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On the existence and Morse index of solutions to the Allen-Cahn equation in two dimensions

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Abstract. In this paper we study transition layers in the solutions to the Allen-Cahn equation in two dimensions. We show that for any straight line segment intersecting the boundary of the domain orthogonally there exists a solution to the Allen-Cahn equation, whose transition layer is located near this segment. In addition we analyze stability of such solutions and show that it is completely determined by a geometric eigenvalue problem associated to the transition layer. We prove the existence of both stable and unstable solutions. In the case of the stable solutions we recover a result of Kohn and Sternberg [13]. As for the unstable solutions we show that their Morse index is either 1 or 2.

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1. Introduction

In this paper we consider the following elliptic problem:

$$\begin{aligned} &\varepsilon^2 \Delta u + f(u) = 0 \qquad && \text{in } \Omega, \\ &\partial_n u = 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $f(u) = u(1-u^2)$, $\Omega \in \mathbf{R}^2$ is a bounded domain with smooth boundary, ε is a small parameter and ∂_n denotes the derivative in the direction of the outward normal. Equation (1.1) is know as the Allen-Cahn equation and was introduced in [2] as a model describing the evolution of antiphase boundaries.

The stationary problem (1.1) and its parabolic counterpart have been a subject of an extensive research for many years and now we have a very complete picture of the development, existence and dynamics of transition layers in the solutions to (1.1). In order to describe some of the known results we define the Allen-Cahn functional

$$J(u) = \int_\Omega \left[\frac{\varepsilon^2}{2} |\nabla u|^2 - F(u) \right], \quad F(u) = -\frac{1}{4}(1-u^2)^2.$$

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By $\mathrm{Per}_\Omega(A)$ we denote the perimeter of the set $A \subset \Omega$. Intuitively the gradient flow of J, in the limit as $\varepsilon \to 0$, reduces to the gradient flow of Per_Ω. It is known that the gradient flow of Per_Ω is simply the motion by mean curvature of ∂A. Summarizing: transition layers in the Allen-Cahn flow evolve, as $\varepsilon \to 0$, by their mean curvature. On the level of formal asymptotics this fact was established in [17].

Chen in [5] and de Mottoni and Schatzman in [7] proved that initially developed interfaces evolve by mean curvature as long as the classical solution to the mean curvature flow exists. In [9, 11] analogous results were proven assuming the existence of mean curvature flow in an appropriately generalized sense. Finally, Soner [18] was able to prove those results for a large class of initial data.

Convergence of the Allen-Cahn equation to the mean curvature motion for the interfaces intersecting the boundary was studied in [5] and in [12].

Since the Allen-Cahn flow shrinks interior interfaces to points, nontrivial stationary states to (1.1) are possible only if their transition layers intersect the boundary and their mean curvature approaches 0 as $\varepsilon \to 0$. This situation was studied in [13]. The authors used Γ-convergence techniques to show a general result that states that in a neighborhood of a local, isolated minimizer of Per_Ω there exists a local minimizer to the functional J. They further used this idea to show the existence of stable solutions for (1.1) in two dimensional, non-convex domains, such as a dumbbell. In [16] Padilla and Tonnegawa proved that local minimizers of the Allen-Cahn equation necessary converge to local minimizers of the perimeter functional.

In this paper we study the Allen-Cahn equation in two dimensions and show that for any smooth, stationary and nondegenerate solution to the mean curvature flow there is a corresponding stationary solution to the Allen-Cahn equation. This result in some sense completes the results described above as it establishes the connection between functionals J and Per_Ω on the level of their critical points.

Throughout this paper we assume that a curve $\gamma \in \Omega$, our candidate for an interface, is such that:

(i) the curvature of γ is 0 (γ is a straight line segment);
(ii) γ intersects $\partial\Omega$ at exactly two points γ_0, γ_1 and at those points $\gamma \perp \partial\Omega$;
(iii) γ is nondegenerate in the sense described below (see (1.6) to follow).

Our first goal is to prove the following:

Theorem 1.1. *Let U be the unique heteroclinic solution to*

$$\begin{aligned} &U_{\eta\eta} + f(U) = 0, \qquad -\infty < \eta < \infty, \\ &U(\pm\infty) = \pm 1, \; U(0) = 0. \end{aligned} \tag{1.2}$$

Let $d(\gamma; x, y)$ denote the signed distance of a point $(x, y) \in \Omega$ to the straight line that contains γ. For each sufficiently small ε there exists a solution u^ε to (1.1) *such that*

$$\|u^\varepsilon(x, y) - U(d(\gamma; x, y)/\varepsilon)\|_{C^0(\Omega)} \le C\varepsilon, \tag{1.3}$$

where $C > 0$ is independent on ε.

Our second goal is to analyze the stability of the solution described in Theorem 1.1. More precisely we study the following eigenvalue problem:

$$\begin{aligned} &\varepsilon^2 \Delta V + f'(u^\varepsilon)V = -\Lambda V && \text{in } \Omega, \\ &\partial_n V = 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

The Morse index of u^ε is simply the number of negative eigenvalues of (1.4).

To state our result we need to define a geometric eigenvalue problem that, as we will see, plays an important role in our considerations. Let $\kappa_{\partial\Omega}(\gamma_i)$, $i = 0, 1$ be the curvatures of $\partial\Omega$ at the points of intersection with γ. Consider the following eigenvalue problem

$$\begin{aligned} -\theta_{ss} &= \lambda\theta, && 0 < s < |\gamma|, \\ \theta\kappa_{\partial\Omega}(\gamma_0) + \theta_s &= 0, && s = 0, \\ -\theta\kappa_{\partial\Omega}(\gamma_1) + \theta_s &= 0, && s = |\gamma|, \end{aligned} \tag{1.5}$$

where γ is parameterized by arclenght in such a way that s increases from γ_0 to γ_1 and $\partial\Omega$ is oriented counterclockwise from γ_0 to γ_1. We say that γ is nondegenerate if (1.5) does not have a zero eigenvalue. This is equivalent to the following condition:

$$\kappa_{\partial\Omega}(\gamma_0) + \kappa_{\partial\Omega}(\gamma_1) - \kappa_{\partial\Omega}(\gamma_0)\kappa_{\partial\Omega}(\gamma_1)|\gamma| \neq 0. \tag{1.6}$$

We can now state our second theorem.

Theorem 1.2. *The Morse index of the solution to* (1.1) *described in Theorem* 1.1 *equals the number of negative eigenvalues of* (1.5). *Moreover for any* $k^* > 0$ *there exists* ε_{k^*} *such that, for all* $\varepsilon \in (0, \varepsilon_{k^*}]$, *if* $\{\Lambda_k\}_{k=1,\dots,k^*}$ *are the first* k^* *eigenvalues of the linearized problem* (1.4) *then* $\Lambda_k = \lambda_k\varepsilon^2 + o(\varepsilon^2)$, *where* $\{\lambda_k\}_{k=1,\dots,k^*}$ *are the first* k^* *eigenvalues of* (1.5).

Remark 1.1. We will distinguish 3 cases:

(1) The spectrum of (1.5) is positive-we refer to this case as the minimum axis case.
(2) The spectrum of (1.5) contains one negative eigenvalue-we refer to this case as the short axis case.
(3) The spectrum of (1.5) contains two negative eigenvalues-we refer to this case as the long axis case.

By an explicit calculation one can show that (1.5) has at most two negative eigenvalues. For example if both curvatures are negative then the spectrum is positive. In this case γ is a local minimizer of the perimeter and this situation was treated in [13].

If both curvatures are positive then:

- If $\frac{1}{\kappa_{\partial\Omega}(\gamma_0)} + \frac{1}{\kappa_{\partial\Omega}(\gamma_1)} > |\gamma|$ then (1.5) has one negative eigenvalue.
- If $\frac{1}{\kappa_{\partial\Omega}(\gamma_0)} + \frac{1}{\kappa_{\partial\Omega}(\gamma_1)} < |\gamma|$ then (1.5) has two negative eigenvalues.

The terminology short and long axis case is used because of the special situation when Ω is an ellipse-the case that originally motivated our study.

Obviously, except a degenerate case, any combination of the curvatures gives rise to one of the three cases described above.

Some key ideas of our present work were greatly motivated by [1]. In this paper the authors considered the dynamics of the mass conserving Allen-Cahn equation. Starting from a one parameter family of approximate interfaces with constant mean curvature intersecting the boundary they were able to construct an approximate invariant manifold to the parabolic PDE consisting of small drops moving along the boundary. Their construction relies on the fact that the interfaces for the mass conserving Allen-Cahn equation evolve by volume-preserving mean curvature flow [4].

At first sight it seems that in our case a family of translates of γ should provide a one parameter family of interfaces, which then could be used to construct an approximate solution to (1.1). This turns out not to be true. Main effort in this paper is to construct a family of approximate interfaces that are appropriate perturbations of γ and this is done in parallel with the process of constructing an approximate invariant manifold to (1.1). Our construction depends heavily on the geometric properties not only of the boundary at two points of intersection with the interface but also on Ω itself.

Once the approximate invariant manifold is constructed the existence of solutions to (1.1) is proved using a spectral estimate and the Lyapunov-Schmidt reduction. Our results regarding the Morse index of the solution and the asymptotic behavior of the eigenvalues are a byproduct of this approach.

This paper is organized as follows: in Section 2 we construct an approximate solution to the invariant manifold equation. In Section 3 we study the linearized eigenvalue problem, and in Section 4 we give proofs of our main results. The paper concludes with some remarks about possible generalizations of our results.

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2. Approximate manifold equation

2.1. Statement of the problem

Throughout this paper C, c will denote generic constants whose values may vary from line to line. Function $U = U(\eta)$ will be the heteroclinic solution to (1.2) such that $U(\pm\infty) = \pm 1$.

In this section we will construct for each $n > 1$ a solution, up to order n in ε, of the following problem:

$$\begin{aligned} &\varepsilon^2 \Delta u + f(u) = \mathbf{c}(\xi)\cdot\nabla_\xi u + g^\varepsilon && \text{in } \Omega, \\ &\partial_n u = h^\varepsilon && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The unknown functions here are: $u = u(x, y, \xi)$, $\mathbf{c} = \mathbf{c}(\xi)$, $g^\varepsilon = g^\varepsilon(x, y, \xi)$ and $h^\varepsilon = h^\varepsilon(x, y, \xi)$. The above equation is to be valid for $(x, y) \in \Omega$, $|\xi| < \delta$, some

$\delta > 0$, with $\xi \in \mathbf{R}$ and $\mathbf{c} \in \mathbf{R}$ in the minimum and short axis case and $\xi \in \mathbf{R}^2$ and $\mathbf{c} \in \mathbf{R}^2$ in the long axis case. We speak of solution up to order n of (2.1) if $g^\varepsilon = O(\varepsilon^n)$ and $h^\varepsilon = O(\varepsilon^{n-1})$. We will denote such a solution by u^n or, if it does not cause a confusion, by u.

The solution to (2.1) up to an arbitrary order will be constructed by using formal asmptotics expansion techniques.

We will consider $|\xi| < \delta$, where $\delta > 0$ is an arbitrary but from now on fixed number.

2.2. Preliminaries

In order to prepare for finding a solution to (2.1) we will first study some auxiliary problems which are important in what follows.

Proposition 2.1. *Equation*

$$V_{\eta\eta} + f'(U)V = \Phi, \qquad -\infty < \eta < \infty,$$

has a unique solution if and only if

$$\int_{-\infty}^{\infty} \Phi U_\eta \, d\eta = 0.$$

Moreover if $|\Phi(\eta)| \le Ce^{-c|\eta|}$ *then* $|V(\eta)| \le Ce^{-c|\eta|}$*, with similar estimates for the derivatives.*

Proposition 2.2. *Consider*

$$\begin{aligned} &V_{\eta\eta} + V_{\mu\mu} + f'(U)V = \Psi, && -\infty < \eta < \infty, 0 < \mu < \infty, \\ &V_\mu = \psi, && -\infty < \eta < \infty, \mu = 0. \end{aligned} \tag{2.2}$$

Assume that $|\Psi(\eta, \mu)| \le C \min\{e^{-c|\eta|}, e^{-c\mu}\}$ *and* $|\psi(\eta)| \le Ce^{-c|\eta|}$*. Problem* (2.2) *has a unique solution if and only if*

$$\int_0^\infty \int_{-\infty}^\infty \Psi U_\eta \, d\mu d\eta = \int_{-\infty}^\infty \psi U_\eta \, d\eta.$$

Moreover

$$|V(\eta, \mu)| \le C \min\{e^{-c|\eta|}, e^{-c\mu}\},$$

with similar estimates for the derivatives.

Proposition 2.1 is a well known result. For the proof of Proposition 2.2 we refer the reader to [1,8].

In the sequel by λ_i we shall denote the eigenvalues and by θ_i, $i = 1, \ldots$, the corresponding eigenfunctions of (1.5).

At each step in the process of constructing an approximate solution to (1.1) solvability conditions in Proposition 2.1 and Proposition 2.2 lead to boundary value problems which are of the following general type:

$$\begin{aligned} &\mathbf{c}_0 \cdot \nabla_\xi \varphi + \mathbf{b} \cdot \Theta - \varphi_{yy} = A(y, \xi), && 0 < y < 1, |\xi| < \delta \\ &h_0'' \varphi + \varphi_y = B_0(\xi), && y = 0, |\xi| < \delta \\ &h_1'' \varphi + \varphi_y = B_1(\xi), && y = 1, |\xi| < \delta, \end{aligned} \tag{2.3}$$

where

$$(\Theta, \mathbf{c}_0) = \begin{cases} (\theta_1, -\lambda_1 \xi) & \text{in the minimum and short axis case,} \\ (\theta_1, \theta_2, -\lambda_1 \xi_1, -\lambda_2 \xi_2), & \text{in the long axis case,} \end{cases} \tag{2.4}$$

and the unknowns are φ and $\mathbf{b}$. Function φ depends on y and the parameter ξ, and $\mathbf{b}$ is a function of ξ. Constants h_0'', h_1'' satisfy the nondegeneracy condition

$$h_0'' - h_1'' + h_0'' h_1'' \neq 0. \tag{2.5}$$

In what follows $h_0'' = \kappa_{\partial \Omega}(\gamma_0)$ and $h_1'' = -\kappa_{\partial \Omega}(\gamma_1)$. Assuming that $|\gamma| = 1$ (which can be done without loss of generality) we see that conditions (1.6) and (2.5) are equivalent.

