Annali di Matematica 182, 457–486 (2003)

Digital Object Identifier (DOI) 10.1007/s10231-003-0076-2

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Non-smooth atoms and pointwise multipliers in function spaces

Received: April 25, 2002
Published online: August 13, 2003 –

Abstract. The aim of this paper is threefold. First we prove a non-smooth atomic decomposition theorem in some special Besov spaces. Secondly, using this result we deal with pointwise multipliers in the respective function spaces. Thirdly, in a larger context we discuss the intimate relationship between pointwise multipliers on the one hand and some fundamental notation of fractal geometry, such as self-similarity and porosity, on the other hand.

Mathematics Subject Classification (2000). 46E35; 42B35, 28A80

Key words. function spaces – atoms – pointwise multipliers – self-similarity

1. Introduction

Let $n \in \mathbb{N}$ and let $B^s_p(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n)$ be a special space of Besov type in $\mathbb{R}^n$ with

$$0 < p \leq \infty \quad \text{and} \quad s > \sigma_p = n\left(\frac{1}{p} - 1\right)_+ . \tag{1.1}$$

Let ψ be an appropriate C^∞ function in $\mathbb{R}^n$ with compact support such that

$$\sum_{l \in \mathbb{Z}^n} \psi(x - l) = 1, \quad x \in \mathbb{R}^n,$$

(resolution of unity). Then $B^s_{p,\,\mathrm{unif}}(\mathbb{R}^n)$ and $B^s_{p,\,\mathrm{selfs}}(\mathbb{R}^n)$ are the collections of functions m such that

$$\left\| m \,|\, B^s_{p,\,\mathrm{unif}}(\mathbb{R}^n) \right\| = \sup_{l \in \mathbb{Z}^n} \left\| \psi(\cdot - l)\, m \,|\, B^s_p(\mathbb{R}^n) \right\| < \infty$$

and

$$\left\| m \,|\, B^s_{p,\,\mathrm{selfs}}(\mathbb{R}^n) \right\| = \sup_{l \in \mathbb{Z}^n,\, j \in \mathbb{N}_0} \left\| \psi(\cdot - l)\, m(2^{-j}\cdot) \,|\, B^s_p(\mathbb{R}^n) \right\| < \infty, \tag{1.2}$$

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respectively. Let $M(B^s_p)$ be the set of all pointwise multipliers of $B^s_p(\mathbb{R}^n)$. One aim of this paper is to prove the following assertion (Theorem 1):

(i) Let p and s as in (1.1). Then

$$\bigcup_{\sigma>s} B^\sigma_{p,\,\mathrm{selfs}}(\mathbb{R}^n) \subset M(B^s_p) \hookrightarrow B^s_{p,\,\mathrm{selfs}}(\mathbb{R}^n). \tag{1.3}$$

(ii) If, in addition, $0 < p \le 1$, then

$$M(B^s_p) = B^s_{p,\,\mathrm{selfs}}(\mathbb{R}^n). \tag{1.4}$$

(iii) If, in addition, either

$$0 < p \le 1,\ s \ge \frac{n}{p} \qquad \text{or} \qquad 1 < p \le \infty,\ s > \frac{n}{p}, \tag{1.5}$$

then

$$M(B^s_p) = B^s_{p,\,\mathrm{selfs}}(\mathbb{R}^n) = B^s_{p,\,\mathrm{unif}}(\mathbb{R}^n). \tag{1.6}$$

All spaces involved in (1.3), (1.4), (1.6), are multiplication algebras. The study of pointwise multipliers is one of the key problems in the theory of function spaces. It has attracted a lot of attention in the decades since starting with [22]. As far as classical Besov spaces and (fractional) Sobolev spaces are concerned we refer to [13] and, in particular, to [14]. Pointwise multipliers in general spaces

$$B^s_{pq}(\mathbb{R}^n) \quad \text{and} \quad F^s_{pq}(\mathbb{R}^n), \quad \text{where} \quad 0 < p \le \infty,\ 0 < q \le \infty,\ s \in \mathbb{R}, \tag{1.7}$$

have been studied in great detail in [17, Ch. 4], where one also finds many references and historical comments, and in the more recent papers [18], and especially [19]. As for our own contributions we refer to [23, 2.8]; [24, 4.2]; where again one finds detailed references. The equalities in (1.6) with (1.5) are more or less known (and easy to prove) in the context of respective assertions for the more general spaces in (1.7). In [14] ($p = 1$) and in [19] ($0 < p \le 1$) one also finds some cases of (1.4), although the formulations given there are different (later on we give more detailed references). Here we offer simple proofs: the main arguments cover only a few lines. This simplicity applies also to the proofs of (1.6) and of the right-hand side of (1.3).

As for the left-hand side of (1.3) we need non-smooth atoms. This is the second aim of this paper. Recall that, say, the C^∞ function $a^{k,l}$ with $k \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$ is called a $(s,p)_K$-atom if

$$\operatorname{supp} a^{k,l} \subset B_{k,l} \quad \text{and} \quad \left| D^\alpha a^{k,l}(x) \right| \le 2^{-k(s-\frac{n}{p})+|\alpha|k}, \tag{1.8}$$

where $B_{k,l}$ is a ball centred at $2^{-k}l$ with radius $c2^{-k}$ (for some $c > 0$) and $|\alpha| \le K$, for every fixed $K \in \mathbb{N}$ with $K > s$. Then $f \in B^s_p(\mathbb{R}^n)$ with (1.1) admits the atomic decomposition

$$f = \sum_{k\in\mathbb{N}_0, l\in\mathbb{Z}^n} \lambda_{kl} a^{k,l}(x), \quad \|\lambda\,|\ell_p\| = \left(\sum_{k,l} |\lambda_{kl}|^p \right)^{\frac{1}{p}} < \infty \tag{1.9}$$

with

$$\| f \,|B^s_p(\mathbb{R}^n)\| \sim \inf \|\lambda \,|\ell_p \|, \tag{1.10}$$

where the infimum is taken over all representations of type (1.9). This is a special case of the atomic decomposition theorem in [25, Theorem 13.8, p. 75]. There one also finds the necessary references to the literature, especially to [9] and [10], which are of special relevance in the context of (1.9), (1.10). In order to prove the left-hand side of (1.3) one can multiply (1.9) with a respective function m and ask whether the functions $ma^{k,l}$ are again building blocks which fit in this scheme. This results in the problem of non-smooth atoms. We give a rough description. Let p and s again be given by (1.1) and let $\sigma > s$. Then $a^{k,l}$ is called a (non-smooth) $(s, p)_\sigma$-atom if (1.8) is generalised by

$$\operatorname{supp} a^{k,l} \subset B_{k,l} \quad \text{and} \quad \|a^{k,l} \,|B^\sigma_p(\mathbb{R}^n)\| \le 2^{k(\sigma-s)}. \tag{1.11}$$

All (smooth) atoms in the literature satisfy these conditions. It is the second aim of this paper to prove an atomic decomposition theorem of type (1.9), (1.10) based on (1.11). This will be done in Theorem 2. [If $p \le 1$ then one can even choose $\sigma = s$ in (1.11) in good agreement with (1.4). But this does not say very much and is out of our interest.] Afterwards one can prove the left-hand side of (1.3) again in a few lines.

There are surely many applications both of non-smooth atomic decompositions and of the simple criteria (1.3), (1.4), (1.6). We restrict ourselves to the question under which circumstances the characteristic function χ_Ω of a, say, bounded domain Ω in $\mathbb{R}^n$ is a pointwise multiplier in $B^s_p(\mathbb{R}^n)$. By (1.2) and (1.3), (1.4), (1.6) *self-similarity* properties of functions naturally enter the scene in connection with pointwise multipliers. Now a second basic notation of fractal geometry also proves to be useful. If $\Gamma = \partial\Omega$ satisfies a *uniform porosity* condition (*uniform ball* condition) then there is a number $\varepsilon > 0$ such that (Corollary 1 based on Theorem 3)

$$\chi_\Omega \in M(B^s_p), \quad 0 < p < \infty, \quad \sigma_p < s < \frac{\varepsilon}{p}. \tag{1.12}$$

It is the main aim of this paper to shed new light on atomic decompositions based on (1.11) and how they are symbiotically related to (s, p)-*self-similarity* of functions, *porosity* of boundaries $\partial\Omega$ of domains Ω in $\mathbb{R}^n$, and *pointwise multipliers*. This broader point of view may justify our restriction to the technically simple spaces $B^s_p(\mathbb{R}^n)$ with (1.1), although there is a good chance that some (but not all) of our arguments can be extended to some spaces of type (1.7) as will be indicated at the very end of this paper.

The plan of the paper is the following. In Section 2 we give the necessary definitions, formulate and discuss our main results, mostly from the just outlined broader point of view. Proofs are shifted to Section 3. In the complementary Section 4 we collect a few assertions for the more general spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ which follow from the results in Section 2 and from known general properties of pointwise multipliers in spaces of type (1.7) available in the literature. Again we wish to emphasise the connections with some notation of fractal geometry. For example, (1.12) can be extended by interpolation, duality and monotonicity to some spaces in (1.7), also with $s = 0$ and $s < 0$. But here we assume that the reader of this complementary Section 4 is familiar with the theory of the spaces in (1.7).

2. Main results

2.1. Notation and definitions

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be a Euclidean n-space where $n \in \mathbb{N}$; put $\mathbb{R} = \mathbb{R}^1$; whereas $\mathbb{C}$ is the complex plane. As usual, $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$, denotes the lattice of all points $l = (l_1, \dots, l_n) \in \mathbb{R}^n$ with $l_j \in \mathbb{Z}$. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices

$$\alpha = (\alpha_1, \dots, \alpha_n) \quad \text{with} \quad \alpha_j \in \mathbb{N}_0 \quad \text{and} \quad |\alpha| = \sum_{j=1}^n \alpha_j .$$

Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$ is the standard quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\| f \,| L_p(\mathbb{R}^n) \| = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

with the obvious modification if $p = \infty$. Let $L_1^{\mathrm{loc}}(\mathbb{R}^n)$ be the collection of all locally Lebesgue-integrable functions in $\mathbb{R}^n$. We use the standard abbreviation

$$\sigma_p = n\left(\frac{1}{p} - 1\right)_+ = n\left(\frac{1}{\min(p,1)} - 1\right) \quad \text{where} \quad n \in \mathbb{N},\, 0 < p \le \infty. \tag{2.1}$$

Furthermore, if $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ then

$$\left(\Delta_h^1 f\right)(x) = f(x+h) - f(x), \quad \text{where} \quad x \in \mathbb{R}^n,\, 0 \ne h \in \mathbb{R}^n,$$

and, iteratively, $\Delta_h^M = \Delta_h^1(\Delta_h^{M-1})$ if $M - 1 \in \mathbb{N}$. We always assume that ψ is a C^∞ function in $\mathbb{R}^n$ with

$$\psi(x) > 0 \quad \text{if} \quad |x| < c, \qquad \psi(y) = 0 \quad \text{if} \quad |y| \ge c, \tag{2.2}$$

and

$$\sum_{l \in \mathbb{Z}^n} \psi(x - l) = 1 \quad \text{if} \quad x \in \mathbb{R}^n, \tag{2.3}$$

(resolution of unity) for some suitable $c > 0$ (one may choose, for example, $c = \sqrt{n}$). In this paper $n \in \mathbb{N}$ is fixed and all the spaces considered are defined on $\mathbb{R}^n$. This may justify our omission of $\mathbb{R}^n$ in the respective notation. Hence we write L_p and L_1^{loc} instead of $L_p(\mathbb{R}^n)$ and $L_1^{\mathrm{loc}}(\mathbb{R}^n)$, respectively, etc.

