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Gérard Bourdaud · Michael Reissig · Winfried Sickel

Hyperbolic equations, function spaces with exponential weights and Nemytskij operators

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Abstract. Different problems in the theory of hyperbolic equations bases on function spaces of Gevrey type are studied. Beside the original Gevrey classes, spaces defined by the behaviour of the Fourier transform were also used to prove basic results about the well-posedness of Cauchy problems for non-linear hyperbolic systems. In these approaches only the algebra property of the function spaces was used to include analytic non-linearities. Here we will generalize this dependence. First we investigate superposition operators in spaces with exponential weights. Then we show in concrete situations how a priori estimates of strictly hyperbolic type lead to the well-posedness of certain semi-linear hyperbolic Cauchy problems in suitable function spaces with exponential weights of Gevrey type.

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1. Introduction

In the theory of quasi-linear hyperbolic equations the use of Gevrey spaces is an essential idea for proving results about the well-posedness of the Cauchy problem. In the basic paper [14] it was shown that there exists a relation between the multiplicity of characteristic roots of the hyperbolic operator and the order of the Gevrey space which guarantees well-posedness of the Cauchy problem. If the multiplicity of characteristic roots does not exceed ν, then one can prove the well-posedness for $1 \le s < \nu/(\nu-1)$. Sometimes the critical case $s = \nu/(\nu-1)$ can be included in possible classes of well-posedness, too. Let us explain this relation with the aid of the Gevrey example [7]

$$u_{tt} - u_x = 0,\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x). \tag{1.1}$$

G. Bourdaud: Université Pierre et Marie Curie, Euipe d'Analyse Fonctionnelle, Case 186, 4 place Jussieu, 75252 Paris Cedex 05, France, e-mail: bourdaud@ccr.jussieu.fr

M. Reissig: TU Bergakademie Freiberg, Institut für Angewandte Mathematik I, Agricolastraße 1, D-09596 Freiberg, Germany, e-mail: reissig@mathe.tu-freiberg.de

W. Sickel: FSU Jena, Mathematisches Institut, Ernst-Abbe-Platz 1–4, D-07743 Jena, Germany, e-mail: sickel@minet.uni-jena.de

For this reason we introduce the Gevrey spaces $\mathcal{A}(s, \beta, m, \ell)$, $s > 1$, as the set

$$\left\{ u \in L_2(\mathbb{R}^n) : \quad \|u\|_{s,\beta,m,\ell} := \left(\int_{\mathbb{R}^n} \langle\xi\rangle_m^{2\ell} \, e^{\beta \langle\xi\rangle_m^{1/s}} \, |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{1/2} < \infty \right\}.$$

Here $\langle \xi \rangle_m^2 = |\xi|^2 + m^2$ and β is a positive constant, whereas ℓ and m are non-negative constants. The constant multiplicity of the characteristics of the hyperbolic operator $\partial_t^2 - \partial_x$ is two. The explicit representation of the solution of (1.1) shows that this Cauchy problem is well-posed iff $s \le 2$, where in the case $s = 2$ the solution exists only locally in t.

To overcome the critical order $\nu/(\nu - 1)$ one has to suppose Levi conditions. These are relations between lower-order terms and the principal part of the operator. To illustrate these relations let us consider

$$u_{tt} - t^{2m} u_{xx} + bt^k u_x = 0, \qquad u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \tag{1.2}$$

where $0 \le k < m - 1$; k, m are natural numbers. Due to [11] and [12] the Cauchy problem (1.2) is well-posed iff $s \le \frac{2m-k}{m-1-k}$.

Thus Gevrey spaces are an important tool for weakly hyperbolic Cauchy problems.

Nevertheless Gevrey spaces appear in a natural way in the strictly hyperbolic theory, too. The paper [4] describes a connection between regularity of the coefficient $a = a(t)$ in

$$u_{tt} - a(t)u_{xx} = 0, \;\; u(x,0) = \varphi(x), \;\; u_t(x,0) = \psi(x) \tag{1.3}$$

and possible Gevrey spaces of well-posedness. If $a(t) \ge C > 0$ and $a \in C^\alpha[0, T]$, $0 < \alpha < 1$, then the Cauchy problem (1.3) is well-posed iff $s \le \frac{1}{1-\alpha}$, where in the case $s = \frac{1}{1-\alpha}$ the solution exists only locally in t. But the regularity of $a = a(t)$ up to $t = 0$ is not necessary as it is shown in the recent paper [5]. If $a(t) \ge C > 0$, $a \in C^1(0, T]$ and the growth restriction $|a'(t)| \le \frac{C}{t^q}$, $q > 1$, is satisfied for $t \to +0$, then (1.3) is well-posed iff $s \le \frac{q}{q-1}$.

For the linear problems (1.1) to (1.3) the solvability behaviour in Gevrey classes is completely known.

In the following we are interested in semi-linear Cauchy problems of type (1.1) to (1.3) in the Gevrey spaces $\mathcal{A}(s, \beta, m, \ell)$. Let us restrict ourselves, for the moment, to

$$u_{tt} - u_x = f(x, t, u), \;\; u(x,0) = u_t(x,0) \equiv 0. \tag{1.4}$$

Vanishing data imply the dependence of f on t and x, too.

One possibility is to allow non-linearities of f with respect to u bases on algebra properties of Gevrey-type spaces. The algebra property for $\mathcal{A}(s, \beta, m, \ell)$ guarantees a possible analytical dependence of f with respect to u. Analyticity means the Taylor expansion and consequently the algebra property brings a priori estimates for such non-linear terms. To weaken the assumption of analyticity needs the study of superposition operators in Gevrey spaces.

Question. Under which conditions on f one can find a continuous function $g : [0, \infty) \mapsto [0, \infty)$ such that

$$\| f(u) \,|\mathcal{A}(s, \beta, m, \ell)\| \leq \| u \,|\mathcal{A}(s, \beta, m, \ell)\| \, g(\| u \,|\mathcal{A}(s, \beta, m, \ell)\|) \tag{1.5}$$

holds for all $u \in \mathcal{A}(s, \beta, m, \ell)$?

There exist several results about compositions of Gevrey functions. The composition of two G^s functions remains G^s if the Gevrey space $G^s(\Omega)$, here Ω is an open set, is defined by the Gevrey behaviour of functions on compact subsets (see Lions and Magenes [16]). The composition of Gevrey functions defined on a fixed compact set is investigated in Kajitani [13]. Furthermore, let us mention that Gramchev and Rodino [8] and Gramchev and Yoshino [9] have studied composition maps in Gevrey-type spaces by using scales of Banach spaces with norms

$$\|u\| = \sum_{\alpha \in \mathbb{Z}^n_+} \frac{T^{|\alpha|}}{(\alpha!)^\sigma} \left\| D^\alpha_x u \,| H^\ell_2(\mathbb{R}^n) \right\|. \tag{1.6}$$

For the applications presented in Chapter 3 we shall need the estimate (1.5) and even more, we will be interested also in the local Lipschitz continuity of the nonlinear operator $u \mapsto f(u)$. All this will be done in Chapter 2. There we shall work with more general spaces than $\mathcal{A}(s, \beta, m, \ell)$. The necessity of using modifications of these classes arises even from the study of the corresponding linear problems. Our approach is based on some special tools from Fourier analysis (division of the phase space, use of micro-energies etc.) and this forces us to switch to the more general scales.

The goal of this paper will be to prove well-posedness results in Gevrey-type spaces for semi-linear problems for (1.1) to (1.3) (see (1.4) as a semi-linear version of (1.1)). We will develop a unified approach to handle these problems. Let us briefly sketch the main steps.

It turns out that we cannot restrict ourselves to the spaces $\mathcal{A}(s, \beta, m, \ell)$, but we have to incorporate an extra function $h = h(\xi, t)$ in the Gevrey-type weight. Thus superposition operators are studied in spaces $\mathcal{A}(s, \beta, \beta', m, \ell, h, T)$ defined in Chapter 2. Here β' represents an additional parameter which, in a certain sense, restricts the admissible functions h and T in a finite time. Each application leads to its own function $h = h(\xi, t)$. This function will be constructed in such a way that the solution of the linear Cauchy problem

$$L\, u = f(x, t), \quad u(x, 0) = u_t(x, 0) \equiv 0 \tag{1.7}$$

satisfies an a priori estimate of strictly hyperbolic type. This means that there exists an, in general, pseudodifferential operator of first order $H = H(D_x, t)$ such that

$$\|u\| + \|H(D_x, t)u\| + \|D_t u\| \leq C(T)\|f\|, \tag{1.8}$$

where $C(T)$ tends to 0 if T tends to 0. For brevity we denote the norm of $\mathcal{A}(s, \beta, \beta', m, \ell, h, T)$ by $\| \cdot \|$. Such an a priori estimate is known for solutions of wave equations (energy estimates), where $H(D_x, t)$ is equal to D_x. The regularity of f coincides with those ones of $D_x u,\ D_t u$. Thus it is reasonable to call it an *a priori estimate of strictly hyperbolic type*. For several classes of weakly hyperbolic problems we will prove such estimates of strictly hyperbolic type (cf. with (3.4), (3.13) and (3.23)).

Now let us demonstrate how (1.8) can be used to study

$$L\,u = f(u),\ \ u(x,0) = \varphi(x),\ \ u_t(x,0) = \psi(x), \tag{1.9}$$

where L is a linear hyperbolic operator. Setting $w := u - \varphi - t\psi$ transforms (1.9) to a Cauchy problem with homogeneous data and right-hand side $f(w + \varphi + t\psi) - L(\varphi + t\psi) =: g(w, \varphi, t\psi)$.

We suppose:

- the linear problem (1.7) is well-posed in $\mathcal{A}(s, \beta, \beta', m, \ell, h, T)$ and its solution satisfies (1.8);
- if $w \in \mathcal{K}_R(0) = \{w \in \mathcal{A}(s, \beta, \beta', m, \ell, h, T) : \|w\| \le R\}$, then $\|f(w + \varphi + t\psi)\| + \|f'(w + \varphi + t\psi)\| \le C(R)$ (assumptions for φ, ψ and f guarantee such an estimate).

Then the application of (1.8) to the linearized problem

$$L\,w = g(v, \varphi, t\psi),\ \ w(x,0) = w_t(x,0) \equiv 0$$

immediately gives the next conclusions:

For a given R there exists a constant $T = T(R)$ such that the following properties hold for the mapping $P := L^{-1} g(\cdot, \varphi, t\psi)$:

- $P : v \in \mathcal{K}_R(0) \to w \in \mathcal{K}_R(0)$ is continuous;
- P is Lipschitz continuous on $\mathcal{K}_R(0)$.

Using $C(T) \to 0$ for $T \to 0$ yields that P becomes a contraction on $\mathcal{K}_R(0)$. By Banach's fixed point theorem we get the existence of a unique solution from $\mathcal{K}_R(0)$ for (1.9).

This approach is standard, thus we restrict our considerations in Chapter 3 to how to derive only a priori estimates of strictly hyperbolic type.

As usual, c and C denote generic constants which may change from line to line. By $B(x, r)$ we denote a ball with centre in x and radius r. Further we will use the notations e^x and $\exp x$ simultaneously.

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2. Composition operators on spaces with exponential weights

2.1. Some general classes of functions

Our energy estimates will force us to deal with the following very general classes of functions.

Definition 2.1. *Let $s > 0$ and suppose $T,\ \beta > 0$ and $\beta',\ \ell,\ m \geq 0$. Furthermore, let*

$$h\ :\ \mathbb{R}^n \times [0, T) \mapsto \mathbb{R}$$

be a measurable function satisfying

$$|h(\xi, t)| \leq \beta' \langle\, \xi\, \rangle_m^{1/s}, \tag{2.1}$$

for all ξ and all $t \in [0, T)$ and with some $\beta' < \beta$.

(i) *$\mathcal{A}(s, \beta, \beta', m, \ell, h, T)$ denotes the set of all complex-valued functions $u = u(x, t)$, defined on $\mathbb{R}^n \times [0, T)$, $u(\cdot, t) \in L_2(\mathbb{R}^n)$ for each $t < T$ and such that*

$$\| \, u \,| \mathcal{A}(s, \beta, \beta', m, \ell, h, T) \| = \sup_{0<t<T} \left(\int_{\mathbb{R}^n} \langle\, \xi\, \rangle_m^{2\ell} \exp\big(\beta \langle\, \xi\, \rangle_m^{1/s} + h(\xi, t)\big) \, |\mathcal{F} u(\xi, t)|^2 d\xi \right)^{1/2}$$

is finite.

(ii) *By $\mathcal{A}^{\mathbb{R}}(s, \beta, \beta', m, \ell, h, T)$ we denote the subspace of real-valued functions in $\mathcal{A}(s, \beta, \beta', m, \ell, h, T)$ equipped with the same norm.*

The symbol $\mathcal{F}$ has always been understood as the Fourier transform with respect to the space variable x (in contrast to the time variable t).

Remark 2.1. The condition (2.1) guarantees that the elements of $\mathcal{A}(s, \beta, \beta', m, \ell, h, T)$ have an exponentially decaying Fourier transform (for fixed t).

Remark 2.2. The scale is monotone with respect to s, β, β', m, h and T in an obvious way.

Remark 2.3. Let us mention that most of our methods used to derive estimates for composition operators defined on $\mathcal{A}^{\mathbb{R}}(s, \beta, \beta', m, \ell, h, T)$ do not extend to $\mathcal{A}(s, \beta, \beta', m, \ell, h, T)$.

Agreement. If there is no danger of confusion we shall write simply $\mathcal{A}$ instead of $\mathcal{A}(s, \beta, \beta', m, \ell, h, T)$.

2.2. Function spaces with exponential weights. I

In Subsections 2.2–2.7 we shall study composition and multiplication in a simplified situation. Later, in Subsections 2.8–2.13 we shall investigate the problem in those spaces we have introduced in Subsection 2.1. Finally, in Subsection 2.14 we add a few comments about the sharpness of the results obtained in the preceeding subsections.

The capital letters A, B, D, E, ... (except C) are used to denote specific constants and are fixed throughout this article.

Here we normalize the Fourier transform as follows:

$$\mathcal{F}u(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx,$$

which results in

$$\mathcal{F}(u * v)(\xi) = (2\pi)^{n/2}\, \mathcal{F}u(\xi)\, \mathcal{F}v(\xi). \tag{2.2}$$

If not otherwise stated, all functions are defined on $\mathbb{R}^n$. So, $\mathbb{R}^n$ is dropped in our notations.

Definition 2.2. *Let* $0 < s < \infty$. *Then* A_s *denotes the set of all complex-valued functions* u *in* L_2 *such that*

$$\| u \,|A_s\| = \left(\int_{\mathbb{R}^n} |\mathcal{F}u(\xi)|^2\, e^{|\xi|^{1/s}}\, d\xi \right)^{1/2} < \infty.$$

The subspace of the real-valued functions is denoted by $A_s^{\mathbb{R}}$.

Remark 2.4. The classes A_s are special cases of the more general scale $\mathcal{B}_{p,k}$ investigated, e.g., in Björck [2].

Remark 2.5. It holds $A_s \subset \mathcal{A}(s, 1, 0, 0, 0, 0, \infty)$, where the zero in place of h denotes the function identically zero (in addition we use the convention $f(x) = f(x,t)$ for all t). At several places in non-linear analysis spaces with a norm

$$\| u \|_\beta = \left(\int_{\mathbb{R}^n} |\mathcal{F}u(\xi)|^2\, e^{\beta|\xi|^{1/s}}\, d\xi \right)^{1/2} < \infty,$$

for a fixed $\beta > 0$ occur. All results derived below for A_s will be true for these more general classes (this may be seen by a simple change of coordinates). Only for simplicity of the presentation do we fix $\beta = 1$.

Remark 2.6. The scale is monotone with respect to s. For any $\varepsilon > 0$ we have

$$A_s \hookrightarrow A_{s+\varepsilon} \hookrightarrow L_2.$$

Here the symbol $\hookrightarrow$ indicates continuous imbedding.

Remark 2.7. Let us mention that these embeddings are not compact. For simplicity consider $A_s \hookrightarrow L_2$. We use the following simple construction. Denote by $\mathcal{X}_I$ the characteristic function of the set I. We put $\psi(t) = \mathcal{X}_{[0,1/2]}(t) - \mathcal{X}_{[-1/2,0]}(t)$, $t \in \mathbb{R}$. Now, for given $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$ consider the family

$$v_j(\xi) := 2^{j/2}\, \psi(2^j \xi_1)\, \varphi(\xi_2, \ldots, \xi_n), \qquad j = 0, 1, \ldots .$$

Define $u_j = \mathcal{F}v_j$, then u_j, $j = 0, 1, \ldots$ is a bounded family in A_s. Because of the pairwise orthogonality of the u_j in $L_2(\mathbb{R}^n)$ it can not contain a convergent subsequence. This remark immediately extends to the general situation of $\mathcal{A}(s, \beta, \beta', m, \ell, h, T) \hookrightarrow \mathcal{A}(s+\varepsilon, \gamma, \gamma', m', \ell', h', T)$.

Remark 2.8. If we have a bounded family $u_j,\ j = 0, 1, \ldots$ in A_s, then these functions are uniformly bounded and equicontinuous. The Arzela–Ascoli theorem yields that for any compact set $\Omega \subset \mathbb{R}^n$ there exists a subsequence $\{u_{j_\ell}\}_{\ell=0}^\infty$ which converges uniformly on K. Of course, one can not replace K by an unbounded set. An example is given by

$$u_j(x) := \mathcal{F}^{-1}\big[\psi(\xi_1)\, \varphi(\xi_2, \ldots, \xi_n)\big](x_1 - j, x_2, \ldots, x_n), \qquad j = 0, 1, \ldots .$$

This argument can be applied in the general situation $\mathcal{A}(s, \beta, \beta', m, \ell, h, T)$, too.

Lemma 2.1. (i) *The space A_s becomes a Banach space with respect to $\| \cdot |A_s\|$.*
(ii) *The elements of A_s are C^∞-functions satisfying*

$$\| D^\alpha u|L_\infty\| \le (2\pi)^{-n/4} \sqrt{s\, \frac{\Gamma((2|\alpha| + n)\, s)}{\Gamma(n/2)}}\, \| u\, |A_s\|, \tag{2.3}$$

for all $u \in A_s$ and all $\alpha \in \mathbb{N}_0^n$.

