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Symmetrization and second-order Sobolev inequalities

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Abstract. A Pólya–Szegö principle for second-order derivatives is established. As a consequence, a new unified approach to second-order Sobolev-type inequalities, via 1-dimensional inequalities, is derived. Applications to some optimal Sobolev embeddings are exhibited.

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1. Introduction and main results

A form of the Pólya–Szegö principle tells us that if G is any open bounded subset of $\mathbb{R}^n$, $n \geq 1$ and u belongs to the Sobolev space $W_0^{1,p}(G)$, with $1 \leq p \leq \infty$, then the symmetric rearrangement $u^\star$ of u belongs to $W_0^{1,p}(G^\star)$ and

$$\||\nabla u^\star|\|_{L^p(G^\star)} \leq \||\nabla u|\|_{L^p(G)} \tag{1.1}$$

(see [4], [5], [8], [17], [20], [24], [28] [30]). Here, ∇u stands for the gradient of u, $|\nabla u| = (\sum_{i=1}^n u_{x_i}^2)^{1/2}$, and $G^\star$ is the ball centered at the origin and having the same measure as G. An equivalent formulation of this principle amounts to the following statement: whenever a function u belongs to $W_0^{1,p}(G)$, its decreasing rearrangement u^* is locally absolutely continuous in $(0, +\infty)$ and

$$\left\| n C_n{}^{1/n} r^{1-1/n} \big(- u^{*'}(r) \big) \right\|_{L^p(0,|G|)} \leq \||\nabla u|\|_{L^p(G)}, \tag{1.2}$$

where $C_n = \pi^{n/2}/\Gamma(1 + n/2)$, the measure of the unit ball in $\mathbb{R}^n$, prime denotes derivative and $|G|$ is the Lebesgue measure of G.

An extension of the Pólya–Szegö principle holds in any rearrangement invariant space, i.e., roughly speaking, in any Banach function space where the norm of a function depends only on the measure of its level sets. Indeed if $X(G)$ is any rearrangement invariant space endowed with the norm $\| \cdot \|_{X(G)}$, and u is any function from the Sobolev space $W_0^1 X(G)$ of those weakly differentiable functions

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in G, vanishing on ∂G, such that $|\nabla u| \in X(G)$, then u^* is locally absolutely continuous in $(0, +\infty)$ and

$$\left\| n C_n{}^{1/n} r^{1-1/n} (-u^{*'}(r)) \right\|_{\overline{X}(0,|G|)} \leq \| |\nabla u| \|_{X(G)}, \tag{1.3}$$

where $\overline{X}(0, |G|)$ is the representation space of $X(G)$ on $(0, |G|)$ (see e.g. ([14])). Moreover, equality holds in (1.3) if u is spherically symmetric. Recall that, besides Lebesgue spaces, customary examples of rearrangement invariant spaces are provided by Lorentz, Zygmund and Orlicz spaces. Precise definitions and properties concerning rearrangement invariant spaces and related topics coming into play in this section and throughout the whole paper are contained in Section 2.

Among other applications, inequality (1.2) and its generalizations have proved to be a very useful tool in the proof of (first-order) Sobolev-type inequalities and embeddings, especially when their sharp form is in question – see e.g. [4], [9], [10], [14], [16], [21], [24], [25], [30]. The reason is in that they enable us to reduce n-dimensional problems to equivalent 1-dimensional ones. Actually, as a consequence of (1.3), it is not difficult to see that the first-order Sobolev-type inequality

$$\|u\|_{Y(G)} \leq K_1 \| |\nabla u| \|_{X(G)}, \tag{1.4}$$

holds for every $u \in W_0^1 X(G)$ if and only if the Hardy-type inequality,

$$\left\| \int_s^{|G|} \phi(r) r^{-1+1/n} dr \right\|_{\overline{Y}(0,|G|)} \leq K_2 \|\phi(s)\|_{\overline{X}(0,|G|)}, \tag{1.5}$$

holds for every $\phi \in \overline{X}(0, |G|)$. Here, $X(G)$ and $Y(G)$ are rearrangement invariant spaces, and K_1 and K_2 are positive constants independent of u and ϕ, respectively.

It is well known that inequality (1.1) is not suitable for extensions to higher-order derivatives. Indeed, second – and, *a fortiori*, higher-order derivatives of $u^\star$ cannot be estimated in terms of derivatives of u just because it may happen that $u^\star$ is not twice weakly differentiable even if u is very smooth. (Incidentally, let us mention that a positive 1-dimensional result in this direction is proved in [13] in the framework of functions whose second-order derivative is a bounded measure.)

The main objective of this paper is to show that, instead, a proper second-order version of the Pólya–Szegö principle holds in a form patterned on (1.3), regarded as an inequality between the $X(G)$ norm of $|\nabla u|$ and a weighted $\overline{X}(0, |G|)$ norm of $u^{*'}$. Such a version is what is really needed in view of applications to second-order Sobolev inequalities. The result is achieved in Theorem 1.1 below and amounts to an inequality between any rearrangement invariant $X(G)$ norm of the second-order derivatives of u and still a weighted $\overline{X}(0, |G|)$ norm of $u^{*'}$, but with a different weight.

In what follows, $W^2X(G)$ denotes the second-order Sobolev space built upon $X(G)$, and $W_0^2X(G)$ the subspace of $W^2X(G)$ of those functions which vanish, together with their gradient, on ∂G (see Subsection 2.3).

Theorem 1.1. *Let G be an open bounded subset of $\mathbb{R}^n$, $n \geq 3$, and let $X(G)$ be a rearrangement invariant space on G. Then a positive constant $C(n)$, depending only on n, exists such that*

$$\left\| C(n) r^{1-2/n}\left(-u^{*'}(r)\right)\right\|_{\overline{X}(0,|G|)} \leq \| |\nabla^2 u| \|_{X(G)}, \tag{1.6}$$

for every $u \in W_0^2 X(G)$. Here, $\nabla^2 u$ is the Hessian matrix of u and $|\nabla^2 u| = (\sum_{i,j=1}^n u_{x_i x_j}^2)^{1/2}$.

The key step in the proof of Theorem 1.1 is the rearrangement inequality provided by the next result. In fact, owing to a well-known property of rearrangement invariant spaces ((2.8), Subsection 2.1), Theorem 1.1 immediately follows from:

Theorem 1.2. *Let u be any compactly supported function from $W^{2,1}(\mathbb{R}^n)$, $n \geq 3$. Then*

$$C(n) \int_0^s \left[(\cdot)^{1-2/n}\left(-u^{*'}(\cdot)\right)\right]^*(r)dr \leq \int_0^s |\nabla^2 u|^*(r)dr \quad \text{for } s \geq 0, \tag{1.7}$$

where $C(n)$ is the same constant as in Theorem 1.1.

Let us notice that, unlike (1.3), inequality (1.6) does not have an isoperimetric nature, in the sense that there does not exist a constant $C(n)$ for which equality holds in (1.6) whenever u is spherically symmetric. Therefore, one should not expect that inequality (1.6) can be applied to find the best constants in Sobolev inequalities. Nevertheless, it can be efficiently used to prove optimal results as far as function spaces involved in Sobolev inequalities are concerned, thus providing a sharp unified approach to second-order Sobolev-type embeddings. Indeed, inequality (1.6) enables us to show that, in analogy with first-order inequalities like (1.4), also second-order Sobolev inequalities involving rearrangement invariant norms are equivalent to suitable 1-dimensional inequalities of a Hardy-type. This is the content of:

Theorem 1.3. *Let G be an open bounded subset of $\mathbb{R}^n$, $n \geq 3$, and let $X(G)$ and $Y(G)$ be rearrangement invariant spaces on G. Then the following statements are equivalent:*

i) A positive constant K_1 exists such that

$$\|u\|_{Y(G)} \leq K_1 \| |\nabla^2 u| \|_{X(G)}, \tag{1.8}$$

for every $u \in W_0^2 X(G)$.

ii) A positive constant K_2 exists such that

$$\left\| \int_s^{|G|} \phi(r) r^{-1+2/n} dr \right\|_{\overline{Y}(0,|G|)} \leq K_2 \|\phi(s)\|_{\overline{X}(0,|G|)}, \tag{1.9}$$

for every $\phi \in \overline{X}(0,|G|)$.

Remark 1. Theorem 1.1 holds even if $G = \mathbb{R}^n$, and functions from $W^2X(\mathbb{R}^n)$ having compact support are taken into account. Hence, a version of Theorem 1.3 can be proved for rearrangement invariant spaces $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ on $\mathbb{R}^n$: one has just to replace G by $\mathbb{R}^n$ in i) and $|G|$ by ∞ in ii), and to consider functions u and ϕ with compact support in i) and ii), respectively.

Remark 2. Inequality (1.8) is equivalent to the continuous embedding

$$W_0^2X(G) \longrightarrow Y(G),$$

since $\|u\|_{W_0^2X(G)}$ and $\|\nabla^2 u\|_{X(G)}$ are equivalent norms on $W_0^2X(G)$ – see Subsection 2.3.

Remark 3. Our results are stated for $n \geq 3$, the case of interest in the present setting. In fact, the problem of second-order Sobolev inequalities is trivial in the 2-dimensional case, since inequality (1.8) holds for every $X(G)$ and $Y(G)$ in that case. This is a consequence of the Sobolev embedding

$$W_0^{2,1}(G) \longrightarrow L^\infty(G),$$

which holds for every bounded subset $G \subset \mathbb{R}^2$ (see [1, Theorem 5.4]), and of the embeddings $X(G) \to L^1(G)$ and $L^\infty(G) \to Y(G)$, which hold for every rearrangement invariant spaces $X(G)$ and $Y(G)$ [6, Chap. 2, Corollary 6.7].

Remark 4. The standard second- (and higher-) order Sobolev inequality for Lebesgue norms can be proved by iterating the first-order inequality. Let us stress, however, that this approach may not lead to optimal results when more general norms are taken into account.

It may be interesting to compare Theorems 1.1–1.3 with results in the same spirit involving the Riesz potential or the Laplace operator. The second-order Riesz potential operator I_2 is defined at any locally integrable function f decaying fast enough at infinity as

$$I_2(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy \qquad \text{for } x \in \mathbb{R}^n.$$

Owing to the O'Neil rearrangement inequality for convolutions [34, Lemma 1.8.8],

$$(I_2(f))^*(s) \leq \frac{n}{2} {C_n}^{1-2/n} \left(s^{-1+2/n} \int_0^s f^*(r)dr + \int_s^\infty f^*(r) r^{-1+2/n} dr \right) \text{ for } s > 0. \tag{1.10}$$

Estimate (1.10) can be used to prove:

Theorem 1.4. *Let G be an open bounded subset of $\mathbb{R}^n$, $n \geq 3$, and let $X(G)$ and $Y(G)$ be rearrangement invariant spaces on G. Then the following statements are equivalent:*

i) *A positive constant* K_1 *exists such that*

$$\|I_2(f)\|_{Y(G)} \leq K_1 \|f\|_{X(G)}, \tag{1.11}$$

for every function $f \in X(G)$ *with compact support in* G.

ii) *A positive constant* K_2 *exists such that inequality* (1.9) *and the inequality*

$$\left\| s^{-1+2/n} \int_0^s \phi(r)dr \right\|_{\overline{Y}(0,|G|)} \leq K_2 \|\phi(s)\|_{\overline{X}(0,|G|)}, \tag{1.12}$$

hold for every $\phi \in \overline{X}(0, |G|)$.

A standard representation formula tells us that

$$u(x) = \frac{1}{(2-n)nC_n} I_2(\Delta u)(x) \qquad \text{for a.e. } x \in \mathbb{R}^n, \tag{1.13}$$

for every compactly supported function u on $\mathbb{R}^n, n \geq 3$, whose distributional Laplacian Δu is an integrable function (see e.g. [33, Theorem 3.2.3/2]). Combining (1.13) and (1.10) yields

$$u^*(s) \leq \frac{1}{2(n-2){C_n}^{2/n}} \left(s^{-1+2/n} \int_0^s |\Delta u|^*(r)dr + \int_s^\infty |\Delta u|^*(r) r^{-1+2/n} dr \right) \quad \text{for } s > 0. \tag{1.14}$$

Hence, if u is any compactly supported function from $W^{2,1}(\mathbb{R}^n)$,

$$u^*(s) \leq \frac{\sqrt{n}}{2(n-2){C_n}^{2/n}} \left(s^{-1+2/n} \int_0^s |\nabla^2 u|^*(r)dr + \int_s^\infty |\nabla^2 u|^*(r) r^{-1+2/n} dr \right) \quad \text{for } s > 0. \tag{1.15}$$

Now, estimate (1.15) implies that the Sobolev inequality (1.8) is true provided that (1.9) and (1.12) are fulfilled. However, Theorem 1.3 ensures that the sole validity of (1.9) is sufficient (and necessary) for (1.8) to hold. In other words, Theorem 1.3 tells us that the first integral term on the right-hand side of (1.15) turns out to be negligible in view of applications to rearrangement invariant norm inequalities.