Definition 1. *Let* $0 < p \le \infty$ *and* $s > \sigma_p$.

(i) *Let, in addition,* $N \in \mathbb{N}$ *with* $N > s$. *Then* B^s_p *is the collection of all* $f \in L^{\mathrm{loc}}_1$ *such that*

$$\|f\,|B^s_p\| = \|f\,|L_p\| + \left(\int\limits_{|h|\le 1} |h|^{-sp}\,\big\|\Delta^N_h f\,|L_p\big\|^p\,\frac{dh}{|h|^n}\right)^{\frac{1}{p}} < \infty \tag{2.4}$$

(with the usual modification if $p = \infty$*).*

(ii) *Let, in addition,* ψ *be the function according to* (2.2), (2.3). *Then* $B^s_{p,\,\mathrm{unif}}$ *is the collection of all* $f \in L^{\mathrm{loc}}_1$ *such that*

$$\|f\,|B^s_{p,\,\mathrm{unif}}\| = \sup_{l\in\mathbb{Z}^n}\|\psi(\cdot - l)\,f\,|B^s_p\| < \infty, \tag{2.5}$$

and $B^s_{p,\,\mathrm{selfs}}$ *is the collection of all* $f \in L^{\mathrm{loc}}_1$ *such that*

$$\|f\,|B^s_{p,\,\mathrm{selfs}}\| = \sup_{l\in\mathbb{Z}^n,\, j\in\mathbb{N}_0}\|\psi(\cdot - l)\,f(2^{-j}\cdot)\,|B^s_p\| < \infty. \tag{2.6}$$

Remark 1. Recall that $B^s_p = B^s_{pp}$ are special spaces of Besov type with the Hölder–Zygmund spaces

$$\mathcal{C}^s = B^s_\infty, \quad s > 0,$$

as distinguished cases. The theory of the more general spaces B^s_{pq} and F^s_{pq} with (1.7) has been developed in detail in [23] and [24]. They are quasi-Banach spaces, always considered in the framework of tempered distributions S' on $\mathbb{R}^n$. As for the above specific formulation in part (i) we refer to [24, Theorem 2.6.1, p. 140, Corollary 2.6.1/1 and Remark 2.6.1/3 on p. 142]. For different values of $N \in \mathbb{N}$ with $N > s$ in (2.4) one gets equivalent quasi-norms. But this is unimportant for our purpose and not indicated on the left-hand side of (2.4) [one may think of the smallest admitted N]. In connection with pointwise multipliers, $B^s_{p,\,\mathrm{unif}}$ is a rather natural construction, we refer to [17, 4.3.1, p. 150]. Finally, $B^s_{p,\,\mathrm{selfs}}$ describes the (s,p)-*self-similarity* behaviour of f, where $\psi(\cdot - l)\,f(2^{-j}\cdot)$ must be understood as usual,

$$x \mapsto \psi(x - l)\,f(2^{-j}x), \quad x \in \mathbb{R}^n,\, l \in \mathbb{Z}^n,\, j \in \mathbb{N}_0.$$

Hence it checks the quality of f in a 2^{-j}-neighbourhood with respect to the lattice $2^{-j}\,\mathbb{Z}^n$. Of course, $B^s_{p,\,\mathrm{unif}}$ and $B^s_{p,\,\mathrm{selfs}}$ are quasi-Banach spaces.

2.2. Multipliers

Let A be one of the spaces in (1.7), where, with the exception of the complementary Section 4, we are only interested in $A = B^s_p$ as introduced in Definition 1(i). Then a locally integrable function m in $\mathbb{R}^n$ is called a *multiplier* for A if

$$f \mapsto mf \quad \text{generates a bounded map in } A. \tag{2.7}$$

Especially in the general case of the spaces from (1.7) one has to say more precisely what this means. But we refer for this technical point to [17, 4.2]. In the literature, multipliers are often denoted as pointwise multipliers, not to be mixed with Fourier multipliers. But there are no Fourier multipliers in this paper. This may justify calling them simply multipliers from now on. The collection of all multipliers with respect to a given space A is denoted by $M(A)$. Naturally quasi-normed, it becomes a quasi-Banach space. It is a *multiplication algebra*: if $m_1 \in M(A)$ and $m_2 \in M(A)$, then $m_1 m_2 \in M(A)$ and for some $c > 0$,

$$\|m_1 m_2 \,|M(A)\| \le c\, \|m_1 \,|M(A)\| \cdot \|m_2 \,|M(A)\|. \tag{2.8}$$

More generally, a linear space M on $\mathbb{R}^n$ is called a *multiplication algebra* if $m_1 m_2 \in M$ for all $m_1 \in M$ and $m_2 \in M$. If, in addition, M is a quasi-Banach space then (2.8) with M in place of $M(A)$ is assumed to be valid.

We complement Definition 1 by

$$b^s_{p,\,\mathrm{selfs}} = \bigcup_{\sigma > s} B^\sigma_{p,\,\mathrm{selfs}}, \quad \text{where} \quad 0 < p \le \infty,\ s > \sigma_p. \tag{2.9}$$

In the proposition below, $|x| + |h| \lesssim 2^{-j}$ means that there is a suitable chosen constant $c > 0$ with $|x| + |h| \le c\, 2^{-j}$ for all admitted x, h, j. Then it is clear what is meant by $\lesssim$, also in other occurences.

Proposition 1. *Let* $0 < p \le \infty$ *and* $s > \sigma_p$*, where* σ_p *is given by* (2.1).

(i) *Let* $N \in \mathbb{N}_0$ *with* $N > s$*. Then* $B^s_{p,\mathrm{selfs}}$ *consists of all* $f \in L_1^{\mathrm{loc}}$ *such that*

$$\sup_{l \in \mathbb{Z}^n,\, j \in \mathbb{N}_0} 2^{-j(s - \frac{n}{p})} \left(\int\limits_{|x|+|h| \lesssim 2^{-j}} |h|^{-sp} \left| \Delta^N_h f(x + l2^{-j}) \right|^p \frac{dx\, dh}{|h|^n} \right)^{\frac{1}{p}}$$

$$+ \sup_{l \in \mathbb{Z}^n,\, j \in \mathbb{N}_0} 2^{j\frac{n}{p}} \left(\int\limits_{|x| \lesssim 2^{-j}} |f(x + l2^{-j})|^p \, dx \right)^{\frac{1}{p}} < \infty \tag{2.10}$$

(equivalent quasi-norms), with the usual modification if $p = \infty$*. Furthermore,*

$$B^s_{p,\mathrm{selfs}} \hookrightarrow L_\infty \qquad \textit{(continuous embedding).} \tag{2.11}$$

(ii) *Both* $B^s_{p,\mathrm{selfs}}$ *and* $b^s_{p,\mathrm{selfs}}$ *are multiplication algebras.*

Remark 2. We shift the proof to 3.1. As usual, $\hookrightarrow$ always stands for continuous embedding between quasi-Banach spaces. Furthermore if A_1 and A_2 are two quasi-Banach spaces then $A_1 = A_2$ means that they coincide as sets and that the corresponding quasi-norms are equivalent to each other.

Theorem 1. *Let* $0 < p \le \infty$ *and* $s > \sigma_p$, *where* σ_p *is given by* (2.1).

(i) *Then*

$$b^s_{p,\mathrm{selfs}} \subset M(B^s_p) \hookrightarrow B^s_{p,\mathrm{selfs}}. \tag{2.12}$$

(ii) *In addition, let* $0 < p \le 1$. *Then*

$$M(B^s_p) = B^s_{p,\mathrm{selfs}}. \tag{2.13}$$

(iii) *In addition, let*

$$\begin{aligned} &\textit{either} \qquad 0 < p \le 1, \quad s \ge \frac{n}{p}, \\ &\textit{or} \qquad 1 < p \le \infty, \quad s > \frac{n}{p}. \end{aligned} \tag{2.14}$$

Then

$$M(B^s_p) = B^s_{p,\mathrm{selfs}} = B^s_{p,\mathrm{unif}}. \tag{2.15}$$

Remark 3. We shift the proof to 3.2. There is only one point where we cannot rely immediately on properties for the spaces B^s_p available in the book literature. This is the left-hand side of (2.12). Here we need non-smooth atomic decompositions as obtained in the next subsection. Afterwards also the left-hand side of (2.12) is a matter of a few lines. Let $1 < p < \infty$ and let, temporarily, A^s_p be either B^s_p or the (fractional) Sobolev spaces H^s_p. Then

$$M(A^s_p) = A^s_{p,\mathrm{unif}}\,, \qquad 1 < p < \infty,\ s > \frac{n}{p}\,, \tag{2.16}$$

is a classical result. In the case of H^s_p it goes back to [22] (1967). This has been complemented by a corresponding assertion for B^s_p in [16, Corollary, p. 151] (1976), essentially by interpolation. (2.16) can be extended to all spaces in (1.7) which are multiplication algebras and which satisfy the so-called localisation principle. We return to this point in our complementary remarks in 4.1. More details may be found in [17, 4.6.4, p. 222, and 4.9.1, p. 247]. If $0 < p \le \infty$ and $s > \sigma_p$ then one always has

$$M(B^s_p) \hookrightarrow L_\infty \cap B^s_{p,\,\mathrm{unif}}. \tag{2.17}$$

This is well known and may be found explicitly stated in [19, Lemma 3, p. 213/214], with references to the literature. In particular,

$$M(B^s_p) = B^s_{p,\,\mathrm{unif}}$$

can only be expected in the case of (2.14), since otherwise B^s_p and $B^s_{p,\mathrm{unif}}$ are not continuously embedded in L_∞, [17, 4.6.4, pp. 221/222], with a reference to [21] (dealing with all spaces (1.7), $p < \infty$ in the F-case). This is the point where one has to step from *uniform* to the more subtle *self-similar*. Then we have at least (2.11) in any case considered. Hence the right-hand side of (2.12) is a refinement of the known necessary condition (2.17). Some assertions of type (2.13) and also of characterisations of $B^s_{p,\,\mathrm{selfs}}$ according to Proposition 1(i) are known, although the formulations given are different (but they can be translated into each other). If $p = 1$ and $0 < s \notin \mathbb{N}$ then we refer to [14, 3.4.2, p. 140] (a formulation of this result may also be found in [18, Theorem 2.9, pp. 299/300]). As for $p < 1$ (and with some restrictions for $p = 1$) we refer to [19, 3.4.1, pp. 234–236, (and 3.4.2, pp. 236–237, respectively)]. There are characterisations of type (2.13) with (2.10) for $M(B^s_p)$ with $\sigma_p < s < 1$, in the framework of the more general case of $M(F^s_{pq})$. But in any case it has been stated clearly in [19] that it is easier to say something about multipliers in spaces with $p < 1$ than in spaces with $p > 1$. The other assertions of Theorem 1 and Proposition 1 seem to be new. But it is not our aim (neither here nor in the complementing Section 4) to deal systematically with multipliers. There is a huge literature spanning over decades which has been quoted especially in [14]; [23, 2.8]; [24, 4.2]; [17, Ch. 4]; [18]; [19]. Our aim is different. On the one hand we wish to make clear that the above assertions can be obtained rather easily using existing results for the spaces in question, complemented by non-smooth atomic decompositions. Secondly, and more importantly, we wish to emphasise that, apparently, multipliers in function spaces are related to some fundamental notation in fractal geometry such as *self-similarity* and (in the applications given in 2.4) *porosity* of sets.