Proof. Because of $\xi^\alpha\, \mathcal{F} u(\xi) \in L_2$ for all α, we conclude the existence of $D^\alpha u$ for all α and its continuity. Let $\omega_n = 2\, \pi^{n/2}\, (\Gamma(n/2))^{-1}$. Then, using

$$D^\alpha u(x) = (2\pi)^{-n/2} \int e^{ix\xi}\, (i\xi)^\alpha\, \mathcal{F} u(\xi)\, d\xi,$$

we find

$$|D^\alpha u(x)| \le (2\pi)^{-n/2} \| u\, |A_s\| \left(\int e^{-|\xi|^{1/s}}\, |\xi|^{2|\alpha|}\, d\xi \right)^{1/2}$$

and

$$\begin{aligned} \int e^{-|\xi|^{1/s}}\, |\xi|^{2|\alpha|}\, d\xi &= \omega_n \int_0^\infty e^{-r^{1/s}}\, r^{2|\alpha|+n-1}\, dr \\ &= \omega_n\, s \int_0^\infty e^{-t}\, t^{s(2|\alpha|+n)-1}\, dt \\ &= \omega_n\, s\, \Gamma(s(2|\alpha| + n)). \end{aligned}$$

This proves (2.3). □

Having established the estimate (2.3) one can compare the spaces A_s with the classical Gevrey spaces.

Definition 2.3. *Let $s \ge 1$. The function u belongs to $\mathcal{G}^s$ if $u \in C^\infty(\mathbb{R}^n)$ and for every compact subset K of $\mathbb{R}^n$ there exists a positive constant C such that, for all α and $x \in K$,*

$$|D^\alpha u(x)| \le C^{|\alpha|+1}\, (\alpha!)^s.$$

Lemma 2.2. *Let $s \ge 1$. Then $A_s \subset \mathcal{G}^s$ holds.*

Proof. The lemma becomes a consequence of elementary calculations based on Stirling's formula. □

Remark 2.9. Functions in $\mathcal{G}^s$ need not be in L_2. For that reason there can't be a direct converse of Lemma 2.2. However, if we restrict ourselves to functions with compact support, then there is a partial converse. Let *supp u* be compact and let $u \in \mathcal{G}^s$, $s > 1$. Then there exists a constant c and some $\varepsilon > 0$ such that

$$|\mathcal{F} u(\xi)| \le c\, e^{-\varepsilon\,|\xi|^{1/s}},$$

cf. Rodino [19, Theorem 1.6.1]. If we put

$$\mathcal{G}_0^s = \big\{ u \in \mathcal{G}^s \ : \quad \mathit{supp}\, u \quad \text{is compact} \big\},$$

then $\mathcal{G}_0^s \subset A_\sigma$ for all $s < \sigma$ follows.

Of crucial importance for us will be a good estimate for products in A_s.

Theorem 2.1. *The spaces A_s are algebras with respect to pointwise multiplication if and only if $s > 1$.*

Proof. Step 1. Let $s > 1$. For u and v in A_s we can apply (2.2). This yields

$$\begin{aligned} \|\, u\, v\, | A_s \| &\le (2\pi)^{-n/2} \Bigg[\bigg(\int \bigg| \int_{|\eta-\xi|\le|\eta|} \mathcal{F} u(\xi-\eta)\, \mathcal{F} v(\eta)\, d\eta \bigg|^2 e^{|\xi|^{1/s}}\, d\xi \bigg)^{1/2} \\ &\qquad + \bigg(\int \bigg| \int_{|\eta-\xi|>|\eta|} \mathcal{F} u(\xi-\eta)\, \mathcal{F} v(\eta)\, d\eta \bigg|^2 e^{|\xi|^{1/s}}\, d\xi \bigg)^{1/2} \Bigg] \\ &= I_1 + I_2. \end{aligned}$$

Let $\delta = \delta_{1/s}$. Lemma 4.2 from the Appendix yields

$$\begin{aligned} I_1 &\le \bigg\| \int |\mathcal{F} u(\xi-\eta)\, e^{\frac12 |\xi-\eta|^{1/s}}|\, |\mathcal{F} v(\eta)\, e^{\frac12|\eta|^{1/s}}|\, e^{-\frac12\,\delta\,|\xi-\eta|^{1/s}}\, d\eta \,\bigg|\, L_2(\mathbb{R}^n,\xi) \bigg\| \\ &= \bigg\| \int |\mathcal{F} u(\tau)\, e^{\frac12|\tau|^{1/s}}|\, |\mathcal{F} v(\xi-\tau)\, e^{\frac12|\xi-\tau|^{1/s}}|\, e^{-\frac12\,\delta\,|\tau|^{1/s}}\, d\tau \,\bigg|\, L_2(\mathbb{R}^n,\xi) \bigg\| \\ &\le \|\, v\, | A_s \| \int |\mathcal{F} u(\tau)|\, e^{\frac12 (|\tau|^{1/s} - \delta\,|\tau|^{1/s})}\, d\tau \\ &\le \|\, u\, | A_s \|\, \|\, v\, | A_s \| \bigg(\int e^{-\delta\,|\tau|^{1/s}}\, d\tau \bigg)^{1/2}, \end{aligned} \tag{2.4}$$

where we used the Minkowski inequality and Hölder inequality in the last two steps. Similarly we can proceed in case of I_2 and obtain

$$I_2 \le \|\, u\, | A_s \|\, \|\, v\, | A_s \| \bigg(\int e^{-\delta\,|\tau|^{1/s}}\, d\tau \bigg)^{1/2}.$$

This proves the 'if' part of the theorem.

Step 2. Let $s \le 1$. We choose $u \equiv v$ and

$$\mathcal{F} u(\xi)\, e^{|\xi|^{1/s}/2} = \frac{1}{(1+|\xi|)^{\alpha}}, \qquad \alpha > \frac{n}{2}.$$

Then $u \in A_s$. First we consider $s < 1$. Now we apply Lemma 4.3 from the Appendix (L is the constant appearing there). Observe that $|\xi - \eta|$ and $|\eta|$ are bounded by $(L + 1/2)|\xi|$ on $B(\xi/2, L|\xi|)$. Consequently we obtain

$$\begin{aligned} &\| u^2 \,|A_s\|^2 \\ &\ge \int_{|\xi|\ge 1} \left| \int_{B(\xi/2, L|\xi|)} e^{(-|\xi-\eta|^{1/s} - |\eta|^{1/s} + |\xi|^{1/s})/2} \frac{1}{(1+|\xi-\eta|)^{\alpha}} \frac{1}{(1+|\eta|)^{\alpha}} d\eta \right|^2 d\xi \\ &\ge c \int_{|\xi|\ge 1} \mathrm{vol}\left(B(\xi/2, L|\xi|\right)^2 \frac{1}{|\xi|^{4\alpha}}\, d\xi \\ &\ge c \int_1^{\infty} r^{3n-1-4\alpha}\, dr = \infty, \end{aligned}$$

if $\alpha \le 3n/4$. Finally we consider $s = 1$. Here we apply Lemma 4.4 with $c = 1$. Arguing similarly as in case $s < 1$, but replacing the ball $B(\xi/2, L|\xi|)$ by the set E_ξ and using $|\eta| + |\xi - \eta| \le |\xi| + 1$ on E_ξ, we find

$$\begin{aligned} \| u^2 \,|A_s\|^2 &\ge e^{-1/2} \int_{|\xi|\ge 1} \left| \int_{E_\xi} \frac{1}{(1+|\xi-\eta|)^{\alpha}} \frac{1}{(1+|\eta|)^{\alpha}} d\eta \right|^2 d\xi \\ &\ge c \int_1^{\infty} r^{2n-4\alpha}\, dr = \infty, \end{aligned}$$

if $\alpha \le (2n+1)/4$. This proves the theorem. □

The algebra property in case $s > 1$ implies the existence of a constant M such that

$$\| u\, v \,|A_s\| \le M \,\| u \,|A_s\| \; \| v \,|A_s\| \tag{2.5}$$

holds for all u and v. Next we are interested in certain subalgebras of A_s which will allow us to replace M by a smaller constant. Let

$$P_R = \{\xi \in \mathbb{R}^n \,:\, |\xi_i| \le R,\ i = 1, \ldots, n\}, \qquad R > 0.$$

We shall decompose the complement of such a cube into 2^n parts by

$$P_R(\varepsilon) = \{\xi \in \mathbb{R}^n \,:\, \mathit{sign}\, \xi_j = (-1)^{\varepsilon_j},\ j = 1, \ldots, n\} \setminus P_R,$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_j \in \{0, 1\}$. Fix ε. Let us suppose $\mathit{supp}\, \mathcal{F} u,\ \mathit{supp}\, \mathcal{F} v \subset P_R(\varepsilon)$. Then

$$\mathit{supp}\, (\mathcal{F} u * \mathcal{F} v) \subset \{\xi + \eta : \quad \xi \in \mathit{supp}\, \mathcal{F} u,\ \eta \in \mathit{supp}\, \mathcal{F} v\}$$

and hence $\mathit{supp}\, \mathcal{F}(u\, v) \subset P_R(\varepsilon)$.

Theorem 2.2. *Let $s > 1$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_j \in \{0, 1\}$, $j = 1, \ldots, n$, be fixed and suppose $R > 0$. Let $\delta = 2 - 2^{1/s}$. The spaces*

$$A_s(\varepsilon, R) = \{u \in A_s : \quad supp\, \mathcal{F} u \subset P_R(\varepsilon)\}$$

are subalgebras of A_s and

$$\| \, u \, v \, | A_s \| \le D \, \| \, u \, | A_s \| \; \| \, v \, | A_s \|$$

holds for all $u, v \in A_s(\varepsilon, R)$, where

$$D = \left(s \, \omega_n \, \delta^{-sn} \int_{\delta R^{1/s}}^{\infty} e^{-y} \, y^{sn-1} \, dy \right)^{1/2} . \tag{2.6}$$

Proof. There is not a big difference between the proofs of Theorem 2.1 and Theorem 2.2. The additional information about u is used in the last step of (2.4) only. One obtains

$$I_1 \le \| \, u \, | A_s \| \, \| \, v \, | A_s \| \left(\int_{P_R(\varepsilon)} e^{-\delta |\tau|^{1/s}} \, d\tau \right)^{1/2} .$$

A similar argument, but now with respect to v, has to be applied for the estimate of I_2. In view of

$$\begin{aligned} \int_{P_R(\varepsilon)} e^{-\delta |\tau|^{1/s}} \, d\tau &\le \int_{|\tau| > R} e^{-\delta |\tau|^{1/s}} \, d\tau \\ &= \omega_n \int_R^{\infty} r^{n-1} \, e^{-\delta r^{1/s}} \, dr \\ &= s \, \omega_n \int_{R^{1/s}}^{\infty} t^{sn-1} \, e^{-\delta t} \, dt \\ &= s \, \omega_n \, \delta^{-sn} \int_{\delta R^{1/s}}^{\infty} e^{-y} \, y^{sn-1} \, dy \end{aligned}$$

the result follows. □

2.3. The mapping $u \mapsto e^{iu} - 1$ in spaces with exponential weights. I

As we shall see later the main part of the theory for the general operator $T_f : u \mapsto f \circ u$ will be taken by the particular case $u \mapsto e^{iu} - 1$. For that reason we study this operation in great detail. It will be of great service for us to investigate $u \mapsto e^{iu} - 1$ under additional assumptions on u in the frequency domain. Up to now we have to restrict ourselves to real-valued functions.

2.3.1. Extra conditions on the spectrum – Part I. We suppose $supp\, \mathcal{F} u \subset P_R(\varepsilon)$ for some fixed $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_j \in \{0, 1\}$, $j = 1, \ldots, n$. Under this assumption Theorem 2.2 implies

$$\| \, e^{iu} - 1 \, | A_s \| = \left\| \sum_{\ell=1}^{\infty} \frac{(iu)^\ell}{\ell!} \, \middle| A_s \right\| \le \sum_{\ell=1}^{\infty} \frac{\| \, u \, | A_s \|^\ell \, D^{\ell-1}}{\ell!} = \frac{1}{D} \left(e^{D \, \| \, u \, | A_s \|} - 1 \right). \tag{2.7}$$

2.3.2. Extra conditions on the spectrum – Part II. Next we suppose *supp* $\mathcal{F} u \subset P_R$ and u to be real-valued. We shall make use of the following splitting:

$$e^{iu(x)} - 1 = \sum_{\ell=1}^{r} \frac{(iu(x))^\ell}{\ell!} + \sum_{\ell=r+1}^{\infty} \frac{(iu(x))^\ell}{\ell!} = g_1(x) + g_2(x).$$

Then it holds that

$$supp\, \mathcal{F} g_1 \subset [-Rr,\ Rr]^n.$$

Let I be the set of all n-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ such that $\varepsilon_j \in \{0, 1\}$, $j = 1, \ldots, n$. In view of

$$\mathcal{F}(e^{iu} - 1)(\eta) = (2\pi)^{-n/2} \int e^{-ix\eta}\, g_2(x)\, dx, \qquad \eta \notin [-Rr,\ Rr]^n,$$

we introduce the following decomposition:

$$\| e^{iu} - 1\, |A_s\| = \left(\sum_{\varepsilon \in I} \int_{P_{Rr}(\varepsilon)} \ldots\, d\eta + \int_{P_{Rr}} \ldots\, d\eta \right)^{1/2} = \left(T_1^2 + T_2^2\right)^{1/2}, \quad (2.8)$$

where

$$T_1^2 = \sum_{\varepsilon \in I} \int_{P_{Rr}(\varepsilon)} \left| \sum_{\ell=r+1}^{\infty} \frac{i^\ell\, (\mathcal{F} u^\ell)(\eta)}{\ell!} \right|^2 e^{|\eta|^{1/s}}\, d\eta$$

and

$$T_2^2 = \int_{[-Rr, Rr]^n} |\, \mathcal{F}(e^{iu(x)} - 1)(\eta)\, |^2\, e^{|\eta|^{1/s}}\, d\eta.$$

The estimate of T_2. The function $h(t) = e^{it} - 1$ is Lipschitz continuous and $h(0) = 0$. Hence, the corresponding composition operator maps L_2 into L_2 and has a norm bounded by the Lipschitz constant of h. We obtain

$$T_2^2 \le e^{(\sqrt{n} Rr)^{1/s}}\, \| e^{iu} - 1\, |L_2\|^2 \le e^{(\sqrt{n} Rr)^{1/s}}\, \| u\, |L_2\|^2 \le e^{(\sqrt{n} Rr)^{1/s}}\, \| u\, |A_s\|^2. \quad (2.9)$$

Remark 2.10. The inequality (2.9) is the only place where we make use of the fact that u has to be real-valued.

The estimate of T_1. To estimate T_1 we apply Theorem 2.1. This leads to

$$T_1 \le \sum_{\ell=r+1}^{\infty} \frac{1}{\ell!}\, \| u^\ell\, |A_s\| \le \sum_{\ell=r+1}^{\infty} \frac{M^{\ell-1}}{\ell!}\, \| u\, |A_s\|^\ell = \frac{1}{M} \sum_{\ell=r+1}^{\infty} \frac{(M\, \| u\, |A_s\|)^\ell}{\ell!}, \quad (2.10)$$

where M is the constant from (2.5). Next we choose r as a function of $\| u\, |A_s\|$.

Step 1. We assume $M \| u \,|A_s\| > 1$. Suppose

$$3\, M \| u \,|A_s\| \le r \le 3\, M \| u \,|A_s\| + 1. \tag{2.11}$$

From Stirling's formula we know $\ell ! = \Gamma(\ell+1) \ge \ell^\ell \, e^{-\ell} \sqrt{2\pi\ell}$. Hence

$$\begin{aligned}\sum_{\ell=r+1}^{\infty} \frac{(M \| u \,|A_s\|)^\ell}{\ell !} &\le \sum_{\ell=r+1}^{\infty} \Big(\frac{r}{\ell}\Big)^\ell \Big(\frac{e}{3}\Big)^\ell \frac{1}{\sqrt{2\pi\ell}} \\ &\le \sum_{\ell=r+1}^{\infty} \Big(\frac{e}{3}\Big)^\ell \\ &\le \frac{3}{3-e}.\end{aligned}$$

Inserting this in our previous estimate we find

$$T_1 \le c \tag{2.12}$$

as long as r is chosen as in (2.11).

Step 2. We assume $M \| u \,|A_s\| \le 1$. Then we may choose $r = 0$ (which means $T_2 = 0$) ending up with

$$T_1 \le \sum_{\ell=1}^{\infty} \frac{(M \| u \,|A_s\|)^\ell}{\ell !} \le e\, M \| u \,|A_s\|. \tag{2.13}$$

Summarizing, we have proved

$$\| e^{iu} - 1 \,|A_s\| \le c \,\| u \,|A_s\| \left(1 + e^{bR^{1/s} \|u|A_s\|^{1/s}}\right)^{1/2}, \tag{2.14}$$

where c and b are positive numbers independent of u, r and R, cf. (2.8)–(2.13).

2.4. Estimates of $e^{iu} - 1$ for general u. I

We start with a decomposition of u in the frequency domain. To this end let $\chi_{R,\varepsilon}(\eta)$ denote the characteristic function of the set $P_R(\varepsilon)$ and $\chi_R(\eta)$ the characteristic function of the cube P_R. Further, let

$$u_\varepsilon(x) = \mathcal{F}^{-1}[\chi_{R,\varepsilon}(\eta)\, \mathcal{F} u(\eta)](x) \qquad \text{and} \qquad u_0(x) = \mathcal{F}^{-1}[\chi_R(\eta)\, \mathcal{F} u(\eta)](x).$$

Hence

$$u(x) = u_0(x) + \sum_{\varepsilon \in I} u_\varepsilon(x).$$

Immediately one obtains

$$\| u \,|A_s\|^2 = \| u_0 \,|A_s\|^2 + \sum_{\varepsilon \in I} \| u_\varepsilon \,|A_s\|^2. \tag{2.15}$$

Moreover, with an appropriate counting of the u_ε we find

$$e^{iu} - 1 = \sum_{\ell=1}^{2^n+1} \sum_{0\le j_1<j_2<\ldots<j_\ell\le 2^n} (e^{iu_{j_1}} - 1)\ldots(e^{iu_{j_\ell}} - 1),$$

cf. Lemma 4.6 from the Appendix. Employing this decomposition, Theorem 2.1 yields

$$\| e^{iu} - 1 \,|A_s\| \le \sum_{\ell=1}^{2^n+1} \sum_{0\le j_1<j_2<\ldots<j_\ell\le 2^n} M^{\ell-1} \| e^{iu_{j_1}} - 1 \,|A_s\| \ldots \| e^{iu_{j_\ell}} - 1 \,|A_s\|. \tag{2.16}$$

In addition, (2.7) and (2.15) give

$$\| e^{iu_{j_r}} - 1 \,|A_s\| \le \frac{1}{D}\left(e^{D\,\| u\,|A_s\|} - 1\right) \tag{2.17}$$

for any admissible choice of j_r and (2.14), (2.15) lead to

$$\| e^{iu_0} - 1 \,|A_s\| \le c\,\| u\,|A_s\| \left(1 + e^{bR^{1/s}\| u\,|A_s\|^{1/s}}\right), \tag{2.18}$$

for appropriate $c, b > 0$.