On the other hand, the first term on the right-hand side of (1.14) cannot be disregarded when dealing with norm inequalities between u and Δu. Actually, equality holds in (1.14) whenever Δu is non-negative and radially decreasing, provided that the constant $\frac{1}{2(n-2){C_n}^{2/n}}$ is replaced by $\frac{1}{n(n-2){C_n}^{2/n}}$. A precise counterpart of Theorem 1.3, when $|\nabla^2 u|$ is replaced by Δu, is given by the following result. In the statement, $W_0^{\Delta} X(G)$ denotes the space of those functions which vanish together with their gradient on ∂G and such that $\Delta u \in X(G)$ (see Subsection 2.3).

Theorem 1.5. *Let* G *be an open bounded subset of* $\mathbb{R}^n$, $n \geq 3$, *and let* $X(G)$ *and* $Y(G)$ *be rearrangement invariant spaces on* G. *Then the following statements are equivalent:*

i) *A positive constant K_1 exists such that*

$$\|u\|_{Y(G)} \leq K_1 \|\Delta u\|_{X(G)}, \tag{1.16}$$

for every $u \in W_0^{\Delta} X(G)$.

ii) *A positive constant K_2 exists such that inequalities* (1.9) *and* (1.12) *hold for every $\phi \in \overline{X}(0, |G|)$.*

In conclusion, Theorems 1.3−1.5 tell us that, as long as rearrangement invariant norms are concerned, potential inequalities and inequalities involving the Laplacian are equivalent, but they are not equivalent to full second-order Sobolev inequalities. This discrepancy between $\nabla^2 u$ and Δu is related to the fact that not every rearrangement invariant norm of $|\nabla^2 u|$ can be estimated in terms of the same norm of Δu. Loosely speaking, this estimate breaks down when the rearrangement invariant norm is either close to the L^1 norm or to the L^∞ norm. To be specific, such an estimate holds, for instance, in L^p for every $p \in (1, \infty)$, by the Calderon–Zygmund theorem on the boundedness of Riesz transforms, but neither holds for $p = 1$ nor for $p = \infty$. More generally, a theorem due to Boyd [6, Chap. 3, Theorem 5.18], implies that the norm of $|\nabla^2 u|$ in a rearrangement invariant space $X(G)$ is equivalent to the same norm of Δu if and only if the lower Boyd index $i_X > 1$ and the upper Boyd index $I_X < \infty$.

Theorems 1.3–1.5 also yield quantitative information about the gap between potential and Laplacian inequalities on one side, and second-order Sobolev inequalities on the other side: they show that this gap is precisely due to the first integral term on the right-hand sides of (1.10) and (1.14). Hence, the rearrangement invariant spaces $X(G)$ and $Y(G)$ for which inequalities (1.8), (1.11) and (1.16) are equivalent are exactly those where (1.9) implies (1.12). This is the case, for instance, when the function h_Y, defined as the norm of the dilation operator on $\overline{Y}(0, |G|)$ (see (2.12), Subsection 2.2) satisfies

$$\int_0 h_Y(t) t^{-2/n} dt < \infty \tag{1.17}$$

[16, Theorem 4.4]. In particular, condition (1.17) is fulfilled when the lower Boyd index i_Y of $Y(G)$ exceeds $\frac{n}{n-2}$.

Proofs of Theorem 1.2 and Theorem 1.3 will be given in Sections 3 and 4, respectively. A sketch of the proofs of Theorems 1.4 and 1.5 can be found at the end of Section 4. Section 5 contains some applications of Theorem 1.3: after recovering a few classical and more recent Sobolev inequalities, new results are derived, including an optimal embedding theorem for Orlicz–Sobolev spaces and a characterization of those spaces of Sobolev-type which are embedded into the space of bounded functions.

2. Background

We collect in this section some definitions, notations and properties about functions and function spaces involved in our discussion.

2.1. Rearrangements

Let f be a real-valued measurable function on a subset G of $\mathbb{R}^n$. Then the *distribution function* μ_f of f is defined as

$$\mu_f(t) = |\{x \in G : |f(x)| > t\}| \qquad \text{for } t \geq 0. \tag{2.1}$$

The *decreasing rearrangement* f^* of f is the right-continuous non-increasing function from $[0, \infty)$ into $[0, \infty]$ which is equimeasurable with f. Namely,

$$f^*(s) = sup\{t \geq 0 : \mu_f(t) > s\} \qquad \text{for } s \geq 0. \tag{2.2}$$

Notice that $supp\, f^* \subseteq [0, |G|]$. Since f^* is non-increasing, the function f^{**}, defined by

$$f^{**}(s) = \frac{1}{s}\int_0^s f^*(r)dr \qquad \text{for } s \geq 0, \tag{2.3}$$

is also non-increasing and $f^* \leq f^{**}$.

It is easily seen that if g is any radial function on $\mathbb{R}^n$ having the form

$$g(x) = \phi(C_n|x|^n)$$

for some real-valued measurable function $\phi : [0, \infty) \to \mathbb{R}$, then

$$g^* = \phi^*. \tag{2.4}$$

Hence, in particular, the *spherically symmetric* rearrangement $f^\star$ of f, given by

$$f^\star(x) = f^*(C_n|x|^n), \tag{2.5}$$

is a radially decreasing function about the origin which is equimeasurable with f.

A special case of the Hardy–Littlewood inequality tells us that

$$\int_E |f|dx \leq \int_0^{|E|} f^*(r)dr, \tag{2.6}$$

for every measurable subset E of G. In fact, the stronger property,

$$\sup_{|E|=s}\int_E |f|dx = \int_0^s f^*(r)dr, \tag{2.7}$$

holds for a.e. $s \in [0, |G|]$.

2.2. *Rearrangement invariant spaces*

Let G be a measurable subset of $\mathbb{R}^n$, and let $X(G)$ be a Banach space of measurable real-valued functions in G, equipped with the norm $\|\cdot\|_{X(G)}$. We say that $X(G)$ is a *rearrangement invariant Banach function space* (briefly an *r.i. space*) if the following properties are fulfilled:

i) if $0 \leq g \leq f$ a.e. in G and $f \in X(G)$, then $g \in X(G)$ and $\|g\|_{X(G)} \leq \|f\|_{X(G)}$;
ii) if f_n is a sequence such that $0 \leq f_n \nearrow f$ a.e. in G and $f \in X(G)$, then $\|f_n\|_{X(G)} \nearrow \|f\|_{X(G)}$;
iii) $\|\chi_E\|_{X(G)} < \infty$ for every $E \subseteq G$ with $|E| < \infty$; here, χ_E denotes the characteristic function of the set E;
iv) for every $E \subseteq G$ with $|E| < \infty$, there exists a constant C such that $\int_E f dx \leq C\|f\|_{X(G)}$ whenever $f \in X(G)$;
v) if $f \in X(G)$ and $g^* = f^*$, then $g \in X(G)$ and $\|g\|_{X(G)} = \|f\|_{X(G)}$.

A property of r.i. spaces states that

$$\text{if } f \in X(G) \text{ and } \int_0^s g^*(r)dr \leq \int_0^s f^*(r)dr \text{ for } s \geq 0, \qquad (2.8)$$
$$\text{then } g \in X(G) \text{ and } \|g\|_{X(G)} \leq \|f\|_{X(G)}.$$

Given any r.i. space $X(G)$, the set

$$X^{'}(G) = \Big\{ g : g \text{ is a real-valued measurable function in } G \text{ and } \int_G |fg|dx < \infty \text{ for all } f \in X(G) \Big\}$$

is again an r.i. space endowed with the norm

$$\|g\|_{X'(G)} = \sup_{f \neq 0} \frac{\int |fg|dx}{\|f\|_{X(G)}}, \qquad (2.9)$$

and is called the *associate space* of $X(G)$. In a sense, $X^{'}(G)$ plays a role of a dual of $X(G)$ in this framework, and it actually agrees with the topological dual of $X(G)$ in many situations. For every r.i. space $X(G)$, $(X^{'})^{'}(G) = X(G)$. Equation (2.9) obviously implies the Hölder-type inequality

$$\int_G |fg|dx \leq \|f\|_{X(G)} \|g\|_{X'(G)}, \qquad (2.10)$$

for every $f \in X(G)$ and $g \in X^{'}(G)$.

The *representation space* $\overline{X}(0, |G|)$ of an r.i. space $X(G)$ is the unique r.i. space on $[0, |G|]$ such that

$$\|f\|_{X(G)} = \|f^*\|_{\overline{X}(0,|G|)}, \qquad (2.11)$$

for every $f \in X(G)$. It is equipped with the norm

$$\|\phi\|_{\overline{X}(0,|G|)} = \sup_{\|g\|_{X'(G)} \le 1} \int_0^{|G|} \phi^*(r) g^*(r) dr.$$

Let us notice that, for customary r.i. spaces, the norm in $\overline{X}(0, |G|)$ can be immediately computed from the norm in the original space $X(G)$.

For each $t > 0$, the *dilation operator* E_t is defined at a measurable function ϕ on $[0, |G|]$ as

$$E_t \phi(s) = \begin{cases} \phi(st) & \text{if } 0 \le s \le \min\{|G|, |G|/t\} \\ 0 & \text{if } \min\{|G|, |G|/t\} < s \le |G|. \end{cases} \tag{2.12}$$

The operator E_t is bounded on $\overline{X}(0, |G|)$ for every r.i. space $X(G)$ and for every $t > 0$; moreover, the norm $h_X(t)$ of E_t on $\overline{X}(0, |G|)$ satisfies

$$h_X(t) \le max\{1, 1/t\}. \tag{2.13}$$

We define the *lower Boyd index* i_X of $X(G)$ as

$$i_X = \lim_{t \to 0+} \frac{log(1/t)}{log(h_X(t))}. \tag{2.14}$$

One has $1 \le i_X \le \infty$ for every r.i. space $X(G)$. The index i_X, and the corresponding *upper index* I_X, defined on taking the limit as $t \to +\infty$ in (2.14), play a role in the theory of interpolation.

Our applications will include Lebesgue, Lorentz and Orlicz spaces. Recall that the *Lorentz space* $L^{p,q}(G)$, with $1 < p < \infty$ and $1 \le q \le \infty$ or $p = q = \infty$, is the r.i. space consisting of all real-valued measurable functions f in G for which the quantity

$$\|f\|_{L^{p,q}(G)} = \begin{cases} \left(\frac{1}{p} \int_0^\infty (r^{1/p} f^*(r))^q \frac{dr}{r} \right)^{1/q} & \text{if } q < \infty \\ \sup_{r>0} r^{1/p} f^*(r) & \text{if } q = \infty \end{cases} \tag{2.15}$$

is finite. This quantity is a norm if $q \le p$; in general, it is equivalent to the norm obtained on replacing f^* by f^{**} on the right-hand side of (2.15). The associate space of $L^{p,q}(G)$ is, up to equivalent norms, $L^{p',q'}(G)$, where $p' = \frac{p}{p-1}$, the Hölder conjugate of p, and q' is defined analogously. The representation space of $L^{p,q}(G)$ is obviously $L^{p,q}(0, |G|)$.

Given any *Young function* A, i.e. a non-decreasing convex function from $[0, \infty)$ into $[0, \infty]$ vanishing at 0, the *Orlicz space* $L^A(G)$ is defined as the collection of all real-valued measurable functions f on G satisfying $\int_G A\left(\frac{|f|}{\lambda}\right) dx < \infty$ for some $\lambda > 0$. $L^A(G)$, endowed with the *Luxemburg norm*

$$\|f\|_{L^A(G)} = inf\left\{\lambda > 0 : \int_G A\left(\frac{|f|}{\lambda}\right) dx \le 1\right\}, \tag{2.16}$$

is an r.i. space. The associate space of $L^A(G)$ is, up to equivalent norms, the Orlicz space $L^{\widetilde{A}}(G)$, where $\widetilde{A}(s) = sup\{rs - A(r) : r \geq 0\}$, the *Young conjugate* of A. Moreover, the representation space of $L^A(G)$ is $L^A(0, |G|)$.

