Annali di Matematica 182, 37–52 (2003)

Digital Object Identifier (DOI) 10.1007/s10231-002-0054-0

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Continuous modified Newton's-type method for nonlinear operator equations

Received: June 29, 2001; in final form: February 26, 2002
Published online: February 20, 2003 –

Abstract. A nonlinear operator equation $F(x) = 0$, $F : H \to H$, in a Hilbert space is considered. Continuous Newton's-type procedures based on a construction of a dynamical system with the trajectory starting at some initial point x_0 and becoming asymptotically close to a solution of $F(x) = 0$ as $t \to +\infty$ are discussed. Well-posed and ill-posed problems are investigated.

Mathematics Subject Classification (2000). 65J15, 58C15, 47H17

Key words. nonlinear problem – integral inequality – Fréchet derivative – Newton method

1. Introduction

The theme of this paper is solving nonlinear operator equations of the form:

$$F(x) = 0, \quad F : H \to H, \tag{1.1}$$

in a real Hilbert space H. We consider a real Hilbert space for the sake of simplicity: numerical algorithms for solving (1.1) in a complex Hilbert space can be treated similarly. In order to approximate a solution to equation (1.1) we use the idea developed in [2]–[4], which consists of constructing a dynamical system with the trajectory starting at some initial point x_0 and converging to a solution of (1.1) as $t \to +\infty$. This idea, in its simplest form goes back to A.Cauchy (steepest descent) and was proposed in [5] for solving some optimization problems by a continuous analog of the gradient method. In [9] a wide class of linear ill-posed problems was studied by the dynamical systems method, and in [11] a recent development of the dynamical systems method (DSM) is presented. In [6] a continuous Newton's scheme,

$$\dot{x}(t) = -[F'(x(t))]^{-1}F(x(t)), \quad x(0) = x_0 \in H, \tag{1.2}$$

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This paper was finished when AGR was visiting Institute for Theoretical Physics, University of Giessen. The author thanks DAAD for support

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was studied and a theorem establishing convergence with the exponential rate was proved. A modified continuous Newton's method is proposed in [1]:

$$(1.3)\quad \dot{x}(t) = -J(t)F(x(t)), \quad x(0) = x_0 \in H, \quad J(0) \in L(H),$$

$$(1.4)\quad \dot{J}(t) = -\mu\big[F'^*(x(t))F'(x(t))J(t) + J(t)F'(x(t))F'^*(x(t))\big] + 2\mu F'^*(x(t)),$$

where μ is a positive constant. System (1.3)–(1.4) avoids the inversion of the Fréchet derivative $F'(x)$, which is numerically difficult in some applications.

The regularized Gauss–Newton's-type algorithm with simultaneous updates of the operator $[F'^*(x(t))F'(x(t)) + \varepsilon(t)I]^{-1}$ was proposed in [8]:

$$(1.5)\qquad \dot{x}(t) = -D(t)\big[F'^*(x(t))F(x(t)) + \varepsilon(t)(x(t) - x_0)\big],$$

$$(1.6)\qquad \dot{D}(t) = -\big[(F'^*(x(t))F'(x(t)) + \varepsilon(t)I)D(t) - I\big],$$

$$x(0) = x_0 \in H, \quad D(0) \in L(H), \quad 0 < \varepsilon(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

It is shown that $x(t)$ converges to a solution of (1.1) at the rate $O(\varepsilon(t))$. The convergence theorem is proved without assuming the monotonicity of F and bounded invertibility of $F'(x)$.

In [3] and [4] a fairly general approach to the analysis of continuous procedures in a Hilbert space was developed. According to this approach one investigates a solution to the Cauchy problem for a nonlinear operator-differential equation by using differential inequalities. In the well-posed case (the Fréchet derivative operator $F'(x)$ is boundedly invertible in a ball, which contains one of the solutions) one investigates the Cauchy problem for an autonomous equation:

$$(1.7)\qquad \dot{x}(t) = \Phi(x(t)), \quad x(0) = x_0.$$

The choice of $\Phi : H \to H$ yields a corresponding continuous process. In the ill-posed case ($F'(x)$ has a nontrivial null-space at the solution (1.1) or is not boundedly invertible) a regularized continuous procedure is required. For this reason the Cauchy problem for the following equation is to be analysed:

$$(1.8)\qquad \dot{x}(t) = \Phi(x(t), t), \quad x(0) = x_0,$$

with $\Phi : H \times [0, +\infty) \to H$. If one takes

$$\Phi(h, t) := -[F'(h) + \varepsilon(t)]^{-1}(F(h) + \varepsilon(t)(h - x_0)),$$

then one arrives at a continuously regularized Newton's scheme (CRNS). The convergence analysis of CRNS is done in [3] under the assumption that $F'(x) \geq 0$ as an operator in H. For

$$\Phi(h, t) := -[F'^*(h)F'(h) + \varepsilon(t)]^{-1}(F'^*(h)F(h) + \varepsilon(t)(h - x_0))$$

one obtains the continuously regularized Gauss–Newton's scheme (CRGNS). The convergence theorems for CRGNS (see [2] and [3]) do not use any assumption about the location of the spectrum of $F'(x)$. The absence of such assumption is made possible by source-type conditions.

In Section 2 of our paper we study a continuous analog of a modified Newton's method:

$$\dot{x}(t) = -[F'(x_0)]^{-1}F(x(t)), \quad x(0) = x_0 \in H, \tag{1.9}$$

for solving the well-posed nonlinear operator equation (1.1). Theorem 2.3 establishes exponential convergence of (1.9) to a solution of (1.1). Process (1.9) can be used in practical computations when the calculating and inverting of $F'(x)$ at each moment of time require a considerable effort. Another continuous algorithm, investigated in Section 2,

$$\dot{x}(t) = -B(t)F(x(t)), \quad x(0) = x_0 \in H, \quad B(0) = B_0 \in L(H), \tag{1.10}$$

$$\dot{B}(t) = -F'^*(x(t))F'(x(t))B(t) + F'^*(x(t)), \tag{1.11}$$

can also be recommended in the above situation. It allows one to update $[F'(x)]^{-1}$ continuously for $t \in [0, +\infty)$ without actual inversion of the Fréchet derivative. In Theorem 2.6 the exponential convergence of (1.10)–(1.11) to a solution of (1.1) is proved.