Lemma 2.1. *Let function $A(y, \xi)$ be a polynomial with respect to ξ with coefficients that are C^k, $k \geq 1$ functions of y, $0 < y < 1$. Assume also that functions $B_0(\xi)$, $B_1(\xi)$ are polynomials with respect to ξ. The maximum degree of polynomials $A(\cdot, \xi)$, $B_0(\xi)$, $B_1(\xi)$ will be denoted by K.*

There exists a solution $(\varphi, \mathbf{b})$ to (2.3) with the following properties:

(i) Function φ is a $C^{1,\alpha}$ function in ξ, with some $\alpha > 0$, and C^{k+2} function in y.
(ii) In the short axis case φ is a polynomial with respect to ξ of degree K and of the following general form

$$\varphi(y, \xi) = \sum_{n=0}^{K} \varphi_n(y) \xi^n.$$

A similar formula holds in the long axis case.
(iii) Function $\mathbf{b}(\xi)$ is also a polynomial in ξ, such that $\mathbf{b}(0) = 0$.

Proof. We claim that without loss of generality we can consider (2.3) with homogeneous boundary conditions. Indeed, let $\bar{\varphi} = \varphi + my + b$ where

$$m = m(\xi) = \frac{-B_1 h_0'' + B_0 h_1''}{h_0'' - h_1'' + h_1'' h_0''}, \quad b = b(\xi) = \frac{B_1 - B_0(h_1'' - 1)}{h_0'' - h_1'' + h_0'' h_1''}.$$

Then $\bar{\varphi}$ satisfies

$$\begin{aligned} &\mathbf{c}_0 \cdot \nabla_\xi \bar{\varphi} + \mathbf{b} \cdot \Theta - \bar{\varphi}_{yy} = \bar{A}(y, \xi), && 0 < y < 1, |\xi| < \delta, \\ &h_0'' \bar{\varphi} + \bar{\varphi}_y = 0, && y = 0, |\xi| < \delta, \\ &h_1'' \bar{\varphi} + \bar{\varphi}_y = 0, && y = 1, |\xi| < \delta, \end{aligned} \tag{2.6}$$

where $\bar{A} = A + \mathbf{c}_0 \cdot \nabla_\xi (my + b)$. We observe that still $\bar{A}$ is a polynomial with respect to ξ. The claim now follows.

Since (2.6) is a linear equation its solution can be built by taking a superposition of a solution to

$$\begin{aligned} -\varphi_{yy} &= A(y,0), && 0 < y < 1, \\ h_0''\varphi + \varphi_y &= 0, && y = 0, \\ h_1''\varphi + \varphi_y &= 0, && y = 1, \end{aligned} \tag{2.7}$$

and a solution to (2.6) with the right hand side replaced by $A(y,\xi) - A(y,0)$. Thus, to solve (2.3), it suffices to show that there exist solutions to (2.7) and to (2.3) with $A(y,0) = 0$.

We will first solve (2.7). Observe that by the Fredholm alternative and (2.5) there exists a unique solution to this equation.

We will now solve (2.3) under the assumption that $A(y,0) = 0$. We chose $\mathbf{b}$ by setting:

$$\mathbf{b}_i = \int_0^1 A\theta_i \, dy,$$

where $i = 1$ in the minimum axis and short axis case and $i = 1, 2$ in the long axis case. Since $A(\cdot,\xi)$ is a polynomial with respect to ξ and $A(\cdot, 0) = 0$ therefore the same is true for $\mathbf{b}$. In addition the choice of $\mathbf{b}$ implies that φ is orthogonal to Θ in $L^2(0,1)$.

To find φ we consider separately the minimum axis, the short axis and the long axis case.

<u>Minimum axis case.</u> We will solve for $\delta > \xi > 0$ first. Introduce new variable $\xi = \delta e^{-\lambda_1 t}$. In terms of t, φ solves

$$\varphi_t - \varphi_{yy} = A(y,t) - \mathbf{b}_1(t)\theta_1.$$

Since we want to find any solution to (2.3) we will take $\varphi(t,y) = 0$, $t = 0$. From the orthogonality condition it follows that

$$\int_0^1 \varphi\theta_1 \, dy = 0.$$

Using the assumption on A we also have

$$|A(y,t) - \mathbf{b}_1(t)\theta_1| \le Ce^{-\lambda_1 t},$$

hence

$$\|\varphi\|_{L^2(0,1)} \le Ce^{-\lambda_1 t},$$

with similar estimates holding for other norms in (0, 1).

To prove the required regularity of φ with respect to ξ observe that we only need to consider small ξ or, equivalently, large t. Let the expansion of φ in terms of eigenvalues of (1.5) be given by

$$\varphi(t,y) = \sum_{n=2}^{\infty} \varphi_n(t)\theta_n(y),$$

and corresponding expansion of the right hand side:

$$A(y,t) - \mathbf{b}_1(t)\theta_1 = \sum_{n=2}^{\infty} a_n(t)\theta_n(y).$$

Observe that since $A(y,\xi)$ is a polynomial with respect to ξ and $A(y,0) = 0$,

$$a_n(t) = a_{n1}\xi(t) + \cdots + a_{nm}\xi^m(t) = \delta a_{n1}e^{-\lambda_1 t} + O(e^{-2\lambda_1 t}), \qquad \text{as } t \to \infty.$$

One can calculate that

$$\varphi_n(t) = \int_0^t a_n(s)e^{\lambda_n(s-t)}\, ds.$$

And in particular, setting $c_n = \min\{2\lambda_1, \frac{\lambda_1+\lambda_n}{2}\}$, we get

$$\varphi_n = \frac{\delta a_{n1}e^{-\lambda_1 t}}{\lambda_n - \lambda_1} + O(e^{-c_n t}), \quad \varphi_{n,t} = \frac{-\lambda_1 \delta a_{n1}e^{-\lambda_1 t}}{\lambda_n - \lambda_1} + O(e^{-c_n t}),$$

as $t \to \infty$. Since $\varphi_{n,t} = (-\lambda_1\xi)\varphi_{n,\xi}$, we get

$$\varphi_{n,\xi} = \frac{a_{n1}}{\lambda_n - \lambda_1} + O(\xi^{c_n/\lambda_1 - 1}),$$

as $\xi \to 0^+$. Exactly the same result holds if we replace δ by $-\delta$, hence the lemma follows.

Short axis case. In this case it is more convenient to work directly with (y,ξ) variables. By the choice of $\mathbf{b}_1$

$$A(y,\xi) - \mathbf{b}_1\theta_1 = \sum_{n=1}^{K} A_n(y)\xi^n, \qquad \int_0^1 \theta_1 A_n\, dy = 0, n = 1,\ldots,K.$$

If we look for φ in the form

$$\varphi(y,\xi) = \sum_{n=1}^{K} \varphi_n(y)\xi^n,$$

then each φ_n solves

$$-n\lambda_1\varphi_n - \varphi_{n,yy} = A_n.$$

From the orthogonality condition we can solve uniquely for φ_1. When $n > 1$ then, since $n\lambda_1 < 0$, we can solve uniquely as well thanks to Fredholm alternative. The rest of the lemma follows now immediately.

Long axis case. We proceed exactly as in the short axis case, i.e. we look for a solution in the form

$$\varphi(y,\xi) = \sum_{n,m=1}^{n+m=K} \phi_{n,m}(y)\xi_1^n\xi_2^m.$$

Again solvability follows from the orthogonality condition and Fredholm alternative. We omit the details.

The proof is complete. □

We observe that in the minimum axis case even though the right hand side and the boundary conditions in (2.3) are polynomials with respect to ξ, the solution φ is not. A slight modification of the proof in the minimum axis case yields the following:

Corollary 2.1. *Consider the minimum axis case and assume that functions A, B_0, B_1 are $C^{1,\alpha'}$ functions with respect to ξ, for some $\alpha' > 0$, and $|A(\cdot, \xi) - A(0, \xi)| \le C|\xi|$, $|\xi| < \delta$. Assume also that A is a C^k, $k \ge 1$, function of y. There exists a solution $(\varphi, \mathbf{b})$ to (2.3) with the following properties:*

(i) Function φ is a $C^{1,\alpha}$, for some $\alpha > 0$, function in ξ and C^{k+2} function in y, which has the following general form

$$\varphi = \tilde{\varphi}(\xi, y) + \varphi^*(y),$$

where $|\tilde{\varphi}| \le C|\xi|$, $|\xi| < \delta$.

(ii) Function $\mathbf{b}(\xi)$ is in $C^{1,\alpha'}$ and satisfies $|\mathbf{b}(\xi)| \le C|\xi|$.

2.3. Inner expansion

From now on we will not distinguish between the minimum, short and long axis cases unless it is specifically stated. As we will see most of the computations in all those cases are exactly the same.

Observe that, after rescaling, we can always assume that $|\gamma| = 1$. We can also assume that γ is contained in the upper half plane, in the half line $y > 0$ and that the intersection points of γ with $\partial\Omega$ are $(x, y) = (0, 0)$ and $(x, y) = (0, 1)$, respectively. Then the usual Cartesian coordinates can be regarded as local coordinates around γ, with $y \in (0, 1)$ being the arc length of γ and $x \in (-\sigma, \sigma)$ being the signed distance to γ.

In what follows we will denote $\sigma = \varepsilon^{1/2}$. We set $\Omega_\sigma = (-\sigma, \sigma) \times (0, 1)$.

By the inner expansion solution up order n we mean a function $w^{in} = w^{in}(x, y, \xi)$ which solves

$$\varepsilon^2 \Delta w^{in} + f(w^{in}) = \mathbf{c}(\xi) \cdot \nabla_\xi w^{in} + g_n^{in} \qquad \text{in } \Omega_\sigma, \quad |\xi| < \delta, \tag{2.8}$$

where $g_n^{in} = O(\varepsilon^{n+1})$ and $\mathbf{c} = \mathbf{c}(\xi)$ is a function to be determined. Observe that in (2.8) we do not insist that w^{in} satisfies the boundary condition. As we will see in order to find the approximate solution to (2.1) we need to consider a boundary layer expansion. This will be done in the next section.

To find the consecutive terms in the inner expansion up to order n we will introduce a stretched variable as follows:

$$\eta = \frac{x - \varepsilon\phi}{\varepsilon}, \qquad \text{where } \phi = \phi(y, \xi, \varepsilon) = \xi \cdot \Theta(y) + \sum_{i=1}^{n-2} \varepsilon^i \phi_i(y, \xi).$$

We will assume that $-\infty < \eta < \infty$ although one should keep in mind that we need (2.8) to be satisfied only for $\eta = \frac{x-\varepsilon\phi}{\varepsilon}$ with $(x, y) \in \Omega_\sigma$.

Functions ϕ_i, $i > 1$ are at this point unknown and determining them is a part of the inner expansion problem. In the sequel it will be convenient to denote

$$\phi_0(y,\xi) = \xi \cdot \Theta(y).$$

We now look for a solution to (2.8) in the form

$$w^{in} = w^{in}(\eta, y, \xi) = \sum_{i=0}^{n} \varepsilon^i w_i^{in}(\eta, y, \xi), \qquad \mathbf{c}(\xi) = \varepsilon^2 \sum_{i=0}^{n-2} \varepsilon^i \mathbf{c}_i(\xi),$$

where $\mathbf{c}_0$ is defined in (2.4).

In terms of the stretched variable η we have

$$\begin{aligned} w^{in}_{\eta\eta} + f(w^{in}) = & -w^{in}_\eta\big(\mathbf{c}\cdot\nabla_\xi\phi - \varepsilon^2\phi_{yy}\big) - \varepsilon^2\phi_y^2 w^{in}_{\eta\eta} + 2\varepsilon^2\phi_y w^{in}_{\eta y} \\ & -\varepsilon^2 w^{in}_{yy} + \mathbf{c}\cdot w^{in}_\xi + g^{in}_n. \end{aligned} \tag{2.9}$$

We set

$$\begin{aligned} P(w^{in},\phi,\mathbf{c}) = & -w^{in}_\eta\big(\mathbf{c}\cdot\nabla_\xi\phi - \varepsilon^2\phi_{yy}\big) - \varepsilon^2\phi_y^2 w^{in}_{\eta\eta} + 2\varepsilon^2\phi_y w^{in}_{\eta y} \\ & -\varepsilon^2 w^{in}_{yy} + \mathbf{c}\cdot w^{in}_\xi. \end{aligned}$$

Observe that P is a polynomial with respect to ε, w^{in}_η, $w^{in}_{\eta\eta}$, $w^{in}_{\eta y}$, w^{in}_{yy}, w^{in}_ξ, $\nabla_\xi\phi$, ϕ_y, ϕ_{yy} and $\mathbf{c}$. In addition if all those functions are polynomials with respect to ξ or $C^{1,\alpha}$ functions of ξ so is P.

We will also expand $f(w^{in})$

$$\begin{aligned} f(w^{in}) = & f\big(w^{in}_0\big) + f'\big(w^{in}_0\big)\big(w^{in} - w^{in}_0\big) + \tfrac{1}{2} f''\big(w^{in}_0\big)\big(w^{in} - w^{in}_0\big)^2 \\ & + \tfrac{1}{6} f'''\big(w^{in}_0\big)\big(w^{in} - w^{in}_0\big)^3. \end{aligned}$$

<u>0th order expansion.</u> Comparing terms of order 0 in ε we obtain that w^{in}_0 satisfies

$$w^{in}_{0,\eta\eta} + f\big(w^{in}_0\big) = 0, \qquad -\infty < \eta < \infty, \tag{2.10}$$

and thus we take $w^{in}_0 = U$.

<u>1st order expansion.</u> Including w^{in}_1 term in the formal expansion for w^{in} and solving an appropriate equation one finds that necessarily $w^{in}_1 \equiv 0$. We omit this calculation here.

<u>2nd order expansion.</u> Taking into account $w^{in}_0 = U$ we get for w^{in}_2 the following equation:

$$w^{in}_{2,\eta\eta} + f'(U) w^{in}_2 = -U_\eta(\mathbf{c}_0\cdot\Theta - \xi\cdot\Theta_{yy}) - U_{\eta\eta}(\xi\cdot\Theta_y)^2, \qquad -\infty < \eta < \infty. \tag{2.11}$$

From Proposition 2.1 a unique solution to (2.11) exists, provided that

$$\int_{-\infty}^{\infty} U_\eta^2 (\mathbf{c}_0\cdot\Theta - \xi\cdot\Theta_{yy})\, d\eta = 0,$$

which holds if $\mathbf{c}_0$ is taken as in (2.4).

It follows that w_2^{in} is a polynomial in ξ and it satisfies the following inequality

$$\left|w_2^{in}\right| \le C|\xi|^2 e^{-c|\eta|}, \tag{2.12}$$

with some $c > 0$. Similar estimates hold for the derivatives of w_2^{in}.

