Annali di Matematica 181, 501–526 (2002)

Digital Object Identifier (DOI) 10.1007/s10231-002-0052-2

Mark Elin · Simeon Reich · David Shoikhet

Asymptotic behavior of semigroups of ρ-non-expansive and holomorphic mappings on the Hilbert Ball

Received: January 3, 2001; in final form: November 28, 2001
Published online: October 30, 2002 – © Springer-Verlag 2002

Abstract. We study generated semigroups of those self-mappings of the Hilbert ball which are non-expansive with respect to the hyperbolic metric. We find optimal convergence rates for such semigroups to interior stationary and boundary sink points. Since the hyperbolic metric is not defined on the boundary, the usual approach treats these two cases separately. In contrast with this practice, we use a special non-Euclidean "distance" (which induces the original topology) to present a unified theory. Our approach leads to new results even in the one-dimensional case. When the semigroups consist of holomorphic self-mappings, we obtain the rather unexpected phenomenon of universal rates of convergence of an exponential type. In particular, in the case of a boundary sink point we establish a continuous analog of the celebrated Julia–Wolff–Carathéodory theorem.

Mathematics Subject Classification (2000). 32F45, 34G20, 46G20, 46T25, 47H20

1. Preliminaries

Let $\mathbb{B}$ be the open unit ball of a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$, and let $\rho : \mathbb{B} \times \mathbb{B} \mapsto \mathbb{R}^+$ be the hyperbolic metric on $\mathbb{B}$ ([9], p. 98), i.e.,

$$\rho(x, y) = \tanh^{-1} \sqrt{1 - \sigma(x, y)},$$
$$\sigma(x, y) = \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{|1 - \langle x, y \rangle|^2}, \quad x, y \in \mathbb{B}. \tag{1.1}$$

We denote by $\mathcal{N}_\rho$ the class of all those self-mappings $F : \mathbb{B} \mapsto \mathbb{B}$ which are non-expansive with respect to ρ (ρ-non-expansive), i.e.,

$$\rho(F(x), F(y)) \leq \rho(x, y). \tag{1.2}$$

Note that the class $\mathcal{N}_\rho$ properly contains the class $\text{Hol}(\mathbb{B})$ of all holomorphic self-mappings of $\mathbb{B}$ ([8, 9]).

M. Elin: Department of Applied Mathematics, ORT Braude College, 21982 Karmiel, Israel, e-mail: `elin_mark@hotmail.com`

S. Reich: Department of Mathematics, The Technion — Israel Institute of Technology, 32000 Haifa, Israel, e-mail: `sreich@tx.technion.ac.il`

D. Shoikhet: Department of Applied Mathematics, ORT Braude College, 21982 Karmiel, Israel, e-mail: `davs@tx.technion.ac.il`

Definition 1.1. *A family* $\mathcal{S} = \{F(t)\}_{t \geq 0}$ *of self-mappings of* $\mathbb{B}$ *is said to be a one-parameter continuous semigroup (flow) on* $\mathbb{B}$ *if*

$$F(t + s) = F(t) \circ F(s), \quad t, s \geq 0,$$

and

$$\lim_{t \to 0^+} F(t) = I, \tag{1.3}$$

where I *is the restriction of the identity mapping of* $\mathcal{H}$ *to* $\mathbb{B}$ *and the limit is taken pointwise with respect to the strong topology of* $\mathcal{H}$.

Definition 1.2. *A flow* $\mathcal{S} = \{F(t)\}_{t \geq 0}$ *on* $\mathbb{B}$ *is said to be generated if, for each* $x \in \mathbb{B}$*, there exists the strong limit*

$$f(x) := \lim_{t \to 0^+} \frac{1}{t}(x - F(t)x). \tag{1.4}$$

In this case the mapping $f : \mathbb{B} \mapsto \mathcal{H}$ *is called the (infinitesimal) generator of* $\mathcal{S}$.

If, in addition, $\mathcal{S} \subset \mathcal{N}_\rho$ *is a flow of* ρ*-non-expansive self-mappings of* $\mathbb{B}$ *generated by* f*, then we will write* $f \in \mathcal{G}\mathcal{N}_\rho(\mathbb{B})$.

It is known (see [13]) that if f is bounded and uniformly continuous on each ρ-ball in $\mathbb{B}$, then $f \in \mathcal{G}\mathcal{N}_\rho(\mathbb{B})$ if and only if it satisfies the following strong range condition:

(SRC) for each $r \geq 0$, the mapping $\mathcal{J}_r := (I + rf)^{-1}$ is a well-defined ρ-non-expansive self-mapping of $\mathbb{B}$.