For many important inverse problems of the form (1.1) the operator $F'(x)$ is not boundedly invertible. For such problems the regularized version of algorithm (1.9) is suggested in Section 3:

$$\begin{aligned} \dot{x}(t) = -[F'(x_0) + \varepsilon(t)I]^{-1}(F(x(t)) + \varepsilon(t)(x(t) - x_0)), \\ x(0) = x_0 \in H, \quad \varepsilon(t) > 0. \end{aligned} \tag{1.12}$$

The convergence analysis of the continuous regularized method (1.12) is done in Theorem 3.1 under the following assumption:

$$F'(x) = F'(x_0)G(x_0, x), \tag{1.13}$$

where

$$||G(x_0, x) - I|| \le C(G)||x_0 - x||, \quad x_0, x \in U(\rho, \hat{x}), \tag{1.14}$$

and $\hat{x}$ is a solution to (1.1). Assumption (1.13)–(1.14) is similar to condition (8) in [7]. It means that the operator $F'(x)$ remains, in principle, the same for all x in a neighborhood of a solution up to some modification by a linear operator $G(x_0, x)$. The reader may consult [7] for several examples of nonlinear inverse problems for which condition (1.13)–(1.14) can be verified. As a consequence of Theorem 3.1 we obtain the stability of process (1.12) towards noise in the data and choose an optimal regularization parameter (the stopping time) such that the method converges to a solution of (1.1) when the noise level tends to zero.

Our main motivations for this investigation are:

1) We think that the DSM (dynamical systems method) that we develop in this paper (and in the earlier publications, cited in the references) is not only of theoretical interest, but also provides a powerful numerical tool for solving a very wide variety of problems, namely all the problems which can be described by equation (1.1) with the nonlinearity satifying the assumptions of our theorems formulated in Sections 2 and 3.

2) We think that the idea of constructing a method for solving equation (1.1) which does not require inverting $F'(u)$ (see, for example, equations (2.21) and (2.22) below) is of both theoretical and practical interest even for well-posed problems.

3) The DSM gives a general approach to constructing convergent iterative methods for solving ill-posed nonlinear problems. We do not address this part of the DSM in our present paper, but it has been addressed in detail in [4].

Finally we note that the DSM was tested numerically (see [2], [3], [10] for example), but it is certainly of interest to study much more the numerical performance of DSM. In this paper, however, the authors deal with the theoretical questions.

2. Continuous modified Newton's schemes for well-posed problems

In this section we solve the nonlinear operator equation (1.1) under the assumption that the Fréchet derivative of the operator F is boundedly invertible in a ball which contains one of the solutions. Let x_0 be an initial approximation for a solution to (1.1) and $x(t)$ be a trajectory of the autonomous dynamical system

$$\dot{x}(t) = \Phi(x(t)), \quad 0 \leq t < +\infty, \quad x(0) = x_0. \tag{2.1}$$

Lemma 2.1 below (see [4]) gives simple sufficient conditions on nonlinear operators F in (1.1) and Φ in (2.1) which guarantee that:

(a) the initial value problem (2.1) is uniquely solvable for all $t \in [0, +\infty)$;
(b) the solution $x(t)$ tends to one of the solutions of (1.1) as $t \to +\infty$.

Lemma 2.1. *Let H be a real Hilbert space, $F, \Phi : H \to H$.*

Suppose that there exist some positive numbers c_1 and c_2 such that F and Φ are Fréchet differentiable in $U(r, x_0) := \{x \in H, \ ||x - x_0|| \leq r\}$, $r := \frac{c_2||F(x_0)||}{c_1}$ and $\forall h \in U(r, x_0)$ the following conditions hold:

$$(F'(h)\Phi(h)), F(h)) \leq -c_1||F(h)||^2, \tag{2.2}$$

and

$$||\Phi(h)|| \leq c_2||F(h)||. \tag{2.3}$$

Then:
1. there exists a global solution $x = x(t)$ to problem (2.1) *in the ball $U(r, x_0)$;*
2. there exists

$$\lim_{t\to+\infty} x(t) = \hat{x}, \tag{2.4}$$

where $\hat{x}$ is a solution to (1.1) *in $U(r, x_0)$, and*

$$||x(t) - \hat{x}|| \leq re^{-c_1 t}, \tag{2.5}$$

$$||F(x(t))|| \leq ||F(x_0)||e^{-c_1 t}. \tag{2.6}$$

Proof. From the Fréchet differentiability of Φ the local existence of a solution to (2.1) follows, and from (2.1) one gets:

$$\frac{d}{dt}\{F(x(t))\} = F'(x(t))\dot{x}(t) = F'(x(t))\Phi(x(t)). \tag{2.7}$$

Let $\lambda(t) := F(x(t))$. Then

$$\dot{\lambda}(t) = F'(x(t))\Phi(x(t)), \quad \lambda(0) = F(x_0). \tag{2.8}$$

At least for sufficiently small t, for which $x(t) \in U(r, x_0)$, one can use estimate (2.2) and get:

$$\frac{1}{2}\frac{d}{dt}||\lambda(t)||^2 = (\dot{\lambda}(t), \lambda(t)) = (F'(x(t))\Phi(x(t)), F(x(t))) \leq -c_1||\lambda(t)||^2. \tag{2.9}$$

