})$ is an almost Hermitian manifold with the Kähler form $F = g \circ J$.

An almost complex submanifold (M^{2m}, J_1) is called **complex** *if the restriction $J = J_{1|TM}$ is an (integrable) complex structure on M.*

A local admissible basis (J_1, J_2, J_3) of Q, defined on a neighbourhood U in $\tilde{M}^{4n}$ of a point $x \in M^{2m}$, is called an **adapted basis** for the almost complex submanifold M^{2m} if $J_{1|(M \cap U)} = J_1^M$.

Definition 2.2. *An almost complex submanifold (M^{2m}, J_1) of $(\tilde{M}^{4n}, \tilde{g}, Q)$ is called* **supercomplex** *if*

$$\tilde{\nabla}_{J_1 X} J_1 - J_1 \tilde{\nabla}_X J_1 = 0, \quad \forall X \in TM$$

where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$.

By using (1) for an adapted basis we get the identity

$$\tilde{\nabla}_{J_1 X} J_1 - J_1 \tilde{\nabla}_X J_1 = (\omega_3 \circ J_1 - \omega_2)(X) J_2 + (\omega_3 \circ J_1 - \omega_2)(JX) J_3$$

$\forall X \in TM$. This implies

Proposition 2.3. *The almost complex submanifold* (M^{2m}, J_1) *of* $(\tilde{M}^{4n}, \tilde{g}, Q)$ *is supercomplex if and only if the locally defined* 1*-form*

$$\psi := (\omega_3 \circ J_1 - \omega_2)_{|TM} = 0.$$

The above condition does not depend on the adapted basis (J_1, J_2, J_3) with the associated 1-forms $(\omega_1, \omega_2, \omega_3)$.

The following result was proved in [2], see (1) of Theorem 1.1.

Proposition 2.4. *If* $m > 1$*, an almost complex submanifold* (M^{2m}, J_1) *of a quaternionic Kähler manifold* $\tilde{M}^{4n}$ *is complex if and only if it is supercomplex.*

Remark that any 2-dimensional almost complex submanifold (M^2, J_1) of $(\tilde{M}^{4n}, \tilde{g}, Q)$ is complex but not necessarily supercomplex.

In Section 4, we characterize supercomplex submanifolds as the projection to $\tilde{M}$ of complex submanifolds of the twistor space $\mathcal{Z}$ (see Corollary 4.6).

Definition 2.5. *An almost complex submanifold* (M^{2m}, J_1) *of a Riemannian manifold* $(\tilde{M}^{4n}, \tilde{g})$ *with a parallel quaternionic structure* Q *is called*

1) **Kähler** *if*

$$\tilde{\nabla}_X J_1 = 0 \quad \forall X \in TM$$

or equivalently

$$\omega_{2|TM} = \omega_{3|TM} = 0$$

where $\tilde{\nabla}$ *is the Levi-Civita connection of* $\tilde{g}$ *and* $\omega_\alpha, \alpha = 1, 2, 3,$ *are the 1-forms associated with a local adapted basis* (J_1, J_2, J_3) *of* Q *and*

2) **totally complex** *if*

$$J_2 TM \perp TM\ .$$

Proposition 2.6. ([2]) *Let* $(\tilde{M}^{4n}, \tilde{g}, Q)$ *be a quaternionic Kähler manifold with non zero scalar curvature. If* $m > 1$*, an almost complex submanifold* (M^{2m}, J_1) *is Kähler if and only if the almost Hermitian manifold* $(M^{2m}, g = \tilde{g}_{|TM}, J = J_{1\,|TM})$ *is Kähler.*

Proposition 2.7. *Let* M^2 *be a* 2*-dimensional oriented submanifold of a Riemannian manifold* $(\tilde{M}^{4n}, \tilde{g})$ *with a quaternionic structure* Q*. Assume that* $dim(T_x^Q M^2) = 4$, $\forall x \in M^2$*, where* $T_x^Q M$ *is the* Q*-invariant subspace of* $T_x\tilde{M}$ *generated by* $T_x M$*. Then* M^2 *is totally complex with respect to a uniquely defined section* $J_1 \in \Gamma(Q_{|M})$ *which induces the given orientation of* M^2*.*

Remark. Such a submanifold M^2 was called *inclusive* in [12] and *pseudoquaternionic* in [32].

Proof. The proof follows from the lemma below. □

Lemma 2.8. *Let V be a $4n$-dimensional vector space with a given (constant) quaternionic structure $Q = span(J_1, J_2, J_3)$. For any oriented 2-plane $U \subset V$ which is included in a quaternionic line $\mathbb{R}v + Qv$ there exists a unique complex structure $J = J_U \in Q$ which preserves U and induces the given orientation.*

Proof. Let (u, u') be an oriented basis of U. The condition that $U \subset \mathbb{R}v + Qv$ means that u' can be written as $u' = a_0 u + a_1 J_1 u + a_2 J_2 u + a_3 J_3 u$. Let define the *complex structure* $J \in Q$ by $J = \frac{1}{b}(a_1 J_1 + a_2 J_2 + a_3 J_3)$, $b = \sqrt{a_1^2 + a_2^2 + a_3^2} > 0$. Then $u' = a_0 u + bJu$ and J leaves U invariant and induces on it the given orientation. The complex structure $J \in Q$ is uniquely determined by these properties: assume that $\hat{J} \in Q$ is a complex structure which leaves U invariant, that is $\hat{J}u = a'u + b'Ju$, and induces the same orientation, that is $b' > 0$; then the linear independence of $(u, J_1 u, J_2 u, J_3 u)$ implies that $a' = 0$, $b' = 1$, that is $\hat{J} = J$. □

Corollary 2.9. *Any oriented 2-dimensional submanifold M^2 of a 4-dimensional oriented Riemannian manifold $(\tilde{M}^4, \tilde{g})$ with the parallel quaternionic structure $Q = \tilde{g}^{-1} \circ \Lambda_+^2$ is a totally complex submanifold with respect to a uniquely defined section J_1 as in Proposition 2.7.*

In general, the totally complex submanifold $M^2 \subset \tilde{M}^4$ is not Kähler. For $n > 1$, the situation is different, as the following proposition shows.

Proposition 2.10. ([41], [2]). *If $n > 1$ and the scalar curvature of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ is non zero, then an almost complex submanifold (M^{2m}, J_1) of $\tilde{M}^{4n}$ is Kähler if and only if it is totally complex.*

It is evident that the existence at a point $p \in M^{2m}$ of two non proportional complex structures $J_1, \tilde{J}_1 \in \mathcal{Z}_p$ both leaving the tangent space T_pM^{2m} invariant would imply that it is Q-invariant, i.e. $Q_pT_pM^{2m} = T_pM^{2m}$.