$k > 2$ order expansion. Expanding P in terms of powers of ε we find that the coefficient of ε^k, P_k, takes form

$$\begin{aligned} P_k = &-[U_\eta(\mathbf{c}_0 \cdot \nabla_\xi \phi_{k-2} + \mathbf{c}_{k-2} \cdot \Theta - \phi_{k-2,yy}) + 2U_{\eta\eta}\xi \cdot \Theta_y \phi_{k-2,y}] \\ &+ Q_k, \end{aligned}$$

where Q_k is a polynomial in $w_{i,\eta}^{in}$, $w_{i,\eta\eta}^{in} w_{i,\eta y}^{in}$, $w_{i,yy}^{in}$, $w_{i,\xi}^{in}$ for $0 \le i \le k-2$, in ϕ_i, $\nabla_\xi \phi_i$, $\phi_{i,y}$, $\phi_{i,yy}$ and in $\mathbf{c}_i$, $0 \le i \le k-3$.

In a similar manner we can write the kth order term in the expansion of $f(w^{in})$, which does not include w_k^{in}, in the form $R_k = R_k(w_0^{in}, \ldots, w_{k-2}^{in})$, where R_k is a polynomial function of its arguments.

We require that w_k^{in} satisfies the following equation

$$\begin{aligned} w_{k,\eta\eta}^{in} + f'(U) w_k^{in} = &-[U_\eta(\mathbf{c}_0 \cdot \nabla_\xi \phi_{k-2} + \mathbf{c}_{k-2} \cdot \Theta \\ &- \phi_{k-2,yy}) + 2U_{\eta\eta}\xi \cdot \Theta_y \phi_{k-2,y}] + Q_k + R_k. \end{aligned} \tag{2.13}$$

Summarizing the above we have

Lemma 2.2. *Assume that w_i, $0 \le i \le k-2$ are smooth functions of (η, y) and that $|w_i| \le Ce^{-c|\eta|}$ with similar estimates holding for partial derivatives. Assume further that ϕ_i, $0 \le i \le k-3$ are smooth functions of y.*

In the minimum axis case we also assume that the functions mentioned above, as well as $\mathbf{c}_i$, $0 \le i \le k-3$ are $C^{1,\alpha}$, for some $\alpha > 0$, functions of ξ and in the short and long axis case we assume that those functions are polynomials in ξ.

Then Q_k, R_k are smooth functions of (η, y) and $C^{1,\alpha}$ functions of ξ in the short axis case and polynomials in ξ in the short and long axis case. Moreover the following estimate holds:

$$|Q_k|, |R_k| \le Ce^{-c|\eta|}.$$

From now on we will assume that all the functions involved in the asymptotic expansion depend on the parameter ξ the way it was described above.

Taking into account the solvability condition for the second order expansion we then obtain the following solvability condition for (2.13):

$$\mathbf{c}_0 \cdot \nabla_\xi \phi_{k-2} + \mathbf{c}_{k-2} \cdot \Theta - \phi_{k-2,yy} = \frac{\int_{-\infty}^{\infty} (Q_k + R_k)\, d\eta}{\int_{-\infty}^{\infty} U_\eta^2\, d\eta}. \tag{2.14}$$

In order to solve (2.14) and determine w_k^{in} uniquely we need to solve for $(\mathbf{c}_{k-2}, \phi_{k-2})$ first. This requires finding boundary conditions for ϕ_{k-2} and we will do it in the next section.

2.4. Boundary layer expansion

We will construct the boundary layer expansion near γ_0. The procedure is the same near γ_1.

In order to define the boundary layer expansion we first have to extend functions ϕ_i for $y < 0$. Denoting the extension by the same symbol we set

$$\phi_i(y, \xi) = \begin{cases} \phi_i(y, \xi), & 1 > y > 0, \\ \sum_{k=0}^{n+1} \phi_i^{(k)}(0, \xi) y^k, & y \leq 0. \end{cases}$$

In a similar manner we can extend ϕ_i, for $y > 1$. Observe that for each $i = 0, \ldots, n$ this extension is a C^{n+1} function in y.

Since $\sigma = \varepsilon^{1/2}$, we can assume that $B(0, \sigma) \cap \partial\Omega$ can be represented as a graph of function $y = h_0(x)$; likewise, in a σ-neighborhood of $(1, 0)$, $\partial\Omega$ can be represented as a graph of $y = h_1(x)$. We clearly have $h_0'(0) = 0 = h_1'(0)$ and $\kappa_{\partial\Omega}(\gamma_0) = h_0''(0)$, $\kappa_{\partial\Omega}(\gamma_1) = -h_1''(0)$.

As calculations near γ_0, γ_1 are very similar we will only consider the boundary layer expansion near γ_0. For brevity we set $h = h_0$.

In $B(0, \sigma) \cap \partial\Omega$ we can change variables

$$\eta = \frac{x - \varepsilon\phi}{\varepsilon}, \quad \mu = \frac{y - h(x)}{\varepsilon},$$

where $\phi = \sum \varepsilon^i \phi_i$ and ϕ_i's are the extended functions defined above. The change of variables we introduced here is not the standard change of variables used in determining boundary layer expansions and sometimes it is referred to as corner expansion.

Lemma 2.3. *Assume that $\phi = \sum_{i\geq 0}^n \varepsilon^i \phi_i$ and ϕ_i are $n + 1$ times differentiable as functions of y. Then for each j, $1 \leq j \leq n$ we have*

$$\left|x - \varepsilon(\eta + \phi_0(0)) - \sum_{i=2}^j \varepsilon^i X_i(\eta, \mu)\right| \leq C_j \varepsilon^{j+1}(1 + |\eta| + \mu)^{j+1},$$
$$\left|y - \varepsilon\mu - \sum_{i=2}^j \varepsilon^i Y_i(\eta, \mu)\right| \leq C_j \varepsilon^{j+1}(1 + |\eta| + \mu)^{j+1},$$

where X_i, Y_i are polynomials in (η, μ) with coefficients depending on $\phi_i(y)$, $i = 0, \ldots, j-1$, $h(x)$ and their derivatives up to order j at $y = 0$, $x = 0$, respectively. Constant C_j depends on L^∞ norms of derivatives of $\phi_i(y)$ $i = 0, \ldots, j$, and $h(x)$ up to order $j + 1$.

We omit a straightforward proof of this lemma. Observe that with the help of Lemma 2.3 we can express functions of (x, y) as functions of (η, μ).

For each $M > 0$ we define a C^∞ cut-off function

$$\tau^M(\mu) = \begin{cases} 1, & \mu \geq 2M, \\ 0 < \tau^M < 1, & M < \mu < 2M, \\ 0, & \mu \leq M. \end{cases}$$

Notice that for each sufficiently large M there exists a constant C_M such that $\mu > M$ implies $y > C_M\varepsilon > 0$, provided that $|x| = \sigma \leq \varepsilon^{1/2}$.

Remark 2.1. In order to find the boundary layer expansion near γ_1 we need to set $\mu = \frac{h_1(x)-y}{\varepsilon}$ and define a cut-off function, analogous to τ^M, appropriately. Below, with a slight abuse of notation, we will use symbols μ and τ^M for those extensions.

We look for the solution to (2.1) near γ_0 in the form

$$w^{bd} = \tau^M w^{in} + (1-\tau^M)U + \varepsilon v^{bd},$$

where w^{in} is the inner expansion contribution from the previous section and v^{bd} is the boundary layer contribution which is still to be determined.

Using (2.1) and substituting we obtain that near γ_0 function $v^{bd} = v^{bd}(\eta, \mu, \xi)$ should satisfy

$$\begin{aligned} \varepsilon^2 \Delta v^{bd} + f'(U)v^{bd} = {} & \mathbf{c}\cdot\nabla_\xi v^{bd} + \varepsilon^{-1}(1-\tau^M)\mathbf{c}\cdot\nabla_\xi U + \varepsilon^{-1}N^{bd} \\ & - 2\varepsilon\nabla\tau^M\cdot\nabla(w^{in}-U) - \varepsilon(w^{in}-U)\Delta\tau^M \\ & + \varepsilon^{-1}g_n^{bd}, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} N^{bd} = {} & \tfrac{1}{2}f''(U)\{\tau^M(w^{in}-U)^2 - [\tau^M(w^{in}-U)+\varepsilon v^{bd}]^2\} \\ & + \tfrac{1}{6}f'''(U)\{\tau^M(w^{in}-U)^3 - [\tau^M(w^{in}-U)+\varepsilon v^{bd}]^3\}, \end{aligned}$$

and $g_n^{bd} = O(\varepsilon^{n+1})$. In addition we also require boundary conditions, namely

$$\partial_n v^{bd} = -\varepsilon^{-1}\big(\partial_n U - h_n^{bd}\big), \qquad \text{near } \gamma_0, \tag{2.16}$$

where $h_n^{bd} = O(\varepsilon^n)$ is a function to be determined.

We will now express (2.15) and (2.16) in terms of the stretched variables (η, μ). Calculating directly we obtain that (2.15) takes form

$$v^{bd}_{\eta\eta} + v^{bd}_{\mu\mu} + f'(U)v^{bd} = p + q + \varepsilon^{-1}g_n^{bd}, \tag{2.17}$$

where

$$\begin{aligned} p(v^{bd}, \phi, \mathbf{c}) = {} & -v^{bd}_\eta\big(\mathbf{c}\cdot\nabla_\xi\phi - \varepsilon^2\phi_{yy}\big) - \varepsilon^2\phi_y^2 v^{bd}_{\eta\eta} - (h')^2 v^{bd}_{\mu\mu} \\ & + 2(h'+\varepsilon\phi_y)v^{bd}_{\eta\mu} + \varepsilon h'' v^{bd}_\mu - \varepsilon^2\phi_{yy}v^{bd}_\eta + \mathbf{c}\cdot v^{bd}_\xi \\ & + \varepsilon^{-1}N_1^{bd}(v^{bd}), \\ N_1^{bd}(v^{bd}) \quad = {} & \tfrac{\varepsilon^2}{2}f''(U)(v^{bd})^2 + \tfrac{\varepsilon^3}{6}f'''(U)(v^{bd})^3, \end{aligned}$$

and

$$\begin{aligned} q(U, w^{in}, \tau^M, \phi, \mathbf{c}) = {} & -\varepsilon^{-1}(1-\tau^M)U_\eta\mathbf{c}\cdot\nabla_\xi\phi \\ & + 2\varepsilon^{-1}\tau^M_\mu\big[(h'+\phi_y)(w^{in}-U)_\eta - w^{in}_y\big] \\ & - \varepsilon^{-1}(w^{in}-U)\big\{[(h')^2+1]\tau^M_{\mu\mu} - \varepsilon h''\tau^M_\mu\big\} \\ & + \varepsilon^{-1}\big(N^{bd} - N_1^{bd}\big). \end{aligned}$$

Lemma 2.4. *Assume that* $w^{in} = \sum_{i=0}^{n} \varepsilon^i w_i^{in}$ *is a smooth function of* η, y *and* $v^{in} = \sum_{i=0}^{n} \varepsilon^i v_i^{in}$ *is a smooth function of* (η, μ). *Moreover suppose that* ϕ *satisfies the assumptions of Lemma* 2.3. *Then* p *and* q *have expansions*

$$p = \sum_{i=i}^{n} \varepsilon^i p_i + \varepsilon^{n+1} \tilde{p}_{n+1}, \qquad q = \sum_{i=1}^{n} \varepsilon^i q_i + \varepsilon^{n+1} \tilde{q}_{n+1}$$

where p_i*'s are polynomials in terms of* η, μ *and* $v_0^{bd}, \dots, v_{i-1}^{bd}$ *and their derivatives, and* q_i*'s are polynomials in terms of* η, μ, $w_2^{in}, \dots, w_{i-1}^{in}$, $v_0^{bd}, \dots, v_{i-1}^{bd}$ *and their derivatives. The coefficients of* p_i *and* q_i *depend on* $\phi_0, \dots, \phi_{i-1}$ *and the derivatives of those functions up to order* n *at* $y = 0$. *In addition if* $v_j^{bd}, w_j^{in}, \phi_j, c_j$, $j = 0, \dots, i-1$ *are* $C^{1,\alpha}$ *functions or polynomials in* ξ *so is* p_i, q_i.

Furthermore if $|w_j^{in}| \le Ce^{-c|\eta|}$ *and* $|v_j^{bd}| \le C \min\{e^{-c|\eta|}, e^{-c\mu}\}$, $0 \le j \le i-1$, *with similar estimates for the derivatives then*

$$|p_i|, |q_i| \le C \min\{e^{-c|\eta|}, e^{-c\mu}\}.$$

Analogous estimates hold for the error terms $\tilde{p}_{n+1}$, $\tilde{q}_{n+1}$.

The proof of this lemma involves expressing functions of (x, y) in terms of (η, μ) by using Lemma 2.3, using Taylor polynomials to expand expressions for p and q in terms of powers of ε, grouping terms with equal powers of ε and finally using induction. We omit the details.

Since near γ_0 we have $n = (h', -1)/\sqrt{1 + (h')^2}$, we get

$$-v_\mu^{bd} + v_\eta^{bd}(h' + \varepsilon\phi_y) - (h')^2 v_\mu^{bd} = -U_\eta(\varepsilon^{-1}h' + \phi_y) + h_n^{bd}\sqrt{1 + (h')^2}. \quad (2.18)$$

Lemma 2.5. *Assuming that* $v^{bd} = \sum_{i=0}^{n} \varepsilon^i v_i^{bd}$ *we have at* $\mu = 0$

$$v_{i,\mu}^{bd} = U_\eta[(\eta + \phi_i(0))h''(0) + \phi_{i,y}(0)] + r_i, i = 0, \dots, n,$$

where functions r_i *are polynomials with respect to* η, v_j^{bd}, $0 \le j \le i-1$ *and their derivatives with coefficients depending on* $\phi_j(y)$, $0 \le j \le i-1$, *as well as their derivatives at* $y = 0$. *Moreover if* $|v_j^{bd}(\eta, 0)| \le Ce^{-c|\eta|}$, *with similar estimates for the derivatives,* $0 \le j \le i-1$, *then* $|r_i| \le Ce^{-c|\eta|}$.

<u>0th order boundary layer expansion.</u> For v_0^{bd} we obtain

$$\begin{aligned} &v_{0,\eta\eta}^{bd} + v_{0,\mu\mu}^{bd} + f'(U)v_0^{bd} = 0, && -\infty < \eta < \infty, 0 < \mu < \infty, \\ &v_{0,\mu}^{bd} = U_\eta[h''(0)(\eta + \xi \cdot \Theta(0)) + \xi \cdot \Theta_y(0)], && -\infty < \eta < \infty, \mu = 0. \end{aligned} \quad (2.19)$$

Using Proposition 2.2 we derive the following compatibility condition

$$h''(0)\xi \cdot \Theta(0) + \xi \cdot \Theta_y(0) = 0. \quad (2.20)$$

Taking into account $h''(0) = \kappa(\gamma_0)$ we obtain the first boundary condition in (1.5). Clearly similar considerations near γ_1 lead to the second boundary condition in (1.5).