Thus, at least for sufficiently small $t \geq 0$, one gets:

$$||\lambda(t)|| \leq ||F(x_0)||e^{-c_1 t}. \tag{2.10}$$

For $0 \leq t_1 \leq t_2$ by (2.3) one has

$$||x(t_2) - x(t_1)|| \leq \left\| \int_{t_1}^{t_2} \dot{x}(s)ds \right\| \leq \int_{t_1}^{t_2} ||\Phi(x(s))||ds \leq c_2 \int_{t_1}^{t_2} ||\lambda(s)||ds$$

$$\leq \frac{c_2||F(x_0)||}{c_1}\left(e^{-c_1 t_1} - e^{-c_1 t_2}\right) < \frac{c_2||F(x_0)||}{c_1}e^{-c_1 t_1}. \tag{2.11}$$

Setting $t_1 = 0$ and $t_2 = t$, one concludes from (2.11) that $x(t) \in U(r, x_0)$ with $r = \frac{c_2||F(x_0)||}{c_1}$ whenever it is defined. Therefore the standard argument yields the existence and uniqueness of a solution to (2.1) on $[0, +\infty)$. Now in (2.11) let $t_1 = t$ and $t_2 \to +\infty$. Then one gets (2.5), and the limit $\hat{x}$ in (2.4) does exist due to (2.11). From (2.10) one concludes that $\hat{x}$ is a solution to (1.1). Inequality (2.6) follows from (2.10). Lemma 2.1 is proved. □

Remark 2.2. The assumptions of Theorem 2.3 do not imply the uniqueness of a solution to equation (1.1). If (1.1) is not uniquely solvable then $x(t)$ converges to one of its solutions in $U(r, x_0)$.

In Lemma 2.1 we have assumed that c_1 and c_2 are some known constants in the ball $U(r, x_0)$, and this assumption allowed us to define r explicitly in terms of the ratio $\frac{c_2}{c_1}$ and $||F(x_0)||$. One may assume that $\frac{c_2}{c_1}$ is not a constant but a function of r, $\frac{c_2}{c_1} := c(r)$. In this case , in order that the argument of Lemma 2.1 be valid, one has to satisfy the inequality $c(r)||F(x_0)|| \leq r$. For example, if $\frac{c(r)}{r} \to 0$ as $r \to \infty$, then there always exists an $r > 0$ such that the conclusion of Lemma 2.1 holds and the assumptions of this lemma are satisfied in the ball $U(r, x_0)$.

Now consider the following continuous modified Newton's scheme:

$$\dot{x}(t) = -[F'(x_0)]^{-1}F(x(t)), \quad x(0) = x_0 \in H. \tag{2.12}$$

Theorem 2.3 below establishes a relation between the asymptotic behavior of a solution $x(t)$ to (2.12) and solutions to equation (1.1). It is a consequence of Lemma 2.1.

Theorem 2.3. *Let H be a real Hilbert space, $F : H \to H$. Assume that F is Fréchet differentiable, its Fréchet derivative F' is Lipschitz-continuous:*

$$||F'(x_1) - F'(x_2)|| \le M_2||x_1 - x_2|| \quad \forall x_1, x_2 \in U(\tilde{r}, x_0), \tag{2.13}$$

where

$$U(\tilde{r}, x_0) := \{x \in H,\ ||x - x_0|| \le \tilde{r}\}, \quad \tilde{r} := \frac{1}{2M_2 m_1}, \quad m_1 := ||[F'(x_0)]^{-1}||, \tag{2.14}$$

and

$$4M_2 m_1^2||F(x_0)|| \le 1. \tag{2.15}$$

Then:
1. there exists an unique solution $x = x(t)$, $t \in [0, +\infty)$, to problem (2.12).
2. $x(t) \in U(\tilde{r}, x_0) \quad \forall t \in [0, +\infty)$, and

$$\lim_{t \to +\infty} x(t) = \hat{x}, \tag{2.16}$$

where $\hat{x}$ is a solution to (1.1).
3. the following estimates hold:

$$||x(t) - \hat{x}|| \le 2m_1||F(x_0)||e^{-\frac{t}{2}}, \tag{2.17}$$

$$||F(x(t))|| \le ||F(x_0)||e^{-\frac{t}{2}}. \tag{2.18}$$

Proof. Take

$$\Phi(x(t)) := -[F'(x_0)]^{-1}F(x(t)). \tag{2.19}$$

Then, under assumptions (2.13) and (2.14) of Theorem 2.3, one gets $\forall h \in U(\tilde{r}, x_0)$

$$\begin{aligned}(F'(h)\Phi(h), F(h)) &= -(F'(h)[F'(x_0)]^{-1}F(h), F(h))\\ &= -||F(h)||^2 + (\{I - F'(h)[F'(x_0)]^{-1}\}F(h), F(h))\\ &= -||F(h)||^2 + (\{F'(x_0) - F'(h)\}[F'(x_0)]^{-1}F(h), F(h))\\ &\le -||F(h)||^2 + M_2 m_1 \tilde{r}||F(h)||^2\\ &= -\frac{1}{2}||F(h)||^2. \end{aligned} \tag{2.20}$$

Also one has $||\Phi(h)|| \le m_1||F(h)||$. Thus conditions (2.2) and (2.3) of Lemma 2.1 hold for any $h \in U(\tilde{r}, x_0)$ with Φ defined in (2.19), $c_1 = \frac{1}{2}$ and $c_2 = m_1$. Hence $r := \frac{c_2||F(x_0)||}{c_1} = 2m_1||F(x_0)||$. From (2.14) and (2.15) one has $2m_1||F(x_0)|| \le \frac{1}{2M_2 m_1} := \tilde{r}$. Therefore (2.2) and (2.3) are satisfied on $U(r, x_0)$, $r := \frac{c_2||F(x_0)||}{c_1}$, $c_1 = \frac{1}{2}$, $c_2 = m_1$. Applying Lemma 2.1, one completes the proof. □