Moreover, in this case the following exponential formula holds:

$$\lim_{n \to \infty} \left[\mathcal{J}_{\frac{t}{n}}\right]^n = F(t), \tag{1.5}$$

where by F^n we denote the n-fold iterate of F and the limit in (1.5) is uniform on each ρ-ball in $\mathbb{B}$.

Definition 1.3. *A mapping* $f : \mathbb{B} \mapsto \mathcal{H}$ *is said to be strongly* ρ*-monotone (respectively,* ρ*-monotone) if for each pair* $x, y \in \mathbb{B}$ *there is* $\epsilon = \epsilon(x, y) > 0$ *(respectively,* $\epsilon = 0$*) such that*

$$\rho(x + rf(x), y + rf(y)) \geq (1 + r\epsilon(x, y))\rho(x, y), \tag{1.6}$$

for all $r \geq 0$ *such that the points* $x + rf(x)$ *and* $y + rf(y)$ *belong to* $\mathbb{B}$.

It was shown in [13] that $f : \mathbb{B} \mapsto \mathcal{H}$ is ρ-monotone if and only if it satisfies the condition

$$\operatorname{Re}\left[\frac{\langle f(x), x\rangle}{1 - \|x\|^2} + \frac{\langle y, f(y)\rangle}{1 - \|y\|^2}\right] \geq \operatorname{Re}\left[\frac{\langle f(x), y\rangle + \langle x, f(y)\rangle}{1 - \langle x, y\rangle}\right]. \tag{1.7}$$

Of course, if $f \in \mathcal{G}\mathcal{N}_\rho(\mathbb{B})$ is bounded and uniformly continuous on each ρ-ball in $\mathbb{B}$, then it is ρ-monotone by (SRC).

As a matter of fact, Theorems 1.3 and 2.1 in [13] show that the following partial converse is also true: if a ρ-monotone f is bounded and uniformly continuous on each ρ-ball in $\mathbb{B}$, and satisfies the weak range condition:

$$\text{(WRC)} \qquad (I + rf)(\mathbb{B}) \supseteq \mathbb{B}$$

for each $r \geq 0$, then $f \in \mathcal{G}\mathcal{N}_\rho(\mathbb{B})$.

Stronger assertions hold for the class $\mathrm{Hol}(\mathbb{B}, \mathcal{H})$ of holomorphic mappings on $\mathbb{B}$ with values in $\mathcal{H}$. Namely,

(I) *A flow $\mathcal{S} \subset \mathrm{Hol}(\mathbb{B})$ is generated if and only if the limit in* (1.3) *is uniform on each ρ-ball in $\mathbb{B}$. Moreover, in this case its generator f is holomorphic and bounded on each ρ-ball in $\mathbb{B}$ (see* [15]*);*

(II) *the following are equivalent for $f \in \mathrm{Hol}(\mathbb{B}, \mathcal{H})$:*
   - *(i) $f \in \mathcal{G}\,\mathrm{Hol}(\mathbb{B}) := \mathcal{G}\mathcal{N}_\rho(\mathbb{B}) \cap \mathrm{Hol}(\mathbb{B}, \mathcal{H})$;*
   - *(ii) f is bounded on each ρ-ball in $\mathbb{B}$ and satisfies the weak range condition (WRC);*
   - *(iii) f satisfies the strong range condition (SRC);*
   - *(iv) f is bounded on each ρ-ball in $\mathbb{B}$ and is ρ-monotone.*

Thus for the finite-dimensional case the classes of holomorphic mappings satisfying (WRC) and (SRC), the class $\mathcal{G}\,\mathrm{Hol}(\mathbb{B})$ and the class of ρ-monotone mappings are one and the same.

Now we turn to the notion of the stationary point set of a flow $\mathcal{S}$ on $\mathbb{B}$.

By $\mathrm{Fix}(F)$ we will denote the fixed point set of a self-mapping F of $\mathbb{B}$; by $\mathrm{Null}(f)$ we denote the null point set of a mapping $f : \mathbb{B} \mapsto \mathcal{H}$ in $\mathbb{B}$. Thus $\mathrm{Fix}(F) = \mathrm{Null}(I - F)$.