For a comprehensive treatment of rearrangement invariant spaces we refer to [6].

2.3. *Spaces of Sobolev functions*

Let G be an open subset of $\mathbb{R}^n$ and let $X(G)$ be an r.i. space on G. We define the *first-order Sobolev space* $W^1X(G)$ built upon $X(G)$ as

$$W^1X(G) = \big\{u : u \text{ is a weakly differentiable function from } X(G) \text{ such that } |\nabla u| \in X(G)\big\}, \tag{2.17}$$

endowed with the norm $\|u\|_{W^1X(G)} = \|u\|_{X(G)} + \||\nabla u|\|_{X(G)}$. Similarly, the *second-order Sobolev space* $W^2X(G)$ is defined as

$$W^2X(G) = \big\{u : u \text{ is a twice weakly differentiable function from } X(G) \text{ such that } |\nabla u|, |\nabla^2 u| \in X(G)\big\}, \tag{2.18}$$

and $\|u\|_{W^2X(G)} = \|u\|_{X(G)} + \||\nabla u|\|_{X(G)} + \||\nabla^2 u|\|_{X(G)}$.

By $W_0^kX(G)$, $k = 1, 2$, we denote the subspace of $W^kX(G)$ of those functions whose continuation by 0 outside G is still k times weakly differentiable in the whole of $\mathbb{R}^n$. Since the Poincaré-type inequality

$$\|u\|_{X(G)} \leq \left(\frac{|G|}{C_n}\right)^{1/n} \||\nabla u|\|_{X(G)} \tag{2.19}$$

holds whenever $|G| < \infty$ and $u \in W_0^1X(G)$ (see [14, Lemma 4.2]), $\||\nabla u|\|_{X(G)}$ is a norm equivalent to $\|u\|_{W^1X(G)}$ on $W_0^1X(G)$. An application of (2.19) with u replaced by $|\nabla u|$ shows that $\||\nabla^2 u|\|_{X(G)}$ is a norm equivalent to $\|u\|_{W^2X(G)}$ on $W_0^2X(G)$.

In the case when $X(G) = L^p(G)$, we adopt the usual notations $W^{k,p}(G)$ and $W_0^{k,p}(G)$, $k = 1, 2$, for $W^kL^p(G)$ and $W_0^kL^p(G)$, respectively. Analogously, when $X(G) = L^A(G)$, we write $W^{k,A}(G)$ and $W_0^{k,A}(G)$, instead of $W^kL^A(G)$ and $W_0^kL^A(G)$.

The notation $W_0^{\Delta}X(G)$ is employed for the space of those functions in G such that the distributional Laplacian of their continuation by 0 outside G is an integrable function in $\mathbb{R}^n$ which belongs to $X(G)$.

2.4. *Spaces of functions of BV type*

Even if not a primary object of this paper, we shall be led, for technical reasons, to deal also with functions of bounded variation.

Let G be an open bounded subset of $\mathbb{R}^n$. A function $u \in L^1(G)$ is called of *bounded variation* if its first-order distributional partial derivatives are signed

Radon measures with finite total variation in G. The class of functions of bounded variation on G will be denoted by $BV(G)$. If $u \in BV(G)$, the distributional gradient Du of u is a vector-valued measure whose total variation $\|Du\|$ is a finite measure on G and

$$\|Du\|(G) = sup \left\{ \int_G u\, div\Theta dx : \Theta \in C_0^\infty(G, \mathbb{R}^n), |\Theta(x)| \leq 1 \text{ for } x \in G \right\}.$$

We denote by $BV^2(G)$ the space of functions $u \in W^{1,1}(G)$ whose second-order distributional derivatives are signed Radon measures having finite total variation in G. Thus, $u \in BV^2(G)$ if and only if $u \in W^{1,1}(G)$ and $u_{x_i} \in BV(G)$ for $i = 1, \dots, n$. The Hessian of a function u from $BV^2(G)$ is a matrix-valued measure denoted by D^2u.

Given $u \in BV(G)$, the measure Du can be split into an absolutely continuous part with respect to the Lebesgue measure and a singular part. The density of the absolutely continuous part with respect to the Lebesgue measure will be denoted by ∇u. Thus, in particular, $dDu = \nabla u\, dx$ if $u \in W^{1,1}(G)$.

The *lower* and *upper approximate limits* u_- and u_+ of a function $u \in BV(G)$ are defined as

$$u_-(x) = sup\{t \in \mathbb{R} : D(\{u < t\}, x) = 0\}$$

and

$$u_+(x) = inf\{t \in \mathbb{R} : D(\{u > t\}, x) = 0\},$$

respectively. Here, $D(E, x)$ denotes the density of a measurable set $E \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ given by $D(E, x) = \lim_{r\to 0+} \frac{|E\cap B(x,r)|}{|B(x,r)|}$, where $B(x, r)$ is the ball of radius r centered at x. The functions u_- and u_+ are finite almost everywhere with respect to H^{n-1}, the $(n-1)$-dimensional Hausdorff measure. In what follows, we adopt the convention that u agrees with its *precise representative*, which is defined as $\frac{u_-(x)+u_+(x)}{2}$ when $u_-(x)$ and $u_+(x)$ are both finite, and equals 0 otherwise. Recall that, if $u \in W^{1,1}(G)$, then $u_-(x) = u_+(x)$ for H^{n-1} a.e. $x \in G$.

A set $E \subseteq \mathbb{R}^n$ is said to have *finite perimeter* if $D\chi_E$ is a (vector-valued) Radon measure with finite total variation on $\mathbb{R}^n$. The *perimeter* of E is defined as

$$P(E) = \|D\chi_E\|(\mathbb{R}^n).$$

A measurable set $E \subseteq \mathbb{R}^n$ has finite perimeter if and only if its *measure theoretic boundary* (also called essential boundary) $\partial_M E$, defined as

$$\partial_M E = \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : D(E; x) = 0 \text{ or } D(\mathbb{R} \setminus E) = 0\},$$

has finite $(n-1)$-dimensional Hausdorff measure. Moreover,

$$P(E) = H^{n-1}(\partial_M E). \tag{2.20}$$

We conclude by recalling two classical results in the theory of BV functions which are basic for our analysis: the isoperimetric inequality and the coarea formula. The classical isoperimetric inequality states that

$$nC_n{}^{1/n}|E|^{1-1/n} \leq P(E), \tag{2.21}$$

for every set $E \subseteq \mathbb{R}^n$ of finite perimeter having finite measure.

The coarea formula tells us that if $u \in BV(\mathbb{R}^n)$, then its level sets $\{x \in \mathbb{R}^n : u(x) > t\}$ have finite perimeter for a.e. $t \in \mathbb{R}$ and

$$\|Du\|(\mathbb{R}^n) = \int_{-\infty}^{+\infty} P(\{u > t\})dt. \tag{2.22}$$

A generalized version of this formula ensures that if $u \in W^{1,1}(\mathbb{R}^n)$ and f is any Borel function on $\mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} f|\nabla u|dx = \int_{-\infty}^{+\infty} \int_{\{u=t\}} f dH^{n-1}(x)dt. \tag{2.23}$$

Standard references for functions of bounded variation include [18], [24], [34]. An updated thorough treatment of this topic is contained in [3].

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is carried out along the following lines. Our point of departure is Lemma 3.3 below, providing an inequality involving $u^{*'}$ which differs from (1.7) essentially in that $\int_0^s r^{1-2/n}(-u^{*'}(r))dr$ appears on the left-hand side instead of $\int_0^s [(\cdot)^{1-2/n}(-u^{*'}(\cdot))]^*(r)dr$. In a sense, such inequality is effective as long as values of s close to 0 are taken into account. In the subsequent Lemma 3.4 we prove a version, under somewhat different assumptions, of a pointwise estimate for $u^{*'}(s)$ in terms of $\int_0^s |\Delta u|^*(r)dr$ contained in [31] (see also [23] for related results). In view of our purposes, this estimate is accurate for values of s far away from 0. An underlying idea behind the proofs of Lemmas 3.3–3.4 is to think of $W^{2,1}(\mathbb{R}^n)$ as embedded in $BV^2(\mathbb{R}^n)$. This allows us to employ certain fine properties of functions of BV type, such as the results of [27] on the truncation of functions from BV^2 and a theorem from [3] concerning piecewise BV functions. The last step of the proof of Theorem 1.2 consists in a careful combined use of Lemmas 3.3–3.4.

We begin with two preliminary results. The first one, Lemma 3.1, substantiates an application of the Gauss–Green theorem on the level sets of functions from $W^{2,1}(\mathbb{R}^n)$. The resulting formula will be used in both proofs of Lemmas 3.3 and 3.4.

Lemma 3.1. *Let u be any compactly supported function from $W^{2,1}(\mathbb{R}^n)$, $n \geq 1$. Then*

$$\int_{\{|u|=t\}} |\nabla u|dH^{n-1} = -\int_{\{|u|>t\}} sign(u)\Delta u\, dx \quad \textit{for a.e. } t > 0. \tag{3.1}$$

Proof. Given $t > 0$, define $\Psi : \mathbb{R} \to [0, \infty)$ as $\Psi(s) = \max\{s - t, 0\}$. It is easily seen that

$$\Psi^{'}(s) = \begin{cases} 1 & \text{if } s > t \\ 1/2 & \text{if } s = t \\ 0 & \text{if } s < t, \end{cases} \tag{3.2}$$

(recall that we are taking into account the precise representative of $\Psi^{'}$) and that

$$D^2\Psi = \delta_t, \tag{3.3}$$

the Dirac mass concentrated at t. Thus, Ψ is a Lipschitz continuous function whose second-order derivative is a measure with finite total variation on $\mathbb{R}$. Then Theorem 2 of [27], applied on a bounded open subset Ω of $\mathbb{R}^n$ having a smooth boundary and containing *supp* u, ensures that $\Psi(u) \in BV^2(\mathbb{R}^n)$. Moreover, formula (0.32) of the same paper yields

$$\Delta(\Psi(u))(\mathbb{R}^n) = \int_{\mathbb{R}} \int_{\{u=s\}} |\nabla u(x)| dH^{n-1}(x) dD^2\Psi(s) + \int_{\mathbb{R}^n} \Psi^{'}(u(x))\Delta u(x) dx. \tag{3.4}$$

Equation (3.3) implies that

$$\int_{\mathbb{R}} \int_{\{u=s\}} |\nabla u(x)| dH^{n-1}(x) dD^2\Psi(s) = \int_{\{u=t\}} |\nabla u(x)| dH^{n-1}(x). \tag{3.5}$$

On the other hand, by (3.2),

$$\int_{\mathbb{R}^n} \Psi^{'}(u)\Delta u\, dx = \int_{\{u>t\}} \Delta u\, dx + \frac{1}{2}\int_{\{u=t\}} \Delta u\, dx = \int_{\{u>t\}} \Delta u\, dx \quad \text{for a.e. } t > 0. \tag{3.6}$$

Notice that the second equality in (3.6) holds because $|\{u = t\}| = 0$, possibly except countably many values of t.

Observe now that $\nabla(\Psi(u))$ is a vector-valued BV function in $\mathbb{R}^n$, since $\Psi(u) \in BV^2(\mathbb{R}^n)$, and that $\nabla(\Psi(u))$ vanishes in a neighborhood of $\partial\Omega$, since u does. By the Gauss–Green theorem for BV vector fields [34, Exercise 5.6],

$$\Delta(\Psi(u))(\mathbb{R}^n) = div(\nabla(\Psi(u))(\Omega) = \int_{\partial\Omega} \nabla(\Psi(u)) \cdot \upsilon\, dH^{n-1} = 0, \tag{3.7}$$

where υ is the outward unit normal to $\partial\Omega$ and "$\cdot$" denotes the scalar product in $\mathbb{R}^n$. From (3.4)–(3.7) we deduce

$$\int_{\{u=t\}} |\nabla u| dH^{n-1} = -\int_{\{u>t\}} \Delta u\, dx \quad \text{for a.e. } t > 0. \tag{3.8}$$

Equation (3.8), applied to $-u$, yields

$$\int_{\{u=-t\}} |\nabla u| dH^{n-1} = \int_{\{u<-t\}} \Delta u\, dx \quad \text{for a.e. } t > 0. \tag{3.9}$$

Equation (3.1) follows from (3.8)–(3.9). □

The second preliminary lemma is a borderline second-order Sobolev inequality (of possible independent interest because of the sharp constant) and will be applied in the proof of Lemma 3.3.