Remark 2.4. (a) Choosing $\Phi(h) = -[F'(h)]^{-1}F(h)$ one gets the continuous Newton's method. In this case $c_1 = 1$, $c_2 = \mu_1 := \sup_{x \in U(r,x_0)} ||[F'(x)]^{-1}||$, and Lemma 2.1 yields the convergence theorem for the continuous Newton's method [6].
(b) Choosing $\Phi(h) = -F(h)$, one gets a simple iteration method, for which condition (2.2) means strict monotonicity of F: $F' \geq c_1 > 0$, and $c_2 = 1$.
(c) $\Phi(h) = -[F'(h)]^* F(h)$ corresponds to the gradient method.
Here $c_2 = M_1 := \sup_{x \in U(r,x_0)} ||F'(x)||$, and $c_1 = \mu_1^{-2}$.
(d) $\Phi(h) = -[F'^*(h)F'(h)]^{-1}F'^*(h)F(h)$ yields the continuous Gauss–Newton's scheme. Here $c_1 = 1$, $c_2 = \mu_1^2 M_1$, where μ_1 is the same as in (a) above, and M_1 is the same as in (c) above.

In order to avoid inversion of the Fréchet derivative $F'(x(t))$ even at the initial moment $t = 0$, one can consider the following algorithm, which is a Cauchy problem for a system of two equations:

$$\dot{x}(t) = -B(t)F(x(t)), \quad x(0) = x_0 \in H, \quad B(0) = B_0 \in L(H), \tag{2.21}$$

$$\dot{B}(t) = -F'^*(x(t))F'(x(t))B(t) + F'^*(x(t)). \tag{2.22}$$

Equation (2.22) is similar to equation (1.3) in [9]. To prove Theorem 2.6 below we use the following lemma, which is an operator-theoretical version of the Gronwall inequality:

Lemma 2.5. *Let*

$$\frac{dV}{dt} + A(t)V(t) = G(t), \quad V(0) = V_0, \tag{2.23}$$

where $A(t)$, $G(t)$, $V(t) \in L(H)$, $L(H)$ is the set of linear bounded operators on H, and H is a real Hilbert space. If there exists a scalar function $\zeta(t) > 0$, $\zeta \in L^1_{loc}(0, \infty)$, such that

$$(A(t)h, h) \geq \zeta(t)||h||^2 \quad \forall h \in H, \tag{2.24}$$

then

$$||V(t)|| \leq e^{-\int_0^t \zeta(p)dp} \left[\int_0^t ||G(s)|| e^{\int_0^s \zeta(p)dp} ds + ||V(0)|| \right]. \tag{2.25}$$

Proof. (see [8]) Take any $h \in H$. Since H is a real Hilbert space one has:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||V(t)h||^2 &= \left(\frac{dV}{dt}h, V(t)h\right) \\ &= -(A(t)V(t)h, V(t)h) + (G(t)h, V(t)h) \\ &\leq -\zeta(t)||V(t)h||^2 + ||G(t)|| \; ||h|| \; ||V(t)h||. \end{aligned} \tag{2.26}$$

Denote $v(t) := ||V(t)h||$. Inequality (2.26) implies

$$v\dot{v} \le -\zeta(t)v^2 + ||G(t)||\ ||h||\ v. \tag{2.27}$$

Divide this inequality by the non-negative v and get a linear first-order differential inequality from which one gets (2.25). Lemma 2.5 is proved. □

Theorem 2.6. *Let H be a real Hilbert space, $F : H \to H$. Assume that:*
1. $U(R, x_0) := \{x \in H,\ ||x - x_0|| \le R\}$, the operator F is twice Fréchet differentiable, $F'(x)$ is boundedly invertible, and

$$||F'(x)|| \le M_1, \quad ||F''(x)|| \le M_2, \quad ||[F'(x)]^{-1}||^2 \le \frac{1}{c} \quad \forall x \in U(R, x_0), \tag{2.28}$$

where

$$R := \frac{\gamma c}{2M_1 M_2 \sigma^2}, \quad \gamma := \frac{1 - ||F'(x_0)B_0 - I||}{2} > 0, \quad \sigma := \frac{M_1}{c} + ||B_0||. \tag{2.29}$$

2. Equation (1.1) *is solvable in $U(R, x_0)$ (not necessarily uniquely), and $\hat{x}$ is a solution.*
3. $F(x_0)$ satisfies the following condition

$$\left\{\frac{2M_1 M_2}{c}\sigma^3 ||F(x_0)||\right\}^{1/2} \le \gamma. \tag{2.30}$$

Then:
1. there exists a unique solution $(x(t), B(t))$, $t \in [0, +\infty)$, to problem (2.21)–(2.22)*;*
2. $x(t) \in U(R, x_0) \quad \forall t \in [0, +\infty)$;
3. the following estimates hold:

$$||x(t) - \hat{x}|| \le \frac{\sigma ||F(x_0)||}{\gamma}\ e^{-\gamma t}, \tag{2.31}$$

$$||F(x(t))|| \le ||F(x_0)||\ e^{-\gamma t}, \tag{2.32}$$

$$||F'(x(t))B(t) - I|| \le (M_2 ||F(x_0)||\sigma^2 t + ||F'(x_0)B_0 - I||)\, e^{-ct}, \quad \text{if} \quad c = \gamma, \tag{2.33}$$

$$||F'(x(t))B(t) - I|| \le \left(\frac{M_2 ||F(x_0)||\sigma^2}{|c - \gamma|} + ||F'(x_0)B_0 - I||\right) e^{-\min\{\gamma, c\}\, t}, \quad \text{if} \quad c \ne \gamma. \tag{2.34}$$

Proof. Under the assumptions of Theorem 2.6 there exists a unique solution $(x(t), B(t))$ to (2.21)–(2.22) on some interval $[0, \tau]$, and, at least for sufficiently small $t > 0$, $x(t) \in U(R, x_0)$. Since $\forall x \in U(R, x_0)$ and $\forall h \in H$

$$(F'(x)F'^*(x)h, h) \geq \frac{1}{||[F'(x)]^{-1}||^2}\, ||h||^2 \geq c||h||^2, \tag{2.35}$$

by (2.22) and Lemma 2.5 one gets

$$||B(t)|| \leq e^{-ct}\left[\int_0^t ||F'^*(x(s))||e^{cs}\, ds + ||B(0)||\right].$$

