Annali di Matematica 182, 143–159 (2003)

Digital Object Identifier (DOI) 10.1007/s10231-002-0058-9

H.A. El-Morshedy · K. Gopalsamy

Oscillation and asymptotic behaviour of a class of higher-order non-linear difference equations

Received: September 5, 2001; in final form: March 7, 2002
Published online: April 14, 2003 –

Abstract. The dynamical characteristics of scalar difference equations of the form

$$x_{n+1} = f_1(x_{n-\tau_1}) + f_2(x_{n-\tau_2}), \qquad n = 0, 1, 2, \dots,$$

are investigated. A necessary and sufficient condition is obtained for all positive solutions to be oscillatory about a unique positive equilibrium point and sufficient criteria for the global attractivity of the equilibrium are established. Also, the stability and periodicity of more general equations are studied via comparison with the corresponding properties of an associated first-order non-linear equation.

Mathematics Subject Classification (2000). 39A11, 39A20, 92D25

Key words. global attractivity – periodicity – oscillation – difference equation

1. Introduction

The dynamical characteristics of the family of non-linear difference equations

$$x_{n+1} = h_\mu(x_n), \qquad n = 0, 1, 2, \dots, \tag{1}$$

where $\mu \in R$ is a parameter, $h_\mu \in C(I)$, $I \subseteq R$, have been investigated extensively by many authors. Some members of this family may look simple but their solutions display complicated behaviour (chaos) as the parameter μ increases beyond a certain critical value (see [5,16,17,21]). The famous result of Li and Yorke [12] confirms the occurrence of that chaotic behaviour if h_μ has period 3 point. On the other hand, Sharkovsky [20] (see also [12]) has pointed out that the lack of period 2 points of the map h_μ implies the absence of all periodic points of higher orders and hence (1) will not exhibit such chaotic behaviour in the sense of Li and Yorke [12]. Moreover, for some members of (1), the lack of period 2 points is also a sufficient condition for the unique equilibrium point to attract all solutions of (1). For instance, Cull [1–3] has studied a prototype of (1), namely

$$x_{n+1} = f(x_n), \qquad n = 0, 1, 2, \dots, \tag{2}$$

H.A. El-Morshedy: Department of Mathematics, Damietta Faculty of Science, New Damietta 34517, Egypt, e-mail: elmorshedy@hotmail.com

K. Gopalsamy: Department of Mathematics and Statistics, The Flinders University of South Australia, GPO Box 2100, Adelaide 5001, Australia

where f is a non-negative continuous function with unique non-negative fixed point $\bar{x}$ such that

$$f(x)\begin{cases} < x, & x > \bar{x} \\ > x, & x < \bar{x}, \end{cases} \tag{3}$$

and if $f'(x^*) = 0$ and $x^* < \bar{x}$, then

$$\begin{cases} f'(x) > 0 & \text{for} \quad 0 \le x < x^* \\ f'(x) < 0 & \text{for} \quad x > x^*. \end{cases} \tag{4}$$

Equation (2) in this case is called a population model. Cull's main result states that $\bar{x}$ is globally attractive, i.e., $\lim_{n\to\infty} x_n = \bar{x}$ regardless the magnitude of the initial value x_0, if and only if f has no period 2 point. He also found the following, more testable result:

Theorem 1. *The equilibrium point of a population model is globally attractive iff either (a) there is no maximum of $f(x)$ in $(0, \bar{x})$; or (b) $x^* < \bar{x}$ and $f(f(x)) > x$ for all $x \in [x^*, \bar{x})$.*

Using this result, Cull [1–3] has determined exact parametric regions of global attractivity in a number of population models. For example, the population model

$$x_{n+1} = x_n e^{r - x_n}, \quad r > 0, \quad n = 0, 1, 2, \ldots \tag{5}$$

has a unique positive equilibrium $\bar{x} = r$ that attracts all solutions of (5) if and only if $r \le 2$. This result improves those in [7, 16] by ensuring the global attractivity when $r = 2$. It is reasonable to ask whether Cull's technique can be extended to higher-order equations or not. In fact it is difficult to give an affirmative answer since the technique used to prove Theorem 1 depends on the fact that a population model is a first-order difference equation. Hence one of the main objectives of this article is to develop a technique that enables us to address the global attractivity for higher-order equations which contain the population models as a special cases. We consider the difference equation

$$x_{n+1} = f_1(x_{n-\tau_1}) + f_2(x_{n-\tau_2}), \qquad n = 0, 1, 2, \ldots, \tag{E}$$

where f_1, f_2 are positive continuous functions defined on $(0, \infty)$ and the delays τ_1, τ_2 are non-negative integers such that $\tau_1 \le \tau_2$. With any solution $\{x_n\}$ of (E) we associate a set of positive initial values $\{x_{-i}\}_{i=0}^{\tau_2}$. Therefore all solutions, considered in this work, will be positive. Without further mention, if nothing else is stated, we assume the existence of a real number x^* such that:

(C_1) f_1, f_2 are increasing on $(0, x^*)$;
(C_2) f_1, f_2 are decreasing on (x^*, ∞),

and the function $f = f_1 + f_2$ obeys condition (3). Also, in addition to the above assumptions, one of the following conditions will be needed:

(I) $x < f_i(x)$ for all $x \in (0, x^*)$, $i = 1, 2$ and $x^* < \bar{x}$;
(II) there exists a positive continuous function h and a positive constants a_1, a_2 such that $f_i(x) = a_i h(x)$, $i = 1, 2$ and $x > 0$.