Lemma 3.2. *Let* $n \geq 3$. *Then*

$$\|u\|_{L^{\frac{n}{n-2},1}(\mathbb{R}^n)} \leq \frac{1}{n(n-1){C_n}^{2/n}} \|D|\nabla u||(\mathbb{R}^n), \tag{3.10}$$

for every $u \in BV^2(\mathbb{R}^n)$. *Equality holds in* (3.10) *when both* u *and* $|\nabla u|$ *are radially decreasing.*

Proof. We have

$$\begin{aligned} \||\nabla u|\|_{L^{\frac{n}{n-1},1}(\mathbb{R}^n)} &= \int_0^\infty \mu_{|\nabla u|}(t)^{1-1/n} dt \leq \frac{1}{n{C_n}^{1/n}} \int_0^\infty P(\{|\nabla u| > t\}) dt \\ &= \frac{1}{n{C_n}^{1/n}} \|D|\nabla u||(\mathbb{R}^n), \end{aligned} \tag{3.11}$$

where the first equation follows from (2.15), via Fubini's theorem, the inequality is due to the isoperimetric inequality (2.21), and the last equation is an application of the coarea formula (2.22), with u replaced by $|\nabla u|$. Notice that in (3.11) we have made use of the fact that $|\nabla u| \in BV(\mathbb{R}^n)$, since ∇u is a vector-valued BV function and $|\cdot|$ is Lipschitz continuous. Notice also that equality holds in (3.11) when the level sets $\{|\nabla u| > t\}$ are balls for a.e. $t > 0$, and hence, in particular, when $|\nabla u|$ is radially decreasing.

On the other hand, a result from [2] ensures that

$$\|u\|_{L^{\frac{n}{n-2},1}(\mathbb{R}^n)} \leq \frac{1}{(n-1){C_n}^{1/n}} \||\nabla u|\|_{L^{\frac{n}{n-1},1}(\mathbb{R}^n)}, \tag{3.12}$$

and that equality holds in (3.12) if u and $|\nabla u|$ are radially decreasing on $\mathbb{R}^n$. The conclusion follows from (3.11)–(3.12). □

Lemma 3.3. *Let* u *be any compactly supported function from* $W^{2,1}(\mathbb{R}^n)$, $n \geq 3$. *Then*

$$n(n-1){C_n}^{2/n} \int_0^s r^{1-2/n} \big(-u^{*'}(r)\big) dr \leq \int_0^s (|\nabla|\nabla u|| + |\Delta u|)^*(r) dr \quad \text{for } s \geq 0. \tag{3.13}$$

Proof. Given $t > 0$, define $\Phi : \mathbb{R} \to \mathbb{R}$ as $\Phi(s) = sign(s) max\{|s| - t, 0\}$, a Lipschitz function. By Theorem 2 of [27], $\Phi(u) \in BV^2(\mathbb{R}^n)$. Thus, we may apply inequality (3.10), with u replaced by $\Phi(u)$, and get

$$\|\Phi(u)\|_{L^{\frac{n}{n-2},1}(\mathbb{R}^n)} \leq \frac{1}{n(n-1){C_n}^{2/n}} \|D|\nabla \Phi(u)||(\mathbb{R}^n). \tag{3.14}$$

Let us first estimate the right-hand side of (3.14) from above. Since $\Phi(u) \in BV^2(\mathbb{R}^n)$, then $\nabla \Phi(u)$ is a vector-valued BV function and, since $|\cdot|$ is Lipschitz continuous, $|\nabla\Phi(u)| \in BV(\mathbb{R}^n)$. By the chain rule for Sobolev functions (see e.g. [34, Theorem 2.1.11]),

$$|\nabla\Phi(u)(x)| = |\nabla u(x)|\chi_{\{|u|>t\}}(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{3.15}$$

The set $\{|u| > t\}$ has finite perimeter for a.e. $t > 0$, since $u \in BV(\mathbb{R}^n)$. Moreover, $|\nabla u| \in W^{1,1}(\mathbb{R}^n)$, inasmuch as $u \in W^{2,1}(\mathbb{R}^n)$. Combining all these facts enables us to apply Theorem 3.84 of [3] and obtain

$$\|D|\nabla\Phi(u)|\|(\mathbb{R}^n) = \int_{\{|u|>t\}} |\nabla|\nabla u||dx + \int_{\partial_M\{|u|>t\}} |\nabla u|dH^{n-1} \text{ for a.e. } t > 0. \tag{3.16}$$

Notice that in (3.16) we have made use of the fact that $\partial_M\{|u| > t\}$ agrees, up to a set of H^{n-1} measure 0, with the so-called reduced boundary of $\{|u| > t\}$ [3, Theorem 3.61].

Since $u \in W^{1,1}(\mathbb{R}^n)$, one has

$$H^{n-1}(\partial_M\{|u| > t\} \setminus \{|u| = t\}) = 0 \quad \text{for a.e. } t > 0 \tag{3.17}$$

[8, Equation (19)]. From (3.16)–(3.17) and from Lemma 3.1 we get

$$\|D|\nabla\Phi(u)|\|(\mathbb{R}^n) \leq \int_{\{|u|>t\}} |\nabla|\nabla u||dx - \int_{\{|u|>t\}} sign(u)\Delta u\, dx \quad \text{for a.e. } t > 0. \tag{3.18}$$

Inequality (3.18) implies, via (2.6),

$$\|D|\nabla\Phi(u)|\|(\mathbb{R}^n) \leq \int_0^{\mu_u(t)} (|\nabla|\nabla u|| + |\Delta u|)^*(r)dr \quad \text{for a.e. } t > 0. \tag{3.19}$$

We next evaluate the left-hand side of (3.14). An application of Fubini's theorem shows that

$$\|\Phi(u)\|_{L^{\frac{n}{n-2},1}(\mathbb{R}^n)} = \int_t^\infty \mu_u(\tau)^{1-2/n}d\tau \quad \text{for } t > 0. \tag{3.20}$$

We claim that

$$\int_t^\infty \mu_u(\tau)^{1-2/n}d\tau = \int_0^{\mu_u(t)} r^{1-2/n}\big(-u^{*'}(r)\big)dr \quad \text{for } t > 0. \tag{3.21}$$

Indeed, it is easily seen, by an application of De la Vallée–Poussin theorem on the change of variables, that

$$\int_t^\infty \mu_u(\tau)^{1-2/n}d\tau = \int_0^{\mu_u(t)} \mu_u(u^*(r))^{1-2/n}\big(-u^{*'}(r)\big)dr \quad \text{for } t > 0. \tag{3.22}$$

Here we have made use of the fact that $u^*(\mu_u(t)) = t$ for $t > 0$, since u^* is continuous on $[0, \infty)$. Now, a set $F \subset [0, \infty)$ exists such that:

i) $[0,\infty)\setminus F=\bigcup_{i\in I}(\alpha_i,\beta_i)$, where I is a countable set of indices;
ii) $(\alpha_i,\beta_i)\cap(\alpha_j,\beta_j)=\varnothing$ if $i\neq j$;
iii) u^* is constant on each (α_i,β_i) and hence, in particular, $u^{*'}\equiv 0$ on each (α_i,β_i);
iv) $\mu_u(u^*(s))=s$ for $s\in F$

(see e.g. [8, proof of Lemma 2.4]). Thus,

$$\int_0^{\mu_u(t)}\mu_u(u^*(r))^{1-2/n}\big(-u^{*'}(r)\big)dr=\int_{F\cap[0,\mu_u(t)]}\mu_u(u^*(r))^{1-2/n}\big(-u^{*'}(r)\big)dr$$
$$=\int_{F\cap[0,\mu_u(t)]}r^{1-2/n}\big(-u^{*'}(r)\big)dr=\int_0^{\mu_u(t)}r^{1-2/n}\big(-u^{*'}(r)\big)dr\quad\text{for } t>0. \tag{3.23}$$

Equation (3.21) follows from (3.22)–(3.23). Combining (3.14), (3.19), (3.20) and (3.21) yields

$$n(n-1)C_n{}^{2/n}\int_0^{\mu_u(t)}r^{1-2/n}\big(-u^{*'}(r)\big)dr \tag{3.24}$$
$$\leq\int_0^{\mu_u(t)}(|\nabla|\nabla u||+|\Delta u|)^*(r)dr\quad\text{for a.e. } t>0.$$

Inequality (3.24) immediately implies (3.13) if $s\in Im(\mu)$, owing to the continuity of μ from the right. If $s\notin Im(\mu)$, then there exists $t_0\geq 0$ such that $\mu(t_0)<s\leq\mu(t_0-)$. Since $u^*\equiv t_0$ on $(\mu(t_0),\mu(t_0-))$, then $u^{*'}\equiv 0$ on $(\mu(t_0),\mu(t_0-))$. Consequently, by (3.24),

$$n(n-1)C_n{}^{2/n}\int_0^{s}r^{1-2/n}\big(-u^{*'}(r)\big)dr$$
$$=n(n-1)C_n{}^{2/n}\int_0^{\mu_u(t_0)}r^{1-2/n}\big(-u^{*'}(r)\big)dr$$
$$\leq\int_0^{\mu_u(t_0)}(|\nabla|\nabla u||+|\Delta u|)^*(r)dr\leq\int_0^{s}(|\nabla|\nabla u||+|\Delta u|)^*(r)dr.$$

Thus, (3.13) holds in this case too. □

Lemma 3.4. *Let u be any compactly supported function from $W^{2,1}(\mathbb{R}^n)$, $n\geq 2$. Then*

$$n^2C_n{}^{2/n}s^{1-2/n}\big(-u^{*'}(s)\big)\leq\frac{1}{s}\int_0^s|\Delta u|^*(r)dr\quad\textit{for a.e. } s\geq 0. \tag{3.25}$$

Proof. Formula (2.23) implies that

$$\mu_u(t)=|\{|\nabla u|=0\}\cap\{|u|>t\}|+\int_t^\infty\int_{\{|u|=\tau\}}\frac{1}{|\nabla u(x)|}dH^{n-1}(x)d\tau\quad\text{for } t>0.$$

Hence, if $t_1 > t_2 > 0$,

$$\mu_u(t_2) - \mu_u(t_1) \geq \int_{t_2}^{t_1} \int_{\{|u|=t\}} \frac{1}{|\nabla u(x)|} dH^{n-1}(x)dt. \tag{3.26}$$

Hölder's inequality gives

$$H^{n-1}(\{|u|=t\})^2 \leq \left(\int_{\{|u|=t\}} \frac{1}{|\nabla u(x)|} dH^{n-1}(x)dt\right) \left(\int_{\{|u|=t\}} |\nabla u(x)| dH^{n-1}(x)dt\right) \text{ for } t > 0. \tag{3.27}$$

Thus, by Lemma 3.1 and by inequality (2.6),

$$\begin{aligned} H^{n-1}(\{|u|=t\})^2 &\leq \left(\int_{\{|u|=t\}} \frac{1}{|\nabla u(x)|} dH^{n-1}(x)dt\right) \left(\int_{\{|u|>t\}} -sign(u)\Delta u dx\right) \\ &\leq \left(\int_{\{|u|=t\}} \frac{1}{|\nabla u(x)|} dH^{n-1}(x)dt\right) \left(\int_0^{\mu_u(t)} |\Delta u|^*(r)dr\right) \\ &\quad \text{for a.e. } t > 0. \end{aligned} \tag{3.28}$$

On the other hand, the isoperimetric inequality (2.21) applied to the set $\{|u| > t\}$ and equations (2.20) and (3.17) yield

$$n{C_n}^{1/n} \mu_u(t)^{1-1/n} \leq H^{n-1}(\{|u|=t\}) \quad \text{for a.e. } t > 0. \tag{3.29}$$

From (3.26), (3.28) and (3.29) one gets

$$\begin{aligned} \mu_u(t_2) - \mu_u(t_1) &\geq n^2 {C_n}^{2/n} \int_{t_2}^{t_1} \frac{\mu_u(t)^{2-2/n}}{\int_0^{\mu_u(t)} |\Delta u|^*(r)dr} dt \\ &\geq \frac{n^2 {C_n}^{2/n}(t_1 - t_2)\mu_u(t_1)^{2-2/n}}{\int_0^{\mu_u(t_2)} |\Delta u|^*(r)dr}. \end{aligned} \tag{3.30}$$