Recall that a submanifold M of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ is called a **quaternionic submanifold** if $QTM \subset TM$. Then the following corollary follows immediately.

Corollary 2.11. *Let M^{2m} be a submanifold of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ with non zero scalar curvature. If M^{2m} is not a quaternionic submanifold then, up to a sign, there exists at most one section $J_1 \in \Gamma(Z_{|M})$ such that (M^{2m}, J_1) is an almost complex submanifold of $\tilde{M}^{4n}$.*

Corollary 2.12. *The maximal (real) dimension of a Kähler submanifold of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ with non zero scalar curvature is $2n$.*

We call a 2n-dimensional Kähler submanifold of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ with non zero scalar curvature a **maximal Kähler submanifold**. In [31] such submanifolds were considered in $\mathbb{H}P^n$ under the name of **complex-Lagrangian** submanifolds; see also [24].

3. Minimal almost complex submanifolds

In this section we calculate the mean curvature vector of an almost complex submanifold of a Riemannian manifold $(\tilde{M}^{4n}, \tilde{g})$ with a parallel quaternionic structure Q and prove that any Kähler submanifold is minimal. We give also some characterizations of Kähler submanifolds.

Let (M^{2m}, J_1) be an almost complex submanifold of a Riemannian manifold $(\tilde{M}^{4n}, \tilde{g})$ with a parallel quaternionic structure Q. We have the following orthogonal decomposition of the tangent space of M at a point $x \in M$:

$$T_x M = \overline{T}_x M \oplus \mathcal{D}_x$$

where $\overline{T}_x M = T_x M \cap J_2 T_x M$ is the maximal Q-invariant subspace of $T_x M$ and $\mathcal{D}_x$ is the J_1-invariant orthogonal complement to $\overline{T}_x M$.

Note that the space $\mathcal{D}'_x = J_2 \mathcal{D}_x$, does not depend on the adapted basis (J_1, J_2, J_3) and

$$T^Q_x M = \overline{T}_x M \oplus \mathcal{D}_x \oplus \mathcal{D}'_x$$

is a direct sum decomposition of the minimal Q-invariant subspace $T^Q_x M$ of $T_x \tilde{M}$ which contains $T_x M$.

Remark that $\mathcal{D}'_x$ is orthogonal to $\overline{T}_x M$ but in general, if $\dim \mathcal{D}_x > 2$, not orthogonal to $\mathcal{D}_x$. Let recall also that, in case $n > 1$, if $\dim \mathcal{D}_x > 2$ for any $x \in M$ then the almost complex structure $J_{1|M}$ is integrable, see [2].

The *Lee form* of an almost Hermitian manifold (M^{2m}, g, J) is defined as the 1-form

$$\theta = -(\delta F) \circ J,$$

where δF is the codifferential of the Kähler form $F = g \circ J$.

For a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ with an admissible basis (J_1, J_2, J_3) of Q the Lee form θ_1 of $(\tilde{g}, J_1)$ is given by

$$\theta_1 = \omega_2 \circ J_2 + \omega_3 \circ J_3$$

where the 1-forms ω_α are associated with (J_1, J_2, J_3), see [5].

Let (M^{2m}, J_1) be an almost complex submanifold of $(\tilde{M}^{4n}, \tilde{g}, Q)$. We denote by

$$\mathbf{t}_1 = \tilde{g}^{-1} \circ \theta_1 \ \in T\tilde{M}|_M$$

the vector field along M, dual to the 1-form θ_1.

Proposition 3.1. *Let $(\tilde{M}^{4n}, \tilde{g})$ be a Riemannian manifold with a parallel quaternionic structure Q. Then the mean curvature vector H of an almost complex submanifold (M^{2m}, J_1) of $\tilde{M}^{4n}$ is given by*

$$H = -\frac{1}{2m}\left[Pr_{\mathcal{D}'}\mathbf{t}_1\right]^\perp \tag{2}$$

where, for any $X \in T_x\tilde{M}$, $Pr_{\mathcal{D}'}(X)$ is the orthogonal projection of X onto the subspace $\mathcal{D}'_x$ and $X^\perp$ means the orthogonal projection of X onto $T_x^\perp M$.

If $m = 1$ the formula can be written as

$$H = \frac{1}{2}\big[(\omega_3 \circ J_1 + \omega_2)(X)J_2X - (\omega_3 \circ J_1 + \omega_2)(J_1X)J_3X\big] \tag{3}$$

where X is any unit vector of TM.

Proof. Let $J = J_{1|TM}$ and (J_1, J_2, J_3) be an adapted basis. For any vectors $X, Y \in TM$ one has

$$\omega_3(X)J_2Y - \omega_2(X)J_3Y = (\tilde{\nabla}_X J_1)Y = (\nabla_X J)Y + h(X, JY) - J_1 h(X, Y).$$

Hence

$$h(X, JY) - J_1 h(X, Y) = \big[\omega_3(X)J_2Y - \omega_2(X)J_3Y\big]^\perp \tag{4}$$

where $\perp$ means the projection on $TM^\perp$. By comparing with the identity where X is exchanged with Y, one gets the identity

$$h(X, JY) - h(Y, JX) = \big[\omega_3(X)J_2Y - \omega_2(X)J_3Y - \omega_3(Y)J_2X + \omega_2(Y)J_3X\big]^\perp$$

that is, by exchanging X with JX,

$$\begin{aligned} h(X, Y) + h(JX, JY) = \big[&\omega_3(J_1X)J_2Y - \omega_2(J_1X)J_3Y + \omega_3(Y)J_3X \\ &+ \omega_2(Y)J_2X\big]^\perp. \end{aligned} \tag{5}$$