Our next step consists of choosing R in dependence of $\| u\,|A_s\|$. Here we try to arrive at a point where the right-hand sides in (2.17) and (2.18) are of the same size. Recall that $D = D(R)$ has been defined in (2.6). The function $D(R)$ is a strictly monotone positive function satisfying $\lim_{R\to\infty} D(R) = 0$ and $D(0) > 1$. Assume $\| u\,|A_s\| > 1$. Then we define R by the equation

$$D(R) = \| u\,|A_s\|^{(1/s)-1}, \tag{2.19}$$

which is always possible by our preceding remarks. Reformulating (2.19) by using the notations from Lemma 4.5 in the Appendix we obtain $\sqrt{c\, f(\delta\, R^{1/s})} = \| u\,|A_s\|^{(1/s)-1}$, i.e. $R = \delta^{-s}\big(g(\|u|A_s\|^{(2/s)-2}/c)\big)^s$. Lemma 4.5 yields

$$R \le C\left(\left(2 - \frac{2}{s}\right)\log \| u\,|A_s\| + \log c\right)^s \tag{2.20}$$

with c and C independent of u. All together we got the following result:

Theorem 2.3. *Let $s > 1$.*

(i) *There exist constants c and a (depending on n and s only) such that*

$$\| e^{iu} - 1|A_s\| \le c \begin{cases} e^{a\,\| u\,|A_s\|^{1/s}\,\log \| u\,|A_s\|} & \text{if} \quad \| u\,|A_s\| > 1\,; \\ \| u\,|A_s\| & \text{if} \quad \| u\,|A_s\| \le 1\,; \end{cases} \tag{2.21}$$

holds for all $u \in A_s^{\mathbb{R}}$.

(ii) *The mapping* $u \mapsto e^{iu} - 1$ *is locally Lipschitz continuous (considered as a mapping of* $A_s^{\mathbb{R}}$ *into* A_s*).*

(iii) *Let* $u \in A_s^{\mathbb{R}}$ *be fixed. Define* $g(\xi) = e^{iu(x)\xi} - 1$, $\xi \in \mathbb{R}^n$. *Then* $g : \mathbb{R}^n \mapsto A_s$ *is continuous.*

Proof. Step 1. Proof of (i). If $\| u \,|A_s\| > 1$ inequality (2.21) becomes a consequence of (2.16)–(2.20). If $\| u \,|A_s\| \le 1$ we choose $R = 1$. Then (2.16)–(2.18) imply the desired inequality.

Step 2. Proof of (ii). Local Lipschitz continuity follows from the identity

$$e^{iu} - e^{iv} = (e^{iv} - 1)\,(e^{i(u-v)} - 1) + (e^{i(u-v)} - 1),$$

the algebra property of A_s and part (i).

Step 3. Proof of (iii). By Step 1 we know that $g(\xi) \in A_s$ (considered as a function in x). The continuity follows from the identity used in Step 2, the algebra property of A_s and part (i). □

Remark 2.11. Observe that (2.5) implies

$$\| e^{iu} - 1 \,|A_s\| \le \frac{1}{M}\left(e^{M\,\| u \,|A_s\|} - 1\right)$$

like in (2.7). The essential difference between (2.21) and the later estimate consists of the exponent of $\| u \,|A_s\|$. If u has a small norm both estimates are comparable but not in the case of large norms.

Remark 2.12. Part (i) of Theorem 2.3 in a somewhat weaker form has been proved by Bourdaud [3], himself inspired by the work of Leblanc [15]. The decomposition technique used in the proof of Theorem 2.3 has been taken over from these papers.

2.5. Composition operators on spaces with exponential weights. I

Now we are in a position to investigate the non-linear operator $T_f : u \mapsto f \circ u$, which we call the composition operator. Our main tool here will be the Minkowski inequality in the context of Banach spaces. In what follows we denote by μ a finite complex measure and by $|\mu|$ its total variation.

Theorem 2.4. *Let* $s > 1$. *Let* μ *be a complex measure on* $\mathbb{R}$ *such that*

$$L_1(\lambda) = \int_{\mathbb{R}} e^{\lambda\,|\xi|^{1/s}\,\log|\xi|}\, d|\mu|(\xi) < \infty, \tag{2.22}$$

for any $\lambda > 0$, *and such that* $\mu(\mathbb{R}) = 0$. *Let* f *be the inverse Fourier transform of* μ. *Then* f *is an infinitely differentiable function and the composition operator* T_f *maps* $A_s^{\mathbb{R}}$ *into* A_s.

Proof. From (2.22) we deduce easily that $\int_{\mathbb{R}} d\,|\mu|(\xi) < \infty$. Hence μ is a finite measure and the condition $\mu(\mathbb{R}) = 0$ makes sense. Let us define

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi t}\, d\mu(\xi).$$

A further use of (2.22) yields $\int_{\mathbb{R}} |(i\xi)^j|\, d\,|\mu|(\xi) < \infty$, for all $j \in \mathbb{N}$. Then f is a C^∞ function and we have

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} [e^{i\xi t} - 1]\, d\mu(\xi).$$

Let $u \in A_s^{\mathbb{R}}$. An application of Theorem 2.3 yields

$$\| e^{iu} - 1\, |A_s\| \le c\, e^{a\, \|u\,|A_s\|^{1/s}\, \log \|u\,|A_s\|},$$

whatever $\|u\,|A_s\|$ is (the above constant c may be larger than that of (2.21)).

Denote by μ_r and μ_i the real and the imaginary part of μ, respectively. These measures are signed measures having a Jordan decomposition. This means

$$\mu(E) = \mu_r(E) + i\, \mu_i(E) = \mu_r^+(E) - \mu_r^-(E) + i\, (\mu_i^+(E) - \mu_i^-(E)),$$

for all measurable sets E, see e.g. [1]. For brevity let ν be one of these four measures on the right-hand side. Then $\nu(E) \le |\mu|(E)$ for all measurable sets E, cf. e.g. [1]. Let g be the function defined in Theorem 2.3(iii). We interpret the integral

$$\int \left[e^{iu(x)\xi} - 1\right] d\nu(\xi) = \int g(\xi)\, d\nu(\xi)$$

as a Bochner integral with values in A_s. Because g is continuous, cf. Theorem 2.3(iii), it is also integrable (the measure ν is finite). In this context we may apply the Minkowski inequality, cf. [6], which implies

$$\| \int \left[e^{iu(x)\xi} - 1\right] d\nu(\xi)\, |A_s\| \le \int_{\mathbb{R}} \|e^{i\xi u} - 1|\, A_s\|\, d|\mu|(\xi).$$

For $|\xi| \ge e$, we have

$$|\xi|^{1/s} \|u\,|A_s\|^{1/s}\, \log \|\xi u\,|A_s\| \le \left(\|u\,|A_s\|^{1/s}\, (1 + \log_+ \|u\,|A_s\|)\right) |\xi|^{1/s} \log |\xi|.$$

Hence

$$\int_{|\xi| \ge e} \|e^{i\xi u} - 1|\, A_s\|\, d|\mu|(\xi) < \infty.$$

The integral for $|\xi| < e$ is easily seen to be finite. Because of

$$\begin{aligned} \sqrt{2\pi}\, f(u(x)) = & \int g(\xi)\, d\mu_r^+(\xi) - \int g(\xi)\, d\mu_r^-(\xi) \\ & + i \int g(\xi)\, d\mu_i^+(\xi) - i \int g(\xi)\, d\mu_i^-(\xi), \end{aligned}$$

an application of the triangle inequality completes the proof. □

Remark 2.13. Observe that we do not require that f is real valued.

For the convenience of the reader we formulate two consequences separately.

Corollary 2.1. *Let $s > 1$. Let $g : \mathbb{R} \mapsto \mathbb{C}$ be a bounded function such that*

$$\lim_{|\xi|\to\infty} \frac{|\xi|^{1/s} \log |\xi|}{\log |g(\xi)|} = 0. \tag{2.23}$$

Assume, moreover, that $\int_{\mathbb{R}} g(\xi)\, d\xi = 0$. Let f be the inverse Fourier transform of g. Then f is an infinitely differentiable function and the composition operator T_f maps $A_s^{\mathbb{R}}$ into A_s.

Proof. We note first that the boundedness of g and the condition (2.23) imply that $\lim_{|\xi|\to\infty} g(\xi) = 0$. Let $\lambda > 0$. By (2.23), there exists $N > 0$ such that

$$2\lambda |\xi|^{1/s} \log |\xi| \le \log \frac{1}{|g(\xi)|} \quad \text{for} \quad |\xi| > N.$$

Then

$$\int_{|\xi|>N} e^{\lambda|\xi|^{1/s}\, \log |\xi|} |g(\xi)|\, d\xi \le \int_{|\xi|>N} e^{-\lambda|\xi|^{1/s}\, \log |\xi|}\, d\xi < \infty.$$

□

Similarly one can proceed in the periodic situation.

Corollary 2.2. *Let $s > 1$. Let $(c_k)_{k\in\mathbb{Z}}$ be a sequence of complex numbers such that*

$$\lim_{|k|\to\infty} \frac{|k|^{1/s} \log |k|}{\log |c_k|} = 0. \tag{2.24}$$

Assume, moreover, that $\sum_{k\in\mathbb{Z}} c_k = 0$ and define $f(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikt}$. Then f is an infinitely differentiable function and the composition operator T_f maps $A_s^{\mathbb{R}}$ into A_s.

Remark 2.14. Corollary 2.2 is mainly contained in Bourdaud [3], see also Leblanc [15]. Let us refer also to Gramchev and Yoshino [9] for some Gevrey composition estimates for functions defined on the unit circle. But as mentioned in the introduction in [9], different scales of Gevrey-type spaces are considered.

2.6. Nemytskij operators on spaces with exponential weights. I

Let $d \ge 2$ be a natural number. Let $f : \mathbb{R}^d \mapsto \mathbb{C}$ be a continuous function. Then we are interested in the action of the Nemytskij operator associated with f and given by

$$T_f(u_1, \dots, u_d) = f(u_1, \dots, u_d).$$

Theorem 2.5. *Let* $s > 1$. *Let*

$$H(\xi_1, \xi_2, \dots, \xi_d) = \exp\Big(\sum_{k=1}^{d} |\xi_k|^{1/s} \log(1 + |\xi_k|)\Big).$$

Let μ *be a complex measure on* $\mathbb{R}^d$ *such that*

$$L_2(\lambda_1, \dots, \lambda_d) = \int_{\mathbb{R}^d} H(\lambda_1\xi_1, \lambda_2\xi_2, \dots, \lambda_d\xi_d)\, d|\mu|(\xi) < \infty, \tag{2.25}$$

for any $(\lambda_1, \dots, \lambda_d)$, $\lambda_k > 0$, $k = 1, \dots, d$ *and suppose* $\mu(\mathbb{R}^d) = 0$. *Let* f *be the inverse Fourier transform of* μ. *Then* f *is an infinitely differentiable function and the associated Nemytskij operator* T_f *maps* $(A_s^{\mathbb{R}})^d$ *into* A_s.

Proof. By assumption

$$\begin{aligned} f(t_1, t_2, \dots, t_d) &= f(t_1, t_2, \dots, t_d) - f(0, \dots, 0) \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \Big[e^{i\sum_{k=1}^{d} \xi_k t_k} - 1\Big]\, d\mu(\xi). \end{aligned}$$

An application of the Minkowski inequality yields

$$\| T_f(u) \,|A_s\| \le 4 \int_{\mathbb{R}^d} \Big\| e^{i\sum_{k=1}^{d} \xi_k u_k(x)} - 1 \,|A_s(\mathbb{R}^n, x)\Big\|\, d|\mu|(\xi).$$

Now we can proceed by means of Lemma 4.6 from the Appendix, and Theorem 2.3. □

2.7. Nemytskij operators on spaces with exponential weights. II

Let d be a natural number. Let $f : \mathbb{R}^{d+n} \mapsto \mathbb{C}$ be a continuous function. Here the operator of interest is defined as

$$T_f(u_1, \dots, u_d)(x) = f(x_1, \dots, x_n, u_1(x), \dots, u_d(x)). \tag{2.26}$$

There is not much difference to the situations treated before.

Theorem 2.6. *Let* $s > 1$. *Let*

$$H(\xi_1, \xi_2, \dots, \xi_{d+n}) = e^{|(\xi_1,\dots,\xi_n)|^{1/s}/2} \exp\Big(\sum_{k=1}^{d} |\xi_k|^{1/s} \log(1 + |\xi_k|)\Big).$$

Let μ *be a complex measure on* $\mathbb{R}^{d+n}$ *such that*

$$L_3(\lambda_1, \dots, \lambda_d) = \int_{\mathbb{R}^{d+n}} H(\xi_1, \dots, \xi_n,\ \lambda_1\xi_{n+1}, \dots,\ \lambda_d\xi_{d+n})\, d|\mu|(\xi) < \infty, \tag{2.27}$$

for all $\lambda_1, \dots, \lambda_d > 0$. *Define* f *as the inverse Fourier transform of* μ *and suppose* $f(x_1, \dots, x_n, 0, \dots, 0) \in A_s$. *Then the Nemytskij operator* T_f, *defined in* (2.26) *maps* $(A_s^{\mathbb{R}})^d$ *into* A_s.

Proof. As above

$$\begin{aligned} f(t_1, \dots, t_{d+n}) &= f(t_1, \dots, t_{d+n}) - f(t_1, \dots, t_n, 0, \dots, 0) \\ &\quad + f(t_1, \dots, t_n, 0, \dots, 0) \\ &= (2\pi)^{-(d+n)/2} \int_{\mathbb{R}^{d+n}} \left[e^{i \sum_{k=n+1}^{d+n} \xi_k t_k} - 1\right] e^{i \sum_{j=1}^{n} \xi_j t_j} d\mu(\xi) \\ &\quad + f(t_1, \dots, t_n, 0, \dots, 0)\,. \end{aligned}$$

Application of the triangle and Minkowski inequalities yields

$$\begin{aligned} \|T_f(u)|A_s\| \le 4 \int_{\mathbb{R}^{d+n}} \left\| \left(e^{i \sum_{k=n+1}^{d+n} \xi_k u_k(x)} - 1\right) e^{i \sum_{j=1}^{n} \xi_j x_j} \left| A_s(\mathbb{R}^n, x) \right\| \, d|\mu|(\xi) \right. \\ + \| f(x, 0)|A_s\|. \end{aligned}$$

The function $e^{i \sum_{j=1}^{n} \lambda_j x_j}$ is a pointwise multiplier for A_s for any $\lambda = (\lambda_1, \dots, \lambda_n)$. More exactly we have

$$\| e^{i\lambda x} u(x) \, |A_s\|^2 = \int |\mathcal{F} u(\xi)|^2 \, e^{|\lambda+\xi|^{1/s}} \, d\xi \le e^{|\lambda|^{1/s}} \, \| u \, |A_s\|^2$$

since $0 < 1/s < 1$. Now we can proceed by means of Theorem 2.3. □

Remark 2.15. As usual, the resulting conditions on $f(x, y)$ to ensure boundedness of the operator $u \mapsto f(x, u(x))$ are weaker than those which guarantee the same property for $(u_1, u_2) \mapsto f(u_1, u_2)$.

2.8. *Function spaces with exponential weights. II*

Here we turn back to the general classes $\mathcal{A}$ introduced in Subsection 2.1.

There are two obvious consequences of the definition. Any $u \in \mathcal{A}$ satisfies

$$\| u(\cdot, t) \, |L_2(\mathbb{R}^n)\| \le \| u \, |\mathcal{A}\|.$$

Moreover, the partial Fourier transform (with respect to the x-variable) of such a function belongs to L_1 (if either $m > 0$ or $m = 0$ and $\ell = 0$) and

$$\int |\mathcal{F} u(\xi, t)| \, d\xi \le \| u \, |\mathcal{A}\| \, \left(\int \langle \xi \rangle_m^{-2\ell} \, \exp\left(- \beta \, \langle \xi \rangle_m^{1/s} - h(\xi, t) \right) d\xi \right)^{1/2}.$$

Consequently, for fixed t, we can use the formula

$$\mathcal{F}\,(u * v)(\xi, t) = (2\pi)^{n/2} \, \mathcal{F} u(\xi, t) \, \mathcal{F} v(\xi, t). \tag{2.28}$$

The role of the parameter β' is to control that $\mathcal{A}(s, \beta, \beta', m, \ell, h, T)$ is not too far from the classes A_s treated before.

Lemma 2.3. (i) *The space $\mathcal{A}$ becomes a Banach space with respect to the norm $\| \cdot \, |\mathcal{A}\|$.*

(ii) *The elements of* $\mathcal{A}$ *are* C^∞*-functions with respect to* x *and satisfy*

$$\sup_{0<t<T} \| D_x^\alpha u(\cdot, t) \,|L_\infty\| \le E \,\| u \,|\mathcal{A}\|,$$

for all $u \in \mathcal{A}$ *and all* $\alpha \in \mathbb{N}_0^n$*. Here* E *does not depend on* u *and* α*.*

Proof. One may follow the proof of Lemma 2.1. □

Again crucial are good estimates for products in $\mathcal{A}$. We need a further abbreviation. We put

$$\| u \,|\mathcal{A}(t)\| = \left(\int_{\mathbb{R}^n} \langle \xi \rangle_m^{2\ell} \exp\big(\beta \langle \xi \rangle_m^{1/s} + h(\xi, t)\big) \, |\mathcal{F} u(\xi, t)|^2 \, d\xi \right)^{1/2}.$$

Observe that $\mathcal{A}(t)$ is a Banach space for each fixed t.