Since the right hand side of the boundary condition in (2.19) decays exponentially in η therefore by Proposition 2.2 we obtain that the unique solution to (2.19) satisfies

$$\left|v_0^{bd}\right| \le C \min\{e^{-c|\eta|}, e^{-c\mu}\}, \tag{2.21}$$

with similar estimates for the derivatives.

<u>$k > 0$ order boundary expansion.</u> For v_k^{bd}, $k > 0$ we get

$$\begin{aligned} v_{k,\eta\eta}^{bd} + v_{k,\mu\mu}^{bd} + f'(U)v_k^{bd} &= p_k + q_k, \quad -\infty < \eta < \infty, 0 < \mu < \infty, \\ v_{k,\mu}^{bd} &= U_\eta[h''(0)(\eta + \phi_k(0)) + \phi_{k,y}(0)] + r_k, \quad -\infty < \eta < \infty, \mu = 0. \end{aligned} \tag{2.22}$$

Thanks to the solvability condition in Proposition 2.2 this leads to:

$$h''(0)\phi_k(0) + \phi_{k,y}(0) = \eta_0 \left[\int_0^\infty \int_\infty^\infty (p_k + q_k) U_\eta \, d\eta d\mu - \int_{-\infty}^\infty r_k U_\eta \, d\eta \right], \tag{2.23}$$

where $\eta_0 = (\int_{-\infty}^\infty U_\eta^2 \, d\eta)^{-1}$. We observe here that solvability condition (2.23) has form

$$h''(0)\phi_k(0) + \phi_{k,y}(0) = B_{k,0}(\xi),$$

where $B_{k,0}$ is a $C^{1,\alpha}$ function (in the minimum axis case) or a polynomial with respect to ξ (in the short and long axis case) provided that all the functions involved in definition of p_k, q_k and r_k are $C^{1,\alpha}$ functions of ξ or polynomials in ξ respectively.

Clearly a similar solvability condition can be derived at $y = 1$ so that we get

$$\begin{aligned} h_0''(0)\phi_k + \phi_{k,y} &= B_{k,0}(\xi), \quad y = 0, \\ h_1''(0)\phi_k + \phi_{k,y} &= B_{k,1}(\xi), \quad y = 1. \end{aligned} \tag{2.24}$$

2.5. *Approximate solution in Ω*

Summarizing the results of the two previous subsections we notice that:

(1) In order to determine kth term w_k^{bd} in the asymptotic expansion of the inner solution w we need to know ϕ_{k-2}, $\mathbf{c}_{k-2}$ first.
(2) Functions ϕ_{k-2}, $\mathbf{c}_{k-2}$ are obtained from solvability conditions involving w_i^{in}, v_i^{bd}, $0 \le i \le k-3$, and lower order terms of ϕ, $\mathbf{c}$.
(3) These solvability conditions allow for solving not only for w_k^{in} but also for v_{k-2}^{bd}.

These facts are the basis for setting up an induction proof of the next lemma.

Lemma 2.6. *Let $\sigma = \varepsilon^{1/2}$ and τ^M be the function defined above. Let C_M be a constant such that $\tau^M(\mu) = 0$ implies either $C_M\varepsilon > y$ or $y > 1 - C_M\varepsilon$.*

For each $n > 2$ there exist $\mathbf{c}(\xi) = \varepsilon^2 \sum_{i=0}^{n-2} \varepsilon^i \mathbf{c}_i(\xi)$, $|\xi| < \delta$ such that:

(i) For each $0 \le i \le n-2$, $\mathbf{c}_i(\xi)$ is a $C^{1,\alpha}$ function of ξ such that $|\mathbf{c}_i(\xi)| \le C|\xi|$ in the minimum axis case and a polynomial in ξ such that $\mathbf{c}_i(0) = 0$ in the short and long axis case.

(ii) Function $w^{in} = \sum_{i=0}^{n} \varepsilon^i w_i^{in}$ solves the inner expansion problem, i.e. for each $|\xi| < \delta$,

$$\Delta w^{in} + f(w^{in}) = \mathbf{c} \cdot \nabla_\xi w^{in} + g_n^{in} \text{ in } (-\sigma, \sigma) \times (C_M\varepsilon, 1 - C_M\varepsilon),$$

with $w^{in} = U(\eta) + \sum_{i=2}^{n} \varepsilon^i w_i^{in}(\eta, y, \xi)$, $g_n^{bd} = O(\varepsilon^{n+1})$. Here

$$\eta = \frac{x - \varepsilon\phi}{\varepsilon}, \quad \phi = \phi(y, \xi) = \xi \cdot \Theta + \sum_{i=1}^{n-2} \varepsilon^i \phi_i,$$

and function ϕ is a smooth function in y and $C^{1,\alpha}$, for some $\alpha > 0$, function in ξ in the minimum axis case, and a polynomial in ξ with smooth coefficients depending on y in the short and long axis case. Moreover the following estimates hold

$$\begin{aligned} |w_i^{in}| &\le Ce^{-c|\eta|}, & i = 2, \ldots, n, \\ |g_n^{in}| &\le C\varepsilon^{n+1} e^{-c|\eta|}, \end{aligned} \tag{2.25}$$

with similar statements for the derivatives of w_i^{in}.

(iii) There exists a positive constant τ, independent on ε such that function $w^{bd} = \tau^M w^{in} + (1 - \tau^M)U + \varepsilon v^{bd}$, solves the boundary layer problem up to order $n - 1$ i.e.

$$\begin{aligned} \Delta w^{bd} + f(w^{bd}) &= \mathbf{c} \cdot \nabla_\xi w^{bd} + g_n^{bd}, & \text{in } \Omega_\sigma \cap \{dist\,(\cdot, \partial\Omega) < 2\tau\}, \\ \partial_n w^{bd} &= h_n^{bd}, & \text{on } \partial\Omega \cap \{|x| < \sigma\}, \end{aligned}$$

where $v^{bd} = \sum_{i=0}^{n-2} \varepsilon^i v_i^{bd}$.
In addition

$$\begin{aligned} \left|v_i^{bd}\right| &\le C \min\{e^{-c|\eta|}, e^{-c\mu}\}, & i = 0, \ldots, n-2, \\ \left|g_n^{bd}\right| &\le C\varepsilon^n \min\{e^{-c|\eta|}, e^{-c\mu}\}, \\ \left|h_n^{bd}\right| &\le C\varepsilon^{n-1} e^{-c|\eta|}, \end{aligned} \tag{2.26}$$

with similar estimates for the derivatives.

Proof. We prove this lemma by induction.

Let $n = 1$. With the choice of $\phi_0 = \xi \cdot \Theta$ and the choice of $\mathbf{c}_0$ in (2.4) we can satisfy the solvability conditions and solve for w_2 and v_0 as it is described in previous sections. The rest of the statements of the lemma are easily verified in this case by using Proposition 2.1, Proposition 2.2 and Lemma 2.1.

Assume now that for some $k > 1$ the lemma is true for all $n < k$. We will show that one can find w_k, v_{k-2}, ϕ_{k-2}, $\mathbf{c}_{k-2}$.

From Lemma 2.2 and (2.14) it follows that in order to solve for ϕ_{k-2}, $\mathbf{c}_{k-2}$ we need to solve

$$\mathbf{c}_0 \cdot \nabla_\xi \phi_{k-2} + \mathbf{c}_{k-2} \cdot \Theta - \phi_{k-2,yy} = A_k(y, \xi) \tag{2.27}$$

where A_k is a function satisfying the assumptions of Corollary 2.1 in the short axis case and Lemma 2.1 in the long axis case.

From Lemma 2.4, Lemma 2.5 and (2.24) we get the boundary conditions for ϕ_{k-2}:

$$\begin{aligned} h_0''(0)\phi_{k-2} + \phi_{k-2,y} &= B_{k-2,0}(\xi), \quad y = 0, \\ h_1''(0)\phi_{k-2} + \phi_{k-2,y} &= B_{k-2,1}(\xi), \quad y = 1, \end{aligned} \tag{2.28}$$

where $B_{k-2,j}(\xi)$, $j = 0, 1$ satisfy the assumption of Corollary 2.1 in the minimum axis case and Lemma 2.1 in the short and long axis case. Consequently we can solve uniquely (2.27) and (2.28) for ϕ_{k-2}, $\mathbf{c}_{k-2}$ and those functions satisfy the assertions of the lemma.

Now we solve for w_k^{in} and v_{k-2}^{bd}. We observe that from Proposition 2.1 and Proposition 2.2 we get exponential decay estimates for w_k^{in} and v_{k-2}^{bd}.

Finally we observe that the reminder term $g_k^{in} = Q_{k+1} + R_{k+1}$ and thus (2.25) is satisfied thanks to Lemma 2.2.

Analogous statements are also easily proven for g_k^{bd} and h_k^{bd}. Notice that $g_k^{bd} = \tau^M g_k^{in} + \tilde{g}_k^{bd}$. It suffices to observe now that functions h_k^{bd}, $\tilde{g}_k^{bd}$ result from replacing h_0, h_1 and ϕ by polynomials in (η, μ) in (2.15) and (2.18). Since functions h_0, h_1, ϕ are smooth, replacing them with Taylor expansions of order $k-1$ introduces error terms of the form $\varepsilon^k O(|\eta|^k + \mu^k)$. In addition those error terms are multiplied by polynomials in terms of U, w_i^{in}, v_{i-2}^{bd}, $i = 2, \ldots, k$ and their derivatives and since those functions decay exponentially in terms of $|\eta|$ and μ we obtain estimates (2.26).

The proof of the lemma is complete. □

We need to define smooth cut-off functions in order to "connect" both expansions. Let τ be the constant in the statement of Lemma 2.6. Let χ_1, χ_2 be $C^\infty(\Omega)$ cut-off functions such that

$$\chi_1 = \begin{cases} 1 & \text{if } |x| < \sigma/2, \text{dist}\,(\cdot, \partial\Omega) > 2\tau/3, \\ 0 \le \chi_1 \le 1 & \text{if } \sigma/2 \le |x| < \sigma, \tau/3 < \text{dist}\,(\cdot, \partial\Omega) < 2\tau/3, \\ 0 & \text{otherwise}, \end{cases}$$

$$\chi_2 = \begin{cases} 1 & \text{if } |x| < \sigma/2, \text{dist}\,(\cdot, \partial\Omega) < \tau/3, \\ 0 \le \chi_2 \le 1 & \text{if } \sigma/2 \le |x| < \sigma, \tau/3 \le \text{dist}\,(\cdot, \partial\Omega) < 2\tau/3, \\ 0 & \text{otherwise}. \end{cases}$$

In addition we assume that $\chi_1(x, y) + \chi_2(x, y) \equiv 1$ for $|x| \le \sigma/2$ and all y.

We define now an approximate solution to (2.1) u by

$$u = \chi_1 w^{in} + \chi_2 w^{bd} + (1 - \chi_1 - \chi_2) U(|x|/\varepsilon).$$

Using Lemma 2.6 we can prove the following:

Proposition 2.3. *Let w^{in} be the solution of the inner expansion problem up to order n and w^{bd} be the solution of the boundary layer problem up to order n. Then function u is the solution of* (2.1) *up to order n with error terms satisfying*

$$|g^\varepsilon| \le C\varepsilon^n e^{-c|x|/\varepsilon}, \quad |h^\varepsilon| \le C\varepsilon^{n-1} e^{-c|x|/\varepsilon}.$$

Moreover $\mathbf{c} = \varepsilon^2(\mathbf{c}_0 + O(\varepsilon))$.

The proof of this proposition is fairly standard and we shall omit it. We only notice that the exponential estimate for v^{bd} in (2.26) plays an important role in matching w^{in} and w^{bd} in the intermediate region where $y \in (\tau/3, 2\tau/3) \cup (1-2\tau/3, 1-\tau/3)$.

3. Eigenvalue analysis

In this section we shall consider the following eigenvalue problem:

$$\begin{aligned} &\varepsilon^2 \Delta V + f'(u)V = -\Lambda V, && \text{in } \Omega, \\ &\partial_n V = 0, && \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where u is a solution to (2.1). We notice that the eigenvalues and eigenvectors of (3.1) are, strictly speaking, different than the eigenvalues and eigenvectors which are the subject of Theorem 1.2. Nevertheless, as we shall see later, the two are close to each other provided that u is the approximation of high enough order.

The analysis of (3.1) follows the lines developed in [1].

In the sequel we will denote

$$\mathcal{L}V = \varepsilon^2 \Delta V + f'(u)V.$$

We will also use Λ_k, V_k, $k = 1, \ldots$ to denote the eigenvalues and eigenvectors of (3.1).

Let $\psi_0 = \varepsilon u_x$. Let $\Omega_\nu = \{\text{dist}((x, y), \gamma) < \nu\} \cap \Omega$. We will take $\nu = \nu(\varepsilon) = \varepsilon(\ln \varepsilon)^2$. We will not emphasize the dependence of ν on ε unless necessary.

Lemma 3.1. *The following formulas hold:*

$$\begin{aligned} &\mathcal{L}\psi_0 = U_{\eta\eta}\mathbf{c} \cdot \Theta + R^\varepsilon && \text{in } \Omega_\nu, \\ &\partial_n \psi_0 = -h_0''(0)\psi_0 + r_0^\varepsilon && \text{on } \partial\Omega_\nu \text{ near } \gamma_0, \\ &\partial_n \psi_0 = h_1''(0)\psi_0 + r_1^\varepsilon && \text{on } \partial\Omega_\nu \text{ near } \gamma_1, \end{aligned}$$

where $|R^\varepsilon| \le C\varepsilon^3 e^{-c|\eta|}$, $|r_i^\varepsilon| \le C\varepsilon e^{-c|\eta|}$, $i = 0, 1$.

Proof. We can write $u = U(\eta) + \varepsilon u_1$, where $u_1 = u_1(\eta, \mu, y)$ and $|u_1| \le C\varepsilon e^{-c|\eta|}$, with similar estimates for the derivatives. Differentiating (2.1) with respect to x and multiplying by ε we get

$$\mathcal{L}\psi_0 = \varepsilon \mathbf{c} \cdot \nabla_\xi u_x + \varepsilon g_x^\varepsilon = U_{\eta\eta}\mathbf{c} \cdot \Theta + R^\varepsilon,$$

as claimed.