Let now $(E_1, \ldots, E_m, J_1E_1, \ldots, J_1E_m)$ be an orthonormal basis of T_xM such that $(E_1, \ldots, E_k, J_1E_1, \ldots, J_1E_k)$ is an orthonormal basis of $\mathcal{D}$ and, hence, $(E_{k+1}, \ldots, E_m, J_1E_{k+1}, \ldots, J_1E_m)$ is an orthonormal basis of $\bar{T}M$. Using the previous identity, we find

$$\begin{aligned} 2mH &= \sum_{i=1}^m \big[h(E_i, E_i) + h(JE_i, JE_i)\big] \\ &= \sum_{i=1}^m \Big\{\big[\omega_3(J_1E_i) + \omega_2(E_i)\big]J_2E_i\big[\omega_2(J_1E_i) - \omega_3(E_i)\big]J_3E_i\Big\}^\perp \end{aligned}$$

and hence

$$\begin{aligned} 2mH &= \sum_{i=1}^m \big\{(\omega_3 \circ J_1 + \omega_2)(E_i)J_2E_i - (\omega_2 \circ J_1 - \omega_3)(E_i)J_3E_i\big\}^\perp \\ &= \sum_{i=1}^m \big\{-(\omega_2 \circ J_2 + \omega_3 \circ J_3)(J_2E_i)J_2E_i - (\omega_2 \circ J_2 + \omega_3 \circ J_3)(J_3E_i)J_3E_i\big\}^\perp \\ &= \sum_{i=1}^k \big\{-(\omega_2 \circ J_2 + \omega_3 \circ J_3)(J_2E_i)J_2E_i - (\omega_2 \circ J_2 + \omega_3 \circ J_3)(J_3E_i)J_3E_i\big\}^\perp \\ &= -\big[\mathcal{P}r_{J_2\mathcal{D}}\mathbf{t}_1\big]^\perp \end{aligned}$$

since $(J_2E_1, \ldots, J_2E_k, J_3E_1, \ldots, J_3E_k)$ is an orthonormal basis of $J_2\mathcal{D}$. □

Corollary 3.2. *An almost complex submanifold $M^{2m} \subset \tilde{M}^{4n}$ is minimal if the 1-form*

$$\chi := (\omega_3 \circ J_1 + \omega_2)_{|TM}$$

vanishes on $\mathcal{D}$ for some adapted basis (J_1, J_2, J_3).

Since for a Kähler submanifold (M^{2m}, J_1), the 1-forms ω_2, ω_3 vanish on M^{2m}, we have the following corollary which was proved for $n > 1$ in [20], [2] and for $n = 1$ in [17].

Corollary 3.3. *A Kähler submanifold (M^{2m}, J_1) of a Riemannian manifold $(\tilde{M}^{4n}, \tilde{g})$ with a parallel quaternionic structure Q is minimal.*

Moreover, as another corollary we get the following result, see also [8], [17], [33].

Corollary 3.4. *Let M^2 be an oriented 2-dimensional submanifold of a 4-dimensional oriented Riemannian manifold $(\tilde{M}^4, \tilde{g})$ with the parallel quaternionic structure $Q = \tilde{g}^{-1} \circ \Lambda^2_+$. Let J_1 be the uniquely defined section of $Q_{|M}$ which leaves TM^2 invariant and induces the given orientation on it (by prop. 2.7). Then the following conditions are equivalent:*

1) (M^2, J_1) is Kähler,
2) (M^2, J_1) is minimal and supercomplex.

Proof. By Proposition 3.3, (M^2, J_1) is supercomplex if and only if $(\omega_3 \circ J_1 - \omega_2)_{|TM} = 0$. By Proposition 3.1 (3) it is minimal if and only if $(\omega_3 \circ J_1 + \omega_2)_{|TM} = 0$. These two conditions imply $\omega_{2|TM} = \omega_{3|TM} = 0$, i.e. (M^2, J_1) is Kähler. The converse statement is clear. □

By the same proof one gets the following corollary.

Corollary 3.5. *Let (M^2, J_1) be a supercomplex surface of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$. Then it is minimal if and only if it is a Kähler submanifold.*

The following proposition, which was proved in [2] (see Theorem 1.9), gives a characterization of Kähler submanifolds between almost complex submanifolds of a quaternionic Kähler manifold.

Recall that a quaternionic submanifold M of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ is totally geodesic.

Proposition 3.6. ([20], [2]) *Let $(\tilde{M}^{4n}, \tilde{g}, Q)$ be a quaternionic Kähler manifold with non zero scalar curvature and (M^{2m}, J_1) an almost complex submanifold of $\tilde{M}$ which is not a quaternionic submanifold. Then (M^{2m}, J_1) is a Kähler submanifold if and only if the shape operator A^ξ verifies the condition*

$$A^{J_1\xi} + J_1 A^\xi = 0 \qquad \forall\, \xi \in TM^\perp$$

or, equivalently, the second fundamental form h of M satisfies the condition

$$h(X, J_1 Y) - J_1 h(X, Y) = 0 \qquad \forall X, Y \in TM. \tag{6}$$

Definition 3.7. *An almost complex submanifold (M^{2m}, J_1) of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ is called* **pluriminimal** *or* $(1,1)$-**geodesic** *if one of the following equivalent conditions holds:*

i) the second fundamental form h of M satisfies

$$h(X,Y) + h(JX,JY) = 0 \quad \forall X, Y \in TM; \tag{7}$$

ii) for any normal direction ξ, the shape operator A^ξ anticommutes with $J = J_{1|TM}$,

$$A^\xi J + JA^\xi = 0 \quad \forall \xi \in TM^\perp; \tag{8}$$

iii) the complexification of the second fundamental form h has type (2,0)+(0,2), i.e. $h^{(1,1)} = 0$;

(For the notion of pluriminimal or $(1,1)$-geodesic isometric immersion of an almost Hermitian, in particular Kähler, manifold into a Riemannian manifold see [35], Remark 2.9, and the references in [36], [14], [15]).

Since the condition (6) implies (7) and (8) implies that $\mathrm{tr} A^\xi = 0$, which shows that the mean curvature vector vanishes, we have the following proposition.

Proposition 3.8. *i) A pluriminimal almost complex submanifold (M^{2m}, J_1) is minimal.*
ii) A Kähler submanifold (M^{2m}, J_1) of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ is pluriminimal.

We do not know if (7) and (6) are equivalent in general but the following proposition shows that it is true for supercomplex submanifolds.