Theorem 2.7. *Let* $\delta_{1/s} = 2 - 2^{1/s}$*. Let either* $m > 0$ *or* $m = \ell = 0$*. Suppose*

$$h(\xi, t) - h(\xi - \eta, t) - h(\eta, t) \le \gamma \,\delta_{1/s} \,\min(\langle \xi - \eta \rangle_m, \langle \eta \rangle_m)^{1/s}, \tag{2.29}$$

for all ξ *and* η *and some* $0 \le \gamma < \beta$*. Then* $\mathcal{A}$ *becomes an algebra with respect to pointwise multiplication. It holds that*

$$\| u\, v \,|\mathcal{A}\| \le F \,\| u \,|\mathcal{A}\| \,\| v \,|\mathcal{A}\|, \tag{2.30}$$

with some constant F*. The estimate* (2.30) *remains true if we replace* $\mathcal{A}$ *by* $\mathcal{A}(t)$*.*

Proof. We follow the arguments used in the proof of Theorem 2.1. Let

$$k(\xi, t) = \langle \xi \rangle_m^{2\ell} \exp\big(\beta \langle \xi \rangle_m^{1/s} + h(\xi, t)\big). \tag{2.31}$$

Then, with C depending on m and ℓ, we find, by using (2.29), the inequality

$$\frac{k(\xi, t)}{k(\xi - \eta, t)\, k(\eta, t)} \le C\, e^{-(\beta - \gamma)\,\delta_{1/s}\, \min(\langle \xi - \eta \rangle_m, \langle \eta \rangle_m)^{1/s}},$$

cf. Lemma 4.2 from the Appendix. For u and v in $\mathcal{A}$ we can apply (2.28). This yields

$$\begin{aligned} \| u\, v \,|\mathcal{A}\| \le & \sup_{0<t<T} \left(\int \left| \int_{|\eta - \xi| \le |\eta|} \mathcal{F} u(\xi - \eta, t)\, \mathcal{F} v(\eta, t)\, d\eta \right|^2 k(\xi, t)\, d\xi \right)^{1/2} \\ & + \sup_{0<t<T} \left(\int \left| \int_{|\eta - \xi| > |\eta|} \mathcal{F} u(\xi - \eta, t)\, \mathcal{F} v(\eta, t)\, d\eta \right|^2 k(\xi, t)\, d\xi \right)^{1/2} \\ = & \sup_{0<t<T} I_1(t) + \sup_{0<t<T} I_2(t)). \end{aligned}$$

Then

$$I_1(t) \le \Big\| \int |\mathcal{F} u(\xi-\eta,t)\,\sqrt{k(\xi-\eta,t)}|\,|\mathcal{F} v(\eta,t)\,\sqrt{k(\eta,t)}| \frac{\sqrt{k(\xi,t)}}{\sqrt{k(\xi-\eta,t)}\,\sqrt{k(\eta,t)}}\,d\eta \,\Big|L_2(\mathbb{R}^n,\xi)\Big\|$$
$$\le C\,\|\,v\,|\mathcal{A}(t)\|\int |\mathcal{F} u(\zeta,t)|\sqrt{k(\zeta,t)\,e^{-(\beta-\gamma)\,\delta_{1/s}\,\langle\,\zeta\,\rangle_m^{1/s}}}\,d\zeta$$
$$\le \|\,u\,|\mathcal{A}(t)\|\,\|\,v\,|\mathcal{A}(t)\|\left(\int e^{-(\beta-\gamma)\,\delta_{1/s}\,\langle\,\zeta\,\rangle_m^{1/s}}\,d\zeta\right)^{1/2}.$$

Similarly we can proceed in the case of I_2 and obtain

$$I_2(t) \le C\,\|\,u\,|\mathcal{A}(t)\|\,\|\,v\,|\mathcal{A}(t)\|\left(\int e^{-(\beta-\gamma)\,\delta_{1/s}\,\langle\,\zeta\,\rangle_m^{1/s}}\,d\zeta\right)^{1/2}.$$

This proves (2.30). □

Remark 2.16. We give a more handsome sufficient condition for h to satisfy (2.29). We suppose $|h(\xi,t)| \le \beta'\,\langle\,\xi\,\rangle_m^{1/s}$ for all ξ and all t (see (2.1)), and

$$|h(\xi,t)-h(\eta,t)| \le \beta'\,\langle\,\xi-\eta\,\rangle_m^{1/s}. \tag{2.32}$$

Then h satisfies (2.29) if

$$2\,\beta' < \beta\,\delta_{1/s}. \tag{2.33}$$

First, let $\min(\langle\,\xi-\eta\,\rangle_m, \langle\,\eta\,\rangle_m) = \langle\,\xi-\eta\,\rangle_m$. We find

$$\begin{aligned}|h(\xi-\eta,t)+h(\eta,t)-h(\xi,t)| &\le |h(\xi-\eta,t)| + |h(\eta,t)-h(\xi,t)|\\ &\le \beta'\,\langle\,\xi-\eta\,\rangle_m^{1/s} + \beta'\,\langle\,\xi-\eta\,\rangle_m^{1/s}.\end{aligned}$$

Second, consider $\min(\langle\,\xi-\eta\,\rangle_m, \langle\,\eta\,\rangle_m) = \langle\,\eta\,\rangle_m$. As above

$$\begin{aligned}|h(\xi-\eta,t)+h(\eta,t)-h(\xi,t)| &\le |h(\eta,t)| + |h(\xi-\eta,t)-h(\xi,t)|\\ &\le \beta'\,\langle\,\eta\,\rangle_m^{1/s} + \beta'\,\langle\,\eta\,\rangle_m^{1/s}.\end{aligned}$$

Again we are interested in certain subalgebras of $\mathcal{A}$. Let P_R and $P_R(\varepsilon)$ be defined as in Subsection 2.1.

Theorem 2.8. *Let $s > 1$. Let either $m > 0$ or $m = \ell = 0$. Further, let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_j \in \{0,1\}$, $j = 1, \ldots, n$, be fixed and suppose $R > 0$. Suppose h satisfies* (2.29). *Then*

$$\mathcal{A}(\varepsilon,R) = \{u \in \mathcal{A} : \quad supp\,\mathcal{F} u \subset P_R(\varepsilon)\}$$

are subalgebras of $\mathcal{A}$ and

$$\|\,u\,v\,|\mathcal{A}\| \le K\,\|\,u\,|\mathcal{A}\|\,\|\,v\,|\mathcal{A}\| \tag{2.34}$$

holds for all $u,\ v \in \mathcal{A}(\varepsilon, R)$*, where*

$$K \le C \left(\omega_n \, ((\beta-\gamma)\,\delta_s)^{-sn} \int_{(\beta-\gamma)\,\delta_{1/s}\,R^{1/s}} e^{-y}\, y^{sn-1}\, dy \right)^{1/2},$$

and C *does not depend on* $u,\ v$ *and* R*. Again the estimate* (2.34) *remains true if we replace* $\mathcal{A}$ *by* $\mathcal{A}(t)$*.*

Proof. There is no difference between the proof of Theorem 2.7 and Theorem 2.8. □

2.9. The mapping $u \mapsto e^{iu} - 1$ *in spaces with exponential weights. II*

We continue with the investigation of $u \mapsto e^{iu}$ for $u \in \mathcal{A}^{\mathbb{R}}$.

2.9.1. Extra conditions on the spectrum – Part I. We suppose $supp\, \mathcal{F} u(\cdot, t) \subset P_R(\varepsilon)$ for fixed t and for some fixed $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_j \in \{0, 1\}$, $j = 1, 2, \dots, n$. Under this assumption Theorem 2.8 implies

$$\| e^{iu} - 1 \,|\mathcal{A}(t)\| = \left\| \sum_{\ell=1}^{\infty} \frac{(iu)^\ell}{\ell!} \middle| \mathcal{A}(t) \right\| \le \sum_{\ell=1}^{\infty} \frac{\| u \,|\mathcal{A}(t)\|^\ell \, K^{\ell-1}}{\ell!} \le \frac{1}{K} \left(e^{K\, \| u \,|\mathcal{A}(t)\|} - 1 \right).$$

2.9.2. Extra conditions on the spectrum – Part II. Next we suppose $supp\, \mathcal{F} u(\cdot, t) \subset P_R$ for fixed t and use the following splitting:

$$e^{iu(x,t)} - 1 = \sum_{\ell=1}^{r} \frac{(iu(x,t))^\ell}{\ell!} + \sum_{\ell=r+1}^{\infty} \frac{(iu(x,t))^\ell}{\ell!} = g_1(x,t) + g_2(x,t),$$

cf. Subsection 2.3.2. Then it holds that

$$supp\, \mathcal{F} g_1(\cdot, t) \subset [-Rr,\ Rr]^n.$$

Again we apply the abbreviation (2.31). With

$$T_1^2(t) = \sum_{\varepsilon \in I} \int_{P_{Rr}(\varepsilon)} \left| \sum_{\ell=r+1}^{\infty} \frac{i^\ell \, (\mathcal{F} u^\ell(\cdot, t))(\eta)}{\ell!} \right|^2 k(\eta, t)\, d\eta$$

and

$$T_2^2(t) = \int_{[-Rr, Rr]^n} |\, \mathcal{F}(e^{iu(x,t)} - 1)(\eta)\, |^2 \, k(\eta, t)\, d\eta,$$

we obtain the decomposition

$$\| e^{iu} - 1 \,|\mathcal{A}(t)\| = \left(T_1^2(t) + T_2^2(t) \right)^{1/2}. \tag{2.35}$$

The estimate of $T_2(t)$. Again we may use the argument that the composition operator associated with the function $h(y) = e^{iy} - 1$ and restricted to real-valued functions maps L_2 into L_2, and we obtain

$$\begin{aligned} T_2^2(t) &\le \sup_{\eta\in[-Rr,Rr]^n} k(\eta,t)\, \big\| \, e^{iu(\cdot,t)} - 1 \, \big| L_2(\mathbb{R}^n, x) \big\|^2 \\ &\le \sup_{\eta\in[-Rr,Rr]^n} k(\eta,t)\, \| \, u(\cdot,t)\, | L_2 \|^2 \\ &\le \sup_{\eta\in[-Rr,Rr]^n} k(\eta,t)\, \| \, u\, | \mathcal{A}(t) \|^2. \end{aligned} \tag{2.36}$$

Observe

$$\sup_{\eta\in[-Rr,Rr]^n} k(\eta,t) \le (m^2 + n\,(Rr)^2)^\ell \, e^{\beta\,(\sqrt{n}Rr)^{1/s}}.$$

The estimate of $T_1(t)$. To estimate T_1 we apply Theorem 2.7. This leads to

$$\begin{aligned} T_1(t) &\le \sum_{\ell=r+1}^{\infty} \frac{1}{\ell!} \, \| \, u^\ell(\cdot,t)\, | \mathcal{A}(t) \| \le \sum_{\ell=r+1}^{\infty} \frac{F^{\ell-1}}{\ell!} \, \| \, u(\cdot,t)\, | \mathcal{A}(t) \|^\ell \\ &\le \frac{1}{F} \sum_{\ell=r+1}^{\infty} \frac{(F \, \| \, u(\cdot,t)\, | \mathcal{A}(t) \|)^\ell}{\ell!}, \end{aligned} \tag{2.37}$$

where F is the constant from (2.30). Next we choose r as a function of $\|u(\cdot,t)|\mathcal{A}(t)\|$.

Step 1. We assume $F \, \| \, u(\cdot,t)\, | \mathcal{A}(t) \| > 1$. Suppose

$$3\,F\, \| \, u\, | \mathcal{A}(t) \| \le r \le 3\,F\, \| \, u\, | \mathcal{A}(t) \| + 1. \tag{2.38}$$

Stirling's formula yields

$$\sum_{\ell=r+1}^{\infty} \frac{(F \, \| \, u(\cdot,t)\, | \mathcal{A}(t) \|)^\ell}{\ell!} \le \frac{3}{3-e}.$$

Inserting this in our previous estimate we find

$$T_1(t) \le c, \tag{2.39}$$

as long as r is chosen as in (2.38).

Step 2. We assume $F \, \| \, u(\cdot,t)\, | \mathcal{A}(t) \| \le 1$. Then we may choose $r = 0$ (which means $T_2 = 0$) ending up with

$$T_1(t) \le \sum_{\ell=r+1}^{\infty} \frac{(F \, \| \, u(\cdot,t)\, | \mathcal{A}(t) \|)^\ell}{\ell!} \le e\,M\, \| \, u(\cdot,t)\, | \mathcal{A}(t) \|. \tag{2.40}$$

Summarizing we have proved

$$\| \, e^{iu(\cdot,t)} - 1 \, | \mathcal{A}(t) \| \le c \, \| \, u(\cdot,t)\, | \mathcal{A}(t) \| \left(1 + e^{bR^{1/s} \|u(\cdot,t)|\mathcal{A}(t)\|^{1/s}}\right)^{1/2},$$

where c and b are positive numbers independent of t, f, r and R, cf. (2.35)–(2.40).

2.10. Estimates of $e^{iu} - 1$ for general u. II

There is no difficulty in applying the same procedure for fixed t as in Subsection 2.4. Similarly to there we obtain

$$\| e^{iu(\cdot,t)} - 1 \,|\mathcal{A}(t)\| \\ \le \sum_{\ell=1}^{2^n+1} \sum_{0\le j_1<\ldots<j_\ell\le 2^n} F^{\ell-1} \, \| e^{iu_{j_1}(\cdot,t)} - 1 \,|\mathcal{A}(t)\| \ldots \| e^{iu_{j_\ell}(\cdot,t)} - 1 \,|\mathcal{A}(t)\|,$$

and

$$\| e^{iu_{j_r}(\cdot,t)} - 1 \,|\mathcal{A}(t)\| \le \frac{1}{K} \left(e^{K \,\| u \,|\mathcal{A}(t)\|} - 1 \right),$$

for any admissible choice of j_r. Further

$$\| e^{iu_0(\cdot,t)} - 1 \,|\mathcal{A}(t)\| \le c \, \| u(\cdot, t) \,|\mathcal{A}(t)\| \left(1 + e^{bR^{1/s} \|u(\cdot,t)|\mathcal{A}(t)\|^{1/s}} \right),$$

for appropriate $c, b > 0$. Keeping t fixed we may proceed in choosing R as before, see Subsection 2.4. All together we got the following:

Proposition 2.1. *Let either $m > 0$ or $m = \ell = 0$. Suppose $s > 1$ and suppose h satisfies* (2.29). *Then there exist constants c and a (depending on n and s only) such that*

$$\| e^{iu(\cdot,t)} - 1|\mathcal{A}(t)\| \le c \begin{cases} e^{a \,\| u(\cdot,t) \,|\mathcal{A}(t)\|^{1/s} \, \log \| u(\cdot,t) \,|\mathcal{A}(t)\|} & \text{if} \quad \| u(\cdot, t) \,|\mathcal{A}(t)\| > 1\,; \\ \| u \,|\mathcal{A}(t)\| & \text{if} \quad \| u(\cdot, t) \,|\mathcal{A}(t)\| \le 1\,; \end{cases}$$

holds for all $u \in \mathcal{A}^{\mathbb{R}}$ and all $0 < t < T$.

2.11. Composition operators on spaces with exponential weights. II

As long as (2.29) is satisfied and t is fixed, all assertions of Subsections 2.5–2.7 have immediate counterparts.

Theorem 2.9. *Let either $m > 0$ or $m = \ell = 0$ and let $s > 1$. Let h satisfy* (2.29). *Suppose μ is a complex measure such that $\mu(\mathbb{R}) = 0$ and such that* (2.22) *is satisfied. Define f as the inverse Fourier transform of μ. Then the composition operator T_f maps $\mathcal{A}^{\mathbb{R}}$ into $\mathcal{A}$.*

Proof. By assumption,

$$f(y) = f(y) - f(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[e^{i\xi y} - 1 \right] d\mu(\xi).$$

An application of Proposition 2.1 and the Minkowski inequality yield

$$\| T_f(u(x,t)) \,|\mathcal{A}(t)\| = \left\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [e^{i\xi u(x,t)} - 1]\, d\mu(\xi) \,\middle|\mathcal{A}(t)\right\|$$
$$\leq 4 \int_{-\infty}^{\infty} \| e^{i\xi u(\cdot,t)} - 1 \,|\mathcal{A}(t)\| \, d|\mu|(\xi)$$
$$\leq c \int \exp\Big(a \,\| \xi\, u(\cdot,t) \,|\mathcal{A}(t)\|^{1/s} \log(1 + \| \xi\, u(\cdot,t) \,|\mathcal{A}(t)\|)\Big)\, d|\mu|(\xi),$$

which is sufficient for our purpose. □

Again we formulate a few consequences separately.

Corollary 2.3. *Let either $m > 0$ or $m = \ell = 0$, and suppose $s > 1$. Let h satisfy* (2.29). *Suppose $g : \mathbb{R} \to \mathbb{C}$ is a function satisfying $\int g(\xi)\, d\xi = 0$ and* (2.23). *Then the composition operator T_f maps $\mathcal{A}^{\mathbb{R}}$ into $\mathcal{A}$.*

Similarly one can proceed in the periodic situation.

Corollary 2.4. *Let either $m > 0$ or $m = \ell = 0$, and suppose $s > 1$. Let h satisfy* (2.29). *Let $\{c_k\}_{k\in\mathbb{Z}}$ be a sequence of complex numbers satisfying* (2.24) *and $\sum_{k=-\infty}^{\infty} c_k = 0$. Define $f(t) = \sum_{k\in\mathbb{Z}} c_k\, e^{ikt}$. Then the composition operator T_f maps $\mathcal{A}^{\mathbb{R}}$ into $\mathcal{A}$.*

For further use we prove the next result:

Corollary 2.5. *Under the conditions of Theorem* 2.9 *the composition operator T_f maps balls of radius R in $\mathcal{A}^{\mathbb{R}}$ into balls of radius R'.*

Proof. In the proof of Theorem 2.9 only $\| u \,|\mathcal{A}\|$ influences the estimate. The resulting condition depends monotonically on this norm. □

2.12. Nemytskij operators on spaces with exponential weights. III

The same method as in the preceding subsection works. Let $d \geq 2$ be a natural number. Let $f : \mathbb{R}^d \mapsto \mathbb{C}$ be a continuous function.

Theorem 2.10. *Let either $m > 0$ or $m = \ell = 0$, and suppose $s > 1$. Let h satisfy* (2.29). *Let*

$$H(\xi_1, \xi_2, \ldots, \xi_d) = \exp\Big(\sum_{k=1}^{d} |\xi_k|^{1/s} \log(1 + |\xi_k|)\Big).$$

Suppose that μ is a complex measure such that $\mu(\mathbb{R}^d) = 0$ and (2.25) *is satisfied. Then the Nemytskij operator T_f maps $(\mathcal{A}^{\mathbb{R}})^d$ into $\mathcal{A}$.*

Similarly as above we can derive the following fact:

Corollary 2.6. *Under the conditions of Theorem* 2.10 *the Nemytskij operator T_f maps a ball in $(\mathcal{A}^{\mathbb{R}})^d$ with radius R into a ball in $\mathcal{A}$ with radius R'.*

2.13. Nemytskij operators on spaces with exponential weights. IV

Let d be a natural number. Let $f : \mathbb{R}^{d+n} \mapsto \mathbb{C}$ be a continuous function.

