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Cylindrical symmetry of extremals of a Hardy–Sobolev inequality

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1. Introduction

In this note we discuss symmetry properties of the extremals of a weighted Hardy–Sobolev inequality due to M. Badiale and G. Tarantello.

A weighted Hardy–Sobolev inequality [2]. Let $n \geq 3$, $2 \leq k \leq n$, $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, $x = (x'x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $0 < \sigma < 1$ and $p := \frac{2n}{n-2+2\sigma}$. There exists a (optimal) constant $S > 0$ such that

$$\left(\int_{\mathbb{R}^n} \frac{|u|^p}{|x'|^{\sigma p}} \right)^{\frac{2}{p}} \leq S \int_{\mathbb{R}^n} |\nabla u|^2 \quad \forall u \in D^{1,2}(\mathbb{R}^n). \quad (1.1)$$

It was also shown in [2] that the best constant in (1.1) is achieved.

Inequality (1.1), which we will refer as (HS) inequality, was introduced in the recent work [2], to handle a nonlinear elliptic PDE with cylindrical symmetry. Such an equation was proposed as a model for describing the dynamics of galaxies, and (1.1) is the basic tool to deal with it by variational methods.

The limiting case $\sigma = 0$ corresponds to the classical Sobolev inequality. It has been exhaustively studied by Talenti [12] who computed exactly the best constant and proved existence of extremal functions, exhibiting them explicitly.

If $\sigma = 1$, (1.1) still holds true (see [2, Remark 2.3]) if $2 < k \leq n$, thus providing an extension of the classical Hardy inequality (which is known not to possess extremal functions). Notice that (1.1) is in general false if $k = 1$.

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When $k = n$, the inequality (1.1) has been thoroughly investigated especially in view of applications to Schrodinger equation. The best constant in this case has been computed in [8] and extremal functions have been identified by Lieb [10]. It is also shown in [8] that when $1 - \frac{n}{2} < \sigma < 0$ no inequality of type (1.1) is possible for all u , but (1.1) holds in the class of radial functions and the best constant is achieved. More general inequalities of type (1.1), still in case $k = n$, have been considered by Caffarelli–Kohn–Nirenberg [4], which include the case $\sigma < 0$ but with a proper weight on the right hand side as well. We refer to [6] for several results and references concerning best constants and corresponding extremal functions. But when $k < n$ the analysis of (1.1) is far from being complete.

We are interested in symmetry properties of extremal functions. If $k = n$, they are radial functions (this is not always the case for more general CKN inequalities, see [6] and [7]). However, if $k < n$, it is clear, from the Euler–Lagrange equation that they cannot be radial. Still, since the inequality (1.1) is symmetric in the variables x' and x'' separately, one expects the extremals to be symmetric in these variables. We prove this is infact the case.

Theorem 1.1. *Extremals of the Hardy–Sobolev inequality (1.1) are cylindrically symmetric, i.e. if the best constant in (1.1) is attained at $u \in D^{1,2}(\mathbb{R}^n)$, then*

- (i) *for any choice of $x'' \in \mathbb{R}^{n-k}$, $|u(\cdot, x'')|$ is symmetric decreasing in $\mathbb{R}^k$.*
- (ii) *there exists $x''_0 \in \mathbb{R}^{n-k}$ such that, for any choice of $x' \in \mathbb{R}^k$, $x' \neq 0$, $|u(x', \cdot)|$ is symmetric decreasing about x''_0 in $\mathbb{R}^{n-k}$.*

Our proof is inspired by the work of Lieb [10], where he studies, among other things, the sharp constants and extremal functions of classical Sobolev and Hardy–Littlewood–Sobolev inequalities using the method of competing symmetries. The basic point in this method is the following: the rearrangement inequality says that the extremals have to be symmetrically decreasing and at the same time we know that the inequality is also conformally invariant. These two facts contradict each other in the sense that a radial function needs not remain radial after the action of a conformal transformation. The extremals appear as consensus candidates of these two requirements. However, (1.1) does not have the full conformal symmetry to carry out Lieb’s programme as such. We make a first step here by proving the expected symmetry. In Section 2 we will prove an equivalent weighted Hardy–Littlewood–Sobolev inequality and then we will establish our theorem by showing the required symmetry for the extremals of the equivalent problem. One may even work directly with (1.1), but this will lead to other questions like when is equality achieved in Polya–Szego type inequalities. Also one may have to do extra work to show the existence of x''_0 mentioned in Theorem 1.1.