We now need to compute $\partial_n u_x$ on $\partial\Omega$. We have near γ_0 with $h_0 = h$,

$$\psi_0 = u_\eta - u_\mu h'(x) = U_\eta + \varepsilon v^{bd}_{0,\eta} - \varepsilon v^{bd}_{0,\mu} h'(x) + O(\varepsilon^2)e^{-c|\eta|}.$$

Using the boundary condition for v_0^{bd} we get

$$\begin{aligned}\partial_n \psi_0|_{\gamma_0} &= U_{\eta\eta}[h_0''(0)(\eta+\phi)+\phi_y] - v^{bd}_{0,\mu\eta} + O(\varepsilon)e^{-c|\eta|}\\ &= U_{\eta\eta}[h_0''(0)(\eta+\xi\cdot\Theta)+\xi\cdot\Theta_y] - v^{bd}_{0,\mu\eta} + O(\varepsilon)e^{-c|\eta|}\\ &= -U_\eta h_0''(0) + O(\varepsilon)e^{-c|\eta|}.\end{aligned}$$

Similarly near γ_1 we get (since $n = (-h_1', 1)/\sqrt{1+(h_1')^2}$)

$$\begin{aligned}\partial_n \psi_0|_{\gamma_1} &= U_\eta h_1''(0) + O(\varepsilon)e^{-c|\eta|}\\ &= \psi_0 h_1''(0) + O(\varepsilon)e^{-c|\eta|}.\end{aligned}$$

This completes the proof. □

We introduce new variables in Ω_ν, $(x, y) \mapsto (x, z)$ by

$$(x, z) = \left(x, \frac{y - h_0(x)}{h_1(x) - h_0(x)}\right).$$

Notice that $z = z(x, y)$; conversely we can write $y = y(x, z)$. For any function $\Phi = \Phi(x, y)$ on Ω_ν we have

$$\int_{\Omega_\nu} \Phi(x, y)\, dxdy = \int_{-\nu}^{\nu}\int_0^1 \Phi(x, z)[1 + j_1(x, z)]\, dxdz, \qquad |j_1(x, z)| \le C\nu^2. \tag{3.2}$$

We also have for any function $\Psi(x, z)$, $z = z(x, y)$,

$$\begin{aligned}\tfrac{\partial\Psi}{\partial y} &= \Psi_z[1 + j_2(x, z)], \qquad |j_2(x, z)| \le C\nu^2,\\ \tfrac{\partial\Psi}{\partial x} &= \Psi_x + \Psi_z j_3(x, z), \qquad |j_3(x, z)| \le C\nu.\end{aligned} \tag{3.3}$$

We define an infinite subspace of $L^2(\Omega_\nu)$ by

$$\mathcal{X} = \left\{\psi \in L^2(\Omega_\nu) \mid \psi(x, y) = \psi_0(x, y)\theta(z), z = z(x, y)\right\}.$$

We will first analyze the bilinear form

$$\mathcal{B}_\nu(\psi_1, \psi_2) = \int_{\Omega_\nu} \left[\varepsilon^2 \nabla\psi_1 \cdot \nabla\psi_2 - f'(u)\psi_1\psi_2\right],$$

for $\psi_i \in \mathcal{X}$.

Lemma 3.2. *Let $\psi^* = \psi_0\theta^*$, $\psi^{**} = \psi_0\theta^{**} \in \mathcal{X}$. The following formula holds:*

$$\mathcal{B}_\nu(\psi^*, \psi^{**}) = \varepsilon^2\alpha_0\Big[h_1''(0)\theta^*(1)\theta^{**}(1) - h_0''(0)\theta^*(0)\theta^{**}(0) + \int_0^1 \theta_z^*\theta_z^{**} + O(\varepsilon)\int_0^1 (|\theta^*|^2 + |\theta^{**}|^2 + |\theta_z^*|^2 + |\theta_z^{**}|^2)\Big],$$

where $\alpha_0 = \int_{\Omega_\nu} \psi_0^2$.

Proof. We have

$$\mathcal{B}_\nu(\psi^*, \psi^{**}) = \varepsilon^2 \int_{\partial\Omega_\nu} \psi^*\partial_n\psi^{**}\, dS - \int_{\Omega_\nu} \psi^*\mathcal{L}\psi^{**} = I + II.$$

We first compute the boundary integral I. Observe that for each $\psi = \psi_0\theta \in \mathcal{X}$ we have by Lemma 3.1 and (3.3)

$$\partial_n\psi = \big[-h_0''(0)\psi_0 + r_0^\varepsilon\big]\theta(0) + \psi_0\partial_n\theta, \qquad \text{near } \gamma_0,$$
$$\partial_n\psi = \big[h_0''(1)\psi_0 + r_1^\varepsilon\big]\theta(1) + \psi_0\partial_n\theta, \qquad \text{near } \gamma_1,$$

hence

$$I = \varepsilon^2\alpha_0\big[\theta^*(1)\theta^{**}(1)h_1''(1) - \theta^*(0)\theta^{**}(0)h_0''(0)\big][1 + O(\varepsilon)] + \varepsilon^2 \int_{\partial\Omega_\nu} \psi_0^2\theta^*\partial_n\theta^{**}\, dS.$$

We also have by Lemma 3.1 and integration by parts

$$\begin{aligned} II &= -\int_{\Omega_\nu} \theta^*\theta^{**}\psi_0\mathcal{L}\psi_0 - \varepsilon^2\int_{\Omega_\nu} [2\nabla\psi_0 \cdot \nabla\theta^{**} + \psi_0\Delta\theta^{**}]\psi_0\theta^* \\ &= -\int_0^1\int_{-\nu}^{\nu} \theta^*\theta^{**}\big[\psi_0 U_{\eta\eta}\mathbf{c} \cdot \nabla_\xi(\xi \cdot \Theta) + R^\varepsilon\psi_0\big][1 + O(\nu^2)]\, dxdz \\ &\quad +\varepsilon^2\int_{\Omega_\nu} \psi_0^2\nabla\theta^* \cdot \nabla\theta^{**} - \varepsilon^2\int_{\partial\Omega_\nu} \psi_0^2\theta^*\partial_n\theta^{**}\, dS \\ &= -II' + II'' - \varepsilon^2\int_{\partial\Omega_\nu} \psi_0^2\theta^*\partial_n\theta^{**}\, dS. \end{aligned}$$

Since $\int_{-\nu}^{\nu} \psi_0 U_{\eta\eta}\, dx = O(e^{-c\nu/\varepsilon})$,

$$|II'| \le C\varepsilon^3\alpha_0 \int_0^1 |\theta^*\theta^{**}|.$$

By a straightforward calculation we get

$$II'' = \varepsilon^2\alpha_0\left(\int_0^1 \theta_z^*\theta_z^{**}\, dz\right)[1 + O(\nu^2)].$$

Combining expressions for I, II, II', II'' we complete the proof of the lemma. □

Corollary 3.1. *The following estimate holds for any* $\psi = \psi_0\theta \in \mathcal{X}$*:*

$$\varepsilon^2 \int_0^1 \theta_z^2 \le C\varepsilon^{-1}\mathcal{B}_\nu(\psi, \psi) + C\varepsilon \int_{\Omega_\nu} \psi^2.$$

This follows directly from Lemma 3.2 if we set $\theta^* = \theta^{**} = \theta$ and use

$$\int_0^1 \theta^2 \le C\varepsilon^{-1} \int_{\Omega_\nu} \psi^2.$$

Let $\mathcal{X}_{k-1}$, $k = 1, \dots$ denote the collection of $k-1$ dimensional subspaces of $\mathcal{X}$. We define

$$\mu_k = \max_{\mathcal{S}\in\mathcal{X}_{k-1}} \min_{\substack{\psi\in\mathcal{S}^\perp \\ \|\psi\|_{L^2(\Omega_\nu)}=1}} \mathcal{B}_\nu(\psi, \psi).$$

Let

$$J(\theta, \theta) = h_1''(0)\theta^2(1) - h_0''(0)\theta^2(0) + \int_0^1 \theta_z^2\, dz.$$

From the Lemma 3.2 above there exists a positive constant C_0 such that

$$\mu_k \le \varepsilon^2\alpha_0 \max_{\mathcal{S}\in\mathcal{X}_{k-1}} \min_{\substack{\psi\in\mathcal{S}^\perp \\ \|\psi\|_{L^2(\Omega_\nu)}=1}} \left[J(\theta, \theta) + C_0\varepsilon\|\theta\|^2_{H^1(0,1)}\right],$$

$$\mu_k \ge \varepsilon^2\alpha_0 \max_{\mathcal{S}\in\mathcal{X}_{k-1}} \min_{\substack{\psi\in\mathcal{S}^\perp \\ \|\psi\|_{L^2(\Omega_\nu)}=1}} \left[J(\theta, \theta) - C_0\varepsilon\|\theta\|^2_{H^1(0,1)}\right].$$

Observe that for any functions $\psi_i = \psi_0\vartheta_i \in \mathcal{X}$, $i = 1, 2$, we have

$$\begin{aligned}\int_{\Omega_\nu} \psi_1\psi_2 &= \int_0^1 \vartheta_1\vartheta_2 \left\{\int_{-\nu}^{\nu} \psi_0^2[1 + j_1(x, z)]\, dx\right\} dz \\ &= \alpha_0 \int_0^1 \vartheta_1\vartheta_2[1 + k^\varepsilon(z)]\, dz,\end{aligned}$$

where $k^\varepsilon = O(\varepsilon)$. It follows that the eigenvalue problem associated to $J(\theta, \theta) - C_0\varepsilon\|\theta\|^2_{H^1(0,1)}$ has form

$$\begin{aligned}-\vartheta_{zz}(1 - C_0\varepsilon) + C_0\varepsilon\vartheta &= \mu^*\vartheta(1 + k^\varepsilon), \\ h_0''(0)\vartheta(0) + \vartheta_z(0) &= 0, \\ h_1''(0)\vartheta(1) + \vartheta_z(1) &= 0.\end{aligned} \tag{3.4}$$

Clearly, a similar problem can be derived for $J(\theta, \theta) + C_0\varepsilon\|\theta\|^2_{H^1(0,1)}$. Since (3.4) is just a regular perturbation of (1.5), one can prove:

Lemma 3.3. *For each positive integer* K *there exists* $\varepsilon_K > 0$ *such that for each* $\varepsilon \in (0, \varepsilon_K]$ *and each* $k = 1, \dots, K$ *the following estimate holds*

$$\mu_k = \varepsilon^2\lambda_k[1 + O(\varepsilon)].$$

We will now analyze $\mathcal{B}_\nu(\psi^\perp, \psi^\perp)$, where $\psi^\perp \in \mathcal{X}^\perp$ and the orthogonal complement is taken in $L^2(\Omega_\nu)$.

We begin with the following lemma:

Lemma 3.4. *Let $\Psi_\varepsilon \in H^1(-\nu/\varepsilon, \nu/\varepsilon)$ be functions such that*

$$\int_{-\nu/\varepsilon}^{\nu/\varepsilon} U_\eta \Psi_\varepsilon = o(1), \qquad \text{as } \varepsilon \to 0.$$

There exists a constant $\delta^* > 0$ such that

$$\int_{-\nu/\varepsilon}^{\nu/\varepsilon} \left[|\Psi_{\varepsilon,\eta}|^2 - f'(U)\Psi_\varepsilon^2\right] \geq \delta^* \int_{-\nu/\varepsilon}^{\nu/\varepsilon} \Psi_\varepsilon^2.$$

Proof. Since this is a well known statement we will only indicate the main steps in the proof. We first notice that U_η is the only element in the kernel of the following linear operator:

$$LV = V_{\eta\eta} + f'(U)V, \qquad -\infty < \eta < \infty.$$

This follows from the fact that $U_\eta > 0$, hence U_η is the principal eigenfunction, which in addition must be simple.

Taking the above for granted we argue by contradiction to prove the assertion of the lemma. We assume that there exists a sequence $\{\varepsilon_n\}_{n=1,\ldots}$ such that

$$\int_{-\nu/\varepsilon_n}^{\nu/\varepsilon_n} \left[|\Psi_{\varepsilon_n,\eta}|^2 - f'(U)\Psi_{\varepsilon_n}^2\right] \leq \frac{1}{n} \int_{-\nu/\varepsilon_n}^{\nu/\varepsilon_n} \Psi_{\varepsilon_n}^2. \tag{3.5}$$

Without loss of generality we can further assume that $\|\Psi_{\varepsilon_n}^2\|_{L^2(-\nu/\varepsilon_n,\nu/\varepsilon_n)} = 1$. We then have $\|\Psi_{\varepsilon_n}^2\|_{H^1(-\nu/\varepsilon_n,\nu/\varepsilon_n)} \leq C$ and thus we can pass to the limit obtaining $\Psi \in H^1(-\infty, \infty)$ such that $\int_{-\infty}^{\infty} U_\eta \Psi = 0$ and

$$\int_{-\infty}^{\infty} \left[|\Psi_\eta|^2 - f'(U)\Psi^2\right] \leq 0.$$

One can also prove using (3.5) that there exist constants $M_0, \alpha_0 > 0$ such that for each $M > M_0$ we have

$$\int_{\eta > M} \Psi_{\varepsilon_n}^2 \leq Ce^{-\alpha_0(M-M_0)} \int_{\eta < M_0} \Psi_{\varepsilon_n}^2$$

(we shall not do it here but refer the reader to the proof of Lemma 3.8 where a similar fact is proven). This guarantees that $\|\Psi\|_{L^2(-\infty,\infty)} > 1/2$. We now obtain a contradiction with the fact that U_η is the unique eigenfunction corresponding to 0, the principal eigenvalue of L. □

We can now state a counterpart of Lemma 3.4 in our setting.