Inequality (3.30) implies that

$$-\frac{u^*(s_2) - u^*(s_1)}{s_2 - s_1} n^2 {C_n}^{2/n} {s_1}^{2-2/n} \leq \int_0^{s_2} |\Delta u|^*(r)dr, \tag{3.31}$$

for every $s_2 > s_1 > 0$. Indeed, let F and $(\alpha_i, \beta_i), i \in I$, be subsets of $[0, \infty)$ satisfying properties i)–iv) as in the proof of Lemma 3.3. If $s_1, s_2 \in F$, then (3.31) follows from (3.30) with $t_i = u^*(s_i), i = 1, 2$, since $\mu_u(u^*(s_i)) = s_i$. Assume now that $s_1, s_2 \in [0, \infty) \setminus F = \cup_{i \in I}(\alpha_i, \beta_i)$. If $s_1, s_2 \in (\alpha_i, \beta_i)$ for some $i \in I$, then (3.31) trivially holds because the left-hand side vanishes. If, on the contrary, $s_1 \in (\alpha_i, \beta_i)$ and $s_2 \in (\alpha_j, \beta_j)$ for some $i \neq j$, then, by (3.31) with $s_2 = \alpha_j \in F$ and $s_1 = \beta_i \in F$, we have

$$-\frac{u^*(\alpha_j) - u^*(\beta_i)}{\alpha_j - \beta_i} n^2 {C_n}^{2/n} {\beta_i}^{2-2/n} \leq \int_0^{\alpha_j} |\Delta u|^*(r)dr.$$

Hence, since $s_2 - s_1 \geq \alpha_j - \beta_i$, and $u^*(\beta_i) = u^*(s_1) \geq u^*(s_2) = u^*(\alpha_j)$ we get (3.31). A similar argument leads to (3.31) also in the case when $s_1 \in F$ and $s_2 \in [0, \infty) \setminus F$, or viceversa. Passing to the limit in (3.31) as $s_2 \to s_1+$ yields the conclusion. □

We are now in a position to accomplish the proof of Theorem 1.2:

Proof of Theorem 1.2. The main step consists in showing that a constant $C(n)$, depending only on n, exists such that

$$C(n) \sum_{i=1}^{m} \int_{a_i}^{b_i} r^{1-2/n} \big(- u^{*'}(r) \big) dr \leq \int_0^{\sum_{i=1}^m (b_i - a_i)} |\nabla^2 u|^*(r) dr, \tag{3.32}$$

for every finite family of intervals $(a_i, b_i), i = 1, \dots, m$, with $0 < a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m < \infty$.

Let j be any fixed index $\in \{1, \dots, m\}$. By Lemma 3.4,

$$\begin{aligned} \sum_{i \geq j} \int_{a_i}^{b_i} r^{1-2/n} \big(- u^{*'}(r) \big) dr &\leq \frac{1}{n^2 {C_n}^{2/n}} \int_0^\infty \chi_{\cup_{i \geq j}(a_i, b_i)}(r) \frac{1}{r} \int_0^r |\Delta u|^*(s) ds\, dr \\ &= \frac{1}{n^2 {C_n}^{2/n}} \int_0^\infty |\Delta u|^*(s) \int_s^\infty \frac{1}{r} \chi_{\cup_{i \geq j}(a_i, b_i)}(r) dr\, ds. \end{aligned} \tag{3.33}$$

We claim that

$$\int_0^\infty |\Delta u|^*(s) \left(\int_s^\infty \frac{1}{r} \chi_{\cup_{i \geq j}(a_i, b_i)}(r) dr \right) ds \leq \left(1 + \sum_{i \geq j} log \left(\frac{b_i}{a_i} \right) \right) \int_0^\infty |\Delta u|^*(s) \chi_{\left[0, \sum_{i \geq j}(b_i - a_i)\right]}(s) ds. \tag{3.34}$$

To verify (3.34), observe that, given any pair of locally integrable functions $\alpha, \beta : (0, \infty) \to [0, \infty)$, the inequality

$$\int_0^\infty h(s) \alpha(s) ds \leq K \int_0^\infty h(s) \beta(s) ds \tag{3.35}$$

holds for every non-increasing function $h : (0, \infty) \to [0, \infty)$ and for some constant K, if and only if, for the same constant,

$$\int_0^R \alpha(s) ds \leq K \int_0^R \beta(s) ds \quad \text{for } R \geq 0. \tag{3.36}$$

Indeed, (3.35) implies (3.36) on choosing $h = \chi_{(0,R)}$; conversely, on assuming, without loss of generality, that $\lim_{s \to +\infty} h(s) = 0$ and writing $h(s) = \int_0^\infty \chi_{(0,\, \mu_h(t))}(s) dt$ for a.e. $s \in (0, \infty)$, inequality (3.35) follows from (3.36) via Fubini's theorem.

Thus, inequality (3.34) will be proved if we show that

$$\int_0^R \int_s^\infty \frac{1}{r} \chi_{\cup_{i\geq j}(a_i,b_i)}(r)drds \\ \leq \left(1 + \sum_{i\geq j} log\left(\frac{b_i}{a_i}\right)\right) \int_0^R \chi_{\left[0,\sum_{i\geq j}(b_i-a_i)\right]}(s)ds \quad \text{for } R \geq 0. \tag{3.37}$$

After an application of Fubini's theorem to the integral on the left-hand side, inequality (3.37) reads

$$\int_0^R \chi_{\cup_{i\geq j}(a_i,b_i)}(s)ds + R\int_R^\infty \frac{1}{s}\chi_{\cup_{i\geq j}(a_i,b_i)}(s)ds \\ \leq \left(1 + \sum_{i\geq j} log\left(\frac{b_i}{a_i}\right)\right) min\left\{R, \sum_{i\geq j}(b_i - a_i)\right\} \text{ for } R \geq 0. \tag{3.38}$$

For any fixed $R > 0$, we may assume, if necessary on splitting one of the intervals (a_i, b_i) into two subintervals, that an index i_R exists such that $b_{i_R} \leq R \leq a_{i_R+1}$. Inequality (3.38) can be written as

$$\sum_{j\leq i\leq i_R} (b_i - a_i) + R \sum_{i>\max\{j-1,i_R\}} log\left(\frac{b_i}{a_i}\right) \\ \leq \left(1 + \sum_{i\geq j} log\left(\frac{b_i}{a_i}\right)\right) min\left\{R, \sum_{i\geq j}(b_i - a_i)\right\}. \tag{3.39}$$

Hereafter sums are taken to be 0 whenever they are extended to an empty set of indices. First, suppose that $min\left\{R, \sum_{i\geq j}(b_i - a_i)\right\} = R$. Then

$$\sum_{j\leq i\leq i_R} (b_i - a_i) + R \sum_{i>\max\{j-1,i_R\}} log\left(\frac{b_i}{a_i}\right) \leq R + R\sum_{i\geq j} log\left(\frac{b_i}{a_i}\right) \\ = \left(1 + \sum_{i\geq j} log\left(\frac{b_i}{a_i}\right)\right) min\left\{R, \sum_{i\geq j}(b_i - a_i)\right\}.$$

Thus, (3.39) holds. Assume now that $min\left\{R, \sum_{i\geq j}(b_i - a_i)\right\} = \sum_{i\geq j}(b_i - a_i)$. One has

$$\sum_{i>max\{j-1,i_R\}} log\left(\frac{b_i}{a_i}\right) = \int_R^\infty \frac{1}{s}\chi_{\cup_{i>max\{j-1,i_R\}}(a_i,b_i)}(s)ds \\ \leq \frac{1}{R}\int_R^\infty \chi_{\cup_{i>max\{j-1,i_R\}}(a_i,b_i)}(s)ds = \frac{1}{R}\sum_{i>max\{j-1,i_R\}} (b_i - a_i).$$

Hence,

$$\sum_{j\le i\le i_R}(b_i-a_i)+R\sum_{i>max\{j-1,i_R\}}log\left(\frac{b_i}{a_i}\right)$$
$$\le\sum_{j\le i\le i_R}(b_i-a_i)+\sum_{i>max\{j-1,i_R\}}(b_i-a_i)=\sum_{i\ge j}(b_i-a_i)$$
$$\le\left(1+\sum_{i\ge j}log\left(\frac{b_i}{a_i}\right)\right)\sum_{i\ge j}(b_i-a_i)$$
$$=\left(1+\sum_{i\ge j}log\left(\frac{b_i}{a_i}\right)\right)min\left\{R,\sum_{i\ge j}(b_i-a_i)\right\}.$$

Thus, (3.39) holds in this case as well. Inequality (3.34) is proved.

Inequalities (3.33)–(3.34) imply that

$$\sum_{i\ge j}\int_{a_i}^{b_i}r^{1-2/n}(-u^{*'}(r))dr\le\frac{1+\sum_{i\ge j}log\left(\frac{b_i}{a_i}\right)}{n^2{C_n}^{2/n}}\int_0^{\sum_{i\ge j}(b_i-a_i)}|\Delta u|^*(r)dr. \tag{3.40}$$

If $\sum_{i=j}^m log\left(\frac{b_i}{a_i}\right)\le 1$, then inequality (3.32) immediately follows on choosing $j=1$ in (3.40), since $|\Delta u|\le\sqrt{n}|\nabla^2 u|$.

If, on the contrary, $\sum_{i=j}^m log\left(\frac{b_i}{a_i}\right)>1$, then it is easily seen that an index j_0 and a positive number c_{j_0} exist such that $a_{j_0}\le c_{j_0}\le b_{j_0}$ and

$$1<log\left(\frac{b_{j_0}}{c_{j_0}}\right)+\sum_{i>j_0}log\left(\frac{b_i}{a_i}\right)\le 2. \tag{3.41}$$

On applying (3.40) with $j=j_0$, replacing a_{j_0} by c_{j_0} and making use of the second of inequalities (3.41), we get

$$\int_{c_{j_0}}^{b_{j_0}}r^{1-2/n}\big(-u^{*'}(r)\big)dr+\sum_{i>j_0}\int_{a_i}^{b_i}r^{1-2/n}\big(-u^{*'}(r)\big)dr$$
$$\le\frac{1+log\left(\frac{b_{j_0}}{c_{j_0}}\right)+\sum_{i>j_0}log\left(\frac{b_i}{a_i}\right)}{n^2{C_n}^{2/n}}\int_0^{(b_{j_0}-c_{j_0})+\sum_{i>j_0}(b_i-a_i)}|\Delta u|^*(r)dr$$
$$\le\frac{3}{n^2{C_n}^{2/n}}\int_0^{(b_{j_0}-c_{j_0})+\sum_{i>j_0}(b_i-a_i)}|\Delta u|^*(r)dr. \tag{3.42}$$

On the other hand,

$$log\left(\frac{b_{j_0}}{c_{j_0}}\right)+\sum_{i>j_0}log\left(\frac{b_i}{a_i}\right)=\int_{c_{j_0}}^{\infty}\frac{1}{s}\chi_{(c_{j_0},b_{j_0})\cup(\cup_{i>j_0}(a_i,b_i))}(s)ds$$
$$\le\frac{1}{c_{j_0}}\left((b_{j_0}-c_{j_0})+\sum_{i>j_0}(b_i-a_i)\right),$$

whence, by the first of inequalities (3.41),

$$c_{j_0} < (b_{j_0} - c_{j_0}) + \sum_{i > j_0} (b_i - a_i).$$

Thus, by Lemma 3.3,

$$\begin{aligned}
&\sum_{1 \le i < j_0} \int_{a_i}^{b_i} r^{1-2/n} \big(- u^{*'}(r) \big) dr + \int_{a_{j_0}}^{c_{j_0}} r^{1-2/n} \big(- u^{*'}(r) \big) dr \\
&\le \int_0^{c_{j_0}} r^{1-2/n} \big(- u^{*'}(r) \big) dr \le \frac{1}{n(n-1) {C_n}^{2/n}} \int_0^{c_{j_0}} (|\nabla |\nabla u|| + |\Delta u|)^*(r) dr \\
&\le \frac{1}{n(n-1) {C_n}^{2/n}} \int_0^{(b_{j_0} - c_{j_0}) + \sum_{i > j_0} (b_i - a_i)} (|\nabla |\nabla u|| + |\Delta u|)^*(r) dr. \qquad (3.43)
\end{aligned}$$

Combining (3.42)–(3.43) yields

$$\begin{aligned}
&\sum_{i=1}^{m} \int_{a_i}^{b_i} r^{1-2/n} \big(- u^{*'}(r) \big) dr \\
&\le \left(\frac{1}{n(n-1) {C_n}^{2/n}} + \frac{3}{n^2 {C_n}^{2/n}} \right) \int_0^{(b_{j_0} - c_{j_0}) + \sum_{i > j_0} (b_i - a_i)} (|\nabla |\nabla u|| + |\Delta u|)^*(r) dr \\
&\le \frac{5}{n^2 {C_n}^{2/n}} \int_0^{\sum_{i=1}^{m} (b_i - a_i)} (|\nabla |\nabla u|| + |\Delta u|)^*(r) dr. \qquad (3.44)
\end{aligned}$$

From a version of the chain rule for vector-valued Sobolev functions [22, Theorem 2.1], one infers that $|\nabla |\nabla u|| \le |\nabla^2 u|$ a.e. on $\mathbb{R}^n$. Hence, inequality (3.44) implies (3.32).