Theorem 2.11. *Let either $m > 0$ or $m = \ell = 0$, and suppose $s > 1$. Let h satisfy* (2.29). *Let*

$$H(\xi_1, \xi_2, \ldots, \xi_{d+n}) = e^{|(\xi_1,\ldots,\xi_n)|^{1/s}/2} \exp\Big(\sum_{k=1}^{d} |\xi_k|^{1/s} \log(1 + |\xi_k|)\Big).$$

The inverse Fourier transform f of the complex measure μ should satisfy $f(x_1, \ldots, x_n, 0, \ldots, 0) \in \mathcal{A}$ and suppose that (2.27) *holds. Then the Nemytskij operator T_f, defined in* (2.26) *maps $(\mathcal{A}^{\mathbb{R}})^d$ into $\mathcal{A}$.*

For later use we add the following:

Corollary 2.7. *Under the conditions of Theorem* 2.11 *the Nemytskij operator T_f maps balls with radius R in $(\mathcal{A}^{\mathbb{R}})^d$ into balls in $\mathcal{A}$ with radius R'.*

As described in the introduction we need at least local Lipschitz continuity of the Nemytskij operator for our applications. It seems that such an argument to derive this property as used in case of the mapping $u \mapsto e^{iu} - 1$, see Theorem 2.3, is not available in the general situation. For that reason we have to strengthen the conditions with respect to f. Let $f(x) = f(x_1, \ldots, x_{n+d})$. To have a compact notation let $x' = (x_1, \ldots, x_{n+d-1})$. Recall the classical identity

$$\begin{aligned}
&f(x', y) - f(x', z) = \\
&\quad (y - z) \int_0^1 \Big(\frac{\partial f}{\partial x_{n+d}}(x', y + \Theta(z - y)) - \frac{\partial f}{\partial x_{n+d}}(x_1, \ldots, x_n, 0, \ldots, 0)\Big)\, d\Theta \\
&\quad + \frac{\partial f}{\partial x_{n+d}}(x_1, \ldots, x_n, 0, \ldots, 0)\, (y - z),
\end{aligned}$$

and of course its obvious counterparts for the other variables. Further, let $M(\mathcal{A})$ denote the set of all pointwise multipliers of $\mathcal{A}$. We equip this space with the natural norm; because of the algebra property we know $\mathcal{A} \hookrightarrow M(\mathcal{A})$. Again as a consequence of the Minkowski inequality and the algebra property of $\mathcal{A}$ we can derive the local Lipschitz continuity of T_f by making assumptions on the first-order derivatives of f.

Corollary 2.8. *Suppose that for $j = n + 1, \ldots, n + d$ the first-order derivatives $\frac{\partial f}{\partial x_j}$ of f satisfy the conditions of Theorem* 2.11 *except $\frac{\partial f}{\partial x_j}(x, 0, \ldots, 0) \in \mathcal{A}$. This condition is replaced by*

$$\frac{\partial f}{\partial x_j}(x, 0, \ldots, 0) \in M(\mathcal{A}). \tag{2.41}$$

Then the Nemytskij operator T_f is locally Lipschitz continuous, in particular there exists a non-decreasing function ψ such that

$$\| T_f(u_1, \dots ,u_d) - T_f(v_1, \dots , v_d) \,|\mathcal{A}\| \\ \leq \max_{j=1,\dots,d} \| u_j - v_j \,|\mathcal{A}\| \, \psi(\max_{j=1,\dots,n} \| u_j \,|\mathcal{A}\| + \| v_j \,|\mathcal{A}\|)$$

holds for all $u_1, \dots , u_d, v_1, \dots , v_d \in \mathcal{A}^{\mathbb{R}}$.

Remark 2.17. Observe that in the most simple situation $f = f(u)$, the condition (2.41) becomes superfluous because constants are always in $M(\mathcal{A})$.

2.14. A few comments on the sharpness of the results obtained in 2.5

Here we shall discuss the sharpness of our sufficient conditions in Theorem 2.4.

Proposition 2.2. *Let $s > 1$. Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that T_f maps $A_s^{\mathbb{R}}(\mathbb{R}^n)$ into $A_s(\mathbb{R}^n)$. Then f belongs to the Gevrey class $\mathcal{G}^s$. In particular, if f is compactly supported, then $\mathcal{F} f(\xi) = O(e^{-\alpha|\xi|^{1/s}})$ for some $\alpha > 0$.*

Proof. Let $Q = [a, b] \times [-1, 1]^{n-1}$. According to the known properties of Gevrey classes, cf. e.g. [19, Example 1.4.9], there exists a function $\varphi \in A_s(\mathbb{R}^n)$ such that $\varphi(x) = x_1$ on Q. By assumption we have $f \circ \varphi \in A_s(\mathbb{R}^n) \subset \mathcal{G}^s(\mathbb{R}^n)$, cf. Lemma 2.2. Since $f(t) = (f \circ \varphi)(t, x_2, \dots , x_n)$ on Q, we see at once that $f \in \mathcal{G}^s(\mathbb{R})$. The behaviour of the Fourier transform of f follows from [19, 1.6.1], cf. Remark 2.9. □

Remark 2.18. Let $s > 1$. Let $f : \mathbb{R} \to \mathbb{C}$ be a function such that T_f maps $A_s^{\mathbb{R}}(\mathbb{R}^n)$ into $A_s(\mathbb{R}^n)$. Then f need not be in $A_s(\mathbb{R})$, cf. Corollary 2.2.

Let E_s (resp. $E_{s,0}$) be the set of functions (resp. compactly supported functions) which act on $A_s^{\mathbb{R}}$. Now consider $F_s := \bigcup_{s'<s} \mathcal{G}_0^{s'}(\mathbb{R})$.

Lemma 2.4. *Let $s > 1$.*

(i) *The inclusions*

$$F_s = \bigcup_{s'<s} \mathcal{G}_0^{s'} \subset E_{s,0} \subset \mathcal{G}_0^s.$$

hold.

(ii) *We have the equivalence*

$$f \in E_s \quad \Leftrightarrow \quad f\,\varphi \in E_{s,0}, \quad \textit{for all} \quad \varphi \in F_s.$$

Proof. By Remark 2.7 and Theorem 2.4, we know that $F_s \subset E_{s,0}$. The second inclusion has been proved in Proposition 2.2. Further, F_s contains cut-off functions (since $\mathcal{G}^{s-\varepsilon}$ contains those functions). Since $A_s \subset L^\infty$ this shows that for each $u \in A_s^{\mathbb{R}}$ there exists a cut-off function φ such that $f \circ u = (f\,\varphi) \circ u$. This is enough to establish (ii). □

Remark 2.19. Let $s > 1$. Let $g : \mathbb{R} \mapsto \mathbb{C}$ be a bounded function such that

$$g(\xi) = e^{-|\xi|^{1/s} \log|\xi| \log\log|\xi|},$$

for large ξ and $\int_{\mathbb{R}} g(\xi)\, d\xi = 0$. Let f be the inverse Fourier transform of g. Then the composition operator T_f maps $A_s^{\mathbb{R}}$ into A_s, cf. Corollary 2.1. Observe that $f \in \mathcal{G}^s$ but $f \notin \mathcal{G}^{s'}$ for any $s' < s$. Further, if we compare this with the necessary condition stated in Proposition 2.2 there remains a gap of logarithmic order only.

3. Applications

Here we follow our philosophy explained in the introduction. We shall derive the a priori estimate (1.8) in certain well-adapted classes of functions. As a second step we shall investigate how these classes fit into our theory developed in Section 2. To avoid confusion we shall denote the spaces induced by the linear problem with $\mathcal{B}(\alpha, \beta, \gamma, \dots, \omega)$, where $\alpha, \beta, \dots$ are certain parameters.

3.1. The Gevrey example

Let us consider

$$u_{tt} - u_x = f(u), \quad u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x). \tag{3.1}$$

This is a model problem for weakly hyperbolic operators with characteristics of constant multiplicity. It remains to study

$$u_{tt} - u_x = f(x,t), \quad u(x,0) = u_t(x,0) \equiv 0. \tag{3.2}$$

But this Cauchy problem is equivalent to

$$v_{tt} - i\xi v = \mathcal{F} f, \quad v(\xi,0) = v_t(\xi,0) \equiv 0, \quad v = \mathcal{F} u. \tag{3.3}$$

We can represent the solution of (3.3) in the form

$$v(\xi,t) = \int_0^t X(t,s,\xi)\mathcal{F} f(\xi,s)ds,$$

where the kernel function $X = X(t,s,\xi)$ satisfies, for $0 \le s \le t$, $0 \le t \le T$, the estimate

$$|X(t,s,\xi)| \le (t-s)\exp\big(\langle\xi\rangle_1^{1/2}(t-s)\big).$$

Choosing

$$\mathcal{N}_{\ell,T}(\xi,t) = \exp\big(\langle\xi\rangle_1^{1/2}(2T-t)\big)\langle\xi\rangle_1^{\ell}$$

and defining $\mathcal{B}(\ell, T)$ as the space of all functions satisfying

$$\|f\|_{\ell,T} := \sup_{0<t<T} \left(\int_{-\infty}^{\infty} \mathcal{N}^2_{\ell,T}(\xi, t) |\mathcal{F} f(\xi, t)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty,$$

straightforward calculations lead to

$$\|u\|_{\ell,T} \le T^2 \, \|f\|_{\ell,T}.$$

Using

$$\partial_t \mathcal{F} u(\xi) = \int_0^t \sum_{k=0}^{\infty} \frac{(\chi(\xi)(t-s))^{2k}}{(2k)!} \, \mathcal{F} f(\xi, s) \, ds,$$

$$\chi(\xi) \mathcal{F} u(\xi) = \int_0^t \sum_{k=0}^{\infty} \frac{(\chi(\xi)(t-s))^{2k+1}}{(2k+1)!} \, \mathcal{F} f(\xi, s) \, ds,$$

then the same estimates as above lead to the next strictly hyperbolic-type estimate for the solution of (3.2):

$$\|u\|_{\ell,T} + \|\langle D_x \rangle^{\frac{1}{2}} u\|_{\ell,T} + \|D_t u\|_{\ell,T} \le C \, T^{1/2} \, \|f\|_{\ell,T}. \tag{3.4}$$

There is no difficulty in identifying $\mathcal{B}(\ell, T)$ as $\mathcal{A}(2, 4T, 2T, 1, \ell, h, T)$, where $h(\xi, t) = -2t \, \langle \xi \rangle_1^{1/2}$. The function h satisfies (2.29). This can be checked directly through Lemma 4.2 from the Appendix.

Using Corollaries 2.3, 2.7 and 2.8 we arrive at the following existence and uniqueness result for (3.1) with life span $[0, T_0]$, where T_0 appears due to the non-linear structure of the problem.

Theorem 3.1. *Let us consider*

$$u_{tt} - u_x = f(u), \; u(x, 0) = \varphi(x), \; u_t(x, 0) = \psi(x),$$

under the following assumptions:

- *the data* φ, ψ *satisfy*

$$\int_{-\infty}^{\infty} \exp\left(T_1 \, \langle \xi \rangle_1^{1/2}\right) \langle \xi \rangle_1^{\ell} \left(|\mathcal{F} \varphi(\xi)|^2 + |\mathcal{F} \psi(\xi)|^2 \right) d\xi < \infty$$

  for some $T_1 > 0$*;*
- $f : \mathbb{R} \to \mathbb{C}$ *is an infinitely differentiable function and* $f(0) = 0$*;*
- *the Fourier transform* $\mathcal{F} f$ *of* f *is an integrable function and there exists a non-negative constant* c *such that*

$$|\mathcal{F} f(\xi)| \le c \, e^{-|\xi|^{1/2} (\log |\xi|) \, \log\log |\xi|}, \qquad \text{for large } \; |\xi|.$$

Then the Cauchy problem has a unique solution $u \in \mathcal{B}(\ell, T_0)$ *for* T_0 *sufficiently small. The derivatives (see* (3.4)*)* $\langle D_x \rangle^{\frac{1}{2}} u$ *and* $D_t u$ *belong to* $\mathcal{B}(\ell, T_0)$*, too.*

3.2. Weakly hyperbolic equations with Gevrey-type Levi conditions

Our weakly hyperbolic model problem with characteristics of variable multiplicity is

$$u_{tt} - \lambda(t)^2 u_{xx} - b(t)u_x = f(u), \quad u(x,0) = \varphi(x), \; u_t(x,0) = \psi(x). \tag{3.5}$$

As in Section 3.1 it remains to derive an a priori estimate of type (1.8) for

$$u_{tt} - \lambda(t)^2 u_{xx} - b(t)u_x = f(x,t), \quad u(x,0) = u_t(x,0) \equiv 0. \tag{3.6}$$

We follow ideas from Ishida and Yagdjian [10]. Levi conditions of Gevrey-type allow us to prove well-posedness in Gevrey spaces of order $s > 2$.

Let us formulate the assumptions for the coefficients $\lambda = \lambda(t)$ and $b = b(t)$:

$(A1)$ $\lambda(0) = \lambda'(0) = 0, \; \lambda'(t) > 0, \; \lambda \in C^1[0,T]$;

$(A2)$ $c_0 \dfrac{\lambda(t)}{\Lambda(t)} \le \dfrac{\lambda'(t)}{\lambda(t)} \le c_1 \dfrac{\lambda(t)}{\Lambda(t)^{\frac{s}{s-1}}}, \; c_0 > \dfrac{s}{2s-2}, \; c_1 \ge c_0, \; \Lambda(t) = \displaystyle\int_0^t \lambda(\tau)d\tau$;

$(A3)$ $|b(t)| \le C \dfrac{\lambda(t)^2}{\Lambda(t)^{\frac{s}{s-1}}}, \quad b \in C[0,T]$.

After partial Fourier transformation we get from (3.6)

$$v_{tt} + \lambda(t)^2\xi^2 v - ib(t)\xi\, v = \mathcal{F} f, \quad v(\xi,0) = v_t(\xi,0) \equiv 0. \tag{3.7}$$

We divide $\{(\xi,t) \in (\mathbb{R}\setminus\{0\}) \times (0,T]\}$ into two zones using the relation $\Lambda(t_\xi)^{\frac{s}{s-1}}\langle\xi\rangle_m = N, \; \langle\xi\rangle_m := (\xi^2 + m^2)^{\frac{1}{2}}$. The relation defines in an implicit way a function $t_\xi = t(\langle\xi\rangle_m)$. To a fixed chosen N and given T we determine m such that $T = t(\langle 0\rangle_m)$.

We define the so-called *pseudodifferential zone* by

$$Z_{\text{pd}} := \Big\{(\xi,t) \in (\mathbb{R}\setminus\{0\}) \times (0,T] : \Lambda(t)^{\frac{s}{s-1}}\langle\xi\rangle_m \le N\Big\},$$

and the so-called *hyperbolic zone* by

$$Z_{\text{hyp}} := \Big\{(\xi,t) \in (\mathbb{R}\setminus\{0\}) \times (0,T] : \Lambda(t)^{\frac{s}{s-1}}\langle\xi\rangle_m \ge N\Big\}.$$

3.2.1. Representations of the solutions. In Z_{pd} we use the auxiliary function $\rho^2(\xi,t) = 1 + \langle\xi\rangle_m \frac{\lambda(t)^2}{\Lambda(t)^{\frac{s}{s-1}}}$. Setting $W = (\rho(t,\xi)v, \partial_t v)^T$ we get from (3.7)

$$\partial_t W - A\,W = F, \quad W(\xi,0) = 0,$$

where

$$A = \begin{pmatrix} \frac{\partial_t\rho}{\rho} & \rho \\ \frac{ib(t)\xi - \lambda(t)^2\xi^2}{\rho} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ \mathcal{F} f \end{pmatrix}.$$

Introducing the fundamental matrix $Z = Z(t, \sigma, \xi)$ as the solution of

$$\partial_t Z - A\, Z = 0,\ \ Z(\sigma, \sigma, \xi) = I, \quad \text{for} \quad 0 \le \sigma \le t \le t_\xi,$$

we obtain the representation

$$W(\xi, t) = \int_0^t Z(t, \sigma, \xi) F(\xi, \sigma) d\sigma, \quad \text{for} \quad 0 \le t \le t_\xi. \tag{3.8}$$

In Z_{hyp} we set $U = (\lambda(t)\xi\ v,\ \ D_t v)^T$ and get from (3.7)

$$D_t U - \begin{pmatrix} \frac{D_t\lambda}{\lambda} & \lambda(t)\xi \\ \lambda(t)\xi - \frac{ib(t)}{\lambda(t)} & 0 \end{pmatrix} U = \begin{pmatrix} 0 \\ -\mathcal{F} f \end{pmatrix}.$$

With the diagonalizer M and the transformation $V := MU$, where

$$M := \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad A := \begin{pmatrix} \frac{D_t\lambda}{\lambda} & \lambda(t)\xi \\ \lambda(t)\xi - \frac{ib(t)}{\lambda(t)} & 0 \end{pmatrix}$$

the last system of the first order can be transferred to

$$D_t V = M\, D_t\, U = MAU + M\begin{pmatrix} 0 \\ -\mathcal{F} f \end{pmatrix} = MAM^{-1}V + M\begin{pmatrix} 0 \\ -\mathcal{F} f \end{pmatrix}.$$

But

$$MAM^{-1} = \begin{pmatrix} -\lambda(t)\xi & 0 \\ 0 & \lambda(t)\xi \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{D_t\lambda}{\lambda} + i\frac{b}{\lambda} & \frac{D_t\lambda}{\lambda} + i\frac{b}{\lambda} \\ \frac{D_t\lambda}{\lambda} - i\frac{b}{\lambda} & \frac{D_t\lambda}{\lambda} - i\frac{b}{\lambda} \end{pmatrix}.$$

Consequently, we arrive at the system with diagonal main part

$$\partial_t V - D\, V - B\, V = F,$$

where

$$D := \begin{pmatrix} -i\lambda(t)\xi & 0 \\ 0 & i\lambda(t)\xi \end{pmatrix}, \ \ B := \frac{1}{2\lambda}\begin{pmatrix} \lambda' - b & \lambda' - b \\ \lambda' + b & \lambda' + b \end{pmatrix}, \ \ F := \frac{1}{2}\begin{pmatrix} \mathcal{F} f \\ -\mathcal{F} f \end{pmatrix}.$$

If $X = X(t, \sigma, \xi)$ denotes the fundamental matrix for this system (compare with $Z = Z(t, \sigma, \xi)$ from (3.8) in Z_{pd}), then the solution $U = U(\xi, t)$ can be represented in the form

$$U(\xi, t) = M^{-1}\int_{t_\xi}^t X(t, \sigma, \xi) F(\xi, \sigma) d\sigma + M^{-1} X(t, t_\xi, \xi) M\, U(\xi, t_\xi).$$