Proposition 3.9. *A supercomplex submanifold (M^{2m}, J_1) of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ with non zero scalar curvature is pluriminimal if and only if it is a Kähler submanifold or a quaternionic (hence totally geodesic) submanifold, and these cases cannot happen simultaneously.*

Proof. Due to Corollary 3.4 and Proposition 3.6, it remains to prove only the necessity of the condition for $m > 1$. Recall that for supercomplex submanifolds one has $(\omega_3 \circ J_1 - \omega_2)_{|TM} = 0$. Hence using (5), the condition (7) can be written as

$$\left[\omega_2(X)J_2Y - \omega_2(J_1X)J_3Y - \omega_2(J_1Y)J_3X + \omega_2(Y)J_2X\right]^\perp = 0$$

for any $X, Y \in TM$. Assume that M is not a Kähler submanifold, i.e. there is a point $x \in M$ such that $\omega_{2|T_xM} \neq 0$. Then for $X = Y \in T_xM$ such that $\omega_2(X) = 1$, $\omega_2(J_1X) = 0$ we get $[J_2X]^\perp = 0$ and hence also $[J_3X]^\perp = 0$. Now we take X as above and Y in the kernel of ω_2 and $\omega_2 \circ J_1$. Then we get $[J_2Y]^\perp = [J_3Y]^\perp = 0$. This shows that $J_2T_xM = T_xM$. Hence the (non empty) open submanifold M_1 of M defined by

$$M_1 = \left\{x \in M \mid \quad \omega_{2|T_xM} \neq 0\right\}$$

is a quaternionic submanifold and $J_2T_xM_1 = T_xM_1$ for $x \in M_1$. On the other hand the complementary set M_2 has no interior point since a neighbourhood U

of such point would be Kähler and hence, by Proposition 2.10, a totally complex submanifold, i.e. $J_2 T_x U \perp T_x U$, $x \in U$. This shows that the closure of M_1 is M and M is a (totally geodesic) quaternionic submanifold. □

4. Natural lift of an almost complex submanifold to the twistor space

4.1. The twistor space $\mathcal{Z}$ of a Riemannian manifold $(\tilde{M}^{4n}, \tilde{g})$ with a parallel quaternionic structure Q. Recall that the twistor space $\mathcal{Z}$ of a Riemannian manifold $(\tilde{M}^{4n}, \tilde{g})$ with a parallel quaternionic structure Q is defined as the manifold $\mathcal{Z} = \{J \in Q \mid J^2 = -Id\}$ of all complex structures from Q. The natural projection $\pi : \mathcal{Z} \to \tilde{M}^{4n}$ is a S^2-bundle over $\tilde{M}^{4n}$. Moreover, $\mathcal{Z}$ has a natural almost complex structure J^Z defined as follows. The connection in $\pi : \mathcal{Z} \to \tilde{M}^{4n}$ induced by Levi–Civita connection defines a decomposition

$$T_z\mathcal{Z} = \mathcal{V}_z \oplus \mathcal{H}_z \tag{9}$$

of the tangent space at a point $z = J \in \mathcal{Z}$ into vertical and horizontal subspaces. The complex structure J_z^Z at z is defined by

$$J_z^Z(X) = J^v X^v + \pi_*^{-1} \circ J \circ \pi_* X^h$$

where $X = X^v + X^h$ is the decomposition of a vector $X \in T_z\mathcal{Z}$, J^v is the natural complex structure of a fiber $S^2 = \mathbb{C}P^1$ and $\pi_* : \mathcal{H}_z \overset{\cong}{\to} T_{\pi(z)}\tilde{M}$ is the natural projection.

The complex structure J^Z is integrable if $n > 1$ or if $n = 1$ and $(\tilde{M}^4, \tilde{g})$ is anti-self-dual.

The horizontal distribution $\mathcal{H}$ is J^Z-invariant. Moreover if $(\tilde{M}^{4n}, \tilde{g}, Q)$ is a quaternionic Kähler manifold with non zero scalar curvature then the horizontal distribution $\mathcal{H}$ is an holomorphic contact distribution, see [37].

4.2. Natural lift of an almost complex submanifold. Let (M^{2m}, J_1) be an almost complex submanifold of a Riemannian manifold $(\tilde{M}^{4n}, \tilde{g})$ with a parallel quaternionic structure Q. Then the map

$$J_1 : M \ni x \mapsto J_{1|_x} \in \mathcal{Z}$$

is a section of the twistor bundle $\mathcal{Z}_{|_M} = \pi^{-1}(M) \to M$. The submanifold $J_1(M) \subset \mathcal{Z}$ is called the **natural lift** of the almost complex submanifold (M^{2m}, J_1). In general $J_1(M)$ is not a complex submanifold of $\mathcal{Z}$. To state the condition when this is true, we need the following definition.

Definition 4.1. *A submanifold N of a manifold Z with an almost complex structure J is called complex if the tangent bundle TN is J-invariant and the restriction $J|TN$ is an integrable complex structure on N.*

Theorem 4.2. *Let M be a $2m$-dimensional submanifold of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ and*

$$\begin{gathered} J_1 : M \to N = J_1(M) \subset \mathcal{Z} \\ M \ni x \mapsto J_{1|_x} \end{gathered}$$

a section of the twistor bundle over M. Then $N = J_1(M)$ is a complex submanifold of $\mathcal{Z}$ if and only if (M, J_1) is a supercomplex submanifold of $\tilde{M}^{4n}$

Proof. The proof is based on the following lemmas.

Lemma 4.3. *Let (M^{2m}, J_1) be an almost complex submanifold of the quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$. Then (M^{2m}, J_1) is a supercomplex submanifold if and only if for any point $x \in M$ there exists a quaternionic connection ∇^U in a neighbourhood U of x in $\tilde{M}$ such that*

$$\nabla^U_X J_1 = 0\,,\ \nabla^U_X J_2 = \omega(X) J_3\,,\ \nabla^U_X J_3 = -\omega(X) J_2 \quad \forall X \in TM. \tag{10}$$

Proof. By Proposition 2.3, (M, J_1) is a supercomplex submanifold if and only if for an adapted basis (J_1, J_2, J_3) defined in a neighbourhood $U \subset \tilde{M}$ and Levi–Civita connection $\tilde{\nabla}$ we have

$$\tilde{\nabla}_X J_1 = \omega_3(X) J_2 - \omega_2(X) J_3, \quad X \in T\tilde{M}$$

where

$$(\omega_3 \circ J_1 - \omega_2)_{|TM} = 0. \tag{11}$$

On the other hand, recall that any quaternionic connection ∇' can be obtained by modifying the connection $\tilde{\nabla}$ as follows

$$\nabla' = \tilde{\nabla} + S^\xi$$

where ξ is a 1-form on U and the (1,2)-tensor S^ξ is defined by

$$S^\xi_X = \xi(X) Id + X \otimes \xi - \sum_\alpha \xi(J_\alpha X) J_\alpha - \sum_\alpha J_\alpha X \otimes (\xi \circ J_\alpha).$$