2. An equivalent weighted Hardy–Littlewood–Sobolev inequality

In this section we prove that the (HS) inequality (1.1) is equivalent to a doubly weighted Hardy–Littlewood–Sobolev inequality (wHLS). Moreover, the corresponding sharp constants are the same and we establish a relation between corresponding extremal functions. We borrow the idea of “duality” between (HS) and

(HLS) inequalities from [10] (see also [11] for an exposition in book form), still proving equivalence in a slightly different way. To start with, let us recall that the best constant in (1.1)

$$S := \sup_{\substack{u \in D^{1,2}(\mathbb{R}^n) \\ \int |\nabla u|^2 = 1}} \left(\int_{\mathbb{R}^n} \frac{|u|^p}{|x'|^{\sigma p}} \right)^{\frac{2}{p}} \quad (2.1)$$

is attained at some $u \in D^{1,2}(\mathbb{R}^n)$. It satisfies the Euler–Lagrange equation

$$\int_{\mathbb{R}^n} \frac{|u|^{p-2} u \varphi}{|x'|^{\sigma p}} = S^{\frac{p}{2}} \int_{\mathbb{R}^n} \nabla u \nabla \varphi \quad \forall \varphi \in D^{1,2}(\mathbb{R}^n). \quad (2.2)$$

It immediately follows from standard elliptic theory that the extremals are smooth away from the subspace $x' = 0$. The regularity near the subspace $x' = 0$ depends on n and σ ; for example the extremals are Hölder continuous when $\sigma < \frac{2}{n+2}$ (see [1] for details). We also note that the extremals of (1.1) does not change sign. This follows because if u is an extremal then $|u|$ is also an extremal and hence, by (2.2) and strong maximum principle $|u| > 0$ away from the subspace $x' = 0$ and $\{x' \neq 0\}$ is connected as $k \geq 2$.

Now, let $G = \frac{1}{(n-2)\omega_n |x|^{n-2}}$ denote the fundamental solutions of $-\Delta$ in $\mathbb{R}^n$. We have:

Theorem 2.1. *Let $\sigma \in [0, 1)$, $p = \frac{2n}{n-2+2\sigma}$, $\frac{1}{q} + \frac{1}{p} = 1$. Let $x = (x', z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. Then*

$$\sup_{h \in L^q, \|h\|_q=1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{h(x)}{|x'|^\sigma} G(x-y) \frac{h(y)}{|y'|^\sigma} dy dx = S. \quad (2.3)$$

Furthermore, u is an extremal for (2.1) iff $f := \frac{|u|^{p-2}u}{|x'|^{\sigma(p-1)}}$ is an extremal for (2.3).

Proof. We first derive from (1.1) a doubly weighted Hardy–Littlewood–Sobolev inequality:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{h(x)}{|x'|^\sigma} G(x-y) \frac{f(y)}{|y'|^\sigma} dy dx \leq S \|h\|_q \|f\|_q \quad \forall h, f \in L^q. \quad (2.4)$$

It is clearly enough to prove (2.4) for non negative C_0^∞ functions. For $f \in C_0^\infty$, $f_\sigma(x) := \frac{f(x)}{|x'|^\sigma}$ is in $L^{\frac{2n}{n+2}}$, and hence, by classical (HLS) inequality $G * f_\sigma$ is in $D^{1,2}(\mathbb{R}^n)$ and

$$\int \nabla \varphi \nabla (G * f_\sigma) = \int \varphi f_\sigma \quad \forall \varphi \in D^{1,2}(\mathbb{R}^n) \quad (2.5)$$

Thus, by Holder and (1.1), we have

$$\int (G * f_\sigma) h_\sigma \leq \|h\|_q \left(\int \frac{(G * f_\sigma)^p}{|x'|^{\sigma p}} \right)^{\frac{1}{p}} \leq \|h\|_q \sqrt[p]{S \int |\nabla(G * f_\sigma)|^2}. \quad (2.6)$$

Similarly, we obtain from (2.5)

$$\left(\int |\nabla(G * f_\sigma)|^2 \right)^{\frac{1}{2}} \leq S^{\frac{1}{2}} \|f\|_q. \quad (2.7)$$

This, jointly with (2.6), gives (2.4).