Finally,

$$M\, U(\xi, t_\xi) = M\begin{pmatrix} \frac{\lambda(t_\xi)\xi}{\rho(\xi, t_\xi)}\ \rho(\xi, t_\xi) v \\ \frac{1}{i}\ \partial_t v \end{pmatrix} = Y(\xi, t_\xi) W(\xi, t_\xi)$$

and the representation (3.8) leads to

$$U(\xi,t) = M^{-1}\int_{t_\xi}^{t} X(t,\sigma,\xi)F(\xi,\sigma)d\sigma$$
$$+ M^{-1}X(t,t_\xi,\xi)Y(\xi,t_\xi)\int_0^{t_\xi} Z(t_\xi,\sigma,\xi)F(\xi,\sigma)d\sigma, \tag{3.9}$$

for $t_\xi \le t \le T$, where

$$Y(\xi,t_\xi) := \frac{1}{2}\begin{pmatrix} \frac{\lambda(t_\xi)\xi}{\rho(\xi,t_\xi)} & -\frac{1}{i} \\ \frac{\lambda(t_\xi)\xi}{\rho(\xi,t_\xi)} & \frac{1}{i} \end{pmatrix}.$$

3.2.2. Energy estimates for the solution. We suppose the existence of L and K such that

$$(A4) \qquad \|Z(t,\sigma,\xi)\| \le C\exp(L(t,\sigma,\xi));$$
$$(A5) \qquad \|X(t,\sigma,\xi)\| \le C\exp(K(t,\sigma,\xi));$$

and

$$(A6) \qquad L(t,\sigma,\xi) + L(\sigma,\tau,\xi) = L(t,\tau,\xi),\ K(t,\sigma,\xi) + K(\sigma,\tau,\xi) = K(t,\tau,\xi) \ \text{ for } \tau \le \sigma \le t.$$

Let us define the weight

$$\mathcal{N}_{s,\beta,l}(\xi,t) = \begin{cases} \exp\left(\beta\langle\xi\rangle_m^{1/s}\right)\langle\xi\rangle_m^l \exp\left(L(t_\xi,t,\xi) + K(T,t_\xi,\xi)\right), & 0 \le t \le t_\xi; \\ \exp\left(\beta\langle\xi\rangle_m^{1/s}\right)\langle\xi\rangle_m^l \exp(K(T,t,\xi)), & t_\xi \le t \le T. \end{cases}$$

Taking account of $(A4)$, $(A6)$ and (3.8) gives

$$\begin{aligned} |W(\xi,t)|^2\mathcal{N}^2_{s,\beta,l}(\xi,t) &\le C(T)\int_0^t \|Z(t,\sigma,\xi)\|^2|F(\xi,\sigma)|^2\mathcal{N}^2_{s,\beta,l}(\xi,t)d\sigma \\ &\le C(T)\int_0^t \exp(2L(t,\sigma,\xi))|F(\xi,\sigma)|^2\mathcal{N}^2_{s,\beta,l}(\xi,t)d\sigma \\ &= C(T)\int_0^t |F(\xi,\sigma)|^2\mathcal{N}^2_{s,\beta,l}(\xi,\sigma)d\sigma. \end{aligned}$$

Integration over $\{\xi : |\xi| \le R_0(t)\}$, where $R_0(t)$ is defined by $\Lambda(t)^{\frac{s}{s-1}}\langle R_0(t)\rangle_m = N$, gives

$$\int_{|\xi|\le R_0(t)} |W(\xi,t)|^2\mathcal{N}^2_{s,\beta,l}(\xi,t)d\xi \le C(T)\int_0^t\int_{\mathbb{R}} |F(\xi,\sigma)|^2\mathcal{N}^2_{s,\beta,l}(\xi,\sigma)d\xi\, d\sigma. \tag{3.10}$$

To get an a priori estimate from (3.9) we need $\|Y(\xi, t_\xi)\| \le C_N$. This follows from the definition of $t_\xi = t(\langle\xi\rangle_m)$ and from

$$\frac{|\lambda(t_\xi)\xi|}{\rho(\xi, t_\xi)} = \frac{\lambda(t_\xi)|\xi|}{\sqrt{1 + \langle\xi\rangle_m \frac{\lambda(t_\xi)^2}{\Lambda(t_\xi)^{\frac{s}{s-1}}}}} \le \frac{\lambda(t_\xi)\langle\xi\rangle_m}{\sqrt{1 + \frac{\langle\xi\rangle_m^2 \lambda(t_\xi)^2}{N}}} \le C_N.$$

This allows us to estimate, for $t_\xi \le t \le T$,

$$|U(\xi,t)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,t) \le C(T) \int_{t_\xi}^{t} \exp(2K(t,\sigma,\xi))|F(\xi,\sigma)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,t)d\sigma$$
$$+ C_N(T)\exp(2K(t,t_\xi,\xi)) \int_0^{t_\xi} \exp(2L(t_\xi,\sigma,\xi))|F(\xi,\sigma)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,t)d\sigma.$$

The assumption $(A6)$ implies

$$\begin{aligned}|U(\xi,t)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,t) &\le C(T) \int_{t_\xi}^{t} |F(\xi,\sigma)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,\sigma)d\sigma \\ &\quad + C_N(T) \int_0^{t_\xi} |F(\xi,\sigma)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,\sigma)d\sigma \\ &\le C_N(T) \int_0^{t} |F(\xi,\sigma)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,\sigma)d\sigma.\end{aligned}$$

Integration over $\{\xi : |\xi| \ge R_0(t)\}$ gives

$$\int_{|\xi|\ge R_0(t)} |U(\xi,t)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,t)d\xi \le C_N(T) \int_0^t \int_{\mathbb{R}} |F(\xi,\sigma)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,\sigma)d\xi. \quad (3.11)$$

The estimates (3.10), (3.11) together yield the following estimate for v:

$$\begin{aligned}|v(\xi,t)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,t) &\le C(T) \int_0^t |\partial_\tau v(\xi,\tau)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,t)d\tau \\ &\le C(T) \int_0^t |\partial_\tau v(\xi,\tau)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,\tau)d\tau,\end{aligned}$$

and

$$\int_{\mathbb{R}} |v(\xi,t)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,t)d\xi \le C(T) \int_0^t \int_{\mathbb{R}} |F(\xi,\sigma)|^2 \mathcal{N}^2_{s,\beta,l}(\xi,\sigma)d\sigma. \quad (3.12)$$

We define $\mathcal{B}(s,\beta,l,T)$ as the space of functions satisfying

$$\|f\|_{s,\beta,l,T} := \sup_{0<t<T} \left(\int_{\mathbb{R}} \mathcal{N}^2_{s,\beta,l}(\xi,t)|\mathcal{F}f(\xi,t)|^2 d\xi \right)^{\frac{1}{2}} < \infty.$$

By means of the estimate (3.12) we derive the next result.

Theorem 3.2. *Let* $\chi = \chi(\tau) \in C^\infty(\mathbb{R}^n)$ *with* $\chi(\tau) = 0$ *for* $|\tau| \le \frac{1}{2}$ *and* $\chi(\tau) = 1$ *for* $|\tau| \ge 2$. *Let* $H = H(D_x,t)$ *be the pseudodifferential operator of first order with the symbol*

$$H(\xi,t) = \lambda(t)|\xi|\chi\left(\frac{\Lambda(t)^{\frac{s}{s-1}}\langle\xi\rangle_m}{N}\right) + \rho(\xi,t)\left(1-\chi\left(\frac{\Lambda(t)^{\frac{s}{s-1}}\langle\xi\rangle_m}{N}\right)\right).$$

Then under the assumptions $(A1)$–$(A6)$ *the solution of*

$$u_{tt} - \lambda(t)^2 u_{xx} - b(t)u_x = f(x,t), \quad u(x,0) = u_t(x,0) \equiv 0,$$

satisfies the strictly hyperbolic-type estimate

$$\|u\|_{s,\beta,l,T} + \|H(D_x,t)u\|_{s,\beta,l,T} + \|D_t u\|_{s,\beta,l,T} \le C_N(T)\|f\|_{s,\beta,l,T}, \quad (3.13)$$

where $C_N(T)$ *tends to zero if* T *tends to zero.*

In the next subsections we investigate how the classes $\mathcal{B}(s,\beta,l,T)$ fit in our scheme developed in Section 2.

3.2.3. Determination of $K = K(t,\sigma,\xi)$ *and* $L = L(t,\sigma,\xi)$. Let us begin with K. The purely imaginary roots $\pm i\lambda(t)\xi$ have no essential influence on K. Influence takes only the terms $\frac{\lambda'}{\lambda}$ and $\frac{b}{\lambda}$.

By assumptions $(A2)$ and $(A3)$ it is enough to take into account the term $C_{\text{hyp}}\frac{\lambda(t)}{\Lambda(t)^{\frac{s}{s-1}}}$. Thus it is reasonable to define

$$K(t,\sigma,\xi) = K(t,\sigma) = \int_\sigma^t C_{\text{hyp}}\frac{\lambda(\tau)}{\Lambda(\tau)^{\frac{s}{s-1}}}d\tau = C_{\text{hyp}}\left(\Lambda(\sigma)^{-\frac{1}{s-1}} - \Lambda(t)^{-\frac{1}{s-1}}\right),$$

where we changed C_{hyp} in the last equation.

More complicated is the definition of $L = L(t,\sigma,\xi)$. The matrix A from Z_{pd} tells us that the terms

$$\frac{\partial_t\rho}{\rho}, \quad \rho, \quad \frac{\lambda(t)^2\xi^2}{\rho}, \quad \frac{|b(t)\xi|}{\rho},$$

should take an influence on L. The assumption $(A2)$ for c_0 guarantees that $\partial_t\rho \ge 0$. That really the first two terms determine L follows from

$$|b(t)\xi| \le C\,\frac{\lambda(t)^2}{\Lambda(t)^{\frac{s}{s-1}}}\langle\xi\rangle_m \le C\,\rho^2(\xi,t);$$

$$\lambda(t)^2\xi^2 \le C\,\lambda(t)^2\langle\xi\rangle_m^2 \le C_N\frac{\lambda(t)^2\langle\xi\rangle}{\Lambda(t)^{\frac{s}{s-1}}} \le C_N\rho^2(\xi,t).$$

Taking account of

$$\int_\sigma^t \frac{\partial_\tau \rho}{\rho} d\tau = \frac{1}{2}\ln \rho^2(\xi,t) - \frac{1}{2}\ln \rho^2(\xi,\sigma)\ ,$$

and

$$\begin{aligned}\int_\sigma^t \rho(\xi,\tau)d\tau &= \int_\sigma^t \sqrt{1+\langle\xi\rangle_m \frac{\lambda(\tau)^2}{\Lambda(\tau)^{\frac{s}{s-1}}}}d\tau \\ &\le \int_\sigma^t \left(1+\langle\xi\rangle_m^{1/2}\frac{\lambda(\tau)}{\Lambda(\tau)^{\frac{s}{2(s-1)}}}\right)d\tau \\ &= t-\sigma+\langle\xi\rangle_m^{1/2} C_{\mathrm{pd}}\left(\Lambda(t)^{\frac{s-2}{2(s-1)}} - \Lambda(\sigma)^{\frac{s-2}{2(s-1)}}\right),\end{aligned}$$

we obtain

$$\begin{aligned}K(t,\sigma) &= C_{\mathrm{hyp}}\left(\Lambda(\sigma)^{-\frac{1}{s-1}} - \Lambda(t)^{-\frac{1}{s-1}}\right), \\ L(t,\sigma,\xi) &= \frac{1}{2}\ln\rho^2(\xi,t) - \frac{1}{2}\ln\rho^2(\xi,\sigma) + t - \sigma \\ &\quad + \langle\xi\rangle_m^{1/2} C_{\mathrm{pd}}\left(\Lambda(t)^{\frac{s-2}{2(s-1)}} - \Lambda(\sigma)^{\frac{s-2}{2(s-1)}}\right).\end{aligned}$$

Both functions satisfy $(A6)$.

3.2.4. Estimate of K and L. In this section we show that the weights $K = K(t,\sigma)$ and $L = L(t,\sigma,\xi)$ are compatible with the weight $\beta\langle\xi\rangle_m^{1/s}$, this means, there exists a constant K_0 such that

$$K(t,\sigma) \le K_0\langle\xi\rangle_m^{1/s} \quad \text{for all} \quad t_\xi \le \sigma \le t \le T, \tag{3.14}$$

$$L(t,\sigma,\xi) \le K_0\langle\xi\rangle_m^{1/s} \quad \text{for all} \quad 0 \le \sigma \le t \le t_\xi,\ \ \xi \in R. \tag{3.15}$$

From the definition of $K(t,\sigma)$ we conclude that immediately (3.14) after

$$K(t,\sigma) \le K(T,t_\xi) \le C_{\mathrm{hyp}}\Lambda(t_\xi)^{-\frac{1}{s-1}} = C_{\mathrm{hyp}} C_N \langle\xi\rangle_m^{1/s}.$$

By using the definitions of $L(t,\sigma,\xi)$ and of t_ξ we conclude that

$$\begin{aligned}L(t,\sigma,\xi) &\le \frac{1}{2}\ln\rho^2(\xi,t_\xi) + t_\xi + \langle\xi\rangle_m^{1/s} C_{\mathrm{pd}}\Lambda(t_\xi)^{\frac{s-2}{2(s-1)}} \\ &\le \frac{1}{2}\ln\left(1+\frac{\lambda(t_\xi)^2\langle\xi\rangle_m^2}{N}\right) + T + C_{\mathrm{pd}}C_N\langle\xi\rangle_m^{1/s}.\end{aligned}$$

To estimate the first term we formulate two assumptions to λ, where exactly one should be satisfied,

$$(A7) \qquad \lambda = \lambda(t) = t^l,$$

$$(A8) \qquad \lambda(t) \sim \Lambda(t)|\ln\Lambda(t)|^L,$$

where $l \in \mathbb{N}$ and L is real.

If $(A7)$ is satisfied, then $\lambda(t_\xi)^2\langle\xi\rangle_m^2 \sim \Lambda(t_\xi)^{\frac{2l}{l+1}}\langle\xi\rangle_m^2 \sim \langle\xi\rangle_m^{2-\frac{2l}{(l+1)}\frac{(s-1)}{s}}$.

Consequently,

$$\ln\left(1+\frac{\lambda(t_\xi)^2\langle\xi\rangle_m^2}{N}\right) \sim \ln\left(1+C_N\langle\xi\rangle_m^{2-\frac{2l}{(l+1)}\frac{(s-1)}{s}}\right) \le C_{N,s}\langle\xi\rangle_m^{1/(2s)}.$$

If $(A8)$ is satisfied, then $\lambda(t_\xi)^2\langle\xi\rangle_m^2 \le C_N\langle\xi\rangle_m^{2-\frac{2(s-1)}{s}+\varepsilon}$ with a small positive ε. But this brings

$$\ln\left(1+\frac{\lambda(t_\xi)^2\langle\xi\rangle_m^2}{N}\right) \le C_{N,s}\langle\xi\rangle_m^{1/(2s)}.$$

Thus (3.14) and (3.15) are satisfied with a suitable positive constant K_0.

3.2.5. Admissibility of h. Our aim consists of an application of Remark 2.16. From the definition of the weight we derive

$$h(\xi,t) = \begin{cases} K(T,t) & \text{for} \quad t_\xi \le t \le T, \\ K(T,t_\xi) + L(t_\xi,t,\xi) & \text{for} \quad 0 < t \le t_\xi, \\ K(T,t_\xi) + \lim\limits_{t\to+0} L(t_\xi,t,\xi) & \text{for} \quad t = 0. \end{cases}$$

The last limit exists due to $(A2)$.

Lemma 3.1. *There exists a constant C_s which is independent of $t \in [0,T]$ and $\xi,\eta \in R$ such that*

$$|h(t,\xi) - h(t,\eta)| \le C_s\langle\xi-\eta\rangle_m^{1/s}.$$

Proof. Without loss of generality we assume $|\xi| \ge |\eta|$, this yields $t_\eta \ge t_\xi$. We divide the proof into three cases. Lemma 4.2(ii) will be applied without further reference.

1^{st} *case:* $\quad T \ge t \ge t_\eta$

In this case we have $h(t,\xi) - h(t,\eta) = K(T,t) - K(T,t) = 0$.

2^{nd} *case:* $\quad 0 \le t \le t_\xi$

Now we have

$$\begin{aligned} h(t,\xi) - h(t,\eta) &= K(T,t_\xi) + L(t_\xi,t,\xi) - K(T,t_\eta) - L(t_\eta,t,\eta) \\ &= K(T,t_\xi) - K(T,t_\eta) + \langle\xi\rangle_m^{1/2} C_{\mathrm{pd}}\Lambda(t_\xi)^{\frac{s-2}{2(s-1)}} \\ &\quad - \langle\eta\rangle_m^{1/2} C_{\mathrm{pd}}\Lambda(t_\eta)^{\frac{s-2}{2(s-1)}} \\ &\quad - \Lambda(t)^{\frac{s-2}{2(s-1)}} C_{\mathrm{pd}}\big(\langle\xi\rangle_m^{1/2} - \langle\eta\rangle_m^{1/2}\big) + t_\xi - t_\eta \\ &\quad + \tfrac{1}{2}\ln\rho^2(\xi,t_\xi) - \tfrac{1}{2}\ln\rho^2(\eta,t_\eta) - \tfrac{1}{2}\big(\ln\rho^2(\xi,t) - \ln\rho^2(\eta,t)\big). \end{aligned}$$

A further splitting is needed here.