Then

$$\nabla'_X J_1 = \omega'_3(X) J_2 - \omega'_2(X) J_3 \quad \forall X \in T\tilde{M}$$

where

$$\omega'_2 = \omega_2 - 2\xi \circ J_2, \quad \omega'_3 = \omega_3 - 2\xi \circ J_3,$$

see [1], Prop. 3.6 and p. 51, and also [4]. Moreover

$$\nabla'_X J_1 = 0,\ \forall\, X \in TM \Leftrightarrow (\omega_2 - 2\xi \circ J_2)_{|TM} = (\omega_3 - 2\xi \circ J_3)_{|TM} = 0$$

that is

$$\nabla'_X J_1 = 0,\ \forall\, X \in TM \Leftrightarrow \omega_3 \circ J_{1|TM} = \omega_{2|TM} = 2\xi \circ J_{2|TM}$$

[1], Prop. 5.3. Hence, the existence of a connection $\nabla' = \nabla^U$ satisfying (10) implies (11). On the other hand let (M, J_1) be a supercomplex submanifold. Then we modify the connection $\tilde{\nabla}$ on the open domain U,

where an adapted basis (J_1, J_2, J_3) is defined, as $\nabla^U = \tilde{\nabla} + S^\theta$ with

$$\theta = -\frac{1}{4}(\omega_2 \circ J_2 + \omega_3 \circ J_3).$$

Then ∇^U verifies (10). □

We need the following known characterization of the covariant derivative of a tensor field.

Lemma 4.4. *Let Z be a manifold with a linear connection ∇ and $\pi : W \to Z$ a ∇-invariant tensor bundle on Z. Let $\gamma(t)$ be a curve in Z and $A(t)$ a section of π along γ. We may consider the vertical component $\dot{A}(t)^{vert}$ of the tangent vector $\dot{A}(t)$ of the curve $A(t) \subset W$ as a vector of the fiber $W_{\gamma(t)}$. Then*

$$\frac{\nabla}{dt}A(t) = \dot{A}(t)^{vert}.$$

Corollary 4.5. *Let $\mathcal{H} \subset TW$ be the horizontal distribution of the connection ∇ in π induced by the linear connection ∇. Let M be a submanifold of Z and $\pi_M : W_M = \pi^{-1}(M) \to M$ the restriction of the bundle $\pi : W \to Z$ to M. A section $A \in \Gamma(W_M)$ is parallel along M, $(\nabla_X A = 0, \forall X \in TM)$, if and only if the image $A(M) = \{A_x, x \in M\} \subset W$ is a horizontal submanifold, that is $TA(M) \subset \mathcal{H}$.*

Proof. The horizontal lift $A(t)$ of a curve $\gamma(t) \subset M$ to W with $A(0) = w \in W$ is naturally identified with a parallel section of π along $\gamma(t)$ and the horizontal subspace $\mathcal{H}_w$ consists of the tangent vectors at w of such lifts. If the section $A \in \Gamma(W_M)$ is parallel, then the horizontal lift of a curve $\gamma(t)$ is the restriction of the section A to $\gamma(t)$. Hence, the tangent space to $A(M)$ at a point $A(x)$, $x \in M$, is a horizontal subspace, $TA(M) \subset \mathcal{H}$. Conversely, if $A(M)$ is a horizontal submanifold, then for any curve $\gamma(t) \subset M$ the restriction of the section A to $\gamma(t)$ is the parallel section of W along $\gamma(t)$. Hence, $\nabla_{\gamma'(t)}A = 0$. This shows that the section A is parallel along M. □

Proof of Theorem 4.2. Let (M, J_1) be a supercomplex submanifold of $\tilde{M}^{4n}$. Then by Lemma 4.3 there exists a local quaternionic connection ∇^U which preserves J_1. It induces a local connection in the twistor bundle $\mathcal{Z}$. Denote by $\mathcal{H}^U$ the local horizontal distribution of ∇^U in the twistor space $\mathcal{Z}$. By S. Salamon [38] the horizontal space $\mathcal{H}^U_z \subset T_z\mathcal{Z}$ is a complex subspace with respect to the complex structure J^Z and the map

$$\pi_* : \mathcal{H}^U_z \to T_{\pi(z)}\tilde{M}$$

is an isomorphism of complex spaces with respect to the complex structures $J^Z|\mathcal{H}^U_z$ and J_z, where J_z is the complex structure in $T_{\pi(z)}\tilde{M}$ associated with $z \in \mathcal{Z}$. For any point $x \in U \cap M$ one has

$$\pi_* T_{J_1(x)} J_1(M) = T_x M \subset T_x\tilde{M}$$

and $T_x M$ is invariant under the complex structure $J_z = J_{1|_x}$. By Lemma 4.4 the lift $J_1(M) \subset \mathcal{Z}$ is a horizontal submanifold with respect to ∇^U, that is $T_{J_1(x)} J_1(M) \subset \mathcal{H}^U_{J_1(x)}$. Since $T_x M$ is a $J_{J_1(x)}$-invariant subspace, the tangent space $T_{J_1(x)} J_1(M)$ is J^Z-invariant subspace. Note that if $n = 1$ then $m = 1$ and the restriction $J^Z_{|TN}$ is automatically an integrable complex structure. If $n > 1$ then J^Z is an integrable complex structure and in both cases $J_1(M)$ is a complex submanifold.

Conversely, assume that $N = J_1(M)$ is a complex submanifold of $(\mathcal{Z}, J^Z)$. The tangent space $T_z\mathcal{Z}$, $z \in J_1(M)$, has the decomposition

$$T_z\mathcal{Z} = T_z^{vert} \oplus \mathcal{H}_z$$

into the direct sum of two J^Z-invariant subspaces. Hence the projection $\mathcal{H}_N \subset \mathcal{H}_z$ of the J^Z-invariant subspace $T_z N$ onto $\mathcal{H}_z$ is J^Z-invariant. This implies, by definition of J^Z, that the projection $\pi_* T_z N = \pi_* \mathcal{H}_N \subset T_x M$, where $x = \pi(z)$, is invariant with respect to the complex structure $J_z = J_1(x) = J_{1|_x}$ associated with $z \in N = J_1(M)$. This shows that (M, J_1) is an almost complex submanifold, that is $J_1 T_x M = T_x M$, $\forall x \in M$. Moreover the almost complex structure $J = J_{1|TM}$ is integrable since $\pi : N \to M$ is a J-equivariant diffeomorphism of the complex manifold $(N, J^Z|TN)$ onto (M, J). □

Corollary 4.6. *Let $N \subset \mathcal{Z}$ be a complex submanifold such that the projection $\pi : N \to M = \pi(N)$ is a diffeomorphism. Then M is a supercomplex submanifold of $(\tilde{M}^{4n}, \tilde{g}, Q)$ and any supercomplex submanifold of $(\tilde{M}^{4n}, \tilde{g}, Q)$ has such form.*

Remark. In case of an oriented Riemannian 4-manifold $(\tilde{M}^4, \tilde{g})$ with the parallel quaternionic structure $Q = \tilde{g}^{-1} \circ \Lambda^2_+$ the natural almost complex structure J^Z on the twistor space $\mathcal{Z} = \{J \in Q | J^2 = -\mathrm{Id}\}$ defined on (9) is integrable if and only if $(\tilde{M}^4, \tilde{g})$ is anti-self-dual [7]. Nevertheless the above proof still works and gives the following result.