Thus by (2.28) and (2.29)

$$||B(t)|| \leq \frac{M_1}{c}\left(1 - e^{-ct}\right) + ||B_0||e^{-ct} \leq \frac{M_1}{c} + ||B_0|| := \sigma. \tag{2.36}$$

Let us analyse the initial value problem for $w(t) := F(x(t))$. One has

$$\dot{w}(t) = F'(x(t))\dot{x}(t) = -F'(x(t))B(t)w(t).$$

Therefore

$$\dot{w}(t) + w(t) + [F'(x(t))B(t) - I]w(t) = 0, \quad w(0) = F(x_0). \tag{2.37}$$

Denote $W(t) := F'(x(t))B(t) - I$. Then

$$\begin{aligned}\dot{W}(t) &= F''(x(t))\dot{x}(t)B(t) + F'(x(t))\dot{B}(t)\\ &= -F''(x(t))B(t)F(x(t))B(t) + F'(x(t))[-F'^*(x(t))F'(x(t))B(t) + F'^*(x(t))]\\ &= -F''(x(t))B(t)F(x(t))B(t) - F'(x(t))F'^*(x(t))W(t).\end{aligned}$$

Consider the problem

$$\dot{W}(t) + F'(x(t))F'^*(x(t))W(t) = -F''(x(t))B(t)F(x(t))B(t), \tag{2.38}$$

$$W(0) = F'(x_0)B(0) - I. \tag{2.39}$$

From (2.35), (2.38)–(2.39) and Lemma 2.5 one obtains the estimate

$$||W(t)|| \leq e^{-ct}\left[\int_0^t ||F''(x(s))B(s)F(x(s))B(s)||e^{cs}\, ds + ||W(0)||\right]. \tag{2.40}$$

Assumptions 1 and 2 of Theorem 2.6 yield

$$||F(x(t))|| \leq ||F(x(t)) - F(x_0)|| + ||F(x_0) - F(\hat{x})|| \leq 2M_1R,$$

for all values of t such that $x(t) \in U(R, x_0)$.

Thus:

$$\begin{aligned}||W(t)|| \leq \frac{2M_1M_2R}{c}\left(\frac{M_1}{c} + ||B(0)||\right)^2 + ||W(0)|| &= \frac{2M_1M_2R\sigma^2}{c} + ||W(0)||\\ &= \gamma + ||W(0)||.\end{aligned}$$

Hence one gets, from (2.29),

$$||W(t)|| \le \frac{1-||W(0)||}{2} + ||W(0)|| = \frac{1+||W(0)||}{2}. \tag{2.41}$$

Now one has the following differential inequality:

$$\frac{1}{2}\frac{d}{dt}||w(t)||^2 = -||w(t)||^2 - (W(t)w(t), w(t)) \le -\frac{1-||W(0)||}{2}||w(t)||^2$$
$$= -\gamma||w(t)||^2.$$

Therefore

$$||w(t)|| \le ||w(0)||\, e^{-\gamma t}, \tag{2.42}$$

for all values of t, such that $x(t) \in U(R, x_0)$. If $0 \le t_1 \le t_2$, one obtains

$$\begin{aligned} ||x(t_2) - x(t_1)|| &\le \left\| \int_{t_1}^{t_2} \dot{x}(s)\, ds \right\| \le \left(\frac{M_1}{c} + ||B(0)|| \right) \int_{t_1}^{t_2} ||w(s)||\, ds \\ &\le \frac{\sigma||F(x_0)||}{\gamma} \left(e^{-\gamma t_1} - e^{-\gamma t_2} \right). \end{aligned} \tag{2.43}$$

From (2.43), by conditions (2.29) and (2.30), one gets:

$$\begin{aligned} ||x(t_2) - x(t_1)|| &\le \frac{\gamma c}{2M_1 M_2 \sigma^2} \frac{2M_1 M_2 \sigma^3 ||F(x_0)||}{c\gamma^2} \left(e^{-\gamma t_1} - e^{-\gamma t_2} \right) \\ &\le R \left(e^{-\gamma t_1} - e^{-\gamma t_2} \right). \end{aligned} \tag{2.44}$$

Since $B(t)$ is bounded whenever it is defined, estimate (2.44) implies that there exists an unique solution $(x(t), B(t))$ to (2.21)–(2.22) on $[0, +\infty)$ and $\forall t \in [0, +\infty)$ $x(t) \in U(R, x_0)$. Setting $t_1 = t$ and $t_2 \to +\infty$ in (2.43) one gets (2.31). Inequality (2.32) now follows from (2.42). Let us go back to (2.40). By (2.42) one has

$$||W(t)|| \le e^{-ct} \left[\int_0^t ||F''(x(s))||\, ||B(s)||^2\, ||F(x_0)||\, e^{(c-\gamma)s}\, ds + ||W(0)|| \right]. \tag{2.45}$$

Estimate (2.45) implies (2.33)–(2.34). This completes the proof. □

3. Ill-posed case. Continuously regularized modified Newton's scheme

In many important applications the Fréchet derivative operator is not boundedly invertible, i.e. the problem is ill-posed. To overcome this difficulty we suggest a regularized version of algorithm (2.12):

$$\begin{aligned} \dot{x}(t) &= -[F'(x_0) + \varepsilon(t)I]^{-1}(F(x(t)) + \varepsilon(t)(x(t) - x_0)), \\ x(0) &= x_0 \in H, \quad 0 < \varepsilon(t), \end{aligned} \tag{3.1}$$

where x_0 is chosen so that $(F'(x_0)h, h) \ge 0\, \forall h \in H$. If such a choice is not possible for the original equation $F(x) = 0$, one may consider an auxiliary problem $\phi(x) := F'^*(x_0)F(x) = 0$. If F is Fréchet differentiable, one has $\phi'(x_0) = F'^*(x_0)F'(x_0)$ and $(\phi'(x_0))h, h) \ge 0\, \forall h \in H$. The last equation, in general, is not equivalent to (1.1). However every solution to (1.1) solves $\phi(x) = 0$. The convergence analysis of (3.1) is done in the following theorem:

Theorem 3.1. *Let H be a real Hilbert space, $F : H \to H$, equation* (1.1) *be solvable (not necessarily uniquely), and $\hat{x}$ be a solution to* (1.1)*. Assume that:*
1. A positive function $\varepsilon(t) \in C^1[0, +\infty)$ converges monotonically to zero as $t \to +\infty$, $\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}$ is nondecreasing, and $\varepsilon(0) > |\dot{\varepsilon}(0)|$.
2. F is Fréchet differentiable, its Fréchet derivative F' is Lipschitz-continuous:

$$||F'(x_1) - F'(x_2)|| \le M_2||x_1 - x_2|| \quad \textit{and} \quad F'(x) = F'(x_0)G(x_0, x), \tag{3.2}$$

where

$$G(x_0, x) \in L(H), \quad ||G(x_0, x) - I|| \le C(G)||x_0 - x||, \quad \forall x_1,\ x_2,\ x \in U(\rho, \hat{x}), \tag{3.3}$$

$$U(\rho, \hat{x}) := \{x \in H : ||x - \hat{x}|| \le \rho\}, \quad C(G) > 0, \quad \rho := \frac{\varepsilon(0) - |\dot{\varepsilon}(0)|}{M_2 + C(G)\varepsilon(0)}. \tag{3.4}$$

3. $F'(x_0)$ is non-negative definite:

$$(F'(x_0)h, h) \ge 0 \quad \forall h \in H, \quad ||x_0 - \hat{x}|| < \rho. \tag{3.5}$$

4. There exist $v \in H$ such that $\hat{x} - x_0 = F'(x_0)v$,

$$\varepsilon(0) - |\dot{\varepsilon}(0)| \ge [M_2 + C(G)\varepsilon(0)]\varepsilon(0)\sqrt{\frac{2||v||}{M_2}}. \tag{3.6}$$

Then a unique solution $x = x(t)$ to problem (3.1) *exists for all $t \in [0, +\infty)$ and*

$$||x(t) - \hat{x}|| \le \frac{\varepsilon(0) - |\dot{\varepsilon}(0)|}{\varepsilon(0)[M_2 + C(G)\varepsilon(0)]}\varepsilon(t). \tag{3.7}$$

Remark 3.2. Inequality (3.6) can always be satisfied if $\sqrt{2||v||M_2} < 1$. Indeed, inequality (3.6) is equivalent to

$$1 - \sqrt{2||v||M_2} \ge \frac{|\dot{\varepsilon}(0)|}{\varepsilon(0)} + \varepsilon(0)C(G)\sqrt{\frac{2||v||}{M_2}}. \tag{3.8}$$

Thus, if $\sqrt{2||v||M_2} < 1$, then inequality (3.8) holds if $\frac{|\dot{\varepsilon}(0)|}{\varepsilon(0)}$ and $\varepsilon(0)$ are sufficiently small. For $\varepsilon(t) = a\,e^{-bt}$, $a, b > 0$, inequality (3.6) holds if the following inequality is valid:

$$b + aC(G)\sqrt{\frac{2||v||}{M_2}} \le 1 - \sqrt{2||v||M_2}.$$

The foregoing inequality holds if a and b are positive and sufficiently small and $1 > \sqrt{2||v||M_2}$. Since a priori $||v||$ is not known, in the numerical applications of the scheme one has to try different functions $\varepsilon(t)$ for (3.6) to be fulfilled.

Remark 3.3. Consider a nonlinear integral equation of the first kind:

$$(3.9)\quad F(x) := \psi(x) - y = 0, \quad \psi(x)(t) := \int_0^1 k(t,s)g(s,x(s))\,ds, \quad t \in [0,1],$$

where $k(t,s) \in L^\infty((0,1)^2)$ and $g(s,u)$ is twice continuously differentiable with respect to u on $0 \le s,t \le 1$, $-\infty < u < +\infty$. Suppose $F: H^1[0,1] \to L^2(0,1)$. Then

$$(F'(x)h)(t) = \int_0^1 k(t,s)g_x(s,x(s))h(s)\,ds.$$

Introduce the nonlinear operator $\phi(x) := F'^*(x_0)F(x)$, $\phi: H^1[0,1] \to H^1[0,1]$, and solve the equation $\phi(x) = 0$. Clearly $\phi'(x_0)$ is non-negative definite, i.e. condition 3 of Theorem 3.1 holds. Under the additional assumptions $|g_x(s,x_0)| \ge \kappa > 0$ for any $s \in (0,1)$, and $g(s,u) \in C^3((0,1)\times(-\infty,+\infty))$, one can take $(G(x_0,x)h)(s) := \frac{g_x(s,x(s))}{g_x(s,x_0(s))}h(s)$ in order to satisfy condition 2 of Theorem 3.1. Indeed,

$$(\phi'(x)h)(t) = (\phi'(x_0)G(x_0,x)h)(t),$$

and for any $h \in H$ the following estimates are used in [7]:

$$\begin{aligned}
&||(G(x_0,x) - I)h||_{L^2} = \\
&\quad = \left\{ \int_0^1 \left[\frac{\int_0^1 g_{xx}(s,(x_0+\theta(x-x_0))(s))\,d\theta(x(s)-x_0(s))h(s)}{g_x(s,x_0(s))} \right]^2 ds \right\}^{1/2} \\
&\quad \le \frac{||g_{xx}||_{L^\infty}}{\kappa}||x-x_0||_{L^\infty}\,||h||_{L^2}.
\end{aligned}$$