Lemma 3.5. *There exists a constant* $\nu_0 > 0$*, independent on* ε *such that*

$$\mathcal{B}_\nu(\psi^\perp, \psi^\perp) > \nu_0 \int_{\Omega_\nu} |\psi^\perp|^2,$$

for $\psi^\perp \in \mathcal{X}^\perp$*. Moreover,* $\nu_0 > 0$ *can be chosen such that*

$$\mathcal{B}_\nu(\psi^\perp, \psi^\perp) > \nu_0 \int_{\Omega_\nu} (\varepsilon^2 |\nabla \psi^\perp|^2 + |\psi^\perp|^2).$$

Proof. We will argue by contradiction. Assume that for each positive integer n there exists ε_n and $\psi_n^\perp \in \mathcal{X}^\perp$, $\|\psi_n^\perp\|^2_{L^2(\Omega_\nu)} = \varepsilon_n$ such that

$$\mathcal{B}_\nu(\psi_n^\perp, \psi_n^\perp) \le \frac{\varepsilon_n}{n}.$$

We then have for each function $\theta = \theta(z)$

$$\int_{\nu/\varepsilon_n}^{\nu/\varepsilon_n} \int_0^1 \psi_n^\perp U_\eta \theta(z)\, dz d\eta = o(1), \qquad \text{as } n \to \infty,$$

hence almost everywhere in (0, 1) we have

$$\int_{\nu/\varepsilon_n}^{\nu/\varepsilon_n} \psi_n^\perp U_\eta\, d\eta = o(1).$$

Likewise we get for all n

$$\int_{-\nu/\varepsilon_n}^{\nu/\varepsilon_n} (\psi_n^\perp)^2 < \infty, \qquad \int_{-\nu/\varepsilon_n}^{\nu/\varepsilon_n} |\psi_{n,\eta}^\perp|^2 < \infty$$

almost everywhere in (0, 1). Using Lemma 3.4 we then have

$$\int_{-\nu/\varepsilon_n}^{\nu/\varepsilon_n} \left[|\psi_{n,\eta}^\perp|^2 - f'(U)(\psi_n^\perp)^2 \right] \ge \delta^* \int_{-\nu/\varepsilon_n}^{\nu/\varepsilon_n} (\psi_n^\perp)^2.$$

Integrating the above formula with respect to z over (0, 1) we get

$$\begin{aligned}\frac{\varepsilon_n}{n} \ge B_\nu(\psi_n^\perp, \psi_n^\perp) &\ge \varepsilon_n \int_0^1 \int_{-\nu/\varepsilon_n}^{\nu/\varepsilon_n} \left[|\psi_{n,\eta}^\perp|^2 - f'(U)(\psi_n^\perp)^2 \right] + o(\varepsilon_n) \\ &\ge \varepsilon_n \delta^* \int_0^1 \int_{-\nu/\varepsilon_n}^{\nu/\varepsilon_n} (\psi_n^\perp)^2 + o(\varepsilon_n) = \delta^* \varepsilon_n + o(\varepsilon_n).\end{aligned}$$

Dividing by ε_n and letting $n \to \infty$ we obtain a contradiction. This proves the first estimate of the lemma. The proof of the second estimate is left to the reader. □

Lemma 3.6. *For any* $\psi \in \mathcal{X} \cap span\{\theta_i\psi_0 \mid i = 1, \ldots, k\}$, $\phi^\perp \in \mathcal{X}^\perp$ *and any* $\kappa \in (0, 1/2)$ *the following estimates hold:*

$$\begin{aligned}\mathcal{B}_\nu(\psi, \phi^\perp) &\le C\varepsilon^\kappa \mathcal{B}_\nu(\psi, \psi) + C\varepsilon^{2+\kappa} \int_{\Omega_\nu} \psi^2 \\ &\quad + \int_{\Omega_\nu} \left(\varepsilon^{2+\kappa}|\nabla\phi^\perp|^2 + \frac{\nu_0}{4}|\phi^\perp|^2\right), \\ \mathcal{B}_\nu(\psi + \phi^\perp, \psi + \phi^\perp) &\ge \mathcal{B}_\nu(\psi, \psi)[1 + O(\varepsilon^{1/4})] + C\varepsilon^{9/4}\|\psi\|^2_{L^2(\Omega_\nu)} \\ &\quad + \frac{\nu_0}{2}\|\phi^\perp\|^2_{H^1_\varepsilon(\Omega_\nu)},\end{aligned}$$

where $\|\phi^\perp\|^2_{H^1_\varepsilon(\Omega_\nu)} = \int_{\Omega_\nu}(\varepsilon^2|\nabla\phi^\perp|^2 + |\phi^\perp|^2)$.

Proof. By a straightforward calculation we have

$$\mathcal{B}_\nu(\psi, \phi^\perp) = \varepsilon^2 \int_{\partial\Omega_\nu} \phi^\perp \partial_n \psi \, dS - \int_{\Omega_\nu} \phi^\perp \mathcal{L}\psi. \tag{3.6}$$

We also get

$$\begin{aligned}\int_{\Omega_\nu} \phi^\perp \Delta\psi &= \int_{\Omega_\nu} [\phi^\perp\theta\Delta\psi_0 + 2\phi^\perp\nabla\theta\cdot\nabla\psi_0 + \phi^\perp\psi_0\Delta\theta] \\ &= \int_{\partial\Omega_\nu} \phi^\perp\psi_0\partial_n\theta \, dS \\ &\quad + \int_{\Omega_\nu} [\phi^\perp\theta\Delta\psi_0 + \phi^\perp\nabla\theta\cdot\nabla\psi_0 - \psi_0\nabla\theta\cdot\nabla\phi^\perp].\end{aligned}$$

We will consider $I = \int_{\Omega_\nu} \psi_0\nabla\theta\cdot\nabla\phi^\perp$. We have, using (3.3)

$$I = \int_{\Omega_\nu} \left\{\psi_0\theta_z\phi^\perp_z z_y^2 + \theta_z\phi^\perp_x O(\nu)\right\} = \int_{\Omega_\nu} \psi_0\theta_z\phi^\perp_z + O(\nu)\int_{\Omega_\nu} |\theta_z||\nabla\phi^\perp|.$$

Since $\phi^\perp \in \mathcal{X}^\perp$ therefore for each function $g = g(z)$ we have

$$0 = \int_{\Omega_\nu} \psi_0\theta_z\phi^\perp g(z) = \varepsilon \int_{-\nu/\varepsilon}^{\nu/\varepsilon}\int_0^1 \psi_0\theta_z\phi^\perp g(z)\big[1 + j_1^\varepsilon(\eta, z)\big]\, dz d\eta,$$

where

$$|j_1^\varepsilon|, |j_{1,z}^\varepsilon| \le C\nu.$$

Since g is arbitrary therefore for each $z \in (0, 1)$ we have

$$\int_{-\nu/\varepsilon}^{\nu/\varepsilon} \psi_0\theta_z\phi^\perp\big[1 + j_1^\varepsilon(\eta, z)\big]\, d\eta = 0.$$

Differentiating this last formula with respect to z we get

$$\begin{aligned}\int_{-\nu/\varepsilon}^{\nu/\varepsilon} \psi_0\theta_z\phi_z^{\perp}\big[1+j_1^{\varepsilon}(\eta,z)\big]\,d\eta &= -\int_{-\nu/\varepsilon}^{\nu/\varepsilon} \psi_0\theta_{zz}\phi^{\perp}\big[1+j_1^{\varepsilon}(\eta,z)\big]\,d\eta\\ &\quad -\int_{-\nu/\varepsilon}^{\nu/\varepsilon} \psi_0\theta_z\phi^{\perp} j_{1,z}^{\varepsilon}(\eta,z)\,d\eta\\ &\quad -\int_{-\nu/\varepsilon}^{\nu/\varepsilon} \psi_{0,z}\theta_z\phi^{\perp}\big[1+j_1^{\varepsilon}(\eta,z)\big]\,d\eta\\ &= -\int_{-\nu/\varepsilon}^{\nu/\varepsilon} \psi_0\theta_z\phi^{\perp} j_{1,z}^{\varepsilon}(\eta,z)\,d\eta\\ &\quad +\int_{-\nu/\varepsilon}^{\nu/\varepsilon} \psi_{0,z}\theta_z\phi^{\perp}\big[1+j_1^{\varepsilon}(\eta,z)\big]\,d\eta.\end{aligned}$$

Integrating now with respect to z we obtain

$$\left|\int_{\Omega_\nu} \psi_0\theta_z\phi_z^{\perp}\right| \le C\int_{\Omega_\nu} |\theta_z||\phi^{\perp}|.$$

Consequently,

$$I \le C\int_{\Omega_\nu} |\theta_z||\phi^{\perp}| + C\nu\int_{\Omega_\nu} |\theta_z||\nabla\phi^{\perp}|.$$

We now need to estimate $II=\int_{\partial\Omega_\nu}\phi^{\perp}\theta\partial_n\psi_0\,dS$. Since near γ_0 we have

$$\partial_n\psi_0 = -h_0''(0)U_\eta + r_0^{\varepsilon} = -h_0''(0)\psi_0 + \tilde r_0^{\varepsilon}, \qquad \text{where } |\tilde r_0^{\varepsilon}| \le \varepsilon e^{-c|\eta|},$$

with a similar estimate near γ_1, therefore

$$\begin{aligned} II &= \sum_{i=1}^{2}\int_{\partial\Omega_\nu\cap B(\gamma_i,\nu)} \phi^{\perp}\theta\partial_n\psi_0\,dS + O(e^{-c\nu/\varepsilon})\int_{\partial\Omega_\nu\cap\{|x|=\nu\}}\phi^{\perp}\theta\,dS\\ &= \sum_{i=1}^{2}\int_{-\nu}^{\nu}(-1)^i\psi_0\phi^{\perp}\theta h_{i-1}''(0)\sqrt{1+(h_{i-1}')^2}\,dx\\ &\quad +O(e^{-c\nu/\varepsilon})\int_{\partial\Omega_\nu\cap\{|x|=\nu\}}\phi^{\perp}\theta\,dS\\ &= \sum_{i=1}^{2}\int_{-\nu}^{\nu}(-1)^i\psi_0\phi^{\perp}\theta h_{i-1}''(0)\Big[\sqrt{1+(h_{i-1}')^2}-1\Big]\,dx\\ &\quad +O(e^{-c\nu/\varepsilon})\int_{\partial\Omega_\nu\cap\{|x|=\nu\}}\phi^{\perp}\theta\,dS\\ &= O(\nu^2)\int_{\partial\Omega_\nu}\phi^{\perp}\theta\,dS.\end{aligned}$$

Using a trace inequality we find, for each $\alpha > 0$,

$$\int_{\partial\Omega_\nu} |\phi^\perp||\theta|\, dS \le \alpha \int_{\partial\Omega_\nu} |\phi^\perp|^2\, dS + \frac{C}{\alpha}\int_{\partial\Omega_\nu} \theta^2\, dS$$
$$\le C\alpha \int_{\Omega_\nu} \left[|\nabla\phi^\perp|^2 + \nu^{-1}|\phi^\perp|^2\right]$$
$$+ \frac{C}{\alpha}\left[\int_0^1 \theta^2 + \theta^2(1) + \theta^2(0)\right],$$

hence

$$II \le C\alpha\nu^2 \int_{\Omega_\nu} \left[|\nabla\phi^\perp|^2 + \nu^{-1}|\phi^\perp|^2\right] + \frac{C\nu^2}{\alpha}\int_0^1 \left(\theta_z^2 + \theta^2\right). \tag{3.7}$$

Furthermore we need to estimate $III = \int_{\Omega_\nu} \phi^\perp \nabla\theta\nabla\psi_0$. We have

$$\nabla\theta\nabla\psi_0 = \theta_z[\nabla z\nabla\psi_0] = O(1)\theta_z,$$

hence

$$III \le \int_{\Omega_\nu} |\theta_z||\phi^\perp|.$$

Taking $\alpha = \varepsilon^{-3\kappa/2}$, $\kappa \in (0, 1/2)$ and using Corollary 3.1 we get from (3.6) and estimates on I, II, III,

$$B_\nu(\psi, \phi^\perp) \le \varepsilon^2 \left|\int_{\partial\Omega_\nu} \phi^\perp\theta\partial_n\psi_0\, dS\right| + \int_{\Omega_\nu} |\phi^\perp||\theta||\mathcal{L}\psi_0|$$
$$+ C\varepsilon^2 \int_{\Omega_\nu} |\theta_z|[\nu|\nabla\phi^\perp| + |\phi^\perp|]$$
$$\le C\varepsilon^2\nu^2 \int_{\partial\Omega_\nu} |\phi^\perp||\theta|\, dS$$
$$+ C\int_{\Omega_\nu} \varepsilon^2[(|\theta_z| + |\theta|)|\phi^\perp| + \nu|\theta_z||\nabla\phi^\perp|]$$
$$\le C\varepsilon^{3+\kappa}\int_0^1 \left(|\theta_z|^2 + |\theta|^2\right) + \varepsilon^{3-2\kappa}\int_{\Omega_\nu} |\nabla\phi^\perp|^2 + \frac{\nu_0}{4}\int_{\Omega_\nu} |\phi^\perp|^2$$
$$\le C\varepsilon^\kappa \mathcal{B}_\nu(\psi, \psi) + C\varepsilon^{2+\kappa}\int_{\Omega_\nu} \psi^2$$
$$+ \int_{\Omega_\nu} \left(\varepsilon^{3-2\kappa}|\nabla\phi^\perp|^2 + \frac{\nu_0}{4}|\phi^\perp|^2\right).$$

The first assertion of the lemma follows from the above. The second follows by taking $\kappa = 1/4$. □

Let Σ_{k-1} denote the collection of $k-1$ dimensional subspaces of $L^2(\Omega_\nu)$. We define

$$\mu_k^* = \max_{S\in\Sigma_{k-1}} \min_{\substack{\Psi\in S^\perp \\ \|\Psi\|_{L^2(\Omega_\nu)}=1}} \mathcal{B}_\nu(\Psi, \Psi).$$

We will decompose $\Psi = \psi + \phi^\perp$, where $\psi \in \mathcal{X}, \phi^\perp \in \mathcal{X}^\perp$. Let $\theta_1, \ldots, \theta_{k-1}$ be the eigenfunctions of (1.5) and let $S_{k-1} = \text{span}\{\psi_0\theta_i, i = 1, \ldots, k-1\}$. If $\Psi \in S_{k-1}^\perp$ then

$$0 = \int_{\Omega_\nu} \Psi\psi_0\theta_i = \int_{\Omega_\nu} \psi\psi_0\theta_i, \qquad i = 1, \ldots, k-1.$$

From Lemma 3.6 it follows then

$$\begin{aligned}
\mu_k^* &\geq \min_{\substack{\Psi\in S_{k-1}^\perp \\ \|\Psi\|_{L^2(\Omega_\nu)}=1}} \mathcal{B}_\nu(\Psi, \Psi) \\
&\geq \min_{\substack{\Psi\in S_{k-1}^\perp \\ \|\psi\|^2_{L^2(\Omega_\nu)}+\|\phi^\perp\|^2_{L^2(\Omega_\nu)}=1}} \left\{ \mathcal{B}_\nu(\psi, \psi)[1 + O(\varepsilon^{1/4})] - \varepsilon^{9/4}\int_{\Omega_\nu} \psi^2 \right. \\
&\qquad\qquad \left. + \frac{\nu_0}{2}\int_{\Omega_\nu} |\phi^\perp|^2 \right\} \\
&\geq \min_{\substack{\Psi\in S_{k-1}^\perp \\ \|\psi\|^2_{L^2(\Omega_\nu)}+\|\phi^\perp\|^2_{L^2(\Omega_\nu)}=1}} \left\{ \lambda_k\varepsilon^2[1 + O(\varepsilon^{1/4})]\int_{\Omega_\nu} \psi^2 + \frac{\nu_0}{2}\int_{\Omega_\nu} |\phi^\perp|^2 \right\} \\
&\geq \lambda_k\varepsilon^2[1 + O(\varepsilon^{1/4})].
\end{aligned}$$