From (C_1) and (C_2) we conclude that x^* is the absolute maximum of both f_1 and f_2 on $(0, \infty)$. Therefore for any solution $\{x_n\}$ of (E), we get

$$x_{n+1} \le f_1(x^*) + f_2(x^*) = f(x^*), \qquad n \ge 0.$$

For an easy reference the above conclusion is formalized as follows:

Proposition 1. *If $\{x_n\}$ is a solution of (E), then $x_n \le f(x^*)$ for all $n \ge 1$.*

A prototype of (E) is the equation

$$x_{n+1} = (1-\alpha)x_n e^{r-ax_n} + \alpha x_{n-1} e^{r-ax_{n-1}}, \quad a > 0,\ 0 \le \alpha \le 1,\ \text{and } r > 0, \quad (6)$$

which has been considered by MacDonald [13, 14] as a possible modification of the model (5). In (6) a fraction of the eggs is allowed to delay their hatching for two generations. If a similar assumption is made to the general population model (2), we get the equation

$$x_{n+1} = (1-\alpha) f(x_n) + \alpha f(x_{n-1}), \quad 0 \le \alpha \le 1. \qquad (7)$$

It follows that the population described by (7) has the ability to survive if one generation of the adult is wiped out in one year by a sudden environmental hardship. In this paper we show that such modification will not change the parametric region of global attractivity which is determined by Theorem 1 for equation (2).

The second objective of this work is to study the impact of the delays in (E) on the oscillatory behaviour of the solutions about the unique positive equilibrium. A solution $\{x_n\}$ $(x_n \not\equiv \bar{x})$ of (E) is said to be oscillatory about the equilibrium $\bar{x}$ (or simply oscillatory) if for every integer $N \ge 0$ there exists $n \ge N$ such that $(x_{n+1} - \bar{x})(x_n - \bar{x}) \le 0$. Otherwise, the solution is called non-oscillatory. This is equivalent to saying that $x_n < \bar{x}$ eventually or $x_n > \bar{x}$ eventually. Equation (E) is called oscillatory if all its solutions are oscillatory. We use a direct method to study the oscillation rather than using the linearizing technique developed in [6]. It will be shown that the delays are harmless on the oscillation of (E).

Finally, the stability of the equilibrium point and the existence of periodic solutions of a generalized version of (E) namely,

$$y_{n+1} = g_1(y_{n-\tau_1}) + g_2(y_{n-\tau_2}), \qquad n = 0, 1, 2, \dots,$$

will be investigated via comparison with a first-order equation where g_1, g_2 are real-valued continuous functions which need not satisfy (C_1) or (C_2). The above equation contains (E) as a special case as well as the equation

$$x_n = \alpha - \frac{\beta}{x_{n-1}} - \frac{\gamma}{x_{n-k}},$$

where $\alpha > 0, \beta, \gamma \in R$ and k is a positive integer, which appears in numerical analysis in constructing preconditioners for banded matrices (see, e.g. [4]).

Let $\Omega_\delta(\bar{x}) = \{x : |x - \bar{x}| < \delta\}$, the equilibrium point $\bar{x}$ of (1) is said to be stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $h_\mu^n(\Omega_\delta(\bar{x})) \subseteq \Omega_\epsilon(\bar{x})$ for all $n \geq 1$. For the above equation the equilibrium point $\bar{y}$ is said to be stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that any initial values $\{y_{-i}\}_{i=0}^{\tau_2} \subset \Omega_\delta(\bar{y})$ imply that $y_n \in \Omega_\epsilon(\bar{y})$, for all $n \geq 1$. If in addition to the stability of $\bar{y}$ there exists $\delta > 0$ such that $\bar{y}$ attracts all solutions that have all its initial values in $\Omega_\delta(\bar{y})$, then $\bar{y}$ is said to be (locally) asymptotically stable. The equilibrium point is said to be globally asymptotically stable if it is stable and attracts all solutions regardless of the magnitudes of their initial values.

2. The oscillatory behaviour

According to Proposition 1 we see that all solutions of equation (E) are bounded. Thus, for convenience, with any solution under consideration in this work, say x_n, we associate non-negative real numbers L, S such that $L = \liminf_{n\to\infty} x_n$, $S = \limsup_{n\to\infty} x_n$.

The following lemma will be a crucial tool in obtaining some of the main results in this paper.

Lemma 1. *Let $\{x_n\}$ be a solution of (E) and*

$$x^* < \bar{x}. \tag{8}$$

If either (I) *(or* (II)*) is satisfied, then*

$$L \geq f(S). \tag{9}$$

Proof. As $x_n > 0$ for all $n \geq 1$, then $L = \liminf_{n\to\infty} x_n \geq 0$. We claim that $L > 0$. To this end, suppose that $L = 0$. Then there exists a subsequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$ such that

$$x_{n_k+1} = \inf\{x_i : \tau_2 \leq i \leq n_k + 1\} \tag{10}$$

and

$$\lim_{k\to\infty} x_{n_k+1} = 0. \tag{11}$$

From (11) and equation (E), we get

$$\lim_{k\to\infty} f_1(x_{n_k-\tau_1}) + f_2(x_{n_k-\tau_2}) = 0,$$

which implies that

$$\lim_{k\to\infty} x_{n_k-\tau_1} = 0 = \lim_{k\to\infty} x_{n_k-\tau_2}. \tag{12}$$

It follows from (10) and (12) that

$$x_{n_k+1} \leq x_{n_k-\tau_1} < x^* \quad \text{and} \quad x_{n_k+1} \leq x_{n_k-\tau_2} < x^*, \quad k > K, \tag{13}$$

where K is a sufficiently large integer. Using (C_1) and (13), (E) yields

$$\begin{aligned} x_{n_k+1} &= f_1(x_{n_k-\tau_1}) + f_2(x_{n_k-\tau_2}) \\ &> f_1(x_{n_k+1}) + f_2(x_{n_k+1}) = f(x_{n_k+1}) > 0, \quad \text{for all} \quad k > K. \end{aligned}$$

According to (3), the above inequality implies that $x_{n_k+1} \geq \bar{x}$ which contradicts (11). Thus, $L > 0$ as desired.