2.3. Non-smooth atomic decompositions

Let $k \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$. Then

$$B_{k,l} = \left\{ x \in \mathbb{R}^n \,:\, |x - 2^{-k} l| \lesssim 2^{-k} \right\},$$

are balls in $\mathbb{R}^n$, where again, here and later on, $\lesssim 2^{-k}$ means $\le c\,2^{-k}$ for some suitably chosen $c > 0$, which is independent of all relevant ingredients, here k and l. Let again $0 < p \le \infty$, $s > \sigma_p$, and $K \in \mathbb{N}$ with $K > s$. Then K times differentiable functions $a^{k,l}$ in $\mathbb{R}^n$ are called *classical* $(s,p)_K$*-atoms* if

$$\operatorname{supp} a^{k,l} \subset B_{k,l} \quad \text{and} \quad |D^\alpha a^{k,l}(x)| \le 2^{-k(s-\frac{n}{p})+|\alpha|k},\ |\alpha| \le K. \tag{2.18}$$

Let

$$\lambda = \left\{ \lambda_{kl} \in \mathbb{C} \,:\, k \in \mathbb{N}_0 \text{ and } l \in \mathbb{Z}^n \right\},$$

and for $0 < p \le \infty$,

$$\|\lambda\,|\ell_p\| = \left(\sum_{k \in \mathbb{N}_0,\, l \in \mathbb{Z}^n} |\lambda_{kl}|^p \right)^{\frac{1}{p}}, \tag{2.19}$$

with the usual modification if $p = \infty$. For fixed K one has the classical atomic decomposition theorem saying that $f \in L^{\mathrm{loc}}_1$ is an element of B^s_p if, and only if, it can be represented as

$$f = \sum_{k,l} \lambda_{kl} a^{k,l}(x), \quad \text{with} \quad \|\lambda\,|\ell_p\| < \infty, \tag{2.20}$$

where $a^{k,l}$ are classical $(s,p)_K$-atoms with (2.18). The series (2.20) coverges absolutely in L_1^{loc} and even in some L_r with $1 < r \le \infty$. Furthermore,

$$\big\| f \,|B^s_p\big\| \sim \inf \|\lambda \,|\ell_p\|, \tag{2.21}$$

where the infimum is taken over all representations (2.20). In this formulation the theorem is a special case of [25, Theorem 13.8, pp. 75/76]. But essentially, it goes back to [9], [10]. We need a non-smooth generalisation of this assertion.

Definition 2. *Let $0 < p \le \infty$ and $s > \sigma_p$ where σ_p is given by* (2.1). *Let $s < \sigma < \infty$, $k \in \mathbb{N}_0$, and $l \in \mathbb{Z}^n$. Then $a^{k,l} \in B^\sigma_p$ is called an $(s,p)_\sigma$-atom if*

$$\operatorname{supp} a^{k,l} \subset B_{k,l} \quad \textit{and} \quad \big\|a^{k,l}\,|B^\sigma_p\big\| \le 2^{k(\sigma-s)}. \tag{2.22}$$

Remark 4. If $\sigma_p < s < \sigma < \frac{n}{p}$, then $(s,p)_\sigma$-atoms might be unbounded. If $\sigma_p < s < \sigma < \frac{1}{p}$ then appropriately normalised characteristic functions of cubes are $(s,p)_\sigma$-atoms and then (2.20) is a representation by step-functions. The above definition makes sense for all $0 < p \le \infty$ and $s > \sigma_p$. However the case of interest in our context here is $1 < p \le \infty$.

Proposition 2. *Let $0 < p \le \infty$ and $s > \sigma_p$, where σ_p is given by* (2.1).

(i) *Let $K \in \mathbb{N}$ and $s < \sigma < K$. Any classical $(s,p)_K$-atom is an $(s,p)_\sigma$-atom.*
(ii) *Let $s < \sigma$ and let $a^{k,l}$ be an $(s,p)_\sigma$-atom. Then*

$$\big\|a^{k,l}\,|B^s_p\big\| \le 1 \quad \textit{and} \quad \big\|a^{k,l}\,|L_p\big\| \le 2^{-ks}. \tag{2.23}$$

Remark 5. Here ≤ 1 in (2.23) means that there are appropriate quasi-norms in B^σ_p and B^s_p with this property. Otherwise one has to replace ≤ 1 by $\lesssim 1$ according to our above agreement. We shift the proof of this proposition to 3.3.

Theorem 2. *Let $0 < p \le \infty$ and $\sigma_p < s < \sigma$, where σ_p is given by* (2.1). *Then L_1^{loc} is an element of B^s_p if, and only if, it can be represented as*

$$f = \sum_{k\in\mathbb{N}_0}\sum_{l\in\mathbb{Z}^n} \lambda_{kl}\, a^{k,l}(x) \quad \textit{with} \quad \|\lambda\,|\ell_p\| < \infty, \tag{2.24}$$

where $a^{k,l}$ are $(s,p)_\sigma$-atoms according to Definition 2. *The series on the right-hand side of* (2.24) *coverges absolutely in some L_r with $1 < r \le \infty$. Furthermore,*

$$\big\| f \,|B^s_p\big\| \sim \inf \|\lambda \,|\ell_p\|, \tag{2.25}$$

where the infimum is taken over all representations (2.24).

Remark 6. We shift the proof of this theorem to 3.4. But we wish to clarify two points now. *Firstly*, the insignificance of the theorem if $p \le 1$. In case of $p \le 1$, it is well known, and also an immediate consequence of (2.4), that B^s_p is a p-Banach spaces, which means that

$$\Big\| \sum_{k=1}^\infty f_k \,|B^s_p\Big\|^p \le \sum_{k=1}^\infty \big\| f_k\,|B^s_p\big\|^p, \qquad f_k \in B^s_p \tag{2.26}$$

(appropriate equivalent quasi-norms). Hence, by (2.24) and (2.23) we have that

$$\| f \,|B^s_p\|^p \le \|\lambda \,|\ell_p\|^p.$$

Then (2.25) follows from part (i) of Proposition 2 and the classical atomic decomposition theorem as described at the beginning of this subsection. Later on we need the above theorem only in the case of $p > 1$.

Secondly, we clarify the convergence of the series in (2.24) in some L_r. Let $p > 1$. Since $s > 0$ it follows by (2.23) and the support properties of $a^{k,l}$ according to (2.22) that

$$\|f \,|L_p\| \le \sum_{k=0}^{\infty} 2^{-ks} \Big(\sum_{l\in\mathbb{Z}^n} |\lambda_{kl}|^p \Big)^{\frac{1}{p}} \lesssim \|\lambda \,|\ell_p\|.$$

In particular, the series in (2.24) converges absolutely in L_p (hence we may choose $r = p$ in the theorem). Let $p \le 1$. Since $s > \sigma_p$ there is a number r with $1 < r \le \infty$ such that

$$B^s_p \hookrightarrow L_r, \quad \text{where} \quad s - \frac{n}{p} \ge -\frac{n}{r}.$$

This is a well-known embedding theorem, [23, p. 129], combined with a Paley–Littlewood assertion for L_r, [23, p. 88] (but it is also an easy consequence of the above classical atomic decomposition theorem). Then again it follows by (2.23), (2.22) that

$$\|f \,|L_r\| \le \sum_{k=0}^{\infty} \Big(\sum_{l\in\mathbb{Z}^n} |\lambda_{kl}|^r \Big)^{\frac{1}{r}} \le \|\lambda \,|\ell_p\|.$$

This shows that the series in (2.24) converges absolutely in L_r.

2.4. *Characteristic functions as multipliers*

Let χ_Ω be the characteristic function of the domain Ω in $\mathbb{R}^n$. One of the outstanding problems in the theory of function spaces of type (1.7) and their special cases such as (fractional) Sobolev spaces, classical Besov spaces, Hölder–Zygmund spaces, Hardy spaces, bmo, etc., is the question under which circumstances and for which of these spaces χ_Ω is a multiplier. As for the now older contributions we refer to [23, 2.8.7, pp. 158–165]; [8]; [10, §13]; and the references given there. The state-of-the-art in the middle of the nineties may be found in [17, 4.6.3, pp. 207–221, 258]. More recent results (and again additional references) are given in [18, Section 4]; [19, Section 4]; [27, Section 5]. Here we wish to contribute to this question mainly as an application of Theorem 1 and, hence, restricted to B^s_p (but we add a few comments concerning more general spaces in the complementary Subsection 4.3). Furthermore we wish to hint on a connection between some recent notation related to fractal geometry and multipliers in function spaces. It is no restriction of generality assuming from the very beginning that the domains Ω

in $\mathbb{R}^n$ considered are bounded. Of interest is the, then, compact boundary $\Gamma = \partial\Omega$. First we introduce some notation with respect to arbitrary compact sets Γ in $\mathbb{R}^n$. A ball centred at $x \in \mathbb{R}^n$ and of radius $r > 0$ is denoted by $B(x,r)$.

Definition 3. *Let Γ be a non-empty compact set in $\mathbb{R}^n$.*

(i) *Let $h : t \mapsto h(t)$ be a positive monotonically increasing function on the interval $(0,1]$. Then Γ is called an h-set if there is a finite Radon measure μ in $\mathbb{R}^n$ with*

$$\operatorname{supp}\mu = \Gamma \quad \text{and} \quad \mu(B(\gamma,r)) \sim h(r), \quad \gamma \in \Gamma,\, 0 < r \le 1. \tag{2.27}$$

(ii) *Then Γ is said to be porous if there is a number η with $0 < \eta < 1$, such that one finds, for any ball $B(x,r)$, centred at $x \in \mathbb{R}^n$ and of radius $0 < r < 1$, a ball $B(y,\eta r)$ with*

$$B(y,\eta r) \subset B(x,r) \quad \text{and} \quad B(y,\eta r) \cap \Gamma = \emptyset. \tag{2.28}$$

(iii) *Then Γ is said to be uniformly porous if it is porous and if it is an h-set according to part (i) with respect to an admitted h.*

Remark 7. We comment on part (i). We followed the notation introduced and studied in [2], [3], [4]. Of course, $\sim$ in (2.27) means that the respective equivalence constants are independent of γ and r. It comes out that for a given function h there is a compact set Γ and a Radon measure μ with (2.27) if, and only if, there exists an equivalent function h^*, $h \sim h^*$, with

$$\frac{h^*(2^{-j-k})}{h^*(2^{-j})} \ge 2^{-kn}, \quad \text{for all} \quad j \in \mathbb{N}_0 \quad \text{and} \quad k \in \mathbb{N}_0. \tag{2.29}$$

We refer to [2], [3]. The classical example of an admitted function h is given by $h(t) = t^d$. Then Γ becomes a so-called d-set and one may choose $\mu = \mathcal{H}^d|\Gamma$, the restriction of the Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^n$ on Γ,

$$\operatorname{supp}\mu = \Gamma, \quad \mu(B(\gamma,r)) \sim r^d, \quad \gamma \in \Gamma, \quad 0 < r \le 1. \tag{2.30}$$

Here $0 < d \le n$, ($d = 0$ makes sense but it is not of interest). These d-sets have been studied with great intensity, both in fractal geometry (with a slightly different notation of what is called a d-set) and in the theory of function spaces. Details may be found in [25, Section 3, pp. 5–7]. Perturbed d-sets, so-called (d,Ψ)-sets, typically with $\Psi(t) = |\log ct|^b$, have been introduced in [6], [7], and considered in detail in [15]. They are special h-sets. The theory of the related function spaces on (d,Ψ)-sets and h-sets may be found in [15] and [4], respectively.