On the other hand assume that for each $\delta > 0$ there exists $k-1$ dimensional subspace S_δ such that

$$\min_{\substack{\Psi\in S_\delta^\perp \\ \|\Psi\|_{L^2(\Omega_\nu)}=1}} \mathcal{B}_\nu(\Psi, \Psi) \geq \varepsilon^2(\lambda_k + \delta).$$

Suppose that $S_\delta = \text{span}\{\psi_0\vartheta_i + \phi_i^\perp, i = 1, \ldots, k-1\}$. (Some of the components ϑ_i's may be equal to 0). It follows that there exists a vector $\tilde{\psi} = \psi_0\tilde{\vartheta}$ such that $\tilde{\psi} \in S_\delta^\perp$, $\|\tilde{\psi}\|_{L^2(\Omega_\nu)} = 1$ and in addition $\int_{\Omega_\nu} \psi_0\vartheta_i\tilde{\vartheta} = 0$. Thus $\tilde{\psi} \in S_{m-1}^\perp$, $S_{m-1} = \text{span}\{\psi_0\vartheta_i, \vartheta_i \neq 0\}$, where $m \leq k$. We can assume that

$$\mathcal{B}_\nu(\tilde{\psi}, \tilde{\psi}) = \min_{\psi\in S_{m-1}} \mathcal{B}_\nu(\psi, \psi) \geq \min_{\substack{\Psi\in S_\delta^\perp \\ \|\Psi\|_{L^2(\Omega_\nu)}=1}} \mathcal{B}_\nu(\Psi, \Psi).$$

We then have by Lemma 3.3

$$\varepsilon^2(\lambda_k + \delta) \leq \mathcal{B}_\nu(\tilde{\psi}, \tilde{\psi}) \leq \varepsilon^2(\lambda_m + C\varepsilon),$$

a contradiction. Thus we have proved:

Lemma 3.7. *The following estimate holds:*

$$\varepsilon^2(\lambda_k - C\varepsilon^{1/4}) \leq \mu_k^* \leq \varepsilon^2(\lambda_k + C\varepsilon).$$

We will now consider the linearized operator and it associated bilinear form on the whole domain Ω. More precisely let (V_k, Λ_k) denote the eigenvectors and eigenvalues of (3.1). Let

$$\mathcal{B}(\psi_1, \psi_2) = \int_\Omega \varepsilon^2 \nabla\psi_1 \nabla\psi_2 - f'(u)\psi_1\psi_2.$$

We will show that for any constant $M > 0$ and any k such that $\lambda_k < M + 1$ the kth eigenvalue of (3.1) satisfies $\Lambda_k = \varepsilon^2\lambda_k + o(\varepsilon^2)$. Before we do that we need a lemma which basically states that the critical eigenvectors of (3.1) (i.e. corresponding to eigenvalues of order ε^2) decay exponentially away from the transition layer.

Lemma 3.8. *Let $M > 0$ be fixed and let Λ_k be an eigenvalue of* (3.1) *satisfying $\Lambda_k < M\varepsilon^2$. There exists positive constant β, independent on ε and such that*

$$\int_{|x|>R} [\varepsilon^2|\nabla V_k|^2 + |V_k|^2] \le Ce^{-\beta R/\varepsilon}\|V_k\|^2_{H^1_\varepsilon(\Omega)}, \qquad \forall R > 0. \tag{3.8}$$

Proof. We fix k and denote $V_k \equiv V$, $\Lambda_k = \Lambda$. For each $\beta > 0$ we define a function

$$V^\beta = e^{\beta x/\varepsilon} V.$$

Calculating directly we obtain

$$\mathcal{L}V^\beta = -(\Lambda + \beta^2)V^\beta + 2\varepsilon\beta\partial_x V^\beta.$$

Multiplying the above equation by V^β and integrating by parts we get

$$-\mathcal{B}(V^\beta, V^\beta) + \varepsilon^2 \int_{\partial\Omega} V^\beta \partial_n V^\beta\, dS = -(\Lambda + \beta^2)\|V^\beta\|_{L^2(\Omega)} + 2\varepsilon\beta \int_\Omega V^\beta \partial_x V^\beta. \tag{3.9}$$

We will first deal with the boundary integral above. Using a trace inequality and $\partial_n V = 0$ we find

$$\begin{aligned}\left|\varepsilon^2 \int_{\partial\Omega} V^\beta \partial_n V^\beta\, dS\right| &= \left|\varepsilon\beta \int_{\partial\Omega} |V^\beta|^2 n_x\, dS\right| \\ &\le C\beta\varepsilon\|V^\beta\|_{H^1(\Omega)}\|V^\beta\|_{L^2(\Omega)} + C\beta\varepsilon\|V^\beta\|^2_{L^2(\Omega)} \\ &\le \frac{1}{4}\varepsilon^2\|V^\beta\|^2_{H^1(\Omega)} + C\beta(\beta+\varepsilon)\|V^\beta\|^2_{L^2(\Omega)}.\end{aligned}$$

From (3.9) we get

$$\frac{\varepsilon^2}{4}\int_\Omega |\nabla V^\beta|^2 - \int_\Omega [f'(u) + \Lambda + C\beta(\beta+\varepsilon)]|V^\beta|^2 \le 0. \tag{3.10}$$

Observe that, since $\Lambda < M\varepsilon^2$, there exist positive constants R_0, β_0 and c_0 such that

$$f'(u) + \Lambda + C\beta(\beta+\varepsilon) < -c_0, \qquad |x| > R_0\varepsilon, \quad 0 < \beta < \beta_0.$$

Thus we have for each $R > R_0\varepsilon$

$$\frac{\varepsilon^2}{4}\int_{|x|>R}|\nabla V^\beta|^2 + c_0\int_{|x|>R}|V^\beta|^2 \le C\int_{|x|\le R_0\varepsilon}\left[\varepsilon^2|\nabla V^\beta|^2 + |V^\beta|^2\right].$$

Taking β_0 smaller if necessary we obtain

$$\int_{x>R} e^{\beta x/\varepsilon}\left[\varepsilon^2|\nabla V|^2 + |V|^2\right] \le C\|V\|^2_{H^1_\varepsilon(\Omega)}. \tag{3.11}$$

Modifying the constant on the right hand side of the above estimate we can extend (3.11) for $0 \le R \le R_0\varepsilon$.

An analogous to (3.11) estimate can be proven in the region $x < -R$ if we take $V^\beta = e^{-\beta x/\varepsilon}$ and repeat the above argument.

Estimate (3.8) follows now immediately. □

Lemma 3.9. *Let $M > \lambda_1$ be fixed and let $k^* = \max\{k \mid \lambda_k < M\}$. Let $\psi_k = \alpha_k\psi_0\theta_k$, where α_k is chosen such that $\|\psi_k\|_{L^2(\Omega_\nu)} = 1$. Then for each $k \le k^*$ we have*

$$\begin{aligned} \Lambda_k &= \lambda_k\varepsilon^2[1 + O(\varepsilon^{1/4})],\\ \|V_k - \psi_k\|^2_{L^2_\varepsilon(\Omega_\nu)} &\le C\nu. \end{aligned} \tag{3.12}$$

Proof. We will prove the result using the induction with respect to k. Let $k = 1 \le k^*$.

We first define a $C^\infty(\Omega)$ cut-off function χ_ν as follows

$$\chi_\nu = \begin{cases} 1 & \Omega \cap \{|x| \le \nu/2\},\\ 0 < \chi_\nu < 1 & \Omega \cap \{\nu/2 < |x| \le \nu\},\\ 0 & \Omega \cap \{\nu < |x|\}. \end{cases}$$

Consider a test function $\Psi = \psi_1\chi_\nu$. Using Lemma 3.3 and Corollary 3.1 we find

$$\begin{aligned} \mathcal{B}(\Psi, \Psi) &\le \mathcal{B}_\nu(\psi_1, \psi_1) + Ce^{-c\nu/\varepsilon}\int_0^1\left[\varepsilon^2|\theta_{1,z}|^2 + |\theta_1|^2\right]\\ &\le \mathcal{B}_\nu(\psi_1, \psi_1)[1 + O(e^{-c\nu/\varepsilon})] + Ce^{-c\nu/\varepsilon}\|\psi_1\|^2_{L^2(\Omega_\nu)}\\ &\le \lambda_1\varepsilon^2[1 + O(\varepsilon^{1/4})]. \end{aligned}$$

Thus we conclude that

$$\Lambda_1 \le \lambda_1\varepsilon^2(1 + O(\varepsilon^{1/4})). \tag{3.13}$$

Observe that, since $\|V_1\|_{L^2(\Omega)} = 1$, we have $\|V_1\|_{H^1_\varepsilon(\Omega)} \le C$. From Lemma 3.8 we get

$$\begin{aligned} \mathcal{B}(V_1, V_1) &\ge \mathcal{B}_\nu(V_1, V_1) - Ce^{-\beta\nu/\varepsilon}\|V_1\|^2_{H^1_\varepsilon(\Omega)}\\ &\ge \lambda_1\varepsilon^2\|V_1\|_{L^2(\Omega_\nu)}[1 + O(\varepsilon^{1/4})] - Ce^{-\beta\nu/\varepsilon}\|V_1\|^2_{L^2(\Omega)}\\ &\ge \lambda_1\varepsilon^2\|V_1\|_{L^2(\Omega)}[1 + O(\varepsilon^{1/4})]. \end{aligned} \tag{3.14}$$

From (3.13) and (3.14) we get

$$\Lambda_1 = \lambda_1 \varepsilon^2 [1 + O(\varepsilon^{1/4})]. \tag{3.15}$$

Hence we have proven the first of the inequalities in (3.12) in the case $k = 1$. To show the rest of (3.12) let $W_1 = V_1 \chi_\nu$. Since $\operatorname{supp} W_1 \subset \Omega_\nu$ therefore we can decompose

$$W_1 = a_1 \psi_1 + \tilde{\psi} + \tilde{\psi}^\perp, \quad \tilde{\psi} \in \mathcal{X}, \quad \int_{\Omega_\nu} \psi_1 \tilde{\psi} = 0, \quad \tilde{\psi}^\perp \in \mathcal{X}^\perp.$$

We can calculate

$$\mathcal{L} W_1 = \Lambda_1 W_1 + \varepsilon^2 P_\nu,$$

where

$$P_\nu = V_1 \Delta \chi_\nu + 2 \nabla V_1 \cdot \nabla \chi_\nu.$$

Multiplying the above equation by $\tilde{\psi} + \tilde{\psi}^\perp$ and integrating by parts we get

$$\begin{aligned} \mathcal{B}_\nu(a_1 \psi_1, \tilde{\psi} + \tilde{\psi}^\perp) + \mathcal{B}_\nu(\tilde{\psi} + \tilde{\psi}^\perp, \tilde{\psi} + \tilde{\psi}^\perp) &= \Lambda_1 \big(\|\tilde{\psi}\|^2_{L^2(\Omega_\nu)} + \|\tilde{\psi}^\perp\|^2_{L^2(\Omega_\nu)} \big) \\ &\quad + \varepsilon^2 \int_{\Omega_\nu} (\tilde{\psi} + \tilde{\psi}^\perp) P_\nu. \end{aligned} \tag{3.16}$$

We have

$$\mathcal{B}_\nu(a_1 \psi_1, \tilde{\psi} + \tilde{\psi}^\perp) = \mathcal{B}_\nu(a_1 \psi_1, \tilde{\psi}) + \mathcal{B}_\nu(a_1 \psi_1, \tilde{\psi}^\perp). \tag{3.17}$$

Using Lemma 3.2 one can show that for any functions $\psi^* = \psi_0 \theta^*$, $\psi^{**} = \psi_0 \theta^{**} \in \mathcal{X}$ we have

$$\begin{aligned} \mathcal{B}_\nu(\psi^*, \psi^{**}) = \varepsilon^2 \alpha_0 & \left[h_1''(0) \theta^*(1) \theta^{**}(1) - h_0''(0) \theta^*(0) \theta^{**}(0) + \int_0^1 \theta^*_z \theta^{**}_z \, dz \right] \\ & + O(\varepsilon) \big[\mathcal{B}_\nu(\psi^*, \psi^*) + \varepsilon^2 \|\psi^*\|^2_{L^2(\Omega_\nu)} + \mathcal{B}_\nu(\psi^{**}, \psi^{**}) \\ & + \varepsilon^2 \|\psi^{**}\|^2_{L^2(\Omega_\nu)} \big], \end{aligned} \tag{3.18}$$

where we have made use of Corollary 3.1 as well. We also have

$$0 = \int_{\Omega_\nu} \psi_1 \tilde{\psi} = \alpha_0 \left\{ \int_0^1 \theta_1 \tilde{\theta} [1 + O(\nu)] \, dz \right\}.$$

It follows from (3.18) with $\psi^* = \psi_1$ and $\psi^{**} = \tilde{\psi}$ that

$$\begin{aligned} \mathcal{B}_\nu(a_1 \psi_1, \tilde{\psi}) &= O(\varepsilon) a_1^2 \big[\mathcal{B}_\nu(\psi_1, \psi_1) + \varepsilon \nu \|\psi_1\|^2_{L^2(\Omega_\nu)} \big] \\ &\quad + O(\varepsilon) \big[\mathcal{B}_\nu(\tilde{\psi}, \tilde{\psi}) + \varepsilon \nu \|\tilde{\psi}\|^2_{L^2(\Omega_\nu)} \big]. \end{aligned} \tag{3.19}$$

From Lemma 3.8 we get

$$\|P_\nu\|_{L^2(\Omega_\nu)} \leq C e^{-\beta \nu / 2\varepsilon}. \tag{3.20}$$

By Lemma 3.6 and (3.16), (3.19), (3.20) we get, since $|a_1| \leq 1$,

$$\left[\lambda_2 \varepsilon^2 (1 + O(\varepsilon^{1/4})) - \Lambda_1\right] \|\tilde{\psi}\|^2_{L^2(\Omega_\nu)} + C \|\tilde{\psi}^\perp\|^2_{H^1_\varepsilon(\Omega_\nu)} \leq C\varepsilon^2 \nu. \tag{3.21}$$

Since $\Lambda_1 = \lambda_1 \varepsilon^2 (1 + O(\varepsilon^{1/4}))$ and $\lambda_2 - \lambda_1 > 0$, we get

$$\|\tilde{\psi}\|^2_{L^2(\Omega_\nu)} + \|\tilde{\psi}^\perp\|^2_{H^1_\varepsilon(\Omega_\nu)} \leq C\nu \tag{3.22}$$

hence

$$\begin{aligned} |a_1| &\geq \|V_1\|_{L^2(\Omega_\nu)} - \|W_1 - V_1\|_{L^2(\Omega_\nu)} - \|\tilde{\psi}\|_{L^2(\Omega_\nu)} - \|\tilde{\psi}^\perp\|_{L^2(\Omega_\nu)} \\ &\geq 1 - O(\nu^{1/2}). \end{aligned} \tag{3.23}$$

The second part of (3.12) follows now from (3.22) and (3.23).