Now consider the following possible cases for L:

$$\text{Case (1): } L \leq x^* \qquad \text{and} \qquad \text{Case (2): } L > x^*.$$

Suppose that Case (1) holds. Then for all $\epsilon > 0$ there exists N_ϵ such that

$$L_\epsilon = L - \epsilon \leq x_n \leq S + \epsilon = S_\epsilon, \qquad n \geq N_\epsilon. \tag{14}$$

First, we note that $S_\epsilon > x^*$ for all $\epsilon > 0$, since otherwise we have $L_\epsilon < S_\epsilon < x^*$ for all $0 < \epsilon < \epsilon_1$ and some $\epsilon_1 > 0$. Hence (C_1) and equation (E) imply that

$$x_{n+1} \geq f_1(L_\epsilon) + f_2(L_\epsilon) = f(L_\epsilon), \quad n \geq N_\epsilon + \tau_2,$$

for all $\epsilon < \epsilon_1$. The inferior limit of the above inequality yields

$$L \geq f(L_\epsilon),$$

then as $\epsilon \to 0$ we get $L \geq f(L)$ which is impossible due to (3) and the assumption that $L \leq x^* < \bar{x}$. Thus $S_\epsilon > x^*$ for all $\epsilon > 0$ as noted. This means that $x^* \in [L_\epsilon,\ S_\epsilon]$, therefore, using (14) it follows that the values of $x_n - \tau_1$ and $x_n - \tau_2$, for each $n \geq N_\epsilon + \tau_2$, have either one of the following cases:

$$x_{n-\tau_1},\ x_{n-\tau_2} \in [L_\epsilon,\ x^*], \quad x_{n-\tau_1},\ x_{n-\tau_2} \in [x^*,\ S_\epsilon],$$
$$x_{n-\tau_1} \in [L_\epsilon,\ x^*],\ x_{n-\tau_2} \in [x^*,\ S_\epsilon],\ \text{or}\ x_{n-\tau_1} \in [x^*,\ S_\epsilon],\ x_{n-\tau_2} \in [L_\epsilon,\ x^*].$$

From (C_1) and (C_2) we conclude that f_1 and f_2 are increasing on $[L_\epsilon,\ x^*]$ and decreasing on $[x^*,\ S_\epsilon]$, accordingly equation (E) implies the following inequalities in each of the above cases, respectively;

$$x_{n+1} \geq f(L_\epsilon), \qquad x_{n+1} \geq f(S_\epsilon), \quad x_{n+1} \geq f_1(L_\epsilon) + f_2(S_\epsilon)$$
$$\text{or} \quad x_{n+1} \geq f_1(S_\epsilon) + f_2(L_\epsilon),$$

for each $n \geq N_\epsilon + \tau_2$. Therefore, for all $n \geq N_\epsilon + \tau_2$, we have

$$x_{n+1} \geq \min\{f(L_\epsilon),\ f(S_\epsilon),\ f_1(L_\epsilon) + f_2(S_\epsilon),\ f_1(S_\epsilon) + f_2(L_\epsilon)\}. \tag{15}$$

Taking the inferior limit of x_{n+1} in (15), we get

$$L \geq \min\{f(L_\epsilon),\ f(S_\epsilon),\ f_1(L_\epsilon) + f_2(S_\epsilon),\ f_1(S_\epsilon) + f_2(L_\epsilon)\}.$$

Since ϵ is arbitrarily small, the above inequality implies that

$$L \geq \min\{f(L),\ f(S),\ f_1(L) + f_2(S),\ f_1(S) + f_2(L)\} = m. \tag{16}$$

If (I) is satisfied, then

$$f(L) > L, \quad f_1(L) + f_2(S) > f_1(L) > L \quad \text{and} \quad f_1(S) + f_2(L) > f_2(L) > L.$$

It follows from (16) that the only possible value of m is $m = f(S)$, i.e., (9) holds, which is our desired conclusion. On the other hand if (II) holds, then

$$\begin{aligned} m &= \min\{Ah(L),\ Ah(S),\ a_1h(L) + a_2h(S),\ a_1h(S) + a_2h(L)\} \\ &\geq A \min\{h(L),\ h(S)\}, \quad A = a_1 + a_2. \end{aligned}$$

By (16), we obtain

$$L \geq A \min\{h(L),\ h(S)\}. \tag{17}$$

If

$$\min\{h(L),\ h(S)\} = h(L),$$

then (17) implies that

$$L \geq Ah(L) = f(L),$$

which is impossible according to (3) and the assumption that $L \leq x^*$ (since $x^* < \bar{x}$ by (8)). Therefore, we have

$$L \geq Ah(S) = f(S),$$

which proves (9) when (II) is satisfied. This completes the proof when Case (1) holds.

Suppose that Case (2) holds. Then one can find a sufficiently large integer N such that $x^* < x_n$ for all $n > N$. Using the decreasing nature of $f_1(x)$, $f_2(x)$ on (x^*, ∞), the inferior limit of both sides of equation (E) yields $L \geq f(S)$. This completes the proof.

In the following result neither (I) nor (II) are assumed to be satisfied.

Lemma 2. *Assume that*

$$\bar{x} \leq x^*. \tag{18}$$

Then equation (E) is non-oscillatory.

Proof. Since f_1, f_2 are increasing on $(0, x^*)$, according to (18) they are also increasing on $(0, \bar{x})$. Choose a solution $\{x_n\}$ with initial values $\{x_{-i}\}_{i=0}^{\tau_2}$ such that $x_{-i} < \bar{x}$ for $i = 0, 1, 2, \ldots, \tau_2$. Then (E) implies that

$$\begin{aligned} x_1 &= f_1(x_{-\tau_1}) + f_2(x_{-\tau_2}) \\ &< f_1(\bar{x}) + f_2(\bar{x}) = f(\bar{x}) \\ &= \bar{x}. \end{aligned}$$

That is $x_1 \in (0,\ \bar{x})$. Taking into account that all initial values are taken from $(0,\ \bar{x})$, it follows that $x_{1-\tau_1},\ x_{1-\tau_2} \in (0,\ \bar{x})$. Thus, (E) yields

$$\begin{aligned} x_2 &= f_1(x_{1-\tau_1}) + f_2(x_{1-\tau_2}) \\ &< f_1(\bar{x}) + f_2(\bar{x}) = f(\bar{x}) \\ &= \bar{x}, \end{aligned}$$

which implies that $x_2 \in (0,\ \bar{x})$. By continuing the above process (or by induction), one can prove that $x_n \in (0,\ \bar{x})$ for all $n \geq 1$. Thus equation (E) has a non-oscillatory solution. The proof is complete.