Remark 8. We comment on parts (ii) and (iii) of Definition 3. Properties of sets of the type (2.28) have been used by several authors and at several occasions under different names. Here we followed [26, Definition 9.16, p. 138], where we called it the *ball condition*. But to call such sets *porous* might be better and more suggestive. It is in good agreement with the porosity of sets as considered in [12, 11.12, p. 156]. We proved in [26, 9.17, p. 138–139], that any compact porous set has Lebesgue measure zero. There and in [26, 9.19, pp. 140–141], one also finds further relevant references. By [26, Proposition 9.18, pp. 139–140], an h-set Γ is porous (and hence Γ is uniformly porous) if and only if, there is a positive number ε such that

$$\frac{h(2^{-j-k})}{h(2^{-j})} \gtrsim 2^{-k(n-\varepsilon)} \quad \text{for all } j \in \mathbb{N}_0 \text{ and } h \in \mathbb{N}_0. \tag{2.31}$$

This is a good agreement with (2.29). Recall that $\gtrsim 2^{-k(n-\varepsilon)}$ means $\geq c\,2^{-k(n-\varepsilon)}$ for some constant $c > 0$ which is independent of j and k. We have in any case $\varepsilon < n$.

Recall that σ_p is always given by (2.1).

Theorem 3. (i) *Let Ω be a bounded domain in $\mathbb{R}^n$. Let* $0 < p < \infty$, $\sigma > \sigma_p$, *and let $\Gamma = \partial\Omega$ be an h-set according to Definition* 3(i) *with*

$$\sup_{j\in\mathbb{N}_0} \sum_{k=0}^{\infty} 2^{k\sigma p} \left(\frac{h(2^{-j})}{h(2^{-j-k})}\, 2^{-kn} \right) < \infty. \tag{2.32}$$

Let $B^{\sigma}_{p,\,\mathrm{selfs}}$ be the space introduced in Definition 1(ii). *Then*

$$\chi_\Omega \in B^{\sigma}_{p,\,\mathrm{selfs}}. \tag{2.33}$$

(ii) *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\Gamma = \partial\Omega$ be uniformly porous according to Definition* 3(iii). *Then there is a positive number ε such that*

$$\chi_\Omega \in B^{\sigma}_{p,\,\mathrm{selfs}}, \quad \textit{where} \quad 0 < p < \infty \quad \textit{and} \quad \sigma_p < \sigma < \frac{\varepsilon}{p}. \tag{2.34}$$

(iii) *Let* $1 \leq n-1 \leq d < n$. *There exists a bounded star-like domain Ω in $\mathbb{R}^n$ such that $\Gamma = \partial\Omega$ is a d-set according to* (2.30),

$$\chi_\Omega \in B^{\sigma}_{p,\,\mathrm{selfs}}, \quad \textit{where} \quad 0 < p < \infty, \quad \sigma_p < \sigma < \frac{n-d}{p}, \tag{2.35}$$

and

$$\chi_\Omega \notin B_p^{\frac{n-d}{p}}, \quad \textit{where} \quad 0 < p < \infty, \quad \sigma_p < \frac{n-d}{p}. \tag{2.36}$$

Remark 9. We shift the proof of this theorem to 3.5. But we add a few comments now. By (2.29) the brackets in (2.32) are always uniformly bounded. Hence, (2.32) asks for a qualified decay of these brackets with respect to k and uniformly in j. Inserting $h(t) = t^d$ one gets just (2.35), as it should be. Since Γ is the boundary of a domain we always have $n-1 \leq d \leq n$ (excluding $d = n$ in part (iii)). If the boundary $\Gamma = \partial\Omega$ is a general h-set then one gets, as a by-product of the proof given below, that

$$\frac{h(2^{-j})}{h(2^{-j-k})} \gtrsim 2^{(n-1)k}, \quad k \in \mathbb{N}_0, \quad j \in \mathbb{N}_0.$$

Hence by (2.32) we obtain that $\sigma_p < \sigma < \frac{1}{p}$, as it should be. The number ε in part (ii) is the same as in (2.31). Finally, part (iii) makes clear that the assertions are sharp, at least in terms of B^s_p spaces and d-sets. Recall that a bounded domain Ω is called *star-like*, say, with respect to the origin, if for any $y \in \mathbb{R}^n$ with $|y| = 1$ there is a positive number $\lambda(y)$ such that λy with $\lambda \geq 0$ belongs to Ω if, and only if, $0 \leq \lambda < \lambda(y)$. The proof given below also shows that

$$\chi_\Omega \in B^{\frac{n-d}{p}}_{p\infty,\,\mathrm{selfs}} \quad \text{and} \quad \chi_\Omega \notin B^{\frac{n-d}{p}}_{pq} \text{ if } 0 < q < \infty,\ \frac{n-d}{p} > \sigma_p\,, \tag{2.37}$$

where the first assertion holds for any bounded domain Ω whose boundary $\Gamma = \partial\Omega$ is a d-set, $n - 1 \leq d < n$. This strengthens both (2.35) and (2.36). As said in the introduction when it comes to spaces B^s_{pq} or F^s_{pq} with $q \neq p$, we assume that the reader is familiar with the underlying theory for these spaces. We mention only that $B^s_{p\infty,\mathrm{selfs}}$ is given by (2.6) with $B^s_{p\infty}$ in place of B^s_p.

Recall again that σ_p is given by (2.1).

Corollary 1. (i) *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $0 < p < \infty$, $\sigma > \sigma_p$, and let $\Gamma = \partial\Omega$ be an h-set according to Definition* 3(i) *satisfying* (2.32). *Then*

$$\chi_\Omega \in M(B^s_p), \quad \textit{where} \quad 1 < p < \infty, \quad 0 < s < \sigma,$$

and

$$\chi_\Omega \in M(B^\sigma_p), \quad \textit{where} \quad 0 < p \leq 1, \quad \sigma > n\left(\frac{1}{p} - 1\right).$$

(ii) *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\Gamma = \partial\Omega$ be uniformly porous according to Definition* 3(iii). *Then there is a positive number ε such that*

$$\chi_\Omega \in M(B^s_p) \quad \textit{where} \quad 0 < p < \infty, \quad \sigma_p < s < \frac{\varepsilon}{p}\,. \tag{2.38}$$

(iii) *Let $1 \leq n - 1 \leq d < n$. There exists a bounded star-like domain Ω in $\mathbb{R}^n$ such that $\Gamma = \partial\Omega$ is a d-set according to* (2.30)*,*

$$\chi_\Omega \in M(B^s_p), \quad \textit{where} \quad 0 < p < \infty, \quad \sigma_p < s < \frac{n-d}{p}\,,$$

and

$$\chi_\Omega \notin M\big(B_p^{\frac{n-d}{p}}\big), \quad \textit{where} \quad 0 < p < \infty, \quad \sigma_p < \frac{n-d}{p}\,.$$

Proof. This corollary is an immediate consequence of Theorem 1 and Theorem 3.

3. Proofs

3.1. Proof of Proposition 1

Step 1. First we prove that (2.6) and (2.10) are equivalent quasi-norms. By (2.6), (2.4), (2.3), and elementary multiplier assertions with respect to functions of type ψ it follows that (with the usual modifications if $p = \infty$)

$$
\begin{aligned}
\left\| f \,|B^s_{p,\,\mathrm{selfs}} \right\|^p &= \sup_{l,j} \left\| \psi\, f(2^{-j}\cdot + l\,2^{-j})\,|B^s_p \right\|^p \\
&\sim \sup_{l,j} \Bigg[\int\limits_{|x|\lesssim 1} \left| f(2^{-j}x + l\,2^{-j}) \right|^p dx \\
&\qquad + \int\limits_{|x|+|h|\lesssim 1} |h|^{-sp} \left| \Delta^N_h [f(2^{-j}\cdot + l\,2^{-j})] \right|^p \frac{dh\,dx}{|h|^n} \Bigg] \\
&\sim \sup_{l,j} \Bigg[2^{jn} \int\limits_{|x|\lesssim 2^{-j}} \left| f(x + l\,2^{-j}) \right|^p dx \\
&\qquad + 2^{-j(sp-n)} \int\limits_{|x|+|h|\lesssim 2^{-j}} |h|^{-sp} \left| \Delta^N_h f(\cdot + l\,2^{-j}) \right|^p \frac{dh\,dx}{|h|^n} \Bigg].
\end{aligned}
$$

Hence, (2.6) and (2.10) are equivalent quasi-norms. This proves the first assertion in part (i). As for (2.11) we have, for some $c > 0$ and all $x \in \mathbb{R}^n$ and $j \in \mathbb{N}_0$,

$$
2^{jn} \int\limits_{|h|\lesssim 2^{-j}} |f(x+h)|^p\, dh \le c \left\| f \,|B^s_{p,\,\mathrm{selfs}} \right\|^p. \tag{3.1}
$$

Hence, the right-hand side of (3.1) is a uniform bound at all Lebesgue points of $|f(x)|^p$. This proves (2.11).

Step 2. We prove part (ii). Let $f \in B^s_{p,\,\mathrm{selfs}}$ and $g \in B^s_{p,\,\mathrm{selfs}}$. According to [17, 4.6.4, Theorem 2, p. 222], and again by elementary multiplier assertions and (2.3) we have

$$
\begin{aligned}
&\sup_{l,j} \left\| \psi(\cdot - l)\, f(2^{-j}\cdot)\, g(2^{-j}\cdot) |B^s_p \right\| \\
&\lesssim \sup_{l,j} \left(\| f\,|L_\infty\| \cdot \left\| \psi(\cdot - l)\, g(2^{-j}\cdot)\,|B^s_p \right\| + \|g\,|L_\infty\| \cdot \left\| \psi(\cdot - l)\, f(2^{-j}\cdot)\,|B^s_p \right\| \right).
\end{aligned}
$$

Using (2.11) it follows that $B^s_{p,\,\mathrm{selfs}}$ is a multiplication algebra. Then one obtains by (2.9) that $b^s_{p,\,\mathrm{selfs}}$ is also a multiplication algebra (we refer to our definition of what is called a multiplication algebra between (2.8) and (2.9)).

3.2. Proof of Theorem 1

Step 1. We prove the right-hand side of (2.12). Let

$$g \in B^s_p, \qquad \operatorname{supp} g \subset \{y : |y| \le 1\}. \tag{3.2}$$

Then

$$\|g \,|B^s_p\| \sim 2^{-j(s-\frac{n}{p})} \,\|g(2^j\cdot)\,|B^s_p\|, \quad j \in \mathbb{N}_0, \tag{3.3}$$

where the equivalence constants are independent of g and $j \in \mathbb{N}_0$. This follows from [26, 5.16, with 5.4, 5.5, pp. 66, 44, 45], and $B^s_p = F^s_{pp}$. Let $m \in M(B^s_p)$ and let ψ be given by (2.2). Then we obtain by (3.3),

$$\begin{aligned} \|\psi\, m(2^{-j}\cdot)\,|B^s_p\| &\sim 2^{-j(s-\frac{n}{p})} \,\|\psi(2^j\cdot)\, m\,|B^s_p\| \\ &\le \|m\,|M(B^s_p)\|\; 2^{-j(s-\frac{n}{p})}\,\|\psi(2^j\cdot)\,|B^s_p\| \\ &\lesssim \|m\,|M(B^s_p)\|. \end{aligned} \tag{3.4}$$

The right-hand side of (2.12) is now a consequence of this estimate and (2.6).