Now, given any open set $A \subset (0, \infty)$ having finite measure, there exists a sequence of sets $A_k \subset (0, \infty)$, which are the finite union of open intervals, satisfying $A_k \subset A_{k+1} \subset A$ for $k \in \mathbb{N}$ and $\lim_{k \to \infty} |A \setminus A_k| = 0$. Thus, by (3.32),

$$C(n) \int_{A_k} r^{1-2/n} \big(- u^{*'}(r) \big) dr \le \int_0^{|A_k|} |\nabla^2 u|^*(r) dr \le \int_0^{|A|} |\nabla^2 u|^*(r) dr.$$

Hence, passing to the limit as $k \to \infty$,

$$C(n) \int_A r^{1-2/n} \big(- u^{*'}(r) \big) dr \le \int_0^{|A|} |\nabla^2 u|^*(r) dr, \qquad (3.45)$$

by the monotone convergence theorem. Finally, on approximating from the outside any given set $E \subset (0, \infty)$ having finite measure by a monotone sequence of open sets, applying (3.45) to these sets and passing to the limit, one gets

$$C(n) \int_E r^{1-2/n} \big(- u^{*'}(r) \big) dr \le \int_0^{|E|} |\nabla^2 u|^*(r) dr. \qquad (3.46)$$

Inequality (1.7) follows from (3.46), owing to equation (2.7). □

4. Proofs of Theorems 1.3–1.5

With Theorem 1.1 in place, the derivation of the second-order Sobolev inequality (1.8) from the 1-dimensional inequality (1.9) is more or less straightforward, and is the same as the derivation of (1.4) from (1.5), via (1.3), in the case of first-order inequalities. In that case, the reverse implication is also not difficult, since equality holds in (1.3) whenever u is sperically symmetric. Instead, the fact that (1.8) implies (1.9) is not obvious. The next two lemmas concerning 1-dimensional inequalities will be needed in the proof of this implication.

Lemma 4.1. *Let $l \in (0, +\infty]$ and let $Z(0, l)$ be any r.i. space on $(0, l)$. Let $p > 1$. Then*

$$\left\| s^{1-1/p} \int_s^l \phi(r)dr \right\|_{Z(0,l)} \le \frac{p}{p-1} \|s^{2-1/p}\phi(s)\|_{Z(0,l)}, \tag{4.1}$$

for every measurable function ϕ such that $s^{2-1/p}\phi(s) \in Z(0, l)$.

Proof. Consider the linear operator T defined as

$$T\psi(s) = s^{1-1/p} \int_s^l r^{-2+1/p}\psi(r)dr \qquad \text{for } s \in (0, l),$$

at any measurable function ψ in $(0, l)$ for which the integral converges. The operator T is bounded in $L^1(0, l)$, with norm $\le \frac{p}{2p-1}$, since

$$\begin{aligned}\|T\psi\|_{L^1(0,l)} &= \int_0^l \left| s^{1-1/p} \int_s^l \psi(r)r^{-2+1/p}dr \right| ds \\ &\le \int_0^l |\psi(r)|r^{-2+1/p} \left(\int_0^r s^{1-1/p}ds \right) dr = \frac{p}{2p-1}\|\psi\|_{L^1(0,l)}.\end{aligned}$$

Moreover, T is bounded on $L^\infty(0, l)$ with norm $\le \frac{p}{p-1}$, since

$$\begin{aligned}\|T\psi\|_{L^\infty(0,l)} &= \sup_{0\le s\le l} s^{1-1/p} \left| \int_s^l \psi(r)r^{-2+1/p}dr \right| \\ &\le \|\psi\|_{L^\infty(0,l)} \sup_{0\le s\le l} s^{1-1/p} \int_s^\infty r^{-2+1/p}dr = \frac{p}{p-1}\|\psi\|_{L^\infty(0,l)}.\end{aligned}$$

A theorem by Calderón (see e.g. [6, Chap. 3, Theorem 2.12]) then ensures that T is bounded on $Z(0, l)$ with norm $\le max\left\{\frac{p}{2p-1}, \frac{p}{p-1}\right\} = \frac{p}{p-1}$. Hence, (4.1) follows. □

Lemma 4.2. *Let $l \in (0, +\infty]$ and let $Z(0, l)$ be any r.i. space on $(0, l)$. Then*

$$\left\| \int_s^l \phi(r)dr \right\|_{Z(0,l)} \le 4 \left\| \int_s^l \phi(r)\left(1 - \frac{s}{r}\right)dr \right\|_{Z(0,l)}, \tag{4.2}$$

for every non-negative measurable function ϕ, vanishing outside $(0, l/2)$, and such that $\int_s^l \left(1 - \frac{s}{r}\right)\phi(r)dr \in Z(0, l)$.

Proof. We begin by showing that

$$\left\| \int_s^l \phi(r)dr \right\|_{Z(0,l)} \leq 2 \left\| \chi_{[0,l/2]}(s) \int_{2s}^l \phi(r)dr \right\|_{Z(0,l)}, \tag{4.3}$$

for every non-negative measurable function ϕ, vanishing outside $(0, l/2)$, which renders the right-hand side finite. In order to prove (4.3), let us introduce a new space, denoted by $Z(0, l/2)$, consisting of those measurable functions ψ on $(0, l/2)$ for which the quantity

$$\|\psi\|_{Z(0,l/2)} = \|\psi^*\|_{Z(0,l)} \tag{4.4}$$

is finite (obviously, $Z(0, l/2) = Z(0, l)$ if $l = \infty$). It is not difficult to verify that $Z(0, l/2)$ is an r.i. space equipped with the norm $\|\cdot\|_{Z(0,l/2)}$ (notice that, in particular, the verification of property ii), Subsection 2.2, requires that $0 \leq {\psi_n}^* \nearrow \psi^*$ when $0 \leq \psi_n \nearrow \psi$, and this is true, owing to Proposition 1.7, Chapter 2 of [6]). The dilation operator $E_{1/2}$ defined as in (2.12) is bounded on $Z(0, l/2)$ and, by (2.13), its norm ≤ 2. Thus,

$$\left\| \int_s^l \phi(r)dr \right\|_{Z(0,l/2)} \leq 2 \left\| \int_{2s}^l \phi(r)dr \right\|_{Z(0,l/2)}. \tag{4.5}$$

Since $\phi \geq 0$ and $\phi \equiv 0$ outside $(0, l/2)$, then, by (4.4),

$$\left\| \int_s^l \phi(r)dr \right\|_{Z(0,l/2)} = \left\| \int_s^l \phi(r)dr \right\|_{Z(0,l)} \tag{4.6}$$

and

$$\left\| \int_{2s}^l \phi(r)dr \right\|_{Z(0,l/2)} = \left\| \chi_{(0,l/2)}(s) \int_{2s}^l \phi(r)dr \right\|_{Z(0,l)}. \tag{4.7}$$

Inequality (4.3) is a consequence of (4.5)–(4.7). The conclusion follows from (4.3) and from the fact that

$$\int_s^l \phi(r) \left(1 - \frac{s}{r}\right) dr \geq \int_{2s}^l \phi(r) \left(1 - \frac{s}{r}\right) dr \geq \frac{1}{2} \int_{2s}^l \phi(r)dr,$$

if $0 \leq s \leq l/2$ and $\phi \geq 0$. □

Proof of Theorem 1.3. Assume that ii) holds. One has

$$\|u\|_{Y(G)} = \|u^*\|_{\overline{Y}(0,|G|)} = \left\| \int_s^{|G|} \left(-{u^*}'(r)\right)dr \right\|_{\overline{Y}(0,|G|)}. \tag{4.8}$$

Notice that the last equality holds because, by the first-order Pólya–Szegö principle, u^* is locally absolutely continuous and vanishes at $|G|$. Inequality (1.9), implies that

$$\left\| \int_s^{|G|} \left(-{u^*}'(r)\right)dr \right\|_{\overline{Y}(0,|G|)} \leq K_2 \left\| r^{1-2/n}\left(-{u^*}'(r)\right)\right\|_{\overline{X}(0,|G|)}. \tag{4.9}$$

Combining (4.8)–(4.9) with Theorem 1.1 yields (1.8) with $K_1 = K_2/C(n)$.

Assume now that i) holds. We may suppose, without loss of generality, that $0 \in G$. Let σ be a positive number not exceeding $|G|/2$ and so small that the ball centered at 0 and having measure σ is contained in G. Given any non-negative function $\phi \in \overline{X}(0,|G|)$, with $supp\,\phi \subseteq [0,\sigma]$, define the function $U : \mathbb{R}^n \to [0,\infty)$ as

$$U(x) = \int_{C_n|x|^n}^{\infty} \int_r^{\infty} \phi(t) t^{-2+2/n} dt\, dr \qquad \text{for } x \in \mathbb{R}^n. \tag{4.10}$$

Since U is radially decreasing,

$$U(x) = U^{\star}(x) = U^*(C_n|x|^n) \qquad \text{for } x \in \mathbb{R}^n \tag{4.11}$$

and

$$U^*(s) = \int_s^{\infty} \int_r^{\infty} \phi(t) t^{-2+2/n} dt\, dr \qquad \text{for } s > 0. \tag{4.12}$$

Clearly, U has compact support in G. Moreover, U is twice weakly differentiable and

$$\nabla U(x) = -\left(nC_n|x|^{n-2} \int_{C_n|x|^n}^{\infty} \phi(r) r^{-2+2/n} dr \right) x, \tag{4.13}$$

$$\begin{aligned}\nabla^2 U(x) = &\Bigg(n^2 {C_n}^2 |x|^{-2} \phi(C_n|x|^n) \\ &- n(n-2)C_n|x|^{n-4} \int_{C_n|x|^n}^{\infty} \phi(r) r^{-2+2/n} dr \Bigg) x \otimes x \\ &- nC_n|x|^{n-2} \left(\int_{C_n|x|^n}^{\infty} \phi(r) r^{-2+2/n} dr \right) I_n,\end{aligned} \tag{4.14}$$

for a.e. $x \in \mathbb{R}^n$. Here, $\otimes$ stands for the tensor product and I_n denotes the $n \times n$ unit matrix. Since $|x \otimes x| = |x|^2$ and $|I_n| = \sqrt{n}$, we get from (4.14),

$$\begin{aligned}|\nabla^2 U(x)| \le\, & n^2 {C_n}^{2/n} \phi(C_n|x|^n) \\ &+ n(n+\sqrt{n}-2)C_n|x|^{n-2} \int_{C_n|x|^n}^{\infty} \phi(r) r^{-2+2/n} dr \quad \text{for a.e. } x \in \mathbb{R}^n.\end{aligned} \tag{4.15}$$

Thus,

$$\begin{aligned}\| |\nabla^2 U| \|_{X(G)} \le\, & n^2 {C_n}^{2/n} \left\| \phi(C_n|x|^n) \right\|_{X(G)} \\ &+ n(n+\sqrt{n}-2)C_n \left\| |x|^{n-2} \int_{C_n|x|^n}^{\infty} \phi(r) r^{-2+2/n} dr \right\|_{X(G)} \\ =\, & n^2 {C_n}^2 \left\| (\phi(C_n|\cdot|^n))^*(s) \right\|_{\overline{X}(0,|G|)} \\ &+ n(n+\sqrt{n}-2)C_n \left\| \left(|\cdot|^{n-2} \int_{C_n|\cdot|^n}^{\infty} \phi(r) r^{-2+2/n} dr \right)^*(s) \right\|_{\overline{X}(0,|G|)}.\end{aligned} \tag{4.16}$$

By (2.4),

$$(\phi(C_n|\cdot|^n))^*(s) = \phi^*(s)$$

and

$$\left(|\cdot|^{n-2}\int_{C_n|\cdot|^n}^{\infty}\phi(r)r^{-2+2/n}dr\right)^*(s)$$
$$= \left({C_n}^{-1+2/n}(\cdot)^{1-2/n}\int_{\cdot}^{\infty}\phi(r)r^{-2+2/n}dr\right)^*(s) \text{ for } s \geq 0.$$