Assume that (18) holds strictly and a solution $\{x_n\}$ is chosen such that some of its initial values are in $(0,\ \bar{x})$ and the rest are in $(\bar{x},\ x^*)$. What can be said about the oscillation of $\{x_n\}$ in this case? We believe that $\{x_n\}$ may oscillate; to explain this let us assume that $\tau_1 = \tau_2 = \tau \neq 0$, equation (E) becomes

$$x_{n+1} = f(x_{n-\tau}). \tag{19}$$

Recalling that f is increasing on $(0,\ x^*)$, and that the inequality $\bar{x} < x^*$ and (3) imply that $f(x^*) < x^*$, then for $x \in (0,\ \bar{x})$ we get

$$0 < f(x) < f(\bar{x}) = \bar{x},$$

while for $x \in (\bar{x},\ x^*)$ we have

$$\bar{x} = f(\bar{x}) < f(x) < f(x^*) < x^*.$$

Thus the intervals $(0,\ \bar{x})$ and $(\bar{x},\ x^*)$ are invariant under the map f, i.e.,

$$f((0,\ \bar{x})) \subseteq (0,\ \bar{x}) \quad \text{and} \quad f((\bar{x},\ x^*)) \subseteq (\bar{x},\ x^*). \tag{20}$$

Now substituting $n = k(\tau + 1)$ and $n = (k + 1)(\tau + 1) - 1$, $k = 0, 1, \ldots,$ in (19), the following two equations are obtained:

$$x_{k(\tau+1)+1} = f(x_{(k-1)\tau+k}) = f^{k+1}(x_{-\tau}), \quad k = 0, 1, \ldots ,$$

and

$$x_{(k+1)(\tau+1)} = f(x_{k(\tau+1)}) = f^{k+1}(x_0), \quad k = 0, 1, \ldots .$$

Let $x_{-\tau} \in (0,\ \bar{x})$ and $x_0 \in (\bar{x},\ x^*)$. Then the invariance property (20) and the above equations imply that

$$x_{k(\tau+1)+1} \in (0,\ \bar{x}) \quad \text{and} \quad x_{(k+1)(\tau+1)} \in (\bar{x},\ x^*), \quad k = 0, 1, \ldots ,$$

which means that the solution $\{x_n\}$ is oscillatory. Thus if (18) holds then equation (E), generally, may have oscillatory and non-oscillatory solutions. It should be mentioned that such property is not possible for the population model (2) because, when (18) holds, any solution will lie in one side of $\bar{x}$ for all $n \geq 1$ depending on the location of the initial value x_0.

Theorem 2. *Suppose that either* (I) *(or* (II)*) is satisfied. Then equation (E) is oscillatory if and only if* (8) *is satisfied.*

Proof. The necessity of (8) is an immediate consequence of Lemma 2. Next, we prove that (8) is also sufficient for the oscillation of all solutions of (E). Assume, for the sake of contradiction, that (E) has a non-oscillatory solution, say $\{x_n\}$. Then there exists an integer $N \geq 0$ such that one of the following is satisfied:

$$x_n > \bar{x}, \quad \text{for all} \quad n \geq N, \tag{21}$$

or

$$x_n < \bar{x}, \quad \text{for all} \quad n \geq N. \tag{22}$$

In view of (C_2) and (8) we see that f_1 and f_2 are decreasing on $(\bar{x}, \infty)$. Thus if (21) holds, equation (E) yields

$$\begin{aligned} \bar{x} < x_{n+1} &< f_1(\bar{x}) + f_2(\bar{x}) \\ &= f(\bar{x}) = \bar{x}, \quad n \geq N + \tau, \end{aligned}$$

which is impossible, therefore we have (22). Taking the superior limit of both sides of (22), we get

$$S \leq \bar{x}.$$

Using (3), the above inequality implies

$$f(S) \geq S. \tag{23}$$

Combining (9) and (23), we conclude that $L \geq f(S) \geq S$, which is possible only if $L = S$. Hence the uniqueness of $\bar{x}$ as a fixed point of f yields

$$\lim_{n\to\infty} x_n = \bar{x}.$$

It follows that for every $\epsilon \in (0,\ \bar{x} - x^*)$ there exists an integer $N_\epsilon > 0$ such that

$$x^* < \bar{x} - \epsilon \leq x_n < \bar{x}, \qquad n \geq N_\epsilon > 0. \tag{24}$$

Using (C_2), (E) implies that

$$\begin{aligned} x_{N_\epsilon+1+\tau_2} &= f_1(x_{N_\epsilon+\tau_2-\tau_1}) + f_2(x_{N_\epsilon}) \\ &> f_1(\bar{x}) + f_2(\bar{x}) = f(\bar{x}) = \bar{x}, \end{aligned}$$

which contradicts (24). Thus (22) is also impossible and hence equation (E) cannot have a non-oscillatory solution. This completes the proof.

3. The global attractivity

First we study the asymptotic behaviour of all solutions of equation (E) when $\bar{x} = 0$, i.e., the zero solution is the unique non-negative equilibrium. It should be observed that (I) cannot hold in this case since for all $x > 0$ we have $x > f(x) = f_1(x) + f_2(x)$. For the sake of generality we assume a continuous function g, $0 < g(x) < x$ on $(0, \infty)$ and $g(0) = 0$ such that

$$\text{(III)} \qquad f_i(x) = a_i g(x), \qquad i = 1, 2.$$

Note that the functions f_i need not have certain monotonic properties, as g is given without such limitation.

Theorem 3. *Assume that* $\{x_n\}$ *is any positive solution of* (E) *such that* (III) *holds. Then* $\lim_{n\to\infty} x_n = 0$.