Proposition 4.7. *Let $(\tilde{M}^4, \tilde{g})$ be an oriented 4-dimensional Riemannian manifold and $\mathcal{Z}$ its twistor space with the projection $\pi : \mathcal{Z} \to \tilde{M}^4$. Then there exists a natural one to one correspondence between complex surfaces $N^2 \subset \mathcal{Z}$ (i.e. $J^Z TN = TN$) such that $\pi_{|N^2} : N^2 \to M^2 = \pi(N^2)$ is a diffeomorphism and supercomplex surfaces $M^2 \subset \tilde{M}^4$: the natural lift $N^2 = J_1(M^2)$ of a supercomplex surface $M^2 \subset \tilde{M}^4$ is a complex surface (called also a pseudo-holomorphic curve) in $\mathcal{Z}$ such that $M^2 = \pi(N^2)$.*

5. Natural lift of Kähler submanifolds to the twistor space

5.1. Now we will assume that $(\tilde{M}^{4n}, \tilde{g}, Q)$ is a quaternionic Kähler manifold with non zero scalar curvature and $\pi : \mathcal{Z} \to \tilde{M}^{4n}$ is its twistor fibration. We will consider $\mathcal{Z}$ as a complex manifold equipped with the natural (pseudo-)Kähler–Einstein metric g^Z, see [7], Th. 14.80, Remark 14.86 b). Then the projection π is a (pseudo-)Riemannian submersion with totally geodesic fibers $\mathcal{Z}_x$, $x \in \tilde{M}^{4n}$, isometric to the standard sphere S^2. The horizontal distribution $\mathcal{H}$ is a holomorphic contact distribution, that is there exists a local holomorphic 1-form θ such that

$\mathcal{H} = \operatorname{Ker}\theta$ and $\theta \wedge d\theta^n$ does not vanish. The following theorems give a characterization of the natural lift of Kähler submanifolds of a quaternionic Kähler manifold with non zero scalar curvature.

Theorem 5.1. (see also [40]) *Let (M^{2m}, J_1) be a Kähler submanifold of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ with non zero scalar curvature. The natural lift $N = J_1(M^{2m}) \subset \mathcal{Z}$ is a holomorphic horizontal submanifold of the twistor space. Conversely, any holomorphic horizontal submanifold $N \subset \mathcal{Z}$ locally defines a Kähler submanifold $M = \pi(N)$: this means that if $U \subset N$ is an open submanifold of N such that the projection $\pi : U \to W = \pi(U)$ is a diffeomorphism then (W, J_1), where J_1 is the section $\pi^{-1} : W \to U$, is a Kähler submanifold of $\tilde{M}^{4n}$.*

Proof. By Theorem 4.2, the natural lift $J_1(M)$ of a Kähler submanifold M^{2m} is a complex submanifold of $\mathcal{Z}$ of complex dimension m and by Corollary 4.5 it is horizontal. Viceversa, let $N \subset \mathcal{Z}$ be a holomorphic horizontal submanifold of $\mathcal{Z}$ such that the projection $\pi : N \to M$ is a diffeomorphism. Then by Theorem 4.2 $(M = \pi(N), J_1)$, where $J_1 = \pi^{-1} : M \to N$, is a complex submanifold of $\tilde{M}^{4n}$ and by Corollary 4.5 the section J_1 of Q is parallel along M, that is (M, J_1) is Kähler. □

5.2. Maximal Kähler submanifolds of a quaternionic Kähler manifold and Legendrian submanifolds of the twistor space. We use Theorem 5.1 to describe maximal Kähler submanifolds (M^{2n}, J_1) of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ with non zero scalar curvature. Recall that a holomorphic n-dimensional submanifold $L \subset \mathcal{Z}$ is called a **Legendrian submanifold** if it is tangent to the contact distribution $\mathcal{H}$ (in other words L is a maximal integral submanifold of the holomorphic contact structure $\mathcal{H}$). According to Theorem 5.1, the natural lift $L = J_1(M)$ is a $2n$-dimensional holomorphic horizontal submanifold of the twistor space that is a holomorphic Legendrian submanifold. Moreover the projection $\pi : L \to M$ is a diffeomorphism. We study such Legendrian submanifolds more carefully.

Proposition 5.2. *Let L be a Legendrian submanifold of $(\mathcal{Z}, \mathcal{H})$. Then the projection $\pi : L \to M = \pi(L)$ is a local isometry and the pre-image $\pi^{-1}(x)$ of any point $x \in M$ consists either from one point J_x or from two points $(J_x, -J_x)$ which are antipodal points of the fiber $\mathcal{Z}_x = S^2_x \subset Q_x$.*

Proof. The projection $\pi_{|L}$ is a local isometry since T_zL, $z \in L$ is a horizontal subspace of $\mathcal{Z}$ and $\pi_* : \mathcal{H}_z \to T_{\pi(z)}\tilde{M}$ is an isometry by the definition of the metric g^Z.

Choose a neighbourhood U of a point $z \in L$ such that $\pi : U \to W = \pi(U)$ is an isometry. Then $U = J_1(W)$ is the natural lift of an almost complex submanifold (W, J_1). Moreover the section J_1 of $Q_{|W}$ is parallel along W by Corollary 4.5 and hence (W, J_1) is a Kähler submanifold. By Corollary 2.11 the section J_1 is canonically defined on W up to a sign. This implies that the pre-image $\pi^{-1}(w) \cap L$ of $w \in W$ is contained in $(J_{1|w}, -J_{1|w})$. □

Definition 5.3. *We say that a* **Legendrian submanifold** $L \subset \mathcal{Z}$ **is of type 1** *(respectively* **type 2***) if all fibers of the projection $\pi : L \to M = \pi(L)$ consist of* 1 *point (respectively,* 2 *points).*

Proposition 5.4. *A Legendrian submanifold $L \subset \mathcal{Z}$ which is complete with respect to the Kähler metric induced by g^Z is either of type* 1 *or type* 2.