Step 2. We prove the left-hand side of (2.12) under the assumption that Theorem 2 is valid (which will be proved in 3.4). Let $m \in B^\sigma_{p,\,\mathrm{selfs}}$ for some $\sigma > s$. Let

$$f(x) = \sum_{k,l} \lambda_{kl}\, a^{k,l}(x), \quad \|\lambda\,|\ell_p\| \sim \|f\,|B^s_p\|, \tag{3.5}$$

be an optimal classical atomic decomposition of $f \in B^s_p$ according to (2.20), (2.19), where $a^{k,l}$ are C^∞ classical $(s,p)_K$-atoms with $\sigma < K$. Then we wish to prove that $m\,a^{k,l}$ in

$$(m f)(x) = \sum_{k,l} \lambda_{kl}\,(m a^{k,l})(x), \tag{3.6}$$

after normalisation, are $(s,p)_\sigma$-atoms according to Definition 2. The support condition in (2.22) is obvious. If $l = 0$ then we put $a^k = a^{k,0}$. We may assume that

$$\operatorname{supp} a^k(2^{-k}\cdot) \subset \left\{y : |y| \le \frac{c}{2}\right\}, \quad k \in \mathbb{N}_0,$$

where c is the same positive constant as in (2.2). In particular it follows by (classical) multiplier assertions as proved in [24, Corollary 4.2.2, p. 205], that for some $C > 0$,

$$\left\| \frac{a^k(2^{-k}\cdot)}{\psi} \;|\, M(B^\sigma_p)\right\| \le C\, 2^{-k(s-\frac{n}{p})} \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Using this observation and (3.2), (3.3) with σ in place of s we obtain that

$$\begin{aligned} \|m a^k\,|B^\sigma_p\| &\sim 2^{k(\sigma-\frac{n}{p})}\,\|m(2^{-k}\cdot)\,a^k(2^{-k}\cdot)\,|B^\sigma_p\| \\ &\lesssim 2^{k(\sigma-s)}\,\|\psi\, m(2^{-k}\cdot)\,|B^\sigma_p\|. \end{aligned} \tag{3.7}$$

In the case of $a^{k,l}$ with $l \in \mathbb{Z}^n$ one has (3.7) with $a^{k,l}$ and $\psi(\cdot - l)$ in place of a^k and ψ, respectively. Hence, by (2.6),

$$\left\| ma^{k,l} \,|B^\sigma_p \right\| \lesssim 2^{k(\sigma - s)} \left\| m \,|B^\sigma_{p,\,\mathrm{selfs}} \right\|, \quad l \in \mathbb{Z}^n,\, k \in \mathbb{N}_0, \tag{3.8}$$

and, according to Definition 2, it follows that $ma^{k,l}$ are $(s,p)_\sigma$-atoms multiplied with the indicated constant. By Theorem 2 and (3.5) we obtain that

$$\left\| mf \,|B^s_p \right\| \lesssim \left\| m \,|B^\sigma_{p,\,\mathrm{selfs}} \right\| \cdot \left\| f \,|B^s_p \right\|, \quad f \in B^s_p.$$

This proves the left-hand side of (2.12).

Step 3. We prove (ii). Let $p \le 1$ and $m \in B^s_{p,\,\mathrm{selfs}}$. We have (3.5), (3.6), and (3.8) now with $\sigma = s$. In particular, $ma^{k,l}$ are uniformly bounded functions in B^s_p. We apply (2.26) and obtain that

$$\begin{aligned} \left\| mf \,|B^s_p \right\|^p &\lesssim \sum_{k,l} |\lambda_{kl}|^p \left\| ma^{k,l} \,|B^s_p \right\|^p \\ &\lesssim \left\| m \,|B^s_{p,\,\mathrm{selfs}} \right\|^p \cdot \left\| f \,|B^s_p \right\|^p. \end{aligned} \tag{3.9}$$

Then (2.13) follows from (3.9) and the right-hand side of (2.12).

Step 4. We prove (iii). In the complementary Subsection 4.1 we wish to extend (2.15) to some other spaces in (1.7). To prepare this comment now we put $A = B^s_p$. If p and s are restricted by (2.14), then A is a *multiplication algebra*,

$$\|fg \,|A\| \lesssim \|f \,|A\| \cdot \|g \,|A\|, \quad f \in A, \quad g \in A, \tag{3.10}$$

and satisfies the *localisation principle*,

$$\|f \,|A\| \sim \left(\sum_{l \in \mathbb{Z}^n} \|\psi(\cdot - l) f \,|A\|^p \right)^{\frac{1}{p}}, \quad f \in A, \tag{3.11}$$

(usual modification if $p = \infty$), where ψ is given by (2.2), (2.3). The first assertion, (3.10), may be found in [23, Theorem 2.8.3, pp. 145–146], including some references. (The formulation given there must be corrected in the case of $s = 0$, $p = \infty$, but this is not of relevance here. We refer in this connection to [21] and also to [17, Theorem 4.6.4/1, pp. 221–222].) The localisation principle (3.11) has been proved in [24, Theorem 2.4.7, pp. 124–125]. Actually, (3.11) is a characterisation:

If $\psi(\cdot - l) f \in A$ and if the right-hand side of (3.11) *is finite, then $f \in A$ and we have* (3.11).

We refer to [24, Remark 2.4.7/4, p. 129]. One may also replace ψ in (3.11) by ψ^2. Based on these observations and (3.10), (3.11), the proof of (2.15) is rather simple. Let $m \in B^s_{p,\mathrm{unif}}$ according to (2.5). Then, assuming $p < \infty$, and again $A = B^s_p$,

$$\begin{aligned} \|mf\,|A\|^p &\sim \sum_{l\in\mathbb{Z}^n} \big\|\psi^2(\cdot - l)mf\,|A\big\|^p \\ &\lesssim \sup_{\widetilde{l}} \|\psi(\cdot - \widetilde{l})m\,|A\|^p \cdot \sum_{l\in\mathbb{Z}^n} \|\psi(\cdot - l)f\,|A\|^p \\ &\lesssim \|m\,|A_{\mathrm{unif}}\|^p \cdot \|f\,|A\|^p \end{aligned} \tag{3.12}$$

(with the obvious modification if $p = \infty$). This proves $m \in M(A)$. Conversely if $m \in M(A)$, then $m \in A_{\mathrm{unif}}$ follows from the right-hand side of (2.12) and the obvious embedding $A_{\mathrm{selfs}} \hookrightarrow A_{\mathrm{unif}}$. This also completes the proof of (2.15).

3.3. *Proof of Proposition* 2

Step 1. We prove (i). Let $a^{k,l}$ be a classical $(s,p)_K$-atom according to (2.18). We may assume $l = 0$ and we put $a^k = a^{k,0}$. Then it follows that

$$\big\|\, a^k(2^{-k}\cdot)\,|B^\sigma_p\big\| \lesssim 2^{-k(s-\frac{n}{p})}, \quad k \in \mathbb{N}_0.$$

Using (3.2), (3.3) with σ in place of s we get

$$\big\|a^k\,|B^\sigma_p\big\| \sim 2^{k(\sigma-\frac{n}{p})}\ \big\|a^k(2^{-k}\cdot)\,|B^\sigma_p\big\| \lesssim 2^{k(\sigma-s)}, \quad k \in \mathbb{N}_0.$$

Hence a^k, and also $a^{k,l}$ with $l \in \mathbb{Z}^n$, are $(s,p)_\sigma$-atoms.

Step 2. We prove (ii). It is sufficient to deal with a^k, hence $l = 0$. Again we apply (3.2), (3.3), and get

$$\begin{aligned} \big\|a^k\,|B^s_p\big\| &\sim 2^{k(s-\frac{n}{p})}\ \big\|a^k(2^{-k}\cdot)\,|B^s_p\big\| \\ &\lesssim 2^{k(s-\frac{n}{p})}\ \big\|a^k(2^{-k}\cdot)\,|B^\sigma_p\big\| \\ &\lesssim 2^{-k(\sigma-s)}\ \big\|a^k\,|B^\sigma_p\big\| \lesssim 1. \end{aligned}$$

This proves the first assertion in (2.23). The proof of the second assertion follows by the same arguments with L_p in place of B^s_p.

3.4. *Proof of Theorem* 2

We have the atomic decomposition theorem (2.20), (2.21) based on classical $(s,p)_K$-atoms according to (2.18). By Proposition 2(i) classical $(s,p)_K$-atoms are special $(s,p)_\sigma$-atoms. Hence it remains to prove that

$$\big\|f\,|B^s_p\big\| \lesssim \|\lambda\,|\ell_p\|, \qquad f = \sum_{k,l} \lambda_{kl}\, a^{k,l}, \tag{3.13}$$

for any atomic decomposition (2.24) based on arbitrary $(s,p)_\sigma$-atoms $a^{k,l}$. For this purpose we expand each function $a^{k,l}(2^{-k}\cdot)$ optimally in B^σ_p with respect to classical $(\sigma,p)_K$-atoms $b^{j,w}_{k,l}$ where $\sigma < K$,

$$a^{k,l}(2^{-k}x) = \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} \eta^{k,l}_{jw}\, b^{j,w}_{k,l}(x), \quad x \in \mathbb{R}^n, \tag{3.14}$$

with

$$\operatorname{supp} b^{j,w}_{k,l} \subset B_{j,w}, \qquad \left| D^\alpha b^{j,w}_{k,l}(x)\right| \le 2^{-j(\sigma-\frac{n}{p})+|\alpha|j}, \quad |\alpha| \le K, \tag{3.15}$$

and

$$\left(\sum_{j,w} |\eta^{k,l}_{jw}|^p \right)^{\frac{1}{p}} \sim \left\| a^{k,l}(2^{-k}\cdot)\,|B^\sigma_p \right\| \lesssim 2^{-k(s-\frac{n}{p})}. \tag{3.16}$$

The last estimate is again a consequence of (3.2), (3.3) with σ in place of s, and (2.22),

$$\left\| a^{k,l}(2^{-k}\cdot)\,|B^\sigma_p\right\| \sim 2^{-k(\sigma-\frac{n}{p})} \left\| a^{k,l}\,|B^\sigma_p\right\| \le 2^{-k(s-\frac{n}{p})}.$$

Hence,

$$a^{k,l}(x) = \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} \eta^{k,l}_{jw}\, b^{j,w}_{k,l}(2^k x),$$

where the functions $b^{j,w}_{k,l}(2^k\cdot)$ are supported by balls of radius $\sim 2^{-k-j}$. By (3.15) we have

$$\begin{aligned} \left| D^\alpha b^{j,w}_{k,l}(2^k x)\right| &= 2^{k|\alpha|} \left| \left(D^\alpha b^{j,w}_{k,l}\right)(2^k x)\right| \\ &\le 2^{(j+k)|\alpha|}\, 2^{-j(\sigma-\frac{n}{p})} \\ &= 2^{(j+k)|\alpha|}\, 2^{-(j+k)(s-\frac{n}{p})}\, 2^{-(j+k)(\sigma-s)}\, 2^{k(\sigma-\frac{n}{p})}. \end{aligned}$$