Proof. Let $M = \max\{x_{-i} : i = 0, 1, 2, \ldots, \tau_2\}$. From (E) and (III) we obtain

$$\begin{aligned} x_1 &= a_1 g(x_{-\tau_1}) + a_2 g(x_{-\tau_2}) \\ &\leq \max\{f(x_{-\tau_1}),\ f(x_{-\tau_2})\} < M. \end{aligned}$$

Similarly

$$\begin{aligned} x_2 &= a_1 g(x_{1-\tau_1}) + a_2 g(x_{1-\tau_2}) \\ &\leq \max\{f(x_{1-\tau_1}),\ f(x_{1-\tau_2})\} < M. \end{aligned}$$

Continuing this process, one can prove that $x_n < M$ for all $n \geq 1$. Thus the superior limit of x_n at ∞ exists. We claim that $S = 0$. Then suppose this is not true; that is

$$S = \limsup_{n\to\infty}\ x_n > 0. \tag{25}$$

Then one can find a subsequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$ and some real numbers $a, b \geq 0$ such that

$$\lim_{k\to\infty} x_{n_k+1} = S,\ \lim_{k\to\infty} x_{n_k-\tau_1} = a,\ \text{and}\ \lim_{k\to\infty} x_{n_k-\tau_2} = b.$$

Then

$$\begin{aligned} S &= a_1 g(a) + a_2 g(b) \\ &\leq \max\{f(a),\ f(b)\} < a \text{ or } b, \end{aligned}$$

which is impossible since $a, b \leq S$. Thus $S = 0$, i.e., $\lim_{n\to\infty} x_n = 0$ as desired. The proof is complete.

If (III) does not hold, we can still find a similar result as Theorem 3 by making use of the monotonic properties of the functions f_1, f_2 as in the following result:

Theorem 4. *Assume that* $x = 0$ *is the unique non-negative fixed point of* f *and* $\{x_n\}$ *be a solution of equation* (E). *Then* $\lim_{n\to\infty} x_n = 0$.

Proof. First, since $f(x) = x$ only if $x = 0$ then the graph of the function f lies entirely on one side of the line $y = x$ in the first quadrant of the plane. Combining this with the fact that f is bounded, we conclude that $f(x) < x$ for all $x > 0$. Suppose, to the contrary, that (25) holds. Then Proposition 1 yields

$$x_n \leq f(x^*) < x^*, \qquad n \geq 1. \tag{26}$$

Using (26) and the increasing nature of f_1 and f_2 on $(0,\ x^*)$, equation (E) implies

$$\begin{aligned} S &\leq \limsup_{n\to\infty}\ f_1(x_{n-\tau_1}) + \limsup_{n\to\infty}\ f_2(x_{n-\tau_2}) \\ &\leq f_1(S) + f_2(S) = f(S). \end{aligned} \tag{27}$$

Thus $S \leq f(S) < S$, which is impossible and hence $S = 0$. The proof is complete.

Next, the case when (E) has a positive equilibrium point, i.e., $\bar{x} \neq 0$, is considered.

Theorem 5. *Assume that* (18) *is satisfied. Then* $\bar{x}$ *is a global attractor of the solutions of* (E).

Proof. Suppose that $\{x_n\}$ is a solution of (E). From (3), Proposition 1 and (18), we conclude that (26) holds. Since f_1 and f_2 are increasing on $(0,\ x^*)$, (27) holds, too. Furthermore, the inferior limit of both sides of (E) implies that

$$\begin{aligned} L &\leq \liminf_{n\to\infty}\ f_1(x_{n-\tau_1}) + \liminf_{n\to\infty}\ f_2(x_{n-\tau_2}) \\ &\leq f_1(L) + f_2(L) = f(L). \end{aligned} \tag{28}$$

From (3), (27), and (28) we obtain

$$S \leq \bar{x} \quad \text{and} \quad L \geq \bar{x}.$$

Thus $L = S = \bar{x}$, i.e., $\lim_{n\to\infty} x_n = \bar{x}$. The proof is complete.

Lemma 3. *Assume that* $\{x_n\}$ *is a solution of* (E). *If* (8), (I) *(or* (II)*) are satisfied, and*

$$f(f(x^*)) > x^*, \tag{29}$$

then

$$L > x^*. \tag{30}$$

Proof. Since all hypotheses of Lemma 1 are satisfied, then (9) holds which implies that $S \geq f(S)$, i.e., $S \geq \bar{x}$. On the other hand we see that $S \leq f(x^*)$ since $f(x^*)$ is an upper bound of all solutions of (E) (see Proposition 1). Thus $S \in (\bar{x},\ f(x^*)) \subset (x^*,\ \infty)$. Now in view of the decreasing nature of f on $(x^*,\ \infty)$ and condition (9) we obtain

$$L \geq f(S) \geq f(f(x^*)) > x^*.$$

The proof is complete.

Theorem 6. *Assume that* (8), (I) *(or* (III)*) are satisfied, and*

$$f(f(x)) > x \quad \textit{for all} \quad x \in [x^*,\ \bar{x}). \tag{31}$$

Then $\bar{x}$ is a global attractor of the solutions of (E).

Proof. Let $\{x_n\}$ be any solution of (E) such that $L \neq S$. Since all assumptions of Lemma 3 are satisfied then (30) holds and hence for every $\epsilon \in (0,\ L - x^*)$ there exists an integer $N_\epsilon > 0$ such that

$$x_n \geq L - \epsilon > x^*, \qquad n > N_\epsilon.$$

Recalling that f_1 and f_2 are decreasing on $(x^*,\ \infty)$, the superior and inferior limits of both sides of equation (E) imply, respectively, that

$$S \leq f(L) \quad \text{and} \quad L \geq f(S). \tag{32}$$

Then $L \leq f(L)$ and $S \geq f(S)$, i.e., $L \leq \bar{x}$ and $S \geq \bar{x} > x^*$. Note that, according to (32), if $L = \bar{x}$ then $S = \bar{x}$, which proves the theorem in this case. So, we assume that $L < \bar{x}$ and hence

$$L \in (x^*,\ \bar{x}) \subset (x^*,\ \infty). \tag{33}$$

Using the decreasing nature of f on $(x^*,\ \infty)$, (32) leads to $L \geq f(f(L))$ which is a contradiction because of (31) and (33), therefore $L = S$, which completes the proof.

Combining Theorems 1, 5, and 6, we obtain the following result:

Corollary 1. *Assume that the unique equilibrium $\bar{x}$ is a global attractor of all solutions of* (2)*. Then $\bar{x}$ is also a global attractor of all solutions of (E) where* (I) *(or* (II)*) is satisfied.*

Since the lack of period 2 points of the map f in $(0,\ \infty)$ is a necessary and sufficient condition for the global attractivity of the unique positive equilibrium of (2) (see [2,3]), Corollary 1 can be restated in the following form:

Corollary 2. *Assume that f has no period 2 point in* $(0,\ \infty)$*. Then $\bar{x}$ is a global attractor of all solutions of (E) where* (I) *(or* (II)*) is satisfied.*

The next result is given for practical purposes. Many similar criteria, that are sufficient for the validity of condition (31), can be found in [2,3].