Notice that (2.4) holds true also if we replace q with $\frac{2n}{n+2}$ and h_σ with any $h \in L^{\frac{2n}{n+2}}$. Hence $G * f_\sigma \in L^{\frac{2n}{n+2}}$ for any $f \in L^q$, and in fact, applying (2.7) to $f_j \rightarrow f$ in L^q , we see that $G * f_\sigma \in D^{1,2}$ and (2.5) holds for all $f \in L^q$.

Now, (2.4) implies

$$\hat{S} := \sup_{\|f\|_q=1} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f(x)}{|x'|^\sigma} G(x-y) \frac{f(y)}{|y'|^\sigma} dx dy \leq S. \quad (2.8)$$

Next, we prove the reverse inequality. Let u be an extremal function for (HS), which we can assume to be non negative. After normalization, we have

$$\int \frac{|u|^p}{|x'|^{\sigma p}} = 1, \quad \int_{\mathbb{R}^n} \frac{|u|^{p-1} \varphi}{|x'|^{\sigma p}} = S \int_{\mathbb{R}^n} \nabla u \nabla \varphi \quad \forall \varphi \in D^{1,2}(\mathbb{R}^n). \quad (2.9)$$

If $f := \frac{u^{p-1}}{|x'|^{\sigma(p-1)}}$, then $\int f^q = \int \frac{u^p}{|x'|^{\sigma p}} = 1$. In particular, as noticed above, we have that $G * f_\sigma \in D^{1,2}(\mathbb{R}^n)$, and hence, using (2.5) and (2.9),

$$S \int u f_\sigma = S \int \nabla(G * f_\sigma) \nabla u = \int_{\mathbb{R}^n} \frac{|u|^{p-1}}{|x'|^{\sigma p}} G * f_\sigma.$$

But $f_\sigma = \frac{|u|^{p-1}}{|x'|^{\sigma p}}$ and hence $S = \int f_\sigma G * f_\sigma \leq \hat{S}$ because $\int f^q = 1$. This proves that $S = \hat{S}$ and that $f = \frac{u^{p-1}}{|x'|^{\sigma(p-1)}}$ is an extremal function for (2.3).

Finally, let f be an extremal function for the (wHLS) quotient (2.3). Clearly, f cannot change sign, so we can assume $f \geq 0$. We have:

$$\|f\|_q = 1 \quad \text{and} \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f(x)}{|x'|^\sigma} G(x-y) \frac{g(y)}{|y'|^\sigma} = S \int f^{q-1} g \quad \forall g \in L^q.$$

Let $u := f^{q-1} |x'|^\sigma$ so that $\frac{u^{p-1}}{|x'|^{\sigma(p-1)}} = f \in L^q$. The Euler–Lagrange equation for f rewrites

$$\int_{\mathbb{R}^n} \frac{g(x)}{|x'|^\sigma} G(x-y) \frac{u^{p-1}}{|y'|^{\sigma p}} = S \int \frac{u}{|x'|^\sigma} g \quad \forall g \in L^q.$$

In particular, we find that $Su = G * f_\sigma$ and hence, as remarked above, $u \in D^{1,2}(\mathbb{R}^n)$ and $-S \Delta u = f_\sigma = \frac{u^{p-1}}{|x'|^{\sigma p}}$. Thus $S \int |\nabla u|^2 = \int \frac{u^p}{|x'|^{\sigma p}} = \int f^q = 1$ i.e. u is an extremal function for (HS). $\square$

3. Rearrangement inequalities and Proof of Theorem 1.1

In this section we prove Theorem 1.1. First, a few preliminaries from symmetrization.