Proof. The proof follows from the lemma below.

Lemma 5.5. ([10], Lemma 1.32) *Let $\pi : L \to M = \pi(L)$ be a local isometry of Riemannian manifolds. If L is complete then π is a covering.* □

The following Proposition gives sufficient conditions for a Legendrian submanifold L of $\mathcal{Z}$ to be of type 1.

Proposition 5.6. *Let $\tilde{M}^{4n}$, $n > 1$, be a quaternionic Kähler manifold with non zero scalar curvature. A complete connected Legendrian submanifold L of the twistor space $\mathcal{Z}$ is of type 1 if the local De Rham decomposition of L has no hyperKähler (in particular, locally flat) factor.*

Proof. From the Berger list of irreducible holonomy groups of Riemannian manifolds, it follows that under the above assumptions the normalizer $N_{O(2n)}H$ of the restricted holonomy group H of L in the orthogonal group is contained in $U(n)$. Hence any Riemannian manifold M which is locally isometric to L is also Kähler. In particular the locally defined parallel section J_1 on $M = \pi(L)$ is globally defined. Since L is connected, it coincides with the natural lift $J_1(M)$ and $\pi : L \to M$ is a diffeomorphism. □

As a corollary of the Theorem 5.1 we have the following description of maximal Kähler submanifolds of a quaternionic Kähler manifold.

Theorem 5.7. *There exists a natural one to one correspondence between maximal Kähler submanifolds (M^{2n}, J_1) of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, Q)$ with non zero scalar curvature and Legendrian submanifolds L of type 1 of the twistor space $\mathcal{Z}$: the natural lift $L = J_1(M)$ of a Kähler submanifold (M^{2n}, J_1) is a Legendrian submanifold of type* 1 *and, conversely, the projection $M = \pi(L)$ of a Legendrian submanifold L of type* 1 *is a Kähler submanifold such that $J_1(M) = L$.*

This theorem reduces the construction of Kähler submanifolds of $\tilde{M}^{4n}$ to the construction of Legendrian submanifolds of $\mathcal{Z}$. To construct Legendrian submanifolds it is sufficient to find local holomorphic coordinates $u, p_1, \dots, p_n, q^1, \dots, q^n$, called **Darboux coordinates**, such that the holomorphic 1-form

$$\theta = du - \sum p_i dq^i \tag{12}$$

is a contact form, that is $\mathrm{Ker}\theta = \mathcal{H}$. Such Darboux coordinates exist by the holomorphic Darboux theorem (see [27]).

In terms of appropriate Darboux coordinates any Legendrian submanifold locally has the form

$$L = L_F = \left\{u = F(q^i), \quad p_i = \partial F/\partial q^i\right\}$$

where $F = F(q^i)$ is a holomorphic function called **generating function** of the Legendrian submanifold. The projection $\pi(L)$ of a Legendrian submanifold L gives a maximal Kähler submanifold of the quaternionic Kähler manifold $\tilde{M}^{4n}$ and the projection $\pi(N)$ of a complex submanifold $N \subset L$ is a Kähler submanifold of $\tilde{M}^{4n}$ and any Kähler submanifold can be obtained by such construction.

6. Maximal Kähler submanifolds of $\mathbb{H}P^n$

We saw that the description of Kähler submanifolds of a quaternionic Kähler manifold of non zero scalar curvature reduces to the construction of Darboux coordinates.

R. Bryant constructed such Darboux coordinates on the twistor space $\mathbb{C}P^3$ of the quaternionic Kähler manifold $\tilde{M} = S^4 \equiv \mathbb{H}P^1$ which led to his famous construction of superminimal surfaces in S^4 (see [8]). Remark that superminimal surfaces M^2 in an oriented 4-dimensional Riemannian manifold $(\tilde{M}^4, \tilde{g})$ are the same as maximal Kähler submanifolds of the manifold $(\tilde{M}^4, \tilde{g}, Q)$ with the parallel quaternionic structure $Q = \tilde{g} \circ \Lambda_+$ (see [17], and also [33], Prop. 1). We will show that Bryant's construction of Darboux coordinates has a natural generalization to the case $\tilde{M}^{4n} = \mathbb{H}P^n$.

Recall that the twistor space $\mathcal{Z}$ of the (right) quaternionic projective space $\mathbb{H}P^n$ is the complex projective space $\mathbb{C}P^{2n+1}$ with the natural projection

$$\pi : \mathbb{C}P^{2n+1} \longrightarrow \mathbb{H}P^n$$

given by

$$\mathbb{C}P^{2n+1} \ni \left[w^0, \dots, w^{2n+1}\right] \mapsto \left[x^0, \dots, x^n\right] \in \mathbb{H}P^n$$

where $x^k = w^{2k} + jw^{2k+1}, k = 0, \dots, n, \ (w^0, \dots, w^{2n+1}) \in \mathbb{C}^{2n+2} - \{0\}$ and $1, i, j, k$ is the standard basis of $\mathbb{H}$.

The Riemannian metric $h = g^{\mathcal{Z}}$ of $\mathbb{C}P^{2n+1}$ is the Fubini-Study metric given by

$$h = \frac{(d\mathbf{w} \cdot d\bar{\mathbf{w}})(\mathbf{w} \cdot \bar{\mathbf{w}}) - (d\mathbf{w} \cdot \bar{\mathbf{w}})(\mathbf{w} \cdot d\bar{\mathbf{w}})}{(\bar{\mathbf{w}} \cdot \mathbf{w})^2}$$

where $\mathbf{w} \cdot \mathbf{t} = \sum_{i=0}^{2n+1} w^i \bar{t^i}$ is the canonical Hermitian product of vectors $\mathbf{w} = (w^0, \dots, w^{2n+1}), \mathbf{t} = (t^0, \dots, t^{2n+1}) \in \mathbb{C}^{2n+1}$.