Definition 1.4. *The stationary point set Z of a flow $\mathcal{S} = \{F(t)\}_{t\geq 0}$ on $\mathbb{B}$ consists of all the points $a \in \mathbb{B}$ such that*

$$F(t)a = a,$$

for all $t \geq 0$.

In other words,

$$Z = \bigcap_{t\geq 0} \mathrm{Fix}(F(t)). \tag{1.8}$$

It is known (see [13] and [15]) that if $\mathcal{S} \subset \mathcal{N}_\rho$ is a flow generated by $f \in \mathcal{G}\mathcal{N}_\rho(\mathbb{B})$ and f is bounded and uniformly continuous on each ρ-ball in $\mathbb{B}$ (hence, satisfies (SRC)), then the following relations hold: for each $r \geq 0$,

$$W = \mathrm{Null}(f) = \mathrm{Fix}(\mathcal{J}_r), \tag{1.9}$$

where $\mathcal{J}_r = (I + rf)^{-1}$.

In the study of the asymptotic behavior of flows of ρ-non-expansive (or holomorphic) self-mappings of $\mathbb{B}$, the two cases $Z \neq \emptyset$ and $Z = \emptyset$ are usually considered separately (cf. [11,14]). In particular, if $f \in \mathcal{GN}_\rho(\mathbb{B})$, then one can look for conditions which would imply the strong ρ-monotonicity of f with $\epsilon(x, y) = \epsilon =$ constant (see formula (1.6)). If this is the case, then (SRC) and (1.5) imply that W contains a unique point τ which is globally attractive with an exponential rate of convergence: $\rho(F(t)x, \tau) \leq \exp(-\epsilon t)\rho(x, \tau)$.

However, such an approach cannot work when $Z = \emptyset$.

In addition, one can ask how to trace the dynamics of the semigroup generators when an attractive stationary point tends to the boundary. In particular, we have in mind the following question.

Let $\{f_n\} \subset \mathcal{GN}_\rho(\mathbb{B})$ be a sequence converging to $f \in \mathcal{GN}_\rho(\mathbb{B})$ in a suitable topology and suppose that, for each n, the mapping f_n has a unique null point $\tau_n \in \mathbb{B}$ which is attractive for the flow $\mathcal{S}_n$ generated by f_n. Assume that $\{\tau_n\}$ converges to a boundary point $\tau \in \partial\mathbb{B}$. What is the asymptotic behavior of the flow $\mathcal{S}$ generated by f?

The difficulty is that for each $x \in \mathbb{B}$, $\rho(x, \tau_n) \to \infty$ when $n \to \infty$.

Therefore, our aim is to find some sufficient (and perhaps necessary) conditions for global convergence of the flow generated by f which will not depend on Z being either empty or non-empty.

Definition 1.5. *Let $\mathcal{S} = \{F(t)\}_{t\geq 0}$ be a flow on $\mathbb{B}$ which is generated by f. We will say that a point $\tau \in \overline{\mathbb{B}}$, the closure of $\mathbb{B}$, is a globally attractive point for $\mathcal{S}$ if for each $x \in \mathbb{B}$ the strong limit*

$$\lim_{t\to\infty} F(t)x = \tau,$$

uniformly on each ρ-ball in $\mathbb{B}$.

If $\tau \in \mathbb{B}$, then τ is the unique asymptotically stable stationary point of $\mathcal{S}$. If $\tau \in \partial\mathbb{B}$, the boundary of $\mathbb{B}$, we will call it the attractive sink point of $\mathcal{S}$.

For the case of holomorphic generators the attractivity of a stationary point can be completely described in terms of their derivatives.

If $f \in \mathcal{G}\,\mathrm{Hol}(\mathbb{B})$ and $\tau \in \mathrm{Null}(f)$, then τ is (globally) attractive if and only if the spectrum of the linear operator $f'(\tau)$, the Fréchet derivative of f at τ, lies strictly in the right half-plane (see, for example, [12]). But, as far as we know, even for holomorphic generators with no null points, the situation was described only for the one-dimensional case, that is, when $\mathbb{B} = \Delta$, the open unit disk in the complex plane $\mathbb{C}$. (However, some information on the finite-dimensional case can be found in [7,1] and [14].)

Namely, it was shown in [6] that $f \in \mathcal{G}\,\mathrm{Hol}(\Delta)$ has no null point in Δ if and only if, for some $\tau \in \partial\Delta$, the so-called angular derivative

$$\angle f'(\tau) = \beta \tag{1.10}$$

exists (finitely) with $\mathrm{Re}\,\beta \geq 0$.