Definition. $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be vanishing at ∞ if it satisfies $|\{|f| > t\}| < \infty$ for all $t > 0$, where $|\cdot|$ denotes the Lebesgue measure.

For an f which vanishes at infinity we denote by f^* its symmetric decreasing rearrangement.

Next we recall the classical Riesz's rearrangement inequality and its case of equality. The equality statement was proved by Lieb [9]. For a proof and more details we refer to [11].

Lemma 3.1 (Riesz's rearrangement inequality). *Let f, g, h be three non negative functions on $\mathbb{R}^n$, vanishing at ∞ , then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f^*(x)g^*(x-y)h^*(y)dxdy. \quad (3.1)$$

Moreover, if g is strictly symmetric decreasing, then equality in (3.1) holds if and only if $f(x) = f^*(x - x_0)$ and $h(x) = h^*(x - x_0)$ for some common $x_0 \in \mathbb{R}^n$.

For our purpose we need a weighted form of the above rearrangement inequality, which is a special case of the Brascamp–Lieb–Luttinger inequality [3]. We also need to know when equality occurs in this inequality.

Lemma 3.2. *Let f, g, h be three non negative functions on $\mathbb{R}^n$, vanishing at ∞ and $a > 0$. Then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)}{|x|^a} g(x-y) \frac{h(y)}{|y|^a} dxdy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f^*(x)}{|x|^a} g^*(x-y) \frac{h^*(y)}{|y|^a} dxdy. \quad (3.2)$$

Moreover, if $g = g^*$ equality holds in (3.2) if and only if $f = f^*$ and $h = h^*$.

Proof. For $s > 0$, let us denote by B_s the ball in $\mathbb{R}^n$ with center at zero and radius $(\frac{1}{s})^{\frac{1}{a}}$. With this notation we have

$$\frac{1}{|x|^a} = \int_0^\infty \chi_{B_s}(x) ds.$$

Using the above representation of $\frac{1}{|x|^a}$, Fubini–Tonelli and Lemma 3.1 we get

$$\begin{aligned}
& \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)}{|x|^a} g(x-y) \frac{h(y)}{|y|^a} dx dy \\
&= \int_0^\infty \int_0^\infty \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f\chi_{B_s})(x) g(x-y) (h\chi_{B_t})(y) dy dx \right) ds dt \\
&\leq \int_0^\infty \int_0^\infty \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f\chi_{B_s})^*(x) g^*(x-y) (h\chi_{B_t})^*(y) dy dx \right) ds dt \\
&\leq \int_0^\infty \int_0^\infty \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f^*\chi_{B_s})(x) g^*(x-y) (h^*\chi_{B_t})(y) dy dx \right) ds dt \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f^*(x)}{|x|^a} g^*(x-y) \frac{h^*(y)}{|y|^a} dx dy. \tag{3.3}
\end{aligned}$$

Here we used the fact that $(f\chi_{B_s})^* \leq f^*\chi_{B_s}$, $\forall s > 0$, which follows from the inclusion between sublevel sets:

$$\{(f\chi_{B_s})^* > t\} = \{f\chi_{B_s} > t\}^* = (\{f > t\} \cap B_s)^* \subseteq \{f > t\}^* \cap B_s = \{f^*\chi_{B_s} > t\}$$

so that $(f\chi_{B_s})^*(x) = \int_0^\infty \chi_{\{(f\chi_{B_s})^* > t\}}(x) dt \leq \int_0^\infty \chi_{\{f^*\chi_{B_s} > t\}}(x) dt = (f^*\chi_{B_s})(x)$. This proves (3.2). $\square$

It follows from (3.3) that equality in (3.2) implies $(f\chi_{B_s})^* = f^*\chi_{B_s}$ for a.e. s . Integration over $\mathbb{R}^n$ leads to $\int_{\mathbb{R}^n} f\chi_{B_s} = \int_{\mathbb{R}^n} (f\chi_{B_s})^* = \int_{\mathbb{R}^n} f^*\chi_{B_s}$. A further integration over s and Fubini-Tonelli gives $\int_{\mathbb{R}^n} \frac{f}{|x|^a} = \int_{\mathbb{R}^n} \frac{f^*}{|x|^a}$. Since $|x|^{-a}$ is strictly symmetric decreasing this implies $f = f^*$ (see [11, Theorem 3.4]). Similarly we have $h = h^*$.