Replacing $j+k$ by j we obtain that

$$a^{k,l}(x) = 2^{k(\sigma-\frac{n}{p})} \sum_{j\ge k} \sum_{w\in\mathbb{Z}^n} \eta^{k,l}_{j-k,w}\, 2^{-j(\sigma-s)}\, d^{j,w}_{k,l}(x), \tag{3.17}$$

where $d^{j,w}_{k,l}$ are classical $(s,p)_K$-atoms supported by balls of radius $\sim 2^{-j}$. We insert (3.17) into the expansion in (3.13). We fix $j \in \mathbb{N}_0$ and $w \in \mathbb{Z}^n$, and collect all non-vanishing terms $d^{j,w}_{k,l}$ in the expansions (3.17). We have $k \le j$. Furthermore multiplying (3.14), if necessary, with suitable cut-off functions it follows that there is a natural number N such that for fixed k only at most N points $l \in \mathbb{Z}^n$ contribute to $d^{j,w}_{k,l}$. We denote this set by (j,w,k). Hence its cardinality is at most N, where N is independent of j, w, k. Then

$$d^{j,w}(x) = \frac{\sum_{k\le j} 2^{k(\sigma-\frac{n}{p})} \sum_{l\in(j,w,k)} \eta^{k,l}_{j-k,w}\, \lambda_{k,l}\, d^{j,w}_{k,l}(x)}{\sum_{k\le j} 2^{k(\sigma-\frac{n}{p})} \sum_{l\in(j,w,k)} \left|\eta^{k,l}_{j-k,w}\right| \cdot |\lambda_{kl}|}$$

are correctly normalised classical $(s,p)_K$-atoms located in a ball of radius $\sim 2^{-j}$ centred at $2^{-j}w$. Let

$$\nu_{j,w} = 2^{-j(\sigma-s)} \sum_{k\le j} 2^{k(\sigma-\frac{n}{p})} \sum_{l\in(j,w,k)} \left|\eta^{k,l}_{j-k,w}\right| \cdot |\lambda_{kl}|. \tag{3.18}$$

Then it follows by (3.17) inserted in the expansion (3.13), the classical atomic decomposition

$$f(x) = \sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} \nu_{jw}\, d^{j,w}(x).$$

Let $0 < \varepsilon < \sigma - s$. Then we obtain by (3.18) that (assuming $p < \infty$)

$$|\nu_{jw}|^p \lesssim \sum_{k\le j} \sum_{l\in(j,w,k)} 2^{-(j-k)p(\sigma-s-\varepsilon)}\, 2^{k(s-\frac{n}{p})p} \left|\eta^{k,l}_{j-k,w}\right|^p |\lambda_{kl}|^p.$$

Summation over $j \in \mathbb{N}_0$ and $w \in \mathbb{Z}^n$ results in

$$\begin{aligned}\sum_{j=0}^{\infty} \sum_{w\in\mathbb{Z}^n} |\nu_{jw}|^p &\lesssim \sum_{k=0}^{\infty} \sum_{l\in\mathbb{Z}^n} |\lambda_{kl}|^p \sum_{j\ge k} \sum_{w\in\mathbb{Z}^n} 2^{k(s-\frac{n}{p})p} \left|\eta^{k,l}_{j-k,w}\right|^p, \\ &\lesssim \|\lambda\,|\ell_p\|^p,\end{aligned}$$

where we used (3.16). Then we obtain (3.13) by the classical atomic decomposition theorem. If $p = \infty$ then one has to modify the arguments in the usual way.

3.5. Proof of Theorem 3

Step 1. We prove part (i). We may assume that diam $\Omega < 1$. Let

$$\Omega^k = \left\{x \in \Omega \;:\; 2^{-k-2} \le \operatorname{dist}(x,\Gamma) \le 2^{-k}\right\}, \quad k \in \mathbb{N}_0,$$

and let

$$\left\{\varphi^k_l \;:\; k \in \mathbb{N}_0\,;\; l = 1,\ldots,M_k\right\} \subset C_0^\infty(\Omega)$$

be a resolution of unity,

$$\sum_{k\in\mathbb{N}_0} \sum_{l=1}^{M_k} \varphi^k_l(x) = 1, \quad \text{if} \quad x \in \Omega, \tag{3.19}$$

with

$$\operatorname{supp}\varphi_l^k \subset \left\{x \,:\, \left|x - x_l^k\right| \lesssim 2^{-k}\right\} \subset \Omega^k,$$

and

$$\left|D^\alpha \varphi_l^k(x)\right| \lesssim 2^{|\alpha|k}, \qquad |\alpha| \le K,$$

where $K \in \mathbb{N}$ with $K > \sigma$. It is well known and can be proved easily that resolutions of unity with the required properties exist. Of interest is that the minimal number $M_k \in \mathbb{N}_0$ is needed. First we remark that μ with (2.27) satisfies the so-called doubling condition; this is a consequence of (2.29). Then it follows by (2.27) that

$$M_k\, h(2^{-k}) \lesssim 1, \qquad k \in \mathbb{N}_0. \tag{3.20}$$

We rewrite (3.19) as

$$\chi_\Omega(x) = \sum_{k=0}^{\infty} 2^{k(\sigma-\frac{n}{p})} \sum_{l=1}^{M_k} 2^{-k(\sigma-\frac{n}{p})}\, \varphi_l^k(x), \quad x \in \mathbb{R}^n, \tag{3.21}$$

where $2^{-k(\sigma-\frac{n}{p})}\varphi_l^k$ are classical $(\sigma, p)_K$-atoms. Hence,

$$\left\|\chi_\Omega \,|B_p^\sigma\right\|^p \le \sum_{k=0}^{\infty} 2^{k(\sigma-\frac{n}{p})p}\, M_k \lesssim \sum_{k=0}^{\infty} 2^{k\sigma p} \left(\frac{2^{-kn}}{h(2^{-k})}\right) < \infty. \tag{3.22}$$

This proves $\chi_\Omega \in B_p^\sigma$. Next we wish to show that $\chi_\Omega \in B^\sigma_{p,\,\mathrm{selfs}}$. By (2.6) it is sufficient to deal with

$$\chi_\Omega(2^{-j}\cdot)\psi, \quad \text{assuming } 0 \in 2^j\Gamma, \text{ where } j \in \mathbb{N}_0.$$

Let μ^j be the image measure of μ with respect to the dilations $y \mapsto 2^j y$. Let B_c be the ball of radius $c > 0$, centred at the origin, where c is the same number as in (2.2). Then we have,

$$\mu^j\left(B_c \cap 2^j\Gamma\right) \sim h\left(2^{-j}\right), \quad j \in \mathbb{N}_0.$$

Now we can repeat the above arguments with $B_c \cap 2^j\Omega$ and $B_c \cap 2^j\Gamma$ in place of Ω and Γ, respectively. Let M_k^j be the counterpart of the above number M_k. Then

$$M_k^j\, h\left(2^{-j-k}\right) \lesssim h\left(2^{-j}\right), \quad j \in \mathbb{N}_0, \quad k \in \mathbb{N}_0,$$

is the generalisation of (3.20) we are looking for. Now we can repeat the arguments resulting in (3.21) and (3.22). By (2.32) we obtain (2.33).

Step 2. We prove (ii). Since Γ is uniformly porous we have (2.31) for some h and some $\varepsilon > 0$. Then (2.32) is satisfied if $\sigma p < \varepsilon$. This proves (2.34).

Step 3. We prove (iii) and (2.37). Let Γ be a d-set with $n-1 \le d < n$. Hence, $h(t) \sim t^d$, and we have (2.31) with $\varepsilon = n-d$. Then (2.35) is a consequence of (2.34). As for the first assertion in (2.37) for arbitrary d-sets $\Gamma = \partial\Omega$ we mention that

$$\sup_{j\in\mathbb{N}_0; k\in\mathbb{N}_0} 2^{k\sigma p} \left(\frac{h\left(2^{-j}\right)}{h\left(2^{-j-k}\right)} \, 2^{-kn} \right) < \infty, \quad \sigma = \frac{n-d}{p}, \quad h(t) \sim t^d,$$

is the adequate counterpart of (2.32) for $B^\sigma_{p\infty}$ in place of B^σ_p. We refer to [25, Theorem 13.8, p. 75], as far as the underlying classical atomic decompositions are concerned. This proves the first assertion in (2.37) by the same arguments as above. As for (2.36) and the second assertion in (2.37) we use the star-like bounded domain Ω constructed in [25, 16.3, 16.4, pp. 122–123], based on the preceding Theorem 16.2 in [25, pp. 120–122]. The boundary $\Gamma = \partial\Omega$ is a d-set with $n-1 < d < n$. (There is no problem to incorporate $d = n-1$ here, but this is the case of a smooth boundary, or of a cube, where the corresponding assertion is well known, [23, Remark 2.8.4/2, p. 149], and [17, Lemma 4.6.3/2, p. 209]. Hence we assume now that $d > n-1$.) By the arguments given in [25, 16.4], it is sufficient to deal with cylindrical domains Ω in $\mathbb{R}^n$,

$$\Omega = \left\{ x = (x', x_n) \,:\, x' \in Q', \, 0 < x_n < f(x') \right\},$$

where Q' is, say, the unit cube in $\mathbb{R}^{n-1}$ centred at the origin, and the interesting part of the boundary, now denoted by Γ, is the graph of the function $f(x')$,

$$\Gamma = \left\{ (x', f(x')) \,:\, x' \in Q' \right\}.$$

Here $f(x')$ is a typical fractal lacunary construction

$$x_n = f(x') = \sum_{j=1}^{\infty} f_j(x'), \quad f_j(x') = 2^{-\nu_j s} \sum_{l\in\mathbb{Z}^{n-1}} \omega(2^{\nu_j} x' - l), \tag{3.23}$$

where $d = n-s$; hence $0 < s < 1$; $\nu_j \sim 2^{cj}$ for some $c > 0$, and ω is the classical bump function,

$$\omega(x') = e^{-\frac{1}{1-|2x'|^2}} \ \text{ if } |x'| < \frac{1}{2}; \quad \omega(x') = 0 \text{ if } |x'| \ge \frac{1}{2}.$$

We refer for details to [25, pp. 121–122]. Let

$$\Gamma^j = \left\{ (x', f_j(x')) \,:\, x' \in Q' \right\}, \quad j \in \mathbb{N}_0.$$

Comparing the near-by slices

$$\Omega_j = \left\{ x = (x', x_n) \in \Omega \,:\, 0 < \gamma_n - x_n < 2^{-\nu_j s} \text{ with } \gamma = (x', \gamma_n) \in \Gamma \right\}$$

and

$$\Omega^j = \left\{ x = (x', x_n) \in \Omega \,:\, 0 < \gamma_n - x_n < 2^{-\nu_j s} \text{ with } \gamma = (x', \gamma_n) \in \Gamma^j \right\}$$

then it follows that

$$|\Omega_j| \sim |\Omega^j| \sim 2^{-\nu_j s}, \quad j \in \mathbb{N}_0, \tag{3.24}$$

where the equivalence constants are independent of j. This follows from the lacunary construction in (3.23): the terms f_k with $k < j$ are too tame and the terms f_k with $k > j$ are too small to influence (3.24). We refer to [25, p. 121–122], where we substantiated these admittedly somewhat vague arguments. Even more, Ω_j and Ω^j differ from each other by a marginal set, compared with (3.24), uniformly in j. Let, temporarily, $Q(x,r)$ be a cube in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of side-length $r > 0$. Let $\Omega^c = \mathbb{R}^n \backslash \Omega$. There is a constant $a > 0$ such that

$$|Q(x,a2^{-\nu_j}) \cap \Omega| \sim |Q(x,a2^{-\nu_j} \cap \Omega^c| \sim 2^{-n\nu_j},\, x \in \Omega_j,\ j \in \mathbb{N}_0.$$