Also $\left\| \frac{d}{ds}(G(x_0,x) - I)h \right\|_{L^2}$

$$\begin{aligned}
&= \left\{ \int_0^1 \left[\left(\frac{\int_0^1 g_{sxx}(s,(x_0+\theta(x-x_0))(s)) + g_{xxx}(s,(x_0+\theta(x-x_0))(s))(x_0'+\theta(x'-x_0'))(s)d\theta}{g_x(s,x_0(s))} \right.\right.\right. \\
&\quad \left. - \frac{\int_0^1 g_{xx}(s,(x_0+\theta(x-x_0))(s))\,d\theta(g_{sx}(s,x(s)) + g_{xx}(s,x(s))x'(s))}{g_x^2(s,x_0(s))} \right) \cdot (x(s)-x_0(s))h(s) \\
&\quad \left.\left. + \frac{\int_0^1 g_{xx}(s,(x_0+\theta(x-x_0))(s))\,d\theta}{g_x(s,x_0(s))}((x'(s)-x_0'(s))h(s) + (x(s)-x_0(s))h'(s)) \right]^2 ds \right\}^{1/2} \\
&\le \frac{||g_{sxx}||_{L^\infty}}{\kappa}||x-x_0||_{L^\infty}||h||_{L^2} + \frac{2||g_{xxx}||_{L^\infty}(||x'||_{L^2} + ||x_0'||_{L^\infty})}{3\kappa}||x-x_0||_{L^\infty}||h||_{L^\infty} \\
&\quad + \frac{||g_{xx}||_{L^\infty}||g_{sx}||_{L^\infty}}{\kappa^2}||x-x_0||_{L^\infty}||h||_{L^2} + \frac{||g_{xx}||^2_{L^\infty}||x'||_{L^2}}{\kappa^2}||x-x_0||_{L^\infty}||h||_{L^\infty} \\
&\quad \frac{||g_{xx}||_{L^\infty}}{\kappa}(||x'-x_0'||_{L^2}||h||_{L^\infty} + ||x-x_0||_{L^\infty}||h'||_{L^2}).
\end{aligned}$$

The $L^\infty(0,1)$-norms of $x - x_0$ and h can be estimated by their $H^1[0,1]$-norms times some constants, due to Sobolev's embedding theorems.

Thus if one assumes that equation (3.9) is solvable, $\hat{x}$ is its solution, and in a neighborhood of $\hat{x}$ there exists x_0 such that

$$\hat{x} - x_0 = \phi'(x_0)v, \quad \sqrt{2||v||M_2} < 1,$$

then a unique solution, $x = x(t)$, to the problem

$$\dot{x}(t) = -[\phi'(x_0)+\varepsilon(t)I]^{-1}(\phi(x(t))+\varepsilon(t)(x(t)-x_0)), \quad x(0) = x_0 \in H, \quad 0 < \varepsilon(t),$$

exists for all $t \in [0, +\infty)$ and

$$||x(t) - \hat{x}|| = O(\varepsilon(t)),$$

provided that the above assumptions on $k(t, s)$ and $g(s, u)$ are satisfied and the choice of $\varepsilon(t)$ is made according to (3.8) with $\frac{\dot{\varepsilon}(t)}{\varepsilon(t)}$ being nondecreasing.

Proof of Theorem 3.1 First, from (3.5) one concludes that the operator $[F'(x_0) + \varepsilon(t)I]^{-1}$ is bounded $\forall t \geq 0$. Let us show that if $x = x(t)$ solves (3.1), then $x(t) \in U(\rho, \hat{x})$ with ρ as introduced in (3.4). Assume the converse: there exists $T > 0$ such that

$$||x(t) - \hat{x}|| < \rho \quad \forall t \in [0, T) \quad \text{and} \quad ||x(T) - \hat{x}|| = \rho. \tag{3.10}$$

For any $t \in [0, T]$ one has

$$\frac{1}{2}\frac{d}{dt}||x(t) - \hat{x}||^2 = -([F'(x_0) + \varepsilon(t)I]^{-1}[F'(\hat{x})(x(t) - \hat{x}) + R_2(x(t), \hat{x}) + \varepsilon(t)(x(t) - x_0)], x(t) - \hat{x}),$$

where $||R_2(x(t), \hat{x})|| \leq \frac{M_2}{2}||x(t) - \hat{x}||^2$. Thus one gets

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||x(t) - \hat{x}||^2 \leq &- ||x(t) - \hat{x}||^2 \\ &- ([F'(x_0) + \varepsilon(t)I]^{-1}(F'(\hat{x}) - F'(x_0))(x(t) - \hat{x}), x(t) - \hat{x}) \\ &- \varepsilon(t)([F'(x_0) + \varepsilon(t)I]^{-1}(\hat{x} - x_0), x(t) - \hat{x}) \\ &+ \frac{M_2}{2\varepsilon(t)}||x(t) - \hat{x}||^3.\end{aligned}$$

Condition 2 of Theorem 3.1 and the estimate $||[F'(x_0)+\varepsilon(t)I]^{-1}F'(x_0)|| \leq 1$ yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||x(t)-\hat{x}||^2 &\leq -||x(t)-\hat{x}||^2+||G(x_0, \hat{x})-I||\,||x(t)-\hat{x}||^2+\varepsilon(t)||v||\,||x(t)-\hat{x}|| \\ &+\frac{M_2}{2\varepsilon(t)}||x(t) - \hat{x}||^3 \leq -(1 - C(G)\rho)||x(t) - \hat{x}||^2 + \varepsilon(t)||v||\,||x(t) - \hat{x}|| \\ &+\frac{M_2}{2\varepsilon(t)}||x(t) - \hat{x}||^3.\end{aligned} \tag{3.11}$$