Corollary 3. *Assume that* (8), (I) *(or* (II)*) are satisfied, and*

$$\frac{d}{dx}\frac{f(f(x))}{x} < 0 \quad \textit{for all} \quad x \in [x^*,\ \bar{x}). \tag{34}$$

Then $\bar{x}$ is a global attractor of the solutions of (E).

Proof. From (34) we conclude that

$$\frac{f(f(x))}{x} > \frac{f(f(\bar{x}))}{\bar{x}} = 1 \quad \text{for all} \quad x \in [x^*,\ \bar{x}).$$

Thus (31) holds and the proof is complete since all assumptions of Theorem 6 are satisfied.

4. Some general results for periodicity and stability

Next, we study the existence of periodic solutions as well as the stability of the equilibrium point of the equation

$$y_{n+1} = g_1(y_{n-\tau_1}) + g_2(y_{n-\tau_2}) \tag{35}$$

via comparison with the corresponding properties of the equilibrium point of the equation

$$x_{n+1} = g(x_n), \tag{36}$$

where $g_1, g_2 : I \to J$, $I, J \subseteq R$, $g = g_1 + g_2$ such that g has a unique fixed point, say $\bar{x}$, in I. In what follows $cd(\tau_1,\ \tau_2)$ stands for the set of all common divisors, greater than one, of the integers τ_1 and τ_2.

Theorem 7. *Assume that $\lambda \in cd(\tau_1,\ \tau_2)$. Then* (35) *has a λ-periodic solution if and only if* (36) *has a λ-periodic solution.*

Proof. Let $\{x_n\}$ be a solution of (36) such that $x_{n+\lambda} = x_n$, $n > 0$. Choose a solution $\{y_n\}$ of (35) with initial values $\{y_{-i}\}_{i=0}^{\tau 2}$ defined by

$$y_{i-\tau_2} = x_i, \quad i = 0, 1, \ldots, \tau_2,$$

hence

$$y_{i-\tau_1} = x_i, \quad i = 0, 1, \ldots, \tau_1.$$

So that (35) yields

$$\begin{aligned} y_{n+1} &= g_1(y_{n-\tau_1}) + g_2(y_{n-\tau_2}) \\ &= g_1(x_n) + g_2(x_n) = g(x_n) \\ &= x_{n+1}, \quad n \le \tau_1. \end{aligned}$$

Consequently, for $\tau_1 \le n \le 2\tau_1$, we have $y_{n-\tau_1} = x_n$ and $y_{n-\tau_1} = x_n$ (since $n - \tau_2 \le 2\tau_1 - \tau_2 \le \tau_1$), equation (35) implies that

$$\begin{aligned} y_{n+1} &= g_1(y_{n-\tau_1}) + g_2(y_{n-\tau_2}) \\ &= g_1(x_n) + g_2(x_n) = g(x_n) \\ &= x_{n+1}, \quad \tau_1 \le n \le 2\tau_1. \end{aligned}$$

Continuing the above process, or by induction, one can see that $y_{n+1} = x_{n+1}$ for all $n \in [m\tau_1,\ (m+1)\tau_1]$, $m = 0, 1, \ldots$, i.e., $y_n = x_n$ for all $n > 0$. Hence

$$\begin{aligned} y_{n+\lambda} &= x_{n+\lambda} = x_n \\ &= y_n, \quad \text{for all} \quad n > 0, \end{aligned}$$

that is (35) has a λ-periodic solution.

Now assume that (35) has a λ-periodic solution, say $\{y_n\}$. It follows from (35), with $n > \tau_2$, that

$$\begin{aligned} y_{n+1} &= g_1(y_{n-\tau_1}) + g_2(y_{n-\tau_2}) \\ &= g_1(y_n) + g_2(y_n) = g(y_n). \end{aligned}$$

Then a solution $\{x_n\}$ of (36) can be defined by

$$x_n = y_{n+\tau_2} \quad \text{for all} \quad n \geq 0.$$

Thus

$$x_{n+\lambda} = y_{n+\tau_2+\lambda} = y_{n+\tau_2} = x_n, \quad n \geq 0,$$

which completes the proof.

The following result is a combination of Corollary 2 and Theorem 7.

Corollary 4. *Assume that* (8) *holds and* $2 \in cd(\tau_1, \tau_2)$. *Then* $\bar{x}$ *is a global attractor of all solutions of* (E) *if and only if* f *has no period 2 point in* $(0, \infty)$.

In the case $2 \notin cd(\tau_1, \tau_2)$, the existence of a periodic solution of (E) is guaranteed by the following result which is derived from Theorem 7 by making use of the fact that f has periodic points of all orders if it has a period 3 point (see [12,20,21]).

Corollary 5. *Assume that* $\lambda \in cd(\tau_1, \tau_2)$. *If* f *has a period 3 point, then* (E) *has a periodic solution of period* λ.

We turn now to the stability properties of the equilibrium point of (35). Our main objective is to find sufficient conditions in terms of the non-linearities g_1, g_2 under which the equilibrium will be stable with respect to (35), provided that it is stable with respect to (36).

Lemma 4. *Assume that* $\bar{x} \in I$ *is a stable equilibrium point of* (36). *Then* $\bar{x}$ *is also a stable equilibrium point of* (35) *provided that*

$$\min\{g(y_{n-\tau_1}),\ g(y_{n-\tau_2})\} \leq y_{n+1} \leq \max\{g(y_{n-\tau_1}),\ g(y_{n-\tau_2})\}, \quad n > 0, \tag{37}$$

for any solution $\{y_n\}$ *of* (35) *such that* $\{y_{-i}\}_{i=0}^{\tau_2}$ *are close enough to* $\bar{x}$.