1^{st} *difference.* We have

$$K(T,t_\xi)-K(T,t_\eta)=C_{\mathrm{hyp}}\big(\Lambda(t_\xi)^{-\frac{1}{s-1}}-\Lambda(t_\eta)^{-\frac{1}{s-1}}\big)=C_{\mathrm{hyp}}\big(\langle\xi\rangle_m^{1/s}-\langle\eta\rangle_m^{1/s}\big)$$
$$\leq C_{\mathrm{hyp}}\,\langle\xi-\eta\rangle_m^{1/s}.$$

2^{nd} *difference.* It remains to estimate

$$\langle\xi\rangle_m^{1/2}\Lambda(t_\xi)^{\frac{s-2}{2(s-1)}}-\langle\eta\rangle_m^{1/2}\Lambda(t_\eta)^{\frac{s-2}{2(s-1)}}=\langle\xi\rangle_m^{1/2}\langle\xi\rangle_m^{-\frac{s-2}{2s}}-\langle\eta\rangle_m^{1/2}\langle\eta\rangle_m^{-\frac{s-2}{2s}}$$
$$=\langle\xi\rangle_m^{1/s}-\langle\eta\rangle_m^{1/s}\leq\langle\xi-\eta\rangle_m^{1/s}.$$

3^{rd} *difference.* We have

$$\Lambda(t)^{\frac{s-2}{2(s-1)}}\left(\langle\xi\rangle_m^{1/2}-\langle\eta\rangle_m^{1/2}\right)\leq\Lambda(t_\xi)^{\frac{s-2}{2(s-1)}}\left(\langle\xi\rangle_m^{1/2}-\langle\eta\rangle_m^{1/2}\right)$$
$$=\langle\xi\rangle_m^{-\frac{s-2}{2s}}\left(\langle\xi\rangle_m^{1/2}-\langle\eta\rangle_m^{1/2}\right)$$
$$\leq\langle\xi\rangle_m^{-\frac{s-2}{2s}}\langle\xi-\eta\rangle_m^{1/2}=\langle\xi-\eta\rangle_m^{1/s}\left(\frac{\langle\xi-\eta\rangle_m}{\langle\xi\rangle_m}\right)^{\frac{s-2}{2s}}$$
$$\leq C_s\,\langle\xi-\eta\rangle_m^{1/s}.$$

4^{th} *difference.* Here we use

$$t_\eta-t_\xi=\int_\xi^\eta\frac{d}{d\langle\rho\rangle_m}t_\rho(\langle\rho\rangle_m)d\langle\rho\rangle_m\leq C\int_\eta^\xi\frac{\Lambda(t_\rho)}{\lambda(t_\rho)\langle\rho\rangle_m}d\langle\rho\rangle_m.$$

Now we distinguish two cases. If λ satisfies $(A7)$, then

$$\int_\eta^\xi\frac{\Lambda(t_\rho)}{\lambda(t_\rho)\langle\rho\rangle_m}d\langle\rho\rangle_m\leq C\int_\eta^\xi\frac{\Lambda(t_\rho)^{\frac{1}{l+1}}}{\langle\rho\rangle_m}d\langle\rho\rangle_m=C\int_\eta^\xi\frac{1}{\langle\rho\rangle_m^{1+\frac{s-1}{s(l+1)}}}d\langle\rho\rangle_m$$
$$=C\left(\langle\eta\rangle_m^{-\frac{(s-1)}{s(l+1)}}-\langle\xi\rangle_m^{-\frac{(s-1)}{s(l+1)}}\right)\leq C_{s,l}\langle\xi-\eta\rangle_m^{1/s}.$$

If λ satisfies $(A8)$, then with a suitable positive L it holds that

$$\int_\eta^\xi\frac{\Lambda(t_\rho)}{\lambda(t_\rho)\langle\rho\rangle_m}d\langle\rho\rangle_m\leq C\int_\eta^\xi\frac{(\ln\langle\rho\rangle_m)^L}{\langle\rho\rangle_m}d\langle\rho\rangle_m\leq C\int_\eta^\xi\frac{1}{s\langle\rho\rangle_m^{1-1/s}}d\langle\rho\rangle_m$$
$$=C\big(\langle\xi\rangle_m^{1/s}-\langle\eta\rangle_m^{1/s}\big)\leq C\,\langle\xi-\eta\rangle_m^{1/s}.$$

5^{th} *difference.* We have

$$\frac{1}{2}\ln\rho^2(\xi,t_\xi)-\frac{1}{2}\ln\rho^2(\eta,t_\eta)=\frac{1}{2}\ln\big(1+\lambda(t_\xi)^2\langle\xi\rangle_m^2\big)-\frac{1}{2}\ln\big(1+\lambda(t_\eta)^2\langle\eta\rangle_m^2\big).$$

If λ satisfies $(A7)$, then the last difference is equal to

$$\frac{1}{2}\ln\left(1+\langle\xi\rangle_m^{2-\frac{2l}{(l+1)}\frac{(s-1)}{s}}\right)-\frac{1}{2}\ln\left(1+\langle\eta\rangle_m^{2-\frac{2l(s-1)}{(l+1)s}}\right)\le C_s\left(\langle\xi\rangle_m^{1/s}-\langle\eta\rangle_m^{1/s}\right)$$
$$\le C_s\langle\xi-\eta\rangle_m^{1/s}$$

and is to be handled like the 4^{th} difference. The case λ that satisfies $(A8)$ can be studied in the same way.

6^{th} *difference.* Now we proceed as follows:

$$\ln\left(1+\langle\xi\rangle_m\frac{\lambda(t)^2}{\Lambda(t)^{\frac{s}{s-1}}}\right)-\ln\left(1+\langle\eta\rangle_m\frac{\lambda(t)^2}{\Lambda(t)^{\frac{s}{s-1}}}\right)$$
$$=\int_\eta^\xi\frac{\lambda(t)^2\Lambda(t)^{-\frac{s}{s-1}}}{1+\langle\rho\rangle_m\lambda(t)^2\Lambda(t)^{-\frac{s}{s-1}}}d\langle\rho\rangle_m$$
$$\le C_s\int_\eta^\xi\frac{1}{s}\langle\rho\rangle_m^{1/s-1}d\langle\rho\rangle_m$$
$$=C_s\left(\langle\xi\rangle_m^{1/s}-\langle\eta\rangle_m^{1/s}\right)\le C_s\langle\xi-\eta\rangle_m^{1/s}.$$

Thus we have also proved in the second case that $|h(t,\xi)-h(t,\eta)|\le C_s\langle\xi-\eta\rangle_m^{1/s}$.

3^{rd} *case:* $\quad t_\xi\le t\le t_\eta$

We have

$$h(t,\xi)-h(t,\eta)=K(T,t)-K(T,t_\eta)-L(t_\eta,t,\eta)$$
$$=C_{\mathrm{hyp}}\left(\Lambda(t)^{-\frac{1}{s-1}}-\Lambda(t_\eta)^{-\frac{1}{s-1}}\right)-(t_\eta-t)$$
$$-\langle\eta\rangle_m^{1/2}C_{\mathrm{pd}}\left(\Lambda(t_\eta)^{\frac{s-2}{2(s-1)}}-\Lambda(t)^{\frac{s-2}{2(s-1)}}\right)$$
$$-\tfrac{1}{2}\big(\ln\rho^2(\eta,t_\eta)-\ln\rho^2(\eta,t)\big).$$

1^{st} *difference.* Compare with 1^{st} difference from 2^{nd} case if we use $K(T,t)\le K(T,t_\xi)$.

2^{nd} *difference.* Compare with 4^{th} difference from 2^{nd} case if we recall $t_\eta-t\le t_\eta-t_\xi$.

3^{rd} *difference.* We have

$$\langle\eta\rangle_m^{1/2}\left(\Lambda(t_\eta)^{\frac{s-2}{2(s-1)}}-\Lambda(t)^{\frac{s-2}{2(s-1)}}\right)\le\langle\eta\rangle_m^{1/2}\left(\Lambda(t_\eta)^{\frac{s-2}{2(s-1)}}-\Lambda(t_\xi)^{\frac{s-2}{2(s-1)}}\right)$$
$$=\langle\eta\rangle_m^{1/2}\Big(\langle\eta\rangle_m^{-\frac{s-2}{2s}}-\langle\xi\rangle_m^{-\frac{s-2}{2s}}\Big)$$

$$= \frac{\langle\xi\rangle_m^{1/2}\langle\eta\rangle_m^{1/s} - \langle\eta\rangle_m^{1/2}\langle\xi\rangle_m^{1/s}}{\langle\xi\rangle_m^{1/2}}$$
$$= \frac{\langle\xi\rangle_m^{1/2}\big(\langle\xi\rangle_m^{1/s} - \langle\eta\rangle_m^{1/s}\big) + \langle\xi\rangle_m^{1/s}\big(\langle\xi\rangle_m^{1/2} - \langle\eta\rangle_m^{1/2}\big)}{\langle\xi\rangle_m^{1/2}}$$
$$\le \langle\xi-\eta\rangle_m^{1/s} + \langle\xi-\eta\rangle_m^{1/s}\left(\frac{\langle\xi-\eta\rangle_m}{\langle\xi\rangle_m}\right)^{\frac{s-2}{2s}}$$
$$\le C_s\langle\xi-\eta\rangle_m^{1/s}.$$

4^{th} *difference.* Finally we have to denote

$$\ln\rho^2(\eta,t_\eta) - \ln\rho^2(\eta,t) = \ln\left(\frac{1+\langle\eta\rangle_m\lambda(t_\eta)^2\Lambda(t_\eta)^{-\frac{s}{s-1}}}{1+\langle\eta\rangle_m\lambda(t)^2\Lambda(t)^{-\frac{s}{s-1}}}\right).$$

If λ satisfies $(A7)$, then

$$\ln\left(\frac{1+\langle\eta\rangle_m\lambda(t_\eta)^2\Lambda(t_\eta)^{-\frac{s}{s-1}}}{1+\langle\eta\rangle_m\lambda(t)^2\Lambda(t)^{-\frac{s}{s-1}}}\right) = \ln\left(\frac{1+C_l\langle\eta\rangle_m\Lambda(t_\eta)^{\frac{2l}{l+1}-\frac{s}{s-1}}}{1+C_l\langle\eta\rangle_m\Lambda(t)^{\frac{2l}{l+1}-\frac{s}{s-1}}}\right).$$

But $(A2)$ implies $\frac{2l}{l+1} - \frac{s}{s-1} > 0$. Hence the last term is estimated by

$$\ln\left(\frac{1+C_l\langle\eta\rangle_m\Lambda(t_\eta)^{\frac{2l}{l+1}-\frac{s}{s-1}}}{1+C_l\langle\eta\rangle_m\Lambda(t_\xi)^{\frac{2l}{l+1}-\frac{s}{s-1}}}\right) = \ln\left(\frac{1+C_l\langle\eta\rangle_m\langle\eta\rangle_m^{-\frac{s-1}{s}\left(\frac{2l}{l+1}-\frac{s}{s-1}\right)}}{1+C_l\langle\eta\rangle_m\langle\xi\rangle_m^{-\frac{s-1}{s}\left(\frac{2l}{l+1}-\frac{s}{s-1}\right)}}\right).$$

Using $\ln\left(\frac{1+a}{1+b}\right) \le \ln\left(\frac{a}{b}\right)$ if $a \ge b > 0$, then the last term can be estimated by

$$\ln\langle\xi\rangle_m^{-\frac{s-1}{s}\left(\frac{2l}{l+1}-\frac{s}{s-1}\right)} - \ln\langle\eta\rangle_m^{-\frac{s-1}{s}\left(\frac{2l}{l+1}-\frac{s}{s-1}\right)} \le C_s\langle\xi-\eta\rangle_m^{1/s}.$$

If λ satisfies $(A8)$, then we follow the same reasoning. Thus we have proved in the third case that

$$|h(t,\xi) - h(t,\eta)| \le C_s\langle\xi-\eta\rangle_m^{1/s}.$$

The statement of the lemma is proved. □

Thus our weight function $\mathcal{N}_{s,\beta,l}(\xi,t)$, satisfies (2.32) and (2.33), and therefore (2.29), if β is sufficiently large (e.g. $\beta > 2\,\delta_{1/s}^{-1}\max(K_0, C_s)$). The spaces $\mathcal{B}(s,\beta,l,T)$ can be identified with the classes $\mathcal{A}(s,\beta,K_0,m,l,\mathcal{N}_{s,\beta,l},T)$, where m is determined from $T = t(\langle 0\rangle_m)$. Applying Corollaries 2.3, 2.7 and 2.8 we can follow the approach which was sketched at the end of Section 1 and arrive at the following existence and uniqueness result for (3.5).

Theorem 3.3. *Let us consider*

$$u_{tt} - \lambda(t)^2u_{xx} - b(t)u_x = f(u),\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),$$

under the assumptions $(A1)$ *to* $(A3)$. *Let us suppose the following additional conditions:*

- *the data* φ, ψ *satisfy*

$$\int_{-\infty}^{\infty} \exp\left(T_1 \langle\xi\rangle_m^{1/s}\right) \langle\xi\rangle_m^{2\ell} \left(|\mathcal{F}\varphi(\xi)|^2 + |\mathcal{F}\psi(\xi)|^2\right) d\xi < \infty,$$

  for some sufficiently large $T_1 > 0$*;*
- $f : \mathbb{R} \to \mathbb{C}$ *is an infinitely differentiable function and* $f(0) = 0$*;*
- *the Fourier transform* $\mathcal{F} f$ *of* f *is an integrable function and there exists a non-negative constant* c *such that*

$$|\mathcal{F} f(\xi)| \le c\, e^{-|\xi|^{1/s} (\log|\xi|) \log\log|\xi|}, \qquad \text{for large } |\xi|.$$

Then, with a suitable positive constant T_0*, the Cauchy problem has a unique solution* $u \in \mathcal{B}(s, \beta, l, T_0)$ *for sufficiently large* β*. The terms* $H(D_x, t)u$ *and* $D_t u$ *(see (3.13)) belong to* $\mathcal{B}(s, \beta, l, T_0)$*, too.*

3.3. Strictly hyperbolic equations with non-Lipschitz coefficients

Our model problem is

$$u_{tt} - a(t)u_{xx} = f(u),\ u(x, 0) = \varphi(x),\ u_t(x, 0) = \psi(x). \tag{3.16}$$

We will derive a strictly hyperbolic-type estimate for the solution of

$$u_{tt} - a(t)u_{xx} = f(x, t),\ u(x, 0) = u_t(x, 0) \equiv 0 \tag{3.17}$$

by using ideas from [5]. In the examples treated in Subsections 3.1 and 3.2 we always worked with an explicit representation formula of the solution which we do not have here. For that reason we shall make a few further remarks but without going into details. For $\varepsilon > 0$ we study the family of Cauchy problems

$$u_{tt} - a(t + \varepsilon)u_{xx} = f(x, t),\ u(x, 0) = u_t(x, 0) \equiv 0,$$

instead of (3.17). Due to the assumption (A10), see below, this is on the one hand a family of strict hyperbolic Cauchy problems and on the other hand there exists a cone of dependence uniformly with respect to ε. Consequently, the Cauchy problems are C^∞ well-posed. If we apply our approach to these modified problems, then we obtain a strictly hyperbolic-type estimate of the same kind as for (3.17). Thus compactness results (see Remark 2.8) are applicable and give a solution of our linear Cauchy problem (3.17) in a weaker space. Finally, the a priori estimate (3.23) leads to the regularity of the solution we have in mind for that one of (3.16), compare with Theorem 3.4. After having this regularity for the solution of (3.17) one can follow the approach which is sketched in the introduction to get a uniquely determined solution of (3.16).

It remains to derive the strictly hyperbolic-type estimate. We suppose

$(A9)$ $\quad a(t) \ge A > 0 \quad \text{on} \quad [0, T],$

$(A10)$ $\quad a \in C[0, T] \cap C^1(0, T], \ \text{with} \ |a'(t)| \le \dfrac{B}{t^q} \ \text{on} \ (0, T], \ \text{where} \ q > 1.$

After a partial Fourier transformation we obtain, from (3.17),

$$v_{tt} + a(t)\xi^2 v = \mathcal{F} f,\ \ v(\xi, 0) = v_t(\xi, 0) \equiv 0. \tag{3.18}$$

Again we divide $\{(\xi, t) \in (\mathbb{R} \setminus \{0\}) \times (0, T]\}$ into two zones. For given s, q, and T we define $m > 0$ by the relation $T\, m^{\frac{1}{s(q-1)}} = 1$. Having fixed m we introduce t_ξ by $t_\xi \langle \xi \rangle_m^{\frac{1}{s(q-1)}} = 1$. Obviously, the function $t(\xi) = t_\xi$ is strictly decreasing in $|\xi|$. Hence, there is an inverse function denoted by ξ_t, defined on $[0, T]$. As in the previous section we introduce the *pseudodifferential zone*

$$Z_{\mathrm{pd}} = \Big\{(\xi, t) \in (\mathbb{R} \setminus \{0\}) \times (0, T] : t\langle \xi \rangle_m^{\frac{1}{s(q-1)}} \le 1\Big\},$$

and the *hyperbolic zone*

$$Z_{\mathrm{hyp}} = \Big\{(\xi, t) \in (\mathbb{R} \setminus \{0\}) \times (0, T] : t\langle \xi \rangle_m^{\frac{1}{s(q-1)}} \ge 1\Big\}.$$

In both zones we define the energy density $E(\xi, t)(v)$ of v as follows (we use the abbreviation $v' = v_t$):

$$E(\xi, t)(v) := (|v'(\xi, t)|^2 + (1 + \tilde{a}(\xi, t)\xi^2)|v(\xi, t)|^2)\mathcal{N}(\xi, t),$$

where

$$\mathcal{N}(\xi, t) = \begin{cases} \exp\Big(- K_1 \int_0^t \alpha(\sigma, \xi)d\sigma + \beta\langle \xi \rangle_m^{1/s}\Big)\langle \xi \rangle_m^l & \text{if } \ t < t_\xi; \\ \exp\Big(- K_1\, K_3\, \langle \xi \rangle_m\, t_\xi - K_2 \int_{t_\xi}^t \alpha(\sigma, \xi)d\sigma + \beta\langle \xi \rangle_m^{1/s}\Big)\langle \xi \rangle_m^l & \text{if } \ t \ge t_\xi. \end{cases} \tag{3.19}$$

The functions $\tilde{a}$ and α will depend on the zone and the constants K_1, K_2, K_3, β, and ℓ will be chosen later on. The energy of the solution of (3.18) is defined by

$$\mathcal{E}(t)(v) := \int_{\mathbb{R}} E(\xi, t)(v)d\xi.$$

Let us mention that, in contrast to the other examples, the weight $\mathcal{N}$ appears here with exponent 1.