The complex contact structure of $\mathbb{C}P^{2n+1}$ is induced by the complex 1-form

$$\psi = \sum_{k=0}^{n} \left(w^{2k} dw^{2k+1} - w^{2k+1} dw^{2k}\right),$$

defined on $\mathbb{C}^{2n+2} - \{0\}$, through the projection

$$\pi : \mathbb{C}^{2n+2} - \{0\} \to \mathbb{C}P^{2n+1},$$

see [26]. More precisely, 1-form ψ induces the local contact form θ^i on the coordinate domain $W_i = \{w^i \neq 0\}, i = 0, 1, \dots, 2n+1$ as follows. Let $(z^0 = w^0(w^i)^{-1}, \dots, \hat{z}^i, \dots, z^{2n+1} = w^{2n+1}(w^i)^{-1})$ be the non-homogeneous coordinates in the domain W_i (where the symbol "hat" means that the corresponding term is omitted). Substituting $w^k = w^i z^k$ for $k \neq i$ in ψ, we conclude that the restriction $\psi|_{W_i}$ is conformal to the following contact form

$$\theta^i = (-1)^{i+1} dz^i - \sum_{k=1, k \neq [i/2]}^{n} \left(z^{2k+1} dz^{2k} - z^{2k} dz^{2k+1}\right).$$

We verify now that the contact distribution $\mathcal{H} = \ker \theta^i$ is the holomorphic distribution on $\mathbb{C}P^{2n+1}$ orthogonal to the fibre of the twistor fibration $\mathbb{C}P^{2n+1} \to \mathbb{H}P^n$.

On the domain $W_0 = \{w^0 \neq 0\} \subset \mathbb{C}P^{2n+1}$ we have the non homogeneous complex coordinates

$$z^i = w^i (w^0)^{-1}, \quad i = 1, \dots, 2n+1,$$

and on the projection $U_0 = \{x^0 \neq 0\} \subset \mathbb{H}P^n$ the non-homogeneous quaternionic coordinates

$$\xi^\alpha = x^\alpha (x^0)^{-1}, \quad \alpha = 1, \dots, n.$$

With respect to the complex coordinates $\mathbf{z} = (z^1, \dots, z^{2n+1})$ on W_0 the Hermitian metric of $\mathbb{C}P^{2n+1}$ can be written as $h = \sum_{i,j} dz^i h_{i\bar{j}} dz^{\bar{j}}$ where $z^{\bar{j}} = \overline{z^j}$ and

$$h_{i\bar{j}} = \frac{(1 + \bar{\mathbf{z}} \cdot \mathbf{z})\delta_{i\bar{j}} - z^{\bar{i}} z^j}{(1 + \bar{\mathbf{z}} \cdot \mathbf{z})^2}.$$

Since two points $[1, z^1, \dots, z^{2n+1}], [1, z^{1'}, \dots, z^{2n+1'}] \in W_0 \subset \mathbb{C}P^{2n+1}$ have the same projection in $\mathbb{H}P^n$, if and only if there exists a quaternion $q = \eta + j\xi$, where $\eta, \xi \in \mathbb{C}$, such that

$$\begin{cases} 1 + jz^{1'} & = \big(1 + jz^1\big)q & \equiv \big(\eta - \bar{z}^1\xi\big) + j\big(z^1\eta + \xi\big) \\ z^{2'} + jz^{3'} & = \big(z^2 + jz^3\big)q & \equiv \big(z^2\eta - z^3\xi\big) + j\big(z^3\eta + \bar{z}^2\xi\big) \\ \dots & \dots & \\ z^{2n'} + jz^{2n+1'} & = \big(z^{2n} + jz^{2n+1}\big)q & \equiv \big(z^{2n}\eta - z^{2n+1}\xi\big) + j\big(z^{2n+1}\eta + \bar{z}^{2n}\xi\big) \end{cases}$$

(and hence, in particular, $\eta = \bar{z}^1\xi + 1$), the parametric equations of the fiber through the point $z_0 = (z_0^1, z_0^2, z_0^3, \cdots, z_0^{2n}, z_0^{2n+1})$ of W_0 are

$$\begin{cases} z^1(\lambda) & = z_0^1 + \big(1 + z_0^1\bar{z}_0^1\big)\lambda, \\ z^2(\lambda) & = z_0^2 + \big(z_0^2\bar{z}_0^1 - \bar{z}_0^3\big)\lambda, \\ z^3(\lambda) & = z_0^3 + \big(z_0^3\bar{z}_0^1 + \bar{z}_0^2\big)\lambda \\ \dots & \dots \\ z^{2n}(\lambda) & = z_0^{2n} + \big(z_0^{2n}\bar{z}_0^1 - \bar{z}_0^{2n+1}\big)\lambda, \\ z^{2n+1}(\lambda) & = z_0^{2n+1} + \big(z_0^{2n+1}\bar{z}_0^1 + \bar{z}_0^{2n}\big)\lambda, \end{cases}$$

where $\lambda \in \mathbb{C}$.

Hence the non vanished holomorphic vector field

$$\mathbf{v} = \big(1 + z^1\bar{z}^1\big)\frac{\partial}{\partial z^1} + \sum_{k=1}^{n} \left[\big(z^{2k}\bar{z}^1 - \bar{z}^{2k+1}\big)\frac{\partial}{\partial z^{2k}} + \big(z^{2k+1}\bar{z}^1 + \bar{z}^{2k}\big)\frac{\partial}{\partial z^{2k+1}}\right]$$

is tangent to the fibers of π and defines a trivialization of the vertical bundle $\mathcal{V}$. It is easy to check that the $(1,0)$ form dual to $\mathbf{v}$ under the Fubini-Study metric is $\theta = \theta_{\mathbf{v}} = h \circ \mathbf{v}$ given by

$$\theta_{\mathbf{v}} = |z \cdot \bar{z}|^{-1} \left[dz^1 - \sum_{k=1}^{n} \left(z^{2k+1} dz^{2k} - z^{2k} dz^{2k+1} \right) \right].$$

where $|z \cdot \bar{z}| = 1 + z^1 z^{\bar{1}} + \cdots + z^{2n+1} \bar{z}^{2n+1}$, and it is proportional to

$$\theta^1 = dz^1 - \sum_{k=1}^{n} \left(z^{2k+1} dz^{2k} - z^{2k} dz^{2k+1} \right)$$

which is the local holomorphic contact form defining on W_0 the complex contact structure of $\mathcal{Z} = \mathbb{C}P^{2n+1}$.

In the new coordinates

$$\begin{cases} u &= z^1 + \sum_{k=1}^{n} z^{2k} z^{2k+1}, \\ p_k &= \frac{1}{2} z^{2k+1}, \\ q^k &= z^{2k} \end{cases}$$

θ^1 takes the Darboux form

$$\theta^1 = du - \sum_{k=1}^{n} p_k dq^k.$$

Hence any holomorphic function $u = f(q^1, \ldots, q^n)$ defines a Legendrian submanifold $L_f = \{u = f(q^k),\, p_k = \frac{\partial f}{\partial q^k}\}$.

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