Proof. Since $\bar{x}$ is stable with respect to the solutions of (36), then for every ϵ there exists $\delta > 0$ such that $g^n(\Omega_\delta) \subseteq \Omega_\epsilon$ for all $n \geq 1$ where $\Omega_z = \{x : |x - \bar{x}| < z\}$. Let $\{y_n\}$ be any solution of (35) such that $\{y_{-i}\}_{i=0}^{\tau_2} \subset \Omega_\delta$. Then (37) yields

$$y_{n+1} \in g(\Omega_\delta) \subseteq \Omega_\epsilon, \quad n = 0, 1, \ldots, \tau_1. \tag{38}$$

For $\tau_1 < n \leq 2\tau_1$, it follows from (37) and (38) that

$$y_{n+1} \in g(\Omega_\delta) \cup g^2(\Omega_\delta) \subseteq \Omega_\epsilon.$$

Similarly, for $2\tau_1 < n \leq 3\tau_1$, we have

$$\begin{aligned} y_{n+1} &\in g\big(\Omega_\delta \cup g^2(\Omega_\delta)\big) \cup g(\Omega_\delta) \\ &= g(\Omega_\delta) \cup g^2(\Omega_\delta) \cup g^3(\Omega_\delta). \end{aligned}$$

By continuing the above process we get

$$y_{n+1} \in \cup_{i=1}^{m} g^i(\Omega_\delta), \quad (m-1)\tau_1 < n \leq m\tau_1, \quad m = 1, 2, \ldots . \tag{39}$$

Since $g^i(\Omega_\delta) \subseteq \Omega_\epsilon$ for all $i > 0$, we have $\cup_{i=1}^{m} g^i(\Omega_\delta) \subseteq \Omega_\epsilon$ for all $m \geq 1$ and hence (39) implies that $y_n \in \Omega_\epsilon$ for all $n \geq 1$, which in turn implies the stability of $\bar{x}$ as an equilibrium point of (35). This completes the proof.

Theorem 8. *Assume that* $\frac{g_1(x)}{a_1} = \frac{g_2(x)}{a_2}$ *for all* $x \in I$ *and some positive constants* a_1, a_2. *If* $\bar{x}$ *is a stable equilibrium point of* (36)*, then* $\bar{x}$ *is also a stable equilibrium point of* (35).

Proof. Consider a function G such that $G(x) = \frac{g_1(x)}{a_1}$ for all $x \in I$. Then (35) can be written in the form

$$y_{n+1} = a_1 G(x_{n-\tau_1}) + a_2 G(x_{n-\tau_2}), \quad n > 0,$$

and hence (37) holds with $g(x) = (a_1 + a_2)G(x)$. Therefore the proof follows by applying Lemma 4.

Note that no differentiability is required in Theorem 8, consequently Theorem 8 can be applied to some equations that cannot be tested by the linearizing technique. If the non-linearities of (35) are differentiable near $\bar{x}$, we have the following result:

Theorem 9. *Let* $\delta_1 > 0$ *be such that* $g_1, g_2 \in C^1(\Omega_{\delta_1})$ *and*

$$g_1'(x)\, g_2'(x) \geq 0, \quad \textit{for all} \quad x \in \Omega_{\delta_1}, \tag{40}$$

such that g_1' g_2' *do not change their signs on* Ω_{δ_1}. *If* $\bar{x}$ *is a stable equilibrium point of* (36)*, then it is also a stable equilibrium point of* (35).

Proof. From (40) we have either

$$g_1'(x) \geq 0, \; g_2'(x) \geq 0, \quad \text{for all} \quad x \in \Omega_{\delta_1}, \tag{41}$$

or

$$g_1'(x) \leq 0, \; g_2'(x) \leq 0, \quad \text{for all} \quad x \in \Omega_{\delta_1}. \tag{42}$$

The stability of $\bar{x}$ as an equilibrium point of (36) implies that for every $\epsilon < \delta_1$, there exists $\delta < \delta_1$ such that

$$g^n(\Omega_\delta) \subseteq \Omega_\epsilon, \quad n = 1, 2, \ldots .$$

If (41) holds, we assume that $\{y_n\}$ be any solution of (35) with $\{y_{-i}\}_{i=0}^{\tau_2} \subset \Omega_\delta$. The non-decreasing nature of g_1, g_2 on Ω_δ implies that

$$g_1(y_{n-\tau_1}) \leq g_1(y_{n-\tau_2}), \quad g_2(y_{n-\tau_1}) \leq g_2(y_{n-\tau_2}), \quad \text{if} \quad y_{n-\tau_1} \leq y_{n-\tau_2}$$

and

$$g_1(y_{n-\tau_1}) \geq g_1(y_{n-\tau_2}), \quad g_2(y_{n-\tau_1}) \geq g_2(y_{n-\tau_2}), \quad \text{if} \quad y_{n-\tau_1} \geq y_{n-\tau_2},$$

for each $n = 0, 1, \ldots, \tau_1$. From the previous two inequalities and (35), we obtain

$$\min\{g(y_{n-\tau_1}),\ g(y_{n-\tau_2})\} \leq y_{n+1} \leq \max\{g(y_{n-\tau_1}),\ g(y_{n-\tau_2})\}, \quad 0 \leq n \leq \tau_1.$$

Therefore,

$$y_{n+1} \in g(\Omega_\delta) \subseteq \Omega_\epsilon, \quad 0 \leq n \leq \tau_1.$$

Similarly one can show that (37) holds for $\tau_1 < n \leq 2\tau_1$ which implies that

$$y_{n+1} \in g(\Omega_\delta) \cup g^2(\Omega_\delta) \subseteq \Omega_\epsilon, \quad \tau_1 < n \leq 2\tau_1.$$

Continuing this process, it follows that (37) holds for $(m-1)\tau_1 < n \leq m\tau_1$, m is any positive integer, which in turn leads to

$$y_{n+1} \in \cup_{i=1}^{m} g^i(\Omega_\delta) \subseteq \Omega_\epsilon, \quad \text{for } (m-1)\tau_1 < n \leq m\tau_1.$$

Thus for every $\epsilon > 0$ a number $\delta > 0$ is found such that any solution $\{y_n\}$ of (35) with initial values inside Ω_δ satisfies that $y_n \in \Omega_\epsilon$ for all $n \geq 1$, i.e., $\bar{x}$ is a stable equilibrium point of (35). The case when (42) holds can be handled in the same fashion as the above case; we omit the details. The proof is complete.