Hence,

$$\begin{aligned}\||\nabla^2 U|\|_{X(G)} &\leq n^2 {C_n}^{2/n}\|\phi^*(s)\|_{\overline{X}(0,|G|)} \\ &\quad + n(n+\sqrt{n}-2)C_n^{2/n}\left\|\left((\cdot)^{1-2/n}\int_{\cdot}^{\infty}\phi(r)r^{-2+2/n}dr\right)^*(s)\right\|_{\overline{X}(0,|G|)} \\ &= n^2 {C_n}^{2/n}\|\phi(s)\|_{\overline{X}(0,|G|)} \\ &\quad + n(n+\sqrt{n}-2)C_n^{2/n}\left\|s^{1-2/n}\int_{s}^{\infty}\phi(r)r^{-2+2/n}dr\right\|_{\overline{X}(0,|G|)}, \qquad (4.17)\end{aligned}$$

inasmuch as $\overline{X}(0,|G|)$ is an r.i. space. Since we are assuming that *supp* $\phi \subseteq [0,\sigma] \subseteq [0,|G|]$, Lemma 4.1 implies that

$$\left\|s^{1-2/n}\int_{s}^{\infty}\phi(r)r^{-2+2/n}dr\right\|_{\overline{X}(0,|G|)} \leq \frac{n}{n-2}\|\phi(s)\|_{\overline{X}(0,|G|)}.$$

Consequently,

$$\||\nabla^2 U|\|_{X(G)} \leq n^2 {C_n}^{2/n}\frac{2n+\sqrt{n}-4}{n-2}\|\phi(s)\|_{\overline{X}(0,|G|)}, \qquad (4.18)$$

whence $U \in W_0^2 X(G)$. By (1.8), (4.12) and (4.18), one gets, after an application of Fubini's theorem,

$$\left\|\int_{s}^{|G|}\phi(r)\left(1-\frac{s}{r}\right)r^{-1+2/n}dr\right\|_{\overline{Y}(0,|G|)} \leq 4K_1 n^2 {C_n}^{2/n}\|\phi(s)\|_{\overline{X}(0,|G|)}, \qquad (4.19)$$

for every non-negative $\phi \in \overline{X}(0,|G|)$ with *supp* $\phi \subseteq [0,\sigma]$. Hence, by Lemma 4.2,

$$\left\|\int_{s}^{|G|}\phi(r)r^{-1+2/n}dr\right\|_{\overline{Y}(0,|G|)} \leq 16K_1 n^2 {C_n}^{2/n}\|\phi(s)\|_{\overline{X}(0,|G|)}, \qquad (4.20)$$

for every non-negative $\phi \in \overline{X}(0,|G|)$ with *supp* $\phi \subseteq [0,\sigma]$.

Now, let ϕ be any function from $\overline{X}(0,|G|)$. Then

$$\begin{aligned}&\left\| \int_s^{|G|} \phi(r) r^{-1+2/n} dr \right\|_{\overline{Y}(0,|G|)} \\ &\leq \left\| \int_s^{|G|} \chi_{[0,\sigma]}(r)\phi(r) r^{-1+2/n} dr \right\|_{\overline{Y}(0,|G|)} \\ &\quad + \left\| \int_s^{|G|} \chi_{[\sigma,|G|]}(r)\phi(r) r^{-1+2/n} dr \right\|_{\overline{Y}(0,|G|)}. \end{aligned} \tag{4.21}$$

We have

$$\begin{aligned}&\left\| \int_s^{|G|} \chi_{[\sigma,|G|]}(r)\phi(r) r^{-1+2/n} dr \right\|_{\overline{Y}(0,|G|)} \leq \sigma^{-1+2/n} \left\| \int_s^{|G|} |\phi(r)| dr \right\|_{\overline{Y}(0,|G|)} \\ &\leq \sigma^{-1+2/n} \|1\|_{\overline{Y}(0,|G|)} \int_0^{|G|} |\phi(r)| dr \leq \sigma^{-1+2/n} \|1\|_{\overline{Y}(0,|G|)} \|1\|_{\overline{X}'(0,|G|)} \|\phi\|_{\overline{X}(0,|G|)}. \end{aligned} \tag{4.22}$$

Notice that the last inequality is due to (2.10). On estimating the first term on the right-hand side of (4.21) by (4.20) (with ϕ replaced by $\chi_{[0,\sigma]}|\phi|$) and the second term by (4.22), inequality (1.9) follows. □

Proof of Theorem 1.4, sketched. Assume that ii) holds. Then, inequalities (1.10), (1.12) and (1.9) imply that (1.11) holds for every function $f \in X(G)$ having support in G.

Conversely, assume that i) holds and suppose, without loss of generality, that $0 \in G$. Let σ be a positive number such that the ball centered at 0 and having measure σ is contained in G. Given any non-negative function $\phi \in \overline{X}(0,|G|)$ with *supp* $\phi \subseteq [0,\sigma]$, define the function $f : \mathbb{R}^n \to [0,\infty)$ as $f(x) = \phi(C_n|x|^n)$, so that *supp* $f \subseteq G$ and, by (2.4), $f^* = \phi^*$. It is not difficult to show that

$$I_2(f)(x) \geq 2^{2-n} {C_n}^{1-2/n} \left(\frac{1}{(C_n|x|^n)^{1-2/n}} \int_0^{C_n|x|^n} \phi(r) dr + \int_{C_n|x|^n}^{|G|} \phi(r) r^{-1+2/n} dr \right) \quad \text{for } |x| \leq (|G|/C_n)^{1/n}. \tag{4.23}$$

On taking the decreasing rearrangement of both sides of (4.23) and making use of (2.4), we get

$$\begin{aligned}&\|(I_2(f))^*(s)\|_{\overline{Y}(0,|G|)} \geq \\ &2^{2-n} {C_n}^{1-2/n} \left\| s^{-1+2/n} \int_0^s \phi(r) dr + \int_s^{|G|} \phi(r) r^{-1+2/n} dr \right\|_{\overline{Y}(0,|G|)}. \end{aligned} \tag{4.24}$$

Combining (1.11) and (4.24) tells us that inequalities (1.12) and (1.9) hold for every non-negative $\phi \in \overline{X}(0,|G|)$ with *supp* $\phi \subseteq [0,\sigma]$. The general case can be treated on writing $\phi(s) = \phi(s)\chi_{[0,\sigma]}(s) + \phi(s)\chi_{[\sigma,|G|]}(s)$ and arguing as at the end of the proof of Theorem 1.3. □

Proof of Theorem 1.5, sketched. Assume that ii) holds. Then, inequalities (1.14), (1.12) and (1.9) imply that (1.16) holds for every $u \in W_0^\Delta X(G)$.

Conversely, assume that i) is in force and, with no loss of generality, that $0 \in G$. Let σ be a positive number such that the ball centered at 0 and having measure 4σ is contained in G. Let ψ be any function from $\overline{X}(0,|G|)$ such that $supp\,\psi \subseteq [0,4\sigma]$, $\psi \geq 0$ in $[0,\sigma]$, $\psi = 0$ in $(\sigma, 2\sigma]$ and $\psi(s) = -\psi(4\sigma - s)$ for $s \in (2\sigma, 4\sigma]$. Hence, in particular, $\int_0^s \psi(r)dr \geq 0$ for $s \in [0,4\sigma]$ and $\int_0^s \psi(r)dr = 0$ for $s \in [4\sigma, |G|]$. Define the function $V : \mathbb{R}^n \to [0,\infty)$ as

$$V(x) = \frac{1}{n^2 {C_n}^{2/n}} \int_{C_n|x|^n}^{|G|} r^{-2+2/n} \int_0^r \psi(t)dt\,dr.$$

Then V has compact support in G and

$$V^*(s) = \frac{1}{n^2 {C_n}^{2/n}} \int_s^{|G|} r^{-2+2/n} \int_0^r \psi(t)dt\,dr.$$

Moreover, $\Delta V(x) = \psi(C_n|x|^n)$ for a.e. $x \in G$, whence $|\Delta V|^* = \psi^*$, by (2.4), and $V \in W_0^\Delta X(G)$. Hence, on applying inequality (1.16) with $u = V$, one gets

$$K_1 \|\psi(s)\|_{\overline{X}(0,|G|)} \geq \left\| \frac{1}{n^2 {C_n}^{2/n}} \int_s^{|G|} r^{-2+2/n} \int_0^r \psi(t)dtdr \right\|_{\overline{Y}(0,|G|)}. \tag{4.25}$$

Our assumptions on ψ ensure that

$$\begin{aligned} &\int_s^{|G|} r^{-2+2/n} \int_0^r \psi(t)dtdr \geq \chi_{[0,2\sigma]}(s) \int_s^{2\sigma} r^{-2+2/n} \int_0^r \psi(t)dtdr \\ &= \frac{n}{n-2} \chi_{[0,2\sigma]}(s) \Bigg((s^{-1+2/n} - (2\sigma)^{-1+2/n}) \int_0^s \psi(r)dr \\ &\quad + \int_s^{2\sigma} \psi(r)(r^{-1+2/n} - (2\sigma)^{-1+2/n})dr \Bigg) \\ &\geq \frac{n(1-2^{-1+2/n})}{n-2} \chi_{[0,\sigma]}(s) \Bigg(s^{-1+2/n} \int_0^s \chi_{[0,\sigma]}(r)\psi(r)dr \\ &\quad + \int_s^{|G|} \chi_{[0,\sigma]}(r)\psi(r) r^{-1+2/n} dr \Bigg) \text{ for } s \in [0,|G|]. \end{aligned} \tag{4.26}$$

Notice that the last two integrals are both non-negative for $s \in [0,|G|]$.

Now, let ϕ be any function from $\overline{X}(0,|G|)$. We have

$$\begin{aligned} &\left\| s^{-1+2/n} \int_0^s \phi(r)dr \right\|_{\overline{Y}(0,|G|)} \\ &\leq \left\| \chi_{[0,\sigma]}(s) s^{-1+2/n} \int_0^s \phi(r)dr \right\|_{\overline{Y}(0,|G|)} + \left\| \chi_{[\sigma,|G|]}(s) s^{-1+2/n} \int_0^s \phi(r)dr \right\|_{\overline{Y}(0,|G|)}. \end{aligned} \tag{4.27}$$

Denote by $\overline{\phi}$ the function defined in $[0,|G|]$ as $\overline{\phi}(s) = |\phi(s)|$ if $s \in [0,\sigma]$, $\overline{\phi}(s) = 0$ if $s \in (\sigma, 2\sigma]$, $\overline{\phi}(s) = -\overline{\phi}(4\sigma - s)$ if $s \in (2\sigma, 4\sigma]$ and $\overline{\phi}(s) = 0$ otherwise. Since

$\overline{\phi}$ satisfies the same assumptions as ψ, we may apply (4.25) and (4.26) with ψ replaced by $\overline{\phi}$ to estimate the first norm on the right-hand side of (4.27) and get

$$\begin{aligned}
\left\| \chi_{[0,\sigma]}(s) s^{-1+2/n} \int_0^s \phi(r)dr \right\|_{\overline{Y}(0,|G|)} &\leq \left\| \chi_{[0,\sigma]}(s) s^{-1+2/n} \int_0^s \chi_{[0,\sigma]}(r)\overline{\phi}(r)dr \right\|_{\overline{Y}(0,|G|)} \\
&\leq \frac{n-2}{n(1-2^{-1+2/n})} \left\| \int_s^{|G|} r^{-2+2/n} \int_0^r \overline{\phi}(t)dtdr \right\|_{\overline{Y}(0,|G|)} \\
&\leq \frac{K_1 n(n-2)C_n{}^{2/n}}{(1-2^{-1+2/n})} \|\overline{\phi}(s)\|_{\overline{X}(0,|G|)} \leq \frac{2K_1 n(n-2)C_n{}^{2/n}}{(1-2^{-1+2/n})} \|\phi(s)\|_{\overline{X}(0,|G|)}. \quad (4.28)
\end{aligned}$$

The second norm on the right-hand side of (4.27) can be estimated in terms of $\|\phi(s)\|_{\overline{X}(0,|G|)}$ via an argument analogous to that used to estimate corresponding quantities in the proofs of Theorems 1.3–1.4. Hence, (1.12) follows.

Inequality (1.9) can be proved in a similar way, on exploiting inequalities (4.25)–(4.26) again. □

5. Applications

The first part of this section is devoted to showing how well-known Sobolev-type inequalities can be easily recovered via Theorem 1.3. New inequalities are derived in the second part.