3.3.1. Estimate for the energy density in Z_{pd}. Here the auxiliary functions $\tilde{a}$ and α will be chosen as $\tilde{a}(\xi, t) = K_3 := \max_{[0,T]} |a(t)|$ and $\alpha(\xi, t) := K_3\, \langle \xi \rangle_m$. Temporarily we fix ξ and drop it in notations. Differentiation of the energy density E with respect to t yields

$$E'(t) = \Big(2\ \mathrm{Re}\ (v''\overline{v}') + 2\big(1 + K_3\xi^2\big)\ \mathrm{Re}\ (v'\overline{v})) \\ -(|v'|^2 + \big(1 + K_3\,\xi^2\big)|v|^2)\ K_1\ K_3\langle\xi\rangle_m\Big)\,\mathcal{N}(t).$$

Using (3.18) leads to

$$E'(t) = \Big(2(K_3 - a(t))\,\xi^2\ \mathrm{Re}\ (v'\overline{v}) + 2\ \mathrm{Re}\ (\mathcal{F} f\overline{v}') + 2\ \mathrm{Re}\ (v'\overline{v}) \\ -K_1\ K_3\ \langle\xi\rangle_m\big(|v'|^2 + K_3\,\xi^2|v|^2\big)\Big)\mathcal{N}(t).$$

Taking into consideration

$$\begin{aligned}(K_3 - a(t))\xi^2\ \ \mathrm{Re}\ (v'\overline{v}) &\le \tfrac{K_3}{2}|\xi|\big(|v'|^2 + \xi^2|v|^2\big)\\ &\le \tfrac{1}{2}\ \max(K_3, 1)\ |\xi|\big(|v'|^2 + K_3\,\xi^2|v|^2\big),\end{aligned}$$

then $K_1 \ge 1/(2\ \min(1, K_3))$ implies

$$E'(t) \le (2\ \mathrm{Re}\ (\mathcal{F} f\overline{v}') + 2\ \mathrm{Re}\ (v'\overline{v}))\mathcal{N}(t).$$

Because of the homogeneous initial conditions, Gronwall's inequality yields

$$E(\xi, t)(v) \le e^{2t}\int_0^t |\mathcal{F} f(\xi, \sigma)|^2\ \mathcal{N}(\xi, \sigma)d\sigma, \qquad 0 \le t \le t_\xi. \tag{3.20}$$

3.3.2. Estimate for the energy density in Z_{hyp}. In Z_{hyp} we take $\tilde{a}(\xi, t) = a(t)$ and $\alpha(\xi, t) := t^{-q}$. Again we fix ξ and drop it in notations. Differentiating $E(\xi, t)(v)$ with respect to t we obtain

$$E'(t) = \Big((2\ \mathrm{Re}\ (\mathcal{F} f\overline{v}') + a'(t)\xi^2|v|^2 + 2\ \mathrm{Re}\ (v\overline{v}')) \\ -\frac{K_2}{t^q}\ (|v'|^2 + (1 + a(t)\xi^2)|v|^2)\Big)\mathcal{N}(t).$$

Suppose $K_2 \ge B/A$, cf. $(A9)$, $(A10)$, and (3.19), then

$$a'(t) \le \frac{K_2}{t^q}a(t).$$

This inequality implies

$$E'(t) \le 2E(t) + |\mathcal{F} f(t)|^2\mathcal{N}(t).$$

As above, after an application of Gronwall's lemma we arrive at

$$E(\xi, t)(v) \le e^{2t}\int_{t_\xi}^t |\mathcal{F} f(\xi, \sigma)|^2\mathcal{N}(\xi, \sigma)d\sigma + E(\xi, t_\xi)(v), \quad t_\xi \le t \le T. \tag{3.21}$$

3.3.3. A strictly hyperbolic-type estimate for (3.17). Clearly, the function $h = h_{s,K_1,K_2,K_3,q,T}(\xi, t)$ is given by

$$h_{s,K_1,K_2,K_3,q,T}(\xi,t) := \begin{cases} K_1\, K_3\, \langle \xi \rangle_m t & \text{for} \quad 0 \le t \le t_\xi \\ K_1\, K_3\, \langle \xi \rangle_m t_\xi + \frac{K_2}{q-1}\left(\frac{1}{t_\xi^{q-1}} - \frac{1}{t^{q-1}}\right) & \text{for} \quad t_\xi \le t \le T \end{cases}. \tag{3.22}$$

To derive an estimate of $\mathcal{E}$ we split the integration over $\mathbb{R}$ (with respect to ξ) into integration over the three intervals $(-\infty, -\xi_t)$, $(-\xi_t, \xi_t)$, and (ξ_t, ∞) and use there the estimates (3.20) and (3.21). This yields

$$\begin{aligned} \mathcal{E}(t)(v) &= \int_{\mathbb{R}} \big(|v'(\xi,t)|^2 + (1 + \tilde{a}(\xi,t)\xi^2)|v(\xi,t)|^2\big)\mathcal{N}(\xi,t)d\xi \\ &\le T\, e^{2T} \sup_{0<\sigma<T} \int_{\mathbb{R}} |\mathcal{F} f(\xi,\sigma)|^2\, \mathcal{N}(\xi,\sigma)d\xi, \end{aligned}$$

as long as $K_1 \ge 1/(2\min(1, K_3))$ and $K_2 \ge B/A$. We define $\mathcal{B}(s, \beta, K_1, K_2, K_3, q, l, T)$ as the set of all functions f satisfying

$$\begin{aligned} \| f \| &:= \| f \|_{s,\beta,K_1,K_2,K_3,q,l,T} \\ &= \sup_{0<t<T} \left(\int_{\mathbb{R}} |\mathcal{F} f(\xi,t)|^2\, \exp\big(\beta\, \langle \xi \rangle_m - h_{s,K_1,K_2,K_3,q,T}(\xi,t)\big)\, \langle \xi \rangle_m^{\ell}\, d\xi \right)^{\frac{1}{2}} \\ &< \infty, \end{aligned}$$

Then our energy estimate yields, for the solution of (3.17), the strictly hyperbolic-type estimate

$$\| u \| + \| D_x u \| + \| D_t u \| \le C(T) \| f \|, \tag{3.23}$$

where $C(T)$ tends to 0 if T tends to 0.

3.3.4. Properties of the weight $\mathcal{N}$. As for the second example we have to clarify whether the weight $\mathcal{N}$ has the properties needed for an application of the results of Section 2.

From (3.22) and the definition of t_ξ and with $h = h_{s,K_1,K_2,K_3,q,T}$ we derive

$$\langle \xi \rangle_m t \le \langle \xi \rangle_m t_\xi = \langle \xi \rangle_m \langle \xi \rangle_m^{-\frac{1}{s(q-1)}} = \langle \xi \rangle_m^{1-\frac{1}{s(q-1)}} \le \langle \xi \rangle_m^{1/s}$$

as long as $s \le \frac{q}{q-1}$. This leads to the restriction of s in dependence of $q > 1$. Moreover,

$$\begin{aligned} K_1\, K_3\, \langle \xi \rangle_m t_\xi + \frac{K_2}{q-1}\left(\frac{1}{t_\xi^{q-1}} - \frac{1}{t^{q-1}}\right) &\le K_1\, K_3\, \langle \xi \rangle_m t_\xi + \frac{K_2}{q-1} \langle \xi \rangle_m^{1/s} \\ &\le K_4 \langle \xi \rangle_m^{1/s}. \end{aligned}$$

with K_4 suitably chosen. Hence

$$|h(\xi,t)| \le K_4 \langle\xi\rangle_m^{1/s}.$$

If $\beta > K_4$ and $s \le \frac{q}{q-1}$, then $\mathcal{N}$ represents a Gevrey-type weight. Finally we investigate the regularity of h.

Lemma 3.2. *There exists a constant C_s which is independent of $t \in [0,T]$ and $\xi, \eta \in R$ such that*

$$|h(\xi,t) - h(\eta,t)| \le C_s \langle\xi-\eta\rangle_m^{1/s}.$$

Proof. As in the proof of Lemma 3.1 we suppose $|\xi| \ge |\eta|$, $t_\eta \ge t_\xi$ respectively. Once again we shall use Lemma 4.2 without further reference.

1^{st} *case:* $t \le t_\xi$

We have

$$\begin{aligned} h(\xi,t) - h(\eta,t) &= K_1 K_3 (\langle\xi\rangle_m - \langle\eta\rangle_m)\, t \le K_1 K_3 \langle\xi-\eta\rangle_m t_\xi \\ &= K_1 K_3 \langle\xi-\eta\rangle_m^{1/s} \langle\xi-\eta\rangle_m^{1-\frac{1}{s}} \langle\xi\rangle_m^{-\frac{1}{s(q-1)}} \\ &\le K_1 K_3 \langle\xi-\eta\rangle_m^{1/s}. \end{aligned}$$

2^{nd} *case:* $t_\eta \le t$

Starting with

$$h(\xi,t) - h(\eta,t) = K_1 K_3 \langle\xi\rangle_m t_\xi - K_1 K_3 \langle\eta\rangle_m t_\eta + \frac{K_2}{q-1}\left(\langle\xi\rangle_m^{1/s} - \langle\eta\rangle_m^{1/s}\right)$$

we have to estimate two differences. But the definition of zones immediately yields

$$|h(\xi,t) - h(\eta,t)| \le C_s \langle\xi-\eta\rangle_m^{1/s}.$$

3^{rd} *case:* $t_\xi \le t \le t_\eta$

In this case we have

$$h(\xi,t) - h(\eta,t) = K_1 K_3 \langle\xi\rangle_m t_\xi - K_1 K_3 \langle\eta\rangle_m t + \frac{K_2}{q-1}\left(\frac{1}{t_\xi^{q-1}} - \frac{1}{t^{q-1}}\right).$$

To estimate the first difference we use

$$K_1 K_3 \langle\xi\rangle_m t_\xi - K_1 K_3 \langle\eta\rangle_m t \le K_1 K_3 (\langle\xi\rangle_m - \langle\eta\rangle_m) t_\xi$$

and it follows the estimates from the first case. The second difference can be treated in the following way:

$$\begin{aligned} \frac{K_2}{q-1}\left(\frac{1}{t_\xi^{q-1}} - \frac{1}{t^{q-1}}\right) &\le \frac{K_2}{q-1}\left(\frac{1}{t_\xi^{q-1}} - \frac{1}{t_\eta^{q-1}}\right) \\ &= \frac{K_2}{q-1}\left(\langle\xi\rangle_m^{1/s} - \langle\eta\rangle_m^{1/s}\right) \\ &\le \frac{K_2}{q-1} \langle\xi-\eta\rangle_m^{1/s}. \end{aligned}$$

In all three cases we have shown the statement of the lemma. □

Summarizing, the weight $\mathcal{N}_{s,\beta,K_1,K_2,K_3,q,l,T}$ satisfies (2.32) and (2.33), and therefore (2.29), as long as β is sufficiently large (e.g. $\beta > 2\,\delta_{1/s}^{-1}\max(K_4, C_s)$). For $\beta > K_4$ the spaces $\mathcal{B}(s, \beta, K_1, K_2, K_3, q, l, T)$ coincide with $\mathcal{A}(s, \beta, K_4, m, l, \mathcal{N}_{s,\beta,K_1,K_2,K_3,q,l,T}, T)$, where $m = T^{-s(q-1)}$. So we may apply Corollaries 2.3, 2.7 and 2.8 and arrive at the following existence and uniqueness result for (3.16):

Theorem 3.4. *Let us consider*

$$u_{tt} - a(t)u_{xx} = f(u),\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),$$

under the assumptions (*A*9) *and* (*A*10) *for* $q > 1$. *Let us suppose the following additional conditions with* $s \le \frac{q}{q-1}$*:*

- *the data* φ, ψ *satisfy*

$$\int_{-\infty}^{\infty} \exp(\beta\,\langle\xi\rangle_m)\,\langle\xi\rangle_m^{\ell}\,(|\mathcal{F}\varphi(\xi)|^2 + |\mathcal{F}\psi(\xi)|^2)\,d\xi < \infty,$$

  with β *sufficiently large;*
- $f : \mathbb{R} \to \mathbb{C}$ *is an infinitely differentiable function and* $f(0) = 0$*;*
- *the Fourier transform* $\mathcal{F} f$ *of* f *is an integrable function and there exists a non-negative constant c such that*

$$|\mathcal{F} f(\xi)| \le c\, e^{-|\xi|^{1/s}(\log|\xi|)\,\log\log|\xi|}, \qquad \textit{for large } |\xi|.$$

  Then, with a suitable positive constant T_0*, the Cauchy problem has a unique solution* $u \in \mathcal{B}(s, \beta, K_1, K_2, K_3, q, l, T_0)$ *for some* $K_1 \ge 1/(2\min(1, K_3))$ *and* $K_2 \ge B/A$. *The derivatives* $D_x u$ *and* $D_t u$ *(see* (3.23)*) belong to* $\mathcal{B}(s, \beta, K_1, K_2, K_3, q, l, T_0)$*, too.*

Concluding remark. The results in form of Theorems 3.1, 3.3 and 3.4 contain sharp statements with respect to the Gevrey property for spatial variables because there exist counter-examples for $s > 2$ (Theorem 3.1), that the statement of Theorem 3.3 doesn't hold if assumption (*A*3) is not satisfied (see [10]) and for $s > \frac{q}{q-1}$ (Theorem 3.4) (see [5]). Moreover, the statements of these theorems contain sufficient conditions for f in terms of the behaviour of their Fourier transforms in the phase space. These conditions are near to optimal ones due to the counter-examples from Subsection 2.14. In Theorems 3.1, 3.3 and 3.4 we have assumed that the functions $f = f(t)$ are defined on $\mathbb{R}$. This is of course not necessary. We need only that these functions are defined in a ball around $\varphi + t\psi$. With a cut-off Gevrey function we reduce this situation to the above case. By the aid of the cone of dependence property we obtain similar statements. About possible generalizations we mention the study of quasi-linear problems of second or even higher order. But then we have to generalize the statements of Theorems 3.1, 3.3 and 3.4 to corresponding linear problems with coefficients depending on x, too. Then we have, instead of Fourier analysis, to apply microlocal analysis (see e.g. [14]). The results of Theorem 2.11 and Corollary 2.7 of the present paper are important tools to generalize the statements of [14] from analytic non-linear dependence to Gevrey non-linear dependence. For applications we restricted ourselves to the Cauchy problem for hyperbolic equations. Let us mention that the results of this paper can also be used to study other questions as e.g. local solvability (see e.g. Olario [17]). Last but not least we wish to direct the attention of the reader to the very recent paper [18]. There the methods from Section 2 have been used to prove local solvability of certain semi-linear pde with multiple characteristics with non-linearities of Gevrey regularity (instead of analytic non-linearities).

4. Appendix

Here we collect a few elementary lemmas.

Lemma 4.1. *Let* $0 < \varrho < 1$ *and* $\delta_\varrho = 2 - 2^\varrho$. *Then*

$$\delta_\varrho \, t^\varrho \leq 1 + t^\varrho - (1+t)^\varrho \leq t^\varrho \tag{4.1}$$

holds for all $t \in [0, 1]$. *Moreover,* δ_ϱ *is the best constant in* (4.1).

Proof. Define

$$f_\varrho(t) = \frac{1 + t^\varrho - (1+t)^\varrho}{t^\varrho}, \qquad 0 < t \leq 1.$$

Then elementary calculus shows that f'_ϱ is negative on $(0, 1]$. □

As an immediate consequence of this lemma one obtains the following:

Lemma 4.2. *Let* $0 < \varrho < 1$ *and let* δ_ϱ *as in Lemma* 4.1.

(i) *For* $u, v \geq 0$ *it holds that*

$$(u+v)^\varrho \leq u^\varrho + v^\varrho - \delta_\varrho \, (\min(u, v))^\varrho$$

and

$$u^\varrho + v^\varrho - (u+v)^\varrho \leq \min(u, v)^\varrho.$$

(ii) *For arbitrary* $\xi, \ \eta \in \mathbb{R}^n$ *and* $m \geq 0$ *it holds that*

$$\langle \xi \rangle_m^\varrho \leq \langle \xi - \eta \rangle_m^\varrho + \langle \eta \rangle_m^\varrho - \delta_\varrho \, (\min(\langle \xi - \eta \rangle_m, \langle \eta \rangle_m))^\varrho$$

and

$$\langle \xi - \eta \rangle_m^\varrho + \langle \eta \rangle_m^\varrho - \langle \xi \rangle_m^\varrho \leq (\min(\langle \xi - \eta \rangle_m, \langle \eta \rangle_m))^\varrho.$$

Lemma 4.3. *Let* $0 < s < 1$ *and put* $L = (1/2)^s - 1/2 > 0$. *Let* $\xi \in \mathbb{R}^n$ *be fixed. Then the function* $g(\eta) = |\xi|^{1/s} - |\xi - \eta|^{1/s} - |\eta|^{1/s}$ *is non-negative on the ball* $B(\xi/2, L\,|\xi|)$.

Proof. If η is an element of the ball defined in the lemma we conclude that

$$|\eta| \le (L+1/2)\,|\xi| \qquad \text{and} \qquad |\eta-\xi| \le (L+1/2)\,|\xi|.$$

Hence

$$g(\eta) \ge |\xi|^{1/s}\,(1-2(L+1/2)^{1/s}) = 0,$$

which completes the proof. □

Lemma 4.4. *Let* $c>0$ *and* $\xi \in \mathbb{R}^n$ *be fixed. Define*

$$E_\xi = \{\eta \in \mathbb{R}^n : \quad |\xi| - |\xi-\eta| - |\eta| \ge -c\}.$$

Then

$$vol\,(E_\xi) = c_n\,|\xi|^n \left(1+\frac{c}{|\xi|}\right)\left(\left(1+\frac{c}{|\xi|}\right)^2 - 1\right)^{(n-1)/2}.$$

Proof. Let $e^1 = (1,0,\dots,0)$ and put

$$F = \left\{\eta \in \mathbb{R}^n : \quad \left|\frac{e^1}{2}+\eta\right| + \left|\frac{e^1}{2}-\eta\right| \le 1+\frac{c}{|\xi|}\right\}.$$

If $\mathcal{R}$ is a rotation such that $\mathcal{R}\,\xi = |\xi|\,e^1$, it holds that

$$\eta \in E_\xi \qquad \Longleftrightarrow \qquad \frac{e^1}{2} - \mathcal{R}(\eta/|\xi|) \in F.$$

Hence $\mathrm{vol}(E_\xi) = |\xi|^n\,\mathrm{vol}(F)$. It remains to compute the volume of F. Using the notations $r = 1+c/|\xi|$, $\eta = (\eta_1,\eta')$, $\eta' \in \mathbb{R}^{n-1}$, then we have

$$\eta \in F \qquad \Longleftrightarrow \qquad \left(\frac{2\,\eta_1}{r}\right)^2 + \left(\frac{2\,|\eta'|}{\sqrt{r^2-1}}\right)^2 \le 1.$$

An easy computation yields

$$\mathrm{vol}\,(F) = c_n\,r\,(r^2-1)^{(n-1)/2},$$

where c_n equals 2^{-n} times the volume of the unit ball. □

Lemma 4.5. *Let* $\alpha > 0$*. Let* $f(t) = \int_t^\infty e^{-y}\,y^{\alpha-1}\,dy$*,* $t \ge 0$*. Denote by* g *the inverse function of* f*. Then* g *maps* $(0,\Gamma(\alpha)]$ *onto* $[0,\infty)$ *and*

$$\lim_{u\to 0}\frac{g(u)}{\log(1/u)} = 1. \tag{4.2}$$

Proof. The proof is an exercise in applying l'Hospital's rule:

$$\lim_{u\to 0}\frac{g(u)}{\log(1/u)} = -\lim_{t\to\infty} g'(f(t))\,f(t) = -\lim_{t\to\infty}\frac{f(t)}{f'(t)} = -\lim_{t\to\infty}\frac{f'(t)}{f''(t)} = 1.$$

□

With mathematical induction one can establish the following:

Lemma 4.6. *Let N be a natural number. Then the following identity holds:*

$$(a_1 \cdot a_2 \cdot \ldots \cdot a_N - 1) = \sum_{\ell=1}^{N} \sum_{\substack{j=(j_1,j_2,\ldots,j_\ell) \\ 0 \le j_1 < j_2 < \ldots < j_\ell \le N}} (a_{j_1} - 1) \ldots (a_{j_\ell} - 1),$$

for arbitrary complex numbers $a_1, \ldots, a_N$.

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