Since $g_1',\ g_2'$ are continuous, it follows that (40) is satisfied provided that

$$g_1'(\bar{x}),\ g_2'(\bar{x}) > 0 \text{ or } g_1'(\bar{x}),\ g_2'(\bar{x}) < 0. \tag{43}$$

This leads to the following result:

Corollary 6. *Assume that $g_1',\ g_2'$ are continuously differentiable in a neighbourhood of $\bar{x}$. If* (43) *holds and $\bar{x}$ is a stable equilibrium point of equation* (36)*, then $\bar{x}$ is also a stable equilibrium point of equation* (35).

Sadaghat [19] has shown that any globally attractive fixed point of (36) must be stable (see also [15]). In view of this result, Corollary 1 and Corollary 6, we obtain the following result for equation (E):

Corollary 7. *Assume that the unique equilibrium $\bar{x}$ is a global attractor of all solutions of* (2). *Then $\bar{x}$ is a globally asymptotically stable equilibrium point of equation (E) where* (I) *(or* (II)*) is satisfied.*

Proof. Since $\bar{x}$ attracts all solutions of (2), then by [19] $\bar{x}$ is stable. Moreover, Corollary 1 implies that the equilibrium $\bar{x}$ of (E) is a global attractor. So, to complete the proof, it is enough to show that $\bar{x}$ is a stable equilibrium of (E). Since $f_1,\ f_2$ have and share a unique critical point x^*, then all requirements of Corollary 6 are satisfied if $\bar{x} \neq x^*$, so that $\bar{x}$ is stable in this case. When $\bar{x} = x^*$, the linear variational equation corresponding to (E) is $y_{n+1} = 0$, which has no solution other than the trivial one and then is asymptotically stable. Using the linearized stability theory we conclude that $\bar{x}$ is stable when $\bar{x} = x^*$. This completes the proof.

The following example is illustrative:

Example 1. Consider the higher-order version of (6) with $r = 2$, $a = 1$, i.e., the equation

$$x_{n+1} = (1-\alpha)x_n e^{2-x_n} + \alpha x_{n-\tau} e^{2-x_{n-\tau}}, \qquad 0 \le \alpha \le 1. \tag{44}$$

The linear variational equation associated with (44) is

$$y_{n+1} + (1-\alpha)y_n + \alpha y_{n-\tau} = 0. \tag{45}$$

The zero solution of (45) is known to be asymptotically stable if and only if all roots of the characteristic equation

$$\lambda^{\tau+1} + (1-\alpha)\lambda^{\tau} + \alpha = 0$$

lie inside the unit disc. Let τ be any even integer, then $\lambda = -1$ is a solution of the characteristic equation. So that the linearized stability theory fails to apply to (44). On the other hand, as mentioned in the introduction of this work, the equilibrium point of

$$x_{n+1} = x_n e^{2-x_n}$$

is a global attractor. Now applying Corollary 7, we get that the equilibrium point $\bar{x} = 2$ of (44) is globally asymptotically stable.

Acknowledgements. The authors would like to thank the referee for his/her helpful comments.

References

1. Cull, P.: Global stability of population models. Bull. Math. Biol. **43**, 47–58 (1981)
2. Cull, P.: Local and global stability of population models. Biol. Cybern. **54**, 141–149 (1986)
3. Cull, P.: Stability of discrete one-dimensional population models. Bull. Math. Biol. **50**, 67–75 (1988)
4. Di Lena, G., Trigiante, D.: Stability and spectral properties of incomplete factorization. Japan J. Appl. Math. **7**, 145–164 (1990)
5. Devany, R.L.: An Introduction to Chaotic Dynamical Systems. Addison-Wesley 1989
6. Györi, I., Ladas, G.: Oscillation Theory of Delay Differential Equations with applications. Clarendon Press, Oxford 1991
7. Goh, B.S.: Stability in stock-recruitment model of an exploited fishery. Math. Biosci. **33**, 359–372 (1977)
8. Huang, Y.N.: A note on stability of discrete population models. Math. Biosci. **95**, 189–198 (1989)
9. Huang, Y.N.: A note on global stability for discrete one dimensional population models. Math. Biosci. **102**, 121–124 (1990)
10. Kruklis, S.A.: The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$. J. Math. Anal. Appl. **188**, 719–731 (1994)
11. Levin, S.H.: Discrete time modeling of ecosystems with applications in environmental enrichment. Math. Biosci. **24**, 307–317 (1975)
12. Li, T., Yorke, J.: Period 3 implies chaos. Am. Math. Mon. **82**, 985–992 (1975)

13. MacDonald, N.: Extended diapause in a discrete generation population model. Math. Biosci. **31**, 255–257 (1976)
14. MacDonald, N.: Time lags in biological models. Lecture Notes in Biomathematics. Springer-Verlag, Berlin 1978
15. Martelli, M., Marshall, D.: Stability and attractivity in discrete dynamical systems. Math. Biosci. **128**, 347–355 (1995)
16. May, R.M.: Biological populations with nonoverlapping generations: stable points, stable cycles, and chaos. Science **186**, 645–647 (1974)
17. May, R.M.: Simple mathematical models with very complicated dynamics. Nature **261**, 459–467 (1976)
18. Rozenkranz, G.: On global stability discrete population models. Math. Biosci. **64**, 227–231 (1983)
19. Sadaghat, H.: The impossibility of unstable, globally attractive fixed points for continuous mappings of the line. Am. Math. Mon. **104**, 356–358 (1997)
20. Sharkovsky, A.N.: Coexistence of cycles of a continuous map of a line into itself. Ukr. Mat. Zh. **16**, 61–71 (1964)
21. Sharkovsky, A.N., Maistrenko Yu.L., Romanenko, E.Yu.: Difference equations and their applications. Kluwer Academic Publishers, Dordrecht 1993
22. Silvert, W.: Anomalous enhancement of mean population levels by harvesting. Math. Biosci. **42**, 253–256 (1978)