Introduce the notation $Q(t) := ||x(t) - \hat{x}||$. Inequality (3.11) implies

(3.12)
$$\dot{Q}(t) \le -(1 - C(G)\rho)Q(t) + \varepsilon(t)||v|| + \frac{M_2}{2\varepsilon(t)}Q^2(t), \quad Q(0) = ||x_0 - \hat{x}||.$$

Take $f(t) = \frac{Q(t)}{\varepsilon(t)}$. By assumption 1 of Theorem 3.1 one obtains:

(3.13)
$$\dot{f}(t) \le -\left(1 - C(G)\rho - \frac{|\dot{\varepsilon}(0)|}{\varepsilon(0)}\right) f(t) + ||v|| + \frac{M_2}{2} f^2(t), \quad f(0) = \frac{||x_0 - \hat{x}||}{\varepsilon(0)}.$$

From (3.4) and (3.13) one concludes that

(3.14)
$$\dot{f}(t) \le -\frac{M_2(\varepsilon(0) - |\dot{\varepsilon}(0)|)}{\varepsilon(0)[M_2 + C(G)\varepsilon(0)]} f(t) + ||v|| + \frac{M_2}{2} f^2(t), \quad f(0) = \frac{||x_0 - \hat{x}||}{\varepsilon(0)}.$$

Let

$$C_1 := \frac{M_2}{2}, \quad C_2 := \frac{M_2(\varepsilon(0) - |\dot{\varepsilon}(0)|)}{\varepsilon(0)[M_2 + C(G)\varepsilon(0)]}, \quad C_3 := ||v||. \tag{3.15}$$

If $g(t)$ is a solution to the initial value problem

$$\dot{g}(t) = C_1 g^2(t) - C_2 g(t) + C_3, \tag{3.16}$$
$$g(0) = f(0), \tag{3.17}$$

then inequality (3.14) yields

$$f(t) \le g(t), \tag{3.18}$$

whenever $g(t)$ and $f(t)$ are both defined. By (3.4), (3.5) and (3.15) one has

$$f(0) = \frac{||x_0 - \hat{x}||}{\varepsilon(0)} < \frac{\rho}{\varepsilon(0)} = \frac{\varepsilon(0) - |\dot{\varepsilon}(0)|}{\varepsilon(0)[M_2 + C(G)\varepsilon(0)]} = \frac{C_2}{2C_1}. \tag{3.19}$$

By (3.6) the equation $C_1 g^2 - C_2 g + C_3 = 0$ has at least one real root. If there are two roots, the smaller root is a stable equilibrium for problem (3.16), which implies $g(t) \le g(0) = f(0) < \frac{C_2}{2C_1}$. Otherwise $\tilde{g} := \frac{C_2}{2C_1}$ is a solution to (3.16), and $g(t) < \frac{C_2}{2C_1}$ since $g(0) < \frac{C_2}{2C_1}$. Therefore from (3.18) and (3.19) one derives:

$$f(t) \le g(t) < \frac{\rho}{\varepsilon(0)}.$$

Hence inequality (3.14) and conditions (3.4) and (3.6) yield:

$$f(t) < \frac{\varepsilon(0) - |\dot{\varepsilon}(0)|}{\varepsilon(0)[M_2 + C(G)\varepsilon(0)]} = \frac{\rho}{\varepsilon(0)}. \tag{3.20}$$

Thus

$$||x(t) - \hat{x}|| < \frac{\rho}{\varepsilon(0)}\varepsilon(t) \le \rho \quad \forall t \in [0, T], \tag{3.21}$$

which contradicts (3.10). Therefore $x(t) \in U(\hat{x}, \rho)$ for any t, and by the standard argument one concludes that $x(t)$ is defined on $[0, +\infty)$. Inequality (3.7) follows from (3.4) and (3.21). □

Corollary 3.4. *In this corollary it is shown that if the data are noisy, then the stopping time can be chosen so that the solution to the Cauchy problem with noisy data approximates a solution to* (1.1) *stably, i.e. with the error going to zero as the noise level goes to zero. Let the operator F in* (1.1) *have the following form:*

$$F(x) := \psi(x) - y. \tag{3.22}$$

Assume that ψ is given exactly and in place of y we know a δ-approximation y_δ, satisfying the inequality

$$||y - y_\delta|| \le \delta. \tag{3.23}$$

Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||x(t) - \hat{x}||^2 \le -(1 - C(G)\rho)||x(t) - \hat{x}||^2 + \left(\varepsilon(t)||v|| + \frac{\delta}{\varepsilon(t)}\right)||x(t) - \hat{x}||\\ + \frac{M_2}{2\varepsilon(t)}||x(t) - \hat{x}||^3.\end{aligned} \tag{3.24}$$

Take τ_δ such that $\varepsilon(\tau_\delta) = \left(\frac{\delta}{||v||}\right)^{\frac{1}{2}}$. For $t = \tau_\delta$ we get $\frac{\delta}{\varepsilon^2(\tau_\delta)} = ||v||$ and therefore $\forall t \in [0, \tau_\delta]$

$$\dot{f}(t) \le -\left(1 - C(G)\rho - \frac{|\dot{\varepsilon}(0)|}{\varepsilon(0)}\right) f(t) + 2||v|| + \frac{M_2}{2} f^2(t), \quad f(0) = \frac{||x_0 - \hat{x}||}{\varepsilon(0)}. \tag{3.25}$$

Thus one gets

$$||x(\tau_\delta) - \hat{x}|| \le \frac{\rho}{\varepsilon(0)||v||^{\frac{1}{2}}}\,\delta^{\frac{1}{2}}, \tag{3.26}$$

provided that conditions 1, 2, 3 of Theorem 3.1 *and inequality*

$$\varepsilon(0) - |\dot{\varepsilon}(0)| \ge 2[M_2 + C(G)\varepsilon(0)]\varepsilon(0)\sqrt{\frac{||v||}{M_2}} \tag{3.27}$$

hold.

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