Assume now that (3.12) holds for $k = 1, \ldots, m$ and that $m < k^*$. We will show that (3.12) holds for (Λ_{m+1}, V_{m+1}).

Let

$$\psi_\nu = \psi_{m+1}\chi_\nu - \sum_{k=1}^{m} V_k \int_{\Omega_\nu} V_k \psi_{m+1} \chi_\nu.$$

From Lemma 3.8 and the inductive assumption it follows that

$$\|\psi_\nu\|^2_{L^2(\Omega_\nu)} = 1 + O(\nu),$$

hence we have

$$\begin{aligned} \Lambda_{m+1} = \min_{\substack{V \perp V_k, k=1,\ldots m \\ \|V\|_{L^2(\Omega)}=1}} \mathcal{B}(V, V) &\leq \mathcal{B}(\psi_\nu, \psi_\nu) \|\psi_\nu\|^{-2}_{L^2(\Omega_\nu)} \\ &\leq \mathcal{B}_\nu(\psi_{m+1}, \psi_{m+1})[1 + O(\nu)] \\ &\leq \lambda_{m+1}\varepsilon^2 [1 + O(\varepsilon^{1/4})]. \end{aligned} \tag{3.24}$$

Define now functions

$$W_\nu = V_{m+1}\chi_\nu - \sum_{k=1}^{m} \psi_k \int_{\Omega_\nu} \psi_k V_{m+1} \chi_\nu.$$

From Lemma 3.8 we get

$$\|W_\nu\|^2_{L^2(\Omega_\nu)} = 1 + O(\nu),$$

and thus we can estimate

$$\begin{aligned} \lambda_{m+1}\varepsilon^2 &\leq \mathcal{B}_\nu(W_\nu, W_\nu)[1 + O(\nu)] \\ &\leq [\mathcal{B}(V_{m+1}, V_{m+1}) + O(e^{-c\nu/\varepsilon})][1 + O(\nu)] \\ &\leq \Lambda_{m+1}[1 + O(\nu)]. \end{aligned} \tag{3.25}$$

Combining (3.24) and (3.25) we obtain the first inequality in (3.12) holds for Λ_{m+1}.

In order to prove the second of the estimates in (3.12) for V_{m+1} we argue similarly as in the proof for V_1. We omit the details.

The proof of the lemma is complete. □

Corollary 3.2. *(i) Let ψ_k, $k = 1, \dots, m$ be functions defined in the statement of Lemma 3.9 and let $\psi_k^\nu = \psi_k \chi_\nu$. Assume that v satisfies $\int_\Omega v\psi_k^\nu = 0$, $k = 1, \dots, m$. Then*

$$\mathcal{B}(v, v) \geq \Lambda_{m+1}[1 + O(\nu)]\|v\|^2_{L^2(\Omega)}.$$

(ii) Assume that v satisfies $\int_\Omega vu_{\xi_i} = 0$, $i = 1$ in the minimum and short axis case and $i = 1, 2$ in the long axis case. Then we have

$$\mathcal{B}(v, v) \geq \Lambda_2[1 + O(\varepsilon^{1/4})]\|v\|^2_{L^2(\Omega)}, \text{ minimum and short axis case},$$

$$\mathcal{B}(v, v) \geq \Lambda_3[1 + O(\varepsilon^{1/4})]\|v\|^2_{L^2(\Omega)}, \text{ long axis case}.$$

Proof. We will prove part (i) of this lemma. Proof of (ii) is similar and is left to the reader. We define

$$\tilde{v} = v - \sum_{k=1}^{m} V_k \int_\Omega vV_k.$$

From our assumption and Lemma 3.9 it follows

$$\left|\int_\Omega vV_k\right| \leq C\nu^{1/2}\|v\|_{L^2(\Omega)}.$$

Furthermore

$$\mathcal{B}(\tilde{v}, \tilde{v}) \geq \Lambda_{m+1}\|\tilde{v}\|^2_{L^2(\Omega)} \geq \Lambda_{m+1}\|v\|^2_{L^2(\Omega)}[1 + O(\nu)].$$

On the other hand

$$\mathcal{B}(\tilde{v}, \tilde{v}) = \mathcal{B}(v, v) - \sum_{k=1}^{m} \Lambda_k \left(\int_\Omega vV_k\right)^2.$$

Our assertion now follows. □

4. Existence of and stability of stationary states

Proof of Theorem 1.1. The main idea of the proof is to use the method of Lyapunov-Schmidt reduction. For that we need two steps:

(1) Solve (1.1) in the orthogonal complement of $\nabla_\xi u$, for each fixed ξ.
(2) Find a particular value of ξ for which the Lagrange multiplier $\mathbf{c}(\xi)$ is equal to 0.

Step 1. Let $F(u) = -\frac{1}{4}(1 - u^2)^2$. It is well know that u is a solution to (1.1) if and only if u is a critical point of

$$J(u) = \int_\Omega \left[\frac{\varepsilon^2}{2}|\nabla u|^2 - F(u)\right].$$

Let u^n be an approximate solution up to order n, $n > 5$ to (2.1). We will show that for each ξ there exists a function $v^n = v^n(x, y, \xi, \varepsilon)$ such that

$$\inf_{\substack{\tilde{v}\perp\nabla_\xi u^n \\ \|\tilde{v}\|_{L^2(\Omega)}\le\varepsilon^4}} J(u^n + \tilde{v}) = J(u^n + v^n)$$

and

$$\|v^n\|_{L^2(\Omega)} \le C\varepsilon^{(n+3)/2}. \tag{4.1}$$

For each $\tilde{v}$ we can write

$$J(u^n + \tilde{v}) = J(u^n) + DJ(u^n)\tilde{v} + \mathcal{B}(\tilde{v}, \tilde{v}) + \mathcal{N}(\tilde{v}),$$

where $\mathcal{N}(\tilde{v}) = \int_\Omega (u^n\tilde{v}^3 + \frac{1}{4}\tilde{v}^4)$. We first observe that

$$\inf_{\substack{\tilde{v}\perp\nabla_\xi u^n \\ \|\tilde{v}\|_{L^2(\Omega)}\le\varepsilon^4}} J(u^n + \tilde{v}) \le J(u^n). \tag{4.2}$$

On the other hand we have

$$\begin{aligned} DJ(u^n)\tilde{v} &= \varepsilon^2 \int_{\partial\Omega} \tilde{v}h^\varepsilon - \int_\Omega \tilde{v}\mathcal{L}u^n \\ &= \varepsilon^2 \int_{\partial\Omega} \tilde{v}h^\varepsilon - \mathbf{c}\cdot \int_\Omega \tilde{v}\nabla_\xi u^n - \int_\Omega \tilde{v}g^\varepsilon \\ &= \varepsilon^2 \int_{\partial\Omega} \tilde{v}h^\varepsilon - \int_\Omega \tilde{v}g^\varepsilon, \end{aligned}$$

hence, using a trace inequality, we get

$$\begin{aligned} DJ(u^n)\tilde{v} &\ge -\varepsilon^2\|h^\varepsilon\|_{L^\infty(\Omega)}\|\tilde{v}\|_{L^1(\partial\Omega)} - \|\tilde{v}\|_{L^2(\Omega)}\|g^\varepsilon\|_{L^2(\Omega)} \\ &\ge -C\varepsilon^2\|h^\varepsilon\|_{L^\infty(\Omega)}\|\tilde{v}\|_{L^2(\Omega)} - \|\tilde{v}\|_{L^2(\Omega)}\|g^\varepsilon\|_{L^2(\Omega)} \\ &\ge -C\varepsilon^{n+5}. \end{aligned} \tag{4.3}$$

Furthermore we have

$$\begin{aligned} \mathcal{N}(\tilde{v}) &\ge -C\big(\|\tilde{v}\|^3_{L^3(\Omega)} + \|\tilde{v}\|^4_{L^4(\Omega)}\big) \\ &\ge -C(1 + \|\tilde{v}\|_{H^1(\Omega)}\|\tilde{v}\|_{L^2(\Omega)})\|\tilde{v}\|_{H^1(\Omega)}\|\tilde{v}\|^2_{L^2(\Omega)} \\ &\ge -C\varepsilon^4\big(1 + \varepsilon^4\|\tilde{v}\|_{H^1(\Omega)}\big)\|\tilde{v}\|_{H^1(\Omega)}\|\tilde{v}\|_{L^2(\Omega)} \\ &\ge -C\varepsilon^3\|\tilde{v}\|^2_{H^1_\varepsilon(\Omega)}. \end{aligned} \tag{4.4}$$

Now, if v^n is a minimizer of $J(u^n+\tilde{v})$ then, from (4.2), (4.3), (4.4) and Corollary 3.2, it follows

$$\begin{aligned} C\varepsilon^{n+5} &\ge \mathcal{B}(v^n, v^n) - C\varepsilon^3\|v^n\|^2_{H^1_\varepsilon(\Omega)} \\ &\ge \Lambda_m[1 + O(\varepsilon^{1/4})]\|v\|^2_{L^2(\Omega)}, \end{aligned} \tag{4.5}$$

where $m = 2$ in the minimum and short axis case and $m = 3$ in the long axis case. From Lemma 3.9 we conclude (4.1). As an easy corollary we obtain

$$\|v^n\|_{H^1(\Omega)} \le C\varepsilon^{(n+1)/2}. \tag{4.6}$$

Step 2. Since, for $n > 5$, we have $\frac{n+3}{2} > 4$ therefore the minimizer v^n obtained in Step 1 is a weak (and, by elliptic regularity, classical) solution to the following problem

$$\begin{aligned} &\varepsilon^2 \Delta(u^n + v^n) + f(u^n + v^n) = \tilde{\mathbf{c}} \cdot \nabla_\xi u^n, && \text{in } \Omega,\\ &\partial_n v^n = -\partial_n u^n, && \text{on } \partial\Omega,\\ &\textstyle\int_\Omega v^n \nabla_\xi u^n = 0, \end{aligned}$$

where $\tilde{\mathbf{c}}$ is a Lagrange multiplier. We can write the above equation in the form

$$\mathcal{L} v^n = (\tilde{\mathbf{c}} - \mathbf{c}) \cdot \nabla_\xi u^n - g^\varepsilon + N(v^n), \tag{4.7}$$

where $N(v^n) = 2u^n(v^n)^2 + (v^n)^3$. Multiplying (4.7) by V_m, $m = 1$ in the minimum and short axis case and $m = 1, 2$ in the long axis case, integrating by parts, using

$$\left| \int_\Omega V_m \mathcal{L} v^n \right| \le C\varepsilon^{(n+5)/2},$$

$$\int_\Omega u^n_{\xi_j} V_m = -\varepsilon^{1/2}\delta_{mj} + O(\varepsilon^{1/2}\nu),$$

and estimates (4.1), (4.6) we obtain

$$|\tilde{\mathbf{c}} - \mathbf{c}| \le C\varepsilon^{(n+4)/2}. \tag{4.8}$$

Clearly $\mathbf{c}_0(0) = 0$. Since, $\lambda_m \neq 0$ (where $m = 1$ in the minimum and short axis case and $m = 1, 2$ in the long axis case) therefore using (4.8) and the definition of $\mathbf{c}_0$, by a standard topological degree argument we obtain $\tilde{\mathbf{c}}(\tilde{\xi}) = 0$ for some $\tilde{\xi}$, $|\tilde{\xi}| < C\varepsilon$. Consequently, $u^n(\tilde{\xi}) + v^n(\tilde{\xi})$ is a solution to (1.1). Moreover $|\phi(\tilde{\xi}, y)| \le C\varepsilon$ and thus

$$\|u^n(\eta, y, \tilde{\xi}) - U(x/\varepsilon)\|_{C^0(\Omega)} \le C\varepsilon.$$

Standard elliptic regularity and (4.6) give

$$\|v^n\|_{W^{k,2}(\Omega)} \le C\varepsilon^{(n+3)/2-k}. \tag{4.9}$$

Setting $k = 2$ we conclude (1.3). The proof of the theorem is complete. □

Proof of Theorem 1.2. Using estimate (4.9) with $k = 2$ and taking $n > 5$ we obtain

$$\|v^n\|_{C^0(\Omega)} \le C\varepsilon^{(n-1)/2} \le C\varepsilon^{5/2}.$$

Consequently the linear operator

$$\varepsilon^2 \Delta V + f'(u^n + v^n)V = \tilde{\Lambda} V$$

is a $O(\varepsilon^{5/2})$ perturbation of $\mathcal{L}$. The statement of the theorem follows from Lemma 3.9 by a straightforward argument. We omit the details. □

5. Concluding remarks-generalizations

In this paper for simplification we have chosen f to be a cubic nonlinearity. Our results can be generalized when f is a bistable, smooth nonlinearity with polynomial growth at ∞ by using Taylor's expansion of f.

Another possible and straightforward generalization is to consider an interface consisting of multiple, isolated straight line segments. One can construct the approximate invariant manifold by "gluing" manifolds of the separate components. It can be easily done due to the fact that away from the interface the solution to the approximate invariant manifold equation converges at an exponential rate to ± 1. In other words-isolated interfaces interact very weakly with one another.

The works of Bronsard and Stoth [3] and Nakashima and Tanaka [14] suggest that there are solutions with interfaces that collapse onto each other as $\varepsilon \to 0$ (phantom interfaces). Their interaction is stronger and should influence the asymptotics.

A more difficult than our setting situation happens if two straight line segments intersect at the right angle-this corresponds to a nonsmooth, stationary solution to the mean curvature flow. We expect that a solution to (1.1) can be constructed in this case, as suggested by [6], however our method in its present form does not apply. Similar in the spirit problem is to consider the vector valued Allen-Cahn equation with triple well potential F. Flores and Padilla constructed in this case stationary solutions of Morse index 2 [10] by using a variational approach.

5.1. Added in proof

In the recent work Pacard and Ritoré [15] proved a generalization of the existence result presented here. Namely, they showed that any nondegenerate minimal hypersurfaces in $(n + 1)$, $n \geq 1$, Riemanian manifold is a nodal set of a solution to the Allen-Cahn equation. The key idea in their approach is to invert the linearized operator in carefully chosen weighted spaces.

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