Let $0 < \frac{n-d}{p} < 1$ (this is always the case if $1 \le p < \infty$). Then in a generalisation of (2.4) the space B^σ_{pq} with $\sigma = \frac{n-d}{p} = \frac{s}{p}$ and $0 < q < \infty$ can be quasi-normed by

$$\begin{aligned}\left\| g \,|B^\sigma_{pq} \right\| &= \|g\,|L_p\| \\ &\quad + \left(\int\limits_{|h|\le 1} |h|^{-\sigma q} \left(\int\limits_{\mathbb{R}^n} |g(x+h) - g(x)|^p \, dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \right)^{\frac{1}{q}}.\end{aligned}$$

Let $g = \chi_\Omega$, $x \in \Omega_j$, $|h| \sim 2^{-\nu_j}$, and $x + h \in \Omega^c$. Then it follows that

$$\int\limits_{\Omega_j} |\chi_\Omega(x+h) - \chi_\Omega(x)|^p \, dx \gtrsim 2^{-s\nu_j}, \quad j \in \mathbb{N}_0.$$

Since $\sigma = \frac{s}{p}$ we obtain that

$$\int\limits_{|h|\sim 2^{-\nu_j}} |h|^{-\sigma q} \left(\int\limits_{\mathbb{R}^n} |\chi_\Omega(x+h) - \chi_\Omega(x)|^p \, dx \right)^{\frac{q}{p}} \frac{dh}{|h|^n} \gtrsim 1, \quad j \in \mathbb{N}_0,$$

and, consequently,

$$\chi_\Omega \notin B^{\frac{n-d}{p}}_{pq}, \quad 0 < q < \infty. \tag{3.25}$$

If $p < 1$ then it might happen that one needs second differences in (2.4) and its generalisation to B^σ_{pq}, but not any more since

$$n\left(\frac{1}{p} - 1\right) < \frac{n-d}{p} = \frac{s}{p}, \quad \text{hence} \quad \frac{s}{p} < \frac{sn}{n-s} < \frac{n}{n-1} \le 2$$

(recall that $n \ge 2$). Now one can repeat the above arguments with second differences and, say, $x \in \Omega$, $|h| \sim 2^{-\nu_j}$, $x + h \in \Omega^c$, $x - h \in \Omega^c$. Then one again gets (3.25). Finally we remark that (2.36) is an immediate consequence of (3.25).

4. Complements

In this section we collect a few results for the more general spaces B^s_{pq} and F^s_{pq} (always defined on $\mathbb{R}^n$, where $n \in \mathbb{N}$ is fixed) which can be obtained easily from the assertions proved so far for the special spaces $B^s_p = B^s_{pp} = F^s_{pp}$, and from known general observations available in the literature. We do not develop any new specific techniques. Furthermore it is assumed that the reader of these complements is familiar with the theory of the spaces B^s_{pq} and F^s_{pq} as developed in the books of the author and in [17]. We recall only that

$$H^s_p = F^s_{p,2}, \quad \text{with } 0 < p < \infty \text{ and } s \in \mathbb{R},$$

are the Sobolev–Hardy spaces.

4.1. Homogeneity, multiplication algebras, and the localisation principle

The proofs of the right-hand side of (2.12) and of (2.15) with (2.14) in Theorem 1 are based on some properties of the spaces $A = B^s_p$: the homogeneity assertion (3.2), (3.3), the localisation principle in (3.11), and the observation that $A = B^s_p$ is a multiplication algebra according to (3.10) if p and s are restricted by (2.14). In other words if the space

$$A = F^s_{pq}, \quad \text{with } 0 < p < \infty, 0 < q \le \infty, s \in \mathbb{R},$$

complemented by

$$F^s_{\infty\infty} = B^s_{\infty\infty} = \mathcal{C}^s, \quad s \in \mathbb{R},$$

has one or several of these properties then the corresponding arguments can be carried over immediately. Recall that $B^s_p = B^s_{pp} = F^s_{pp}$. Obviously, A_{unif} and A_{selfs} are defined by (2.5) and (2.6) with A in place of B^s_p. As before, $M(A)$ is the collection of all multipliers m according to (2.7).

Corollary 2. (i) *Let*

$$A = F^s_{pq} \quad \text{with} \quad 0 < p \le \infty, \quad 0 < q \le \infty, \quad s > n\left(\frac{1}{\min(p,q)} - 1\right)_+, \tag{4.1}$$

where $q = \infty$ if $p = \infty$. Then

$$M(A) \hookrightarrow A_{\mathrm{selfs}} \hookrightarrow A_{\mathrm{unif}}. \tag{4.2}$$

(ii) *Let*

$$A = F^s_{pq} \quad \text{with (2.14) and } 0 < q \le \infty, \tag{4.3}$$

where $q = \infty$ if $p = \infty$. Then

$$M(A) = A_{\mathrm{unif}}. \tag{4.4}$$

(iii) *Let A be given by* (4.3) *and let, in addition,* $s > n(\frac{1}{q} - 1)_+$. *Then*

$$M(A) = A_{\text{selfs}} = A_{\text{unif}}.$$

Proof. Step 1. By [26, 5.16, with 5.4, 5.5, pp. 66, 44, 45], we have the homogeneity property (3.2), (3.3) with A given by (4.1) ($q = \infty$ always, if $p = \infty$) in place of B^s_p. Then the first inclusion in (4.2) follows from (3.4) with A in place of B^s_p. The second inclusion is obvious by definition.

Step 2. By [17, 4.6.4, pp. 221–222], and the references given there, F^s_{pq} is a multiplication algebra if, and only if, s, p, q are restricted as in (4.3). Furthermore by [24, 2.4.7, p. 124], all spaces F^s_{pq} satisfy the localisation principle. Then (4.4) follows from (3.12) and (3.4) with $j = 0$ and A in place of B^s_p. Finally, part (iii) is a consequence of parts (i) and (ii).

Remark 10. References to the relevant literature may be found in Remark 3. We repeat that part (ii), including the above proof, is known, [17, Theorem 4.9.1/1, p. 247], and [1]. One may ask whether (4.4) also holds for other spaces. Then $A \hookrightarrow L_\infty$ is necessary. But in the case of $A = F^s_{pq}$ this coincides with (4.3); we refer again to [17, Theorem 4.6.4/1, pp. 221–222]. If $A = B^s_{pq}$ then, of course, again $A \hookrightarrow L_\infty$, and hence $s \geq \frac{n}{p}$ is necessary for (4.4). But the above arguments cannot be applied since B^s_{pq} satisfies the localisation principle if, and only if, $p = q$, [24, Theorem 2.4.7, pp. 124–125]. Nevertheless it has been proved in [20] that

$$M(B^s_{pq}) = B^s_{pq,\,\text{unif}}, \quad \text{if} \quad 1 \leq p \leq q \leq \infty \quad \text{and} \quad s > \frac{n}{p}. \tag{4.5}$$

On the other hand one can apply the above arguments to other spaces which are multiplication algebras and which satisfy the localisation principle, for example, to $\mathring{\mathcal{C}}^s$ being the completion of the Schwartz space $S = S(\mathbb{R}^n)$ in $\mathcal{C}^s$, where $s > 0$. Then it comes out that

$$M(\mathring{\mathcal{C}}^s) = \mathring{\mathcal{C}}^s_{\text{unif}} \subsetneqq \mathcal{C}^s_{\text{unif}} = M(\mathcal{C}^s) = \mathcal{C}^s, \quad s > 0. \tag{4.6}$$

In particular, $M(\mathring{\mathcal{C}}^s)$ and $M(\mathcal{C}^s) = \mathcal{C}^s$ are different. This observation can be extended to (in obvious notation)

$$M\left(\mathring{F}^s_{p\infty}\right) = \mathring{F}^s_{p\infty,\,\text{unif}} \subsetneqq F^s_{p\infty,\,\text{unif}} = M\left(F^s_{p\infty}\right), \quad 0 < p < \infty,\, s > \frac{n}{p}, \tag{4.7}$$

and

$$M\left(\mathring{B}^s_{p\infty}\right) \subsetneqq M\left(B^s_{p\infty}\right), \quad 0 < p \leq \infty,\, s > \frac{n}{p}. \tag{4.8}$$

On the other hand, we have

$$M\left(\mathring{B}^s_{\infty q}\right) = M\left(B^s_{\infty q}\right), \quad 0 < q < \infty,\, s \in \mathbb{R}. \tag{4.9}$$

Here $\subsetneq$ in (4.7), (4.8) comes from the fact that there are functions in $B^s_{p\infty}$ and $F^s_{p\infty}$ which cannot be approximated locally by smooth functions in the respective spaces: we may assume $n = 1$. Let

$$f(x) = |x|^{s-\frac{1}{p}}\chi(x) \quad \text{with} \quad \chi \in C_0^\infty(\mathbb{R}), \quad \chi(0) = 1.$$

By [23, Proposition 2.8.4, pp. 147–148], or by atomic arguments,

$$f \in B^s_{p\infty}(\mathbb{R}) \hookrightarrow \mathcal{C}^{s-\frac{1}{p}}(\mathbb{R}), \quad 0 < p \le \infty,$$

but it cannot be approximated by smooth functions in these spaces. In the case of the F-spaces we refer, for a corresponding assertion, to [26, Proposition 5.6, p. 46]. On the other hand by atomic or subatomic decompositions it follows that functions belonging to $B^s_{\infty q}$ with $q < \infty$ can be approximated locally by smooth functions, which results in (4.9). These observations correct corresponding assertions in [17, Lemma 4.9, pp. 246–247], within a larger context.

4.2. *Duality and interpolation*

Let $A = B^s_{pq}$ or $A = F^s_{pq}$. If $p < \infty$ and $q < \infty$, then the dual A' of A can be interpreted in the usual way within the dual pairing (S, S') of the Schwartz space $S = S(\mathbb{R}^n)$ and its dual, the space of tempered distributions $S' = S'(\mathbb{R}^n)$. We refer to [23, 2.11], and [17, 2.1.5]. This can be extended to the spaces with $p = \infty$ and/or $q = \infty$, where one has to replace A by $\mathring{A}$, the completion of S in A. Extending the notation $\mathring{A}$ to all spaces, and hence $\mathring{A} = A$ if $p < \infty$, $q < \infty$, one has

$$M\big(\mathring{A}\big) \hookrightarrow M\left(\left(\mathring{A}\right)'\right)$$

by standard arguments. If, in addition, A is a Banach space, and hence $p \ge 1$ and $q \ge 1$, one may ask whether

$$M\big(\mathring{A}\big) = M\left(\left(\mathring{A}\right)'\right)$$

or not. This is the case if $\mathring{A} = A$ is reflexive, and hence $1 < p < \infty$ and $1 < q < \infty$, but otherwise this is not true in general, in contrast with a corresponding remark in [16, p. 144]. Since

$$\left(\mathring{\mathcal{C}}^s\right)' = B_1^{-s} \quad \text{and} \quad \left(B_1^{-s}\right)' = \mathcal{C}^s, \quad s \in \mathbb{R}, \tag{4.10}$$

(4.6) is a counter-example. As for (4.10) we refer to [23, Theorem 2.11.2, p. 178, and formula (12), p. 180]. Another possibility is to use real or complex interpolation: if m is a multiplier for two spaces A_0 and A_1 of the scales B^s_{pq} and F^s_{pq} then it is also a multiplier for all complex interpolation spaces $[A_0, A_1]_\theta$ and all real interpolation spaces $(A_0, A_1)_{\theta,q}$ with $0 < \theta < 1$ and $0 < q \le \infty$. All this is well known and we do not describe in detail what can be done if one applies duality and interpolation to the results in Section 2. We restrict ourselves to two assertions which complement Theorem 1(i) and Corollary 1(ii).