Proof of Theorem 1.1. In view of Theorem 2.1 it is enough to prove that the extremals of (2.3) have the required symmetry properties.

Let $f \in L^q(\mathbb{R}^n)$ be such that

$$\int_{\mathbb{R}^n} f^q = 1 \text{ and } \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|^\sigma} G(x-y) \frac{f(y)}{|y'|^\sigma} dy dx = S.$$

Let $f^{*'}$ denote the function obtained by taking the k -dim rearrangement of $f(\cdot, x'')$ for all $x'' \in \mathbb{R}^{n-k}$, i.e. $f^{*'}(\cdot, x'') = f(\cdot, x'')^*$, where $*$ denotes the rearrangement of functions in $\mathbb{R}^k$. Then, from the properties of symmetrization, Lemma 3.2 and the

fact that f is an extremal for (2.3), we get $\int_{\mathbb{R}^n} (f^*)^q = 1$ and

$$\begin{aligned}
 S &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)}{|x'|^\sigma} G(x-y) \frac{f(y)}{|y'|^\sigma} dy dx \\
 &= \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} \left(\int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \frac{f(x', x'')}{|x'|^\sigma} G(x' - y', x'' - y'') \frac{f(y', y'')}{|y'|^\sigma} \right) \\
 &\leq \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} \left(\int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \frac{f^{*'}(x', x'')}{|x'|^\sigma} G(x' - y', x'' - y'') \frac{f^{*'}(y', y'')}{|y'|^\sigma} \right) \\
 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f^{*'}(x)}{|x'|^\sigma} G(x-y) \frac{f^{*'}(y)}{|y'|^\sigma} dy dx \leq S.
 \end{aligned}$$

Hence we get equality in all the steps and therefore from Lemma 3.2, we obtain $f(., x'')$ is symmetrically decreasing for almost all $x'' \in \mathbb{R}^{n-k}$. Now let us prove the symmetry in the x'' variable. Let $f^{*''}$ be the function obtained by taking the $(n-k)$ dim symmetrization of $f(x', .)$ for a.e. $x' \in \mathbb{R}^k$. i.e. $f^{*''}(x', .) = (f(x', .))^*$, where $*$ denotes the rearrangement in $\mathbb{R}^{n-k}$. Then we have from Lemma 3.1 $\int_{\mathbb{R}^n} (f^{*''})^q = 1$ and

$$\begin{aligned}
 S &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \frac{1}{|x'|^\sigma} \frac{1}{|y'|^\sigma} \left(\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} f(x', x'') G(x' - y', x'' - y'') f(y', y'') \right) \\
 &\leq \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \frac{1}{|x'|^\sigma} \frac{1}{|y'|^\sigma} \left(\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^{n-k}} f^{*''}(x', x'') G(x' - y', x'' - y'') f^{*''}(y', y'') \right) \\
 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f^{*''}(x)}{|x'|^\sigma} G(x-y) \frac{f^{*''}(y)}{|y'|^\sigma} \leq S.
 \end{aligned}$$

Therefore we have equality in all the steps and in particular in the second step. Hence from Lemma 3.1 we obtain that for a.e. $x', y' \in \mathbb{R}^k$ the functions $f(x', .)$ and $f(y', .)$ are symmetrically decreasing with respect to a common point in $\mathbb{R}^{n-k}$. This is possible only when the common point is same for all pairs of points x', y' . In other words, there exists $x''_0 \in \mathbb{R}^{n-k}$ such that $x'' \rightarrow f(x', x'' + x''_0)$ is symmetrically decreasing in $\mathbb{R}^{n-k}$. This completes the proof of Theorem 1.1. $\square$

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