The classical second-order Sobolev inequality

$$\|u\|_{L^{\frac{np}{n-2p}}(G)} \leq K_1 \| |\nabla^2 u| \|_{L^p(G)} \quad (5.1)$$

for $u \in W_0^{2,p}(G)$, with $1 \leq p < n/2$, follows from Theorem 1.3 thanks to the Bliss inequality

$$\left\| \int_s^{|G|} \phi(r) r^{-1+2/n} dr \right\|_{L^{\frac{np}{n-2p}}(0,|G|)} \leq K_2 \|\phi(s)\|_{L^p(0,|G|)}$$

(see e.g. [30]).

The Sobolev inequality in the finer scale of Lorentz spaces takes the form

$$\|u\|_{L^{\frac{np}{n-2p},q}(G)} \leq K_1 \| |\nabla^2 u| \|_{L^{p,q}(G)}, \quad (5.2)$$

for $u \in W_0^2 L^{p,q}(G)$, with $1 < p < n/2$ and $1 \leq q \leq \infty$. When $q = 1$, inequality (5.2) can be derived from Theorem 1.3 by an application of Fubini's theorem; when $q > 1$, one may replace f^* by f^{**} in (2.15) and make use of the two Hardy inequalities

$$\left(\int_0^{|G|} \left| s^{\frac{n-2p}{np}} s^{-1+2/n} \int_0^s \phi(r)dr \right|^q \frac{ds}{s} \right)^{1/q} \leq K_2 \left(\int_0^{|G|} \left| s^{1/p}\phi(s) \right|^q \frac{ds}{s} \right)^{1/q}$$

and

$$\left(\int_0^{|G|}\left|s^{\frac{n-2p}{np}}\int_s^{|G|}\phi(r)r^{-1+2/n}dr\right|^q\frac{ds}{s}\right)^{1/q}\leq K_2\left(\int_0^{|G|}\left|s^{1/p}\phi(s)\right|^q\frac{ds}{s}\right)^{1/q}$$

(see e.g. [6, Chap.3, Lemma 3.9])

A limiting case of inequality (5.1) when $p=n/2$ is provided by the Trudinger–Strichartz inequality

$$\|u\|_{ExpL^{\frac{n}{n-2}}(G)}\leq K_1\||\nabla^2u|\|_{L^{n/2}(G)}, \tag{5.3}$$

for $u\in W_0^{2,n/2}(G)$ [29], where $\|\cdot\|_{ExpL^{\frac{n}{n-2}}(G)}$ stands for the Luxemburg norm associated with the Young function $A(s)=e^{s^{\frac{n}{n-2}}}-1$. This norm is equivalent to the quantity $\sup_{0<s<|G|}\frac{u^*(s)}{(1+log(|G|/s))^{1-2n}}$ [6, Chap. 4, Lemma 6.12]. Thus, inequality (5.3) follows from Theorem 1.3, inasmuch as

$$\sup_{0<s<|G|}\frac{\left|\int_s^{|G|}r^{-1+2/n}\phi(r)dr\right|}{(1+log(|G|/s))^{1-2n}}\leq\|\phi\|_{L^{n/2}(0,|G|)},$$

by Hölder's inequality.

A slightly stronger result than (5.3) (the best possible, in fact, in the framework of r.i. spaces, as far as the target space is concerned – see [15]) is the inequality

$$\left(\int_0^{|G|}\left(\frac{u^*(s)}{1+log(|G|/s)}\right)^{n/2}\frac{ds}{s}\right)^{2/n}\leq K_1\||\nabla^2u|\|_{L^{n/2}(G)}, \tag{5.4}$$

for $u\in W_0^{2,n/2}(G)$, which appears in [7],[19] (see also [24] for results related to a first-order version). By Theorem 1.3, inequality (5.4) is reduced to the 1-dimensional inequality

$$\left(\int_0^{|G|}\left|\int_s^{|G|}\phi(r)r^{-1+2/n}dr\right|^{n/2}(1+log(|G|/s))^{-n/2}\frac{ds}{s}\right)^{2/n}$$
$$\leq K_2\left(\int_0^{|G|}|\phi(s)|^{n/2}ds\right)^{2/n},$$

a weighted Hardy inequality (see e.g. Lemma 2.10.3 of [34]).

We now present some new results relying on Theorem 1.3. We shall give only an outline of the proofs, and we shall refer to other papers where a detailed treatment of the 1-dimensional inequalities that one is lead to consider can be found.

Let us begin with inequalities in Orlicz–Sobolev spaces. Given any Young function A and any real number $p>1$ such that the function

$$H_p(r)=\left(\int_0^r\left(\frac{t}{A(t)}\right)^{\frac{1}{p-1}}dt\right)^{\frac{p-1}{p}} \tag{5.5}$$

is finite for $r \geq 0$, define the Young function A_p as

$$A_p(s) = A \circ H_p{}^{-1}(s) \quad \text{for}\, s \geq 0. \tag{5.6}$$

Then we have:

Theorem 5.1. *Let G be an open bounded subset of $\mathbb{R}^n$, $n \geq 3$. Let A be a Young function such that the function $H_{n/2}$ is finite in $[0, \infty)$. Then there exists a constant K_1, depending only on n, such that*

$$\|u\|_{L^{A_{n/2}}(G)} \leq K_1 \||\nabla^2 u|\|_{L^A(G)}, \tag{5.7}$$

for every $u \in W_0^{2,A}(G)$. Moreover, $L^{A_{n/2}}(G)$ cannot be replaced by any smaller Orlicz space on the left-hand side of (5.7).

Theorem 5.1 is a consequence of Theorem 1.3 and of the Hardy-type inequality

$$\left\| \int_s^{|G|} \phi(r) r^{-1+1/p} dr \right\|_{L^{A_p}(0,|G|)} \leq K_2 \|\phi(s)\|_{L^A(0,|G|)}, \tag{5.8}$$

which is sharp as far as the target Orlicz space $L^{A_p}(0, |G|)$ is concerned. Inequality (5.8) is proved in Lemma 1 of [10] with A_p replaced by an equivalent Young function (for such equivalence, see e.g. Lemma 2 of [12]).

Remark 5. Theorem 5.1 can be extended to the case when the integral on the right-hand side of (5.5) diverges: one has just to replace A by any Young function, equivalent near infinity, which makes the integral converge. Indeed, since $|G| < \infty$, the norm in the Orlicz space $L^A(G)$ is turned into an equivalent one after such a replacement. However, in that case, the constant K_1 appearing in (5.7) also depends on $|G|$.

Remark 6. Inequality (5.7) is equivalent to the integral inequality

$$\int_G A_{n/2}\left(\frac{|u(x)|}{K_1 \left(\int_G A(|\nabla^2 u|) dy \right)^{2/n}} \right) dx \leq \int_G A(|\nabla^2 u|) dx, \tag{5.9}$$

for every $u \in W_0^{2,A}(G)$. Actually, (5.7) follows from (5.9) by the very definition of the Luxemburg norm, whereas (5.9) follows on replacing $A(s)$ by $\frac{A(s)}{\int_G A(|\nabla^2 u|) dx}$ in (5.7).

Remark 7. The first-order version of Theorem 5.1 is proved in [11] (see also [10] for an equivalent formulation). The conclusion is that there exists a constant K_1 such that

$$\|u\|_{L^{A_n}(G)} \leq K_1 \||\nabla u|\|_{L^A(G)},$$

for every $u \in W_0^{1,A}(G)$.

A special case of Theorem 5.1 is singled out in the next theorem, whose first-order version goes back to [32].

Theorem 5.2. *Let G and A be as in Theorem* 5.1. *Then there exists a constant* K_1 *such that*

$$\|u\|_{L^\infty(G)} \le K_1 \| |\nabla^2 u| \|_{L^A(G)},$$

for every $u \in W_0^{2,A}(G)$, *if and only if*

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{2}{n-2}} dt < \infty. \tag{5.10}$$

Indeed, $L^{A_{n/2}}(G) = L^\infty(G)$ if and only if $A_{n/2}(s) \equiv \infty$ for large s, and this is equivalent to (5.10).

More generally, the following characterization of those r.i. spaces $X(G)$, for which $W_0^2 X(G)$ is continuously embedded into $L^\infty(G)$, holds.

Theorem 5.3. *Let G be an open bounded subset of* $\mathbb{R}^n$, $n \ge 3$, *and let* $X(G)$ *be an r.i. space on G. Then:*

i) A positive constant K_1 *exists such that*

$$\|u\|_{L^\infty(G)} \le K_1 \| |\nabla^2 u| \|_{X(G)}, \tag{5.11}$$

for every $W_0^2 X(G)$, *if and only if*

$$\|r^{-1+2/n}\|_{\overline{X}'(0,|G|)} < \infty.$$

ii) $L^{n/2,1}(G)$ *is the largest r.i. space* $X(G)$ *which renders* (5.11) *true.*

After reducing inequality (5.11) to a 1-dimensional inequality by means of Theorem 1.3, the proof of Theorem 5.3 is analogous to that of the first-order version contained in Theorem 3.5 of [14].

We conclude with two theorems extending to second-order Sobolev inequalities some recent results from [16] and [26]. They provide a characterization of the optimal range $Y(G)$ in inequality (1.8) when the domain space $X(G)$ is given, and, under some additional assumption, of the optimal domain when the range is given. In some concrete situations, they enable us to find an explicit representation of the norms in the optimal spaces.

Let us first take into account the optimal range problem. Given any set $G \subset \mathbb{R}^n$ having finite measure, any r.i. space $X(G)$ and any real number $\alpha > 1$, define the space $Z_{X,\alpha}(G)$ as the collection of those measurable functions f on G such that

$$\|r^{1/\alpha} f^{**}(r)\|_{\overline{X}'(0,|G|)} < \infty.$$

The space $Z_{X,\alpha}(G)$, endowed with the norm

$$\|f\|_{Z_{X,\alpha}(G)} = \|r^{1/\alpha} f^{**}(r)\|_{\overline{X}'(0,|G|)},$$

is an r.i. space. This is an easy consequence of properties i)−v), Subsection 2.2, of $X(G)$ (see [16, Theorem 4.5], for details). Then we have:

Theorem 5.4. *Let G be any open bounded subset of* $\mathbb{R}^n$, $n \geq 3$, *and let* $X(G)$ *be an r.i. space on G. Set*

$$X_{n/2}(G) = Z'_{X,n/2}(G).$$

Then a constant K_1 *exists such that*

$$\|u\|_{X_{n/2}(G)} \leq K_1 \| |\nabla^2 u| \|_{X(G)}, \tag{5.12}$$

for every $u \in W_0^2 X(G)$. *Moreover,* $X_{n/2}(G)$ *cannot be replaced by any smaller r.i. space on the left-hand side of* (5.12).

We finally consider the optimal domain problem. Let G be any subset of $\mathbb{R}^n$ having finite measure. Let $Y(G)$ be an r.i. space on G and let β be a real number > 1 such that the operator T_β, defined as

$$T_\beta \phi(s) = s^{-1/\beta} \sup_{s \leq r \leq |G|} r^{1/\beta} \phi^*(r) \quad \text{for } s \in (0, |G|)$$

at a measurable function ϕ on $[0, |G|]$, is bounded on $\overline{Y}'(0, |G|)$. Then [26, Theorem 5] the space $Y^\beta(G)$ of those functions f on G satisfying

$$\left\| \int_s^{|G|} f^*(r) r^{-1+1/\beta} dr \right\|_{\overline{Y}(0,|G|)} < \infty \tag{5.13}$$

is an r.i. space endowed with a norm $\|f\|_{Y^\beta(G)}$ equivalent to the quantity on the left-hand side of (5.13).

Theorem 5.5. *Let G be any open bounded subset of* $\mathbb{R}^n$, $n \geq 3$, *and let* $Y(G)$ *be any r.i. space on G such that* $T_{n/2}$ *is bounded on* $\overline{Y}'(0, |G|)$. *Then a constant* K_1 *exists such that*

$$\|u\|_{Y(G)} \leq K_1 \| |\nabla^2 u| \|_{Y^{n/2}(G)}, \tag{5.14}$$

for every $u \in W_0^2 X(G)$. *Moreover,* $Y^{n/2}(G)$ *cannot be replaced by any larger r.i. space on the right-hand side of* (5.14).

Starting from Theorem 1.3, the proofs of Theorem 5.4 and Theorem 5.5 follow the same steps as in Theorem 4.5 of [16] and in Theorem 5 of [26], respectively.

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