Corollary 3. *Let* $0 < p \le \infty$ *and* $0 < q \le \infty$.

(i) *Let* $s > \sigma_p$, *where* σ_p *is given by* (2.1). *Then*

$$b^s_{p,\,\mathrm{selfs}} \subset M(B^s_{pq}).$$

(ii) *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and let* $\Gamma = \partial\Omega$ *be uniformly porous according to Definition* 3(iii). *Then there is a number* ε, $0 < \varepsilon < n$, *such that*

$$\chi_\Omega \in M\left(B^s_{pq}\right), \quad \textit{where} \quad \max\left(\varepsilon\left(\frac{1}{p}-1\right), n\left(\frac{1}{p}-1\right)\right) < s < \frac{\varepsilon}{p}. \tag{4.11}$$

Proof. Step 1. Let $m \in b^s_{p,\,\mathrm{selfs}}$ and hence $m \in B^\sigma_{p,\,\mathrm{selfs}}$ for some $\sigma > s$ according to (2.9). By Theorem 1(i) this function m generates a linear and bounded operator in all spaces B^s_p with $\sigma_p < s < \sigma$, and also in all interpolation spaces. This covers, in particular, all spaces B^s_{pq} with $\sigma_p < s < \sigma$ and $0 < q \le \infty$, [23, Theorem 2.4.2, p. 64].

Step 2. By (2.38) and

$$\left(B^s_p\right)' = B^{-s}_{p'}, \quad 1 \le p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad s \in \mathbb{R},$$

[23, 2.11.2, p. 178], it follows that

$$\chi_\Omega \in M\left(B^s_p\right), \quad 1 < p \le \infty, \quad -\varepsilon\left(1 - \frac{1}{p}\right) < s < 0.$$

The rest is again a matter of interpolation.

Remark 11. In [17, 4.9], and in [18], [19], one finds what is known about multipliers in the spaces B^s_{pq} and F^s_{pq}. Of special interest is the case $s = 0$. Here we refer to the recent paper [11]. By the above arguments one obtains the following assertion: let $b^0_{p,\,\mathrm{selfs}}$ be given by (2.9) with $s = 0$. Then

$$b^0_{p,\,\mathrm{selfs}} \subset M\left(B^0_{pq}\right), \quad 1 < p < \infty, \quad 0 < q \le \infty.$$

4.3. Porosity and monotonicity

Let $1 \le p \le \infty$ and let $m \in M(B^s_{pq})$ for some $s > 0$ and some q with $0 < q \le \infty$. Then it follows by real interpolation between B^s_{pq} and L_p (recall $M(L_p) = L_\infty$) that

$$m \in M\left(B^\sigma_{pu}\right) \quad \text{with} \quad 0 < \sigma < s, \quad 0 < u \le \infty.$$

In the case of the spaces F^s_{pq} one can use complex interpolation between F^s_{pq} and L_p, but the outcome is unsatisfactory. Nevertheless there are also some monotonicity assertions for $M(F^s_{pq})$ based on other means. We refer to [18, Lemma 2.14, p. 300], and [19, Lemma 7, p. 222]. Our aim here is different. We wish to complement the Corollaries 1(ii) and 3(ii), and to shed more light on the interplay between porosity and characteristic functions as multipliers. Recall that in the case of F-spaces we always assume $q = \infty$ if $p = \infty$.

Corollary 4. *Let Ω be a bounded domain in $\mathbb{R}^n$.*

(i) *Let $\Gamma = \partial\Omega$ be uniformly porous according to Definition* 3(iii). *Then there is a number ε, $0 < \varepsilon < n$, such that*

$$\chi_\Omega \in M\left(F^s_{pq}\right), \text{ if } 0 < p \le q \le \infty, \max\left(\varepsilon\left(\frac{1}{p}-1\right), n\left(\frac{1}{p}-1\right)\right) < s < \frac{\varepsilon}{p}, \tag{4.12}$$

and

$$\chi_\Omega \in M\left(F^s_{pq}\right), \text{ if } 1 < p < \infty, 1 \le q \le \infty, \varepsilon\left(\frac{1}{p}-1\right) < s < \frac{\varepsilon}{p}. \tag{4.13}$$

(ii) *Let $\Gamma = \partial\Omega$ be porous according to Definition* 3(ii) *(not necessarily uniformly porous). Then*

$$\chi_\Omega \in M\big(F^0_{pq}\big), \quad \text{if} \quad 1 < p < \infty, \quad 1 \le q \le \infty. \tag{4.14}$$

Proof. We prove part (i). Recall that $F^s_{pp} = B^s_{pp} = B^s_p$. Hence, (4.12) with $p = q$ follows from (4.11). Since Γ is porous we can apply the monotonicity assertion in [27, Proposition 5.7(i)], extending (4.12) from $q = p$ to all q with $p \le q \le \infty$. Furthermore, (4.13) and part (ii) of the above corollary follow from the parts (ii) and (iii) of Proposition 5.7 in [27].

Remark 12. Since

$$H^s_p = F^s_{p,2}, \quad 1 < p < \infty, \quad s \in \mathbb{R},$$

are the (fractional) Sobolev spaces, it follows by (4.13) that the uniform porosity of $\Gamma = \partial\Omega$ ensures that χ_Ω is a multiplier in some of these spaces provided that

$$\varepsilon\left(\frac{1}{p}-1\right) < s < \frac{\varepsilon}{p}, \quad \text{for some } \varepsilon > 0 \text{ and } 1 < p < \infty.$$

In the case of $s = 0$ we have $H^0_p = L_p$, hence (4.14) with $q = 2$ is obvious.

4.4. Self-similarity and multipliers

Let A be one of the spaces B^s_{pq} or F^s_{pq} in $\mathbb{R}^n$ with

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \ge \sigma_p = n\left(\frac{1}{p}-1\right)_+$$

($q = \infty$ if $p = \infty$ in the F-case). Recall that $M(A)$ is the algebra of all multipliers, whereas A_{unif} and A_{selfs} are defined by (2.5) and (2.6) with A in place of B^s_p. By the above considerations it makes sense to ask for which of these spaces one has

$$M(A) = A_{\text{selfs}} \quad \text{and/or} \quad A_{\text{unif}} = A_{\text{selfs}}.$$

Corollary 5. (i) *Let A be either B^s_{pq} or F^s_{pq} with $0 < p \le \infty$, $0 < q \le \infty$ (where $q = \infty$ if $p = \infty$ in the F-case) and $s > \sigma_p$. Then A is a multiplication algebra if, and only if,*

$$A_{\mathrm{unif}} = A_{\mathrm{selfs}}. \tag{4.15}$$

(ii) *One has*

$$M(A) = A_{\mathrm{selfs}} \tag{4.16}$$

in each of the following cases:

$$A = B^s_p \text{ with } 0 < p \le 1,\ s > n\left(\frac{1}{p} - 1\right); \tag{4.17}$$

$$A = F^s_{pq} \text{ with (2.14)},\ 0 < q \le \infty,\ (q = \infty \text{ if } p = \infty); \tag{4.18}$$

$$A = B^s_{pq} \text{ with } 1 \le p \le q \le \infty,\ s > \frac{n}{p}; \tag{4.19}$$

$$A = L_p \text{ with } 1 \le p \le \infty.$$

Proof. Step 1. We prove (i). Let $s > \sigma_p$ and let A be one of the spaces B^s_{pq} or F^s_{pq}. Then

$$\|f\,|A\| = \|f\,|\dot{A}\| + \|f\,|L_p\|, \quad f \in A,$$

is an equivalent quasi-norm in A, where $\|f\,|\dot{A}\|$ is the quasi-norm of the corresponding homogeneous space $\dot{A}$. We refer for details and explanations to [24, Theorem 2.3.3, p. 98], and [5, p. 33]. Then

$$\|f(\lambda\cdot)\,|A\| \sim \lambda^{s-\frac{n}{p}}\,\|f\,|\dot{A}\| + \lambda^{-\frac{n}{p}}\|f\,|L_p\|, \quad f \in A, \quad \lambda > 0. \tag{4.20}$$

Let now A be a multiplication algebra. Then we have

$$A \hookrightarrow L_\infty \quad \text{and} \quad \|\psi(2^j\cdot - l)\,|A\| \lesssim 2^{j(s-\frac{n}{p})}, \quad j \in \mathbb{N}_0,\ l \in \mathbb{Z}^n. \tag{4.21}$$

As for the first embedding we refer to [17, Theorem 1 in 4.6.4, pp. 221–222], and [21, Remark 3.3.2, p. 114]. The second assertion follows from (3.2), (3.3) and interpolation. Then we obtain by (4.20), (4.21) that

$$\begin{aligned} \|\psi(\cdot - l)\, f(2^{-j}\cdot)\,|A\| &\sim 2^{-j(s-\frac{n}{p})}\,\|\psi(2^j\cdot - l) f\,|\dot{A}\| + 2^{j\frac{n}{p}}\,\|\psi(2^j\cdot - l) f\,|L_p\| \\ &\lesssim 2^{-j(s-\frac{n}{p})}\,\|\psi(2^j\cdot - l)\,|A\| \cdot \|f\,|A_{\mathrm{unif}}\| + \|f\,|L_\infty\| \\ &\lesssim \|f\,|A_{\mathrm{unif}}\|. \end{aligned} \tag{4.22}$$

This proves (4.15). Conversely, if we have (4.15), then it follows by the first equivalence in (4.22) that

$$2^{jn}\int_{\mathbb{R}^n} |\psi(2^j\cdot - l)|^p\,|f(y)|^p\,dy \lesssim \|f\,|A_{\mathrm{unif}}\|^p, \quad f \in A,$$

$p < \infty$. Hence, as at the end of Step 1 in Subsection 3.1, the right-hand side is a uniform bound at the Lebesgue points of $|f(x)|^p$, and, consequently, $A \hookrightarrow L_\infty$. By the same references as in connection with (4.21) it follows that A is a multiplication algebra.

Step 2. We prove (ii). We have (4.16) in the cases (4.17)–(4.19) by (2.13), (4.4), (4.5), combined with (4.15) in the latter cases. Finally,

$$L_{p,\,\mathrm{selfs}} = L_\infty = M(L_p), \quad 1 \le p \le \infty,$$

follows as above again from the arguments at the end of Step 1 in Subsection 3.1.

Remark 13. If A is either B^s_{pq} or F^s_{pq} with

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \ge n\left(\frac{1}{p} - 1\right)_+ ,$$

($q = \infty$ if $p = \infty$ in the F-case) then the question of whether (4.16) is valid or not makes sense. But if $s > \frac{n}{p}$ then we have (4.15) and if, in addition, $1 \le q < p \le \infty$, then (4.16) is not valid. We refer to [1, p. 162], and to [20, 4.5]. In other words possible candidates for (4.16) are B^s_{pq} with $q \ge p$ and F^s_{pq}.

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