Moreover, if $\mathcal{S} = \{F(t)\}_{t\geq 0}$ is the flow generated by f, then

$$\frac{|F(t)z-\tau|^2}{1-|F(t)z|^2} \leq \exp\left(-t\operatorname{Re}\beta\right)\frac{|z-\tau|^2}{1-|z|^2},$$

i.e., the point τ is unique and a (globally) attractive sink point of $\mathcal{S}$.

This assertion is an infinitesimal version of the Julia–Wolff–Carathéodory theorem.

However, in both cases ($Z \neq \emptyset$ and $Z = \emptyset$), the characteristics of the derivatives are not relevant in general, since f' does not exist for $f \in \mathcal{G}\mathcal{N}_\rho(\mathbb{B})$ in the complex sense if the mapping f is not holomorphic.

At the same time one can show that the number β in (1.10) is actually real and is equal to

$$\inf_{z\in\Delta} 2\operatorname{Re} f(z)\bar{z}^*,$$

where

$$z^* = \frac{z}{1-|z|^2} - \frac{\tau}{1-\bar{z}\tau}.$$

It turns out that this expression can serve as a characterization of the asymptotic behavior of flows of ρ-non-expansive mappings in a general Hilbert space both in the cases of an interior stationary point and a boundary sink point. This will be explained in the next section.

2. General approach

As above, let $\mathbb{B}$ be the open unit ball in a complex Hilbert space $\mathcal{H}$. For a fixed $\tau \in \overline{\mathbb{B}}$, the closure of $\mathbb{B}$, and an arbitrary $x \in \mathbb{B}$, we define a non-Euclidean "distance" between x to τ by the formula

$$d_\tau(x) = \frac{|1-\langle x,\tau\rangle|^2}{1-\|x\|^2}\left(1-\sigma\left(x,\tau\right)\right), \tag{2.1}$$

where $\sigma(x,\tau)$ is defined by formula (1.1).

Geometrically, the sets

$$E(\tau,s) = \{x \in \mathbb{B} : d_\tau(x) < s\}, \quad s > 0,$$

are ellipsoids. If $\tau \in \mathbb{B}$, then these sets are exactly the ρ-balls

$$E(\tau,s) = \{x \in \mathbb{B} : \rho(x,\tau) < r\}$$

centered at $\tau \in \mathbb{B}$ and of radius $r = \tanh^{-1}\sqrt{\frac{s}{s+1-\|\tau\|^2}}$. If $\tau \in \partial\mathbb{B}$, the boundary of $\mathbb{B}$, then these sets,

$$E(\tau,s) = \left\{x \in \mathbb{B} : d_\tau(x) = \frac{|1-\langle x,\tau\rangle|^2}{1-\|x\|^2} < s\right\}, \quad s > 0,$$

are ellipsoids which are internally tangent to the unit sphere $\partial\mathbb{B}$ at τ.

For fixed $\tau \in \overline{\mathbb{B}}$ and $x \in \partial E(\tau, s)$, $x \neq \tau$, now consider the non-zero vector

$$x^* = \frac{1}{1 - \sigma(x, \tau)} \left(\frac{1}{1 - \|x\|^2} x - \frac{1}{1 - \langle \tau, x \rangle} \tau \right). \tag{2.2}$$

As in [3], it can be shown that x^* is a support functional of the smooth convex set $E(\tau, s)$ at x, normalized by the condition

$$\lim_{x \to \tau} \left\langle x - \tau, x^* \right\rangle = 1.$$

For a mapping $f : \mathbb{B} \to \mathcal{H}$, the so-called "flow-invariance condition",

$$\operatorname{Re} \left\langle f(x), x^* \right\rangle \geq 0, \tag{2.3}$$

is necessary for f to be a generator of a continuous flow for which the sets $E(\tau, s)$ are invariant.

In our situation, when $f \in \mathcal{GN}_\rho(\mathbb{B})$, this is exactly the case if $\tau \in \mathbb{B}$ is a null point of f, since

$$\rho(F(t)x, \tau) = \rho(F(t)x, F(t)\tau) \leq \rho(x, \tau). \tag{2.4}$$

Note also that condition (2.3) can be obtained directly from the ρ-monotonicity of f if we substitute $f(\tau) = 0$ into (1.7).

In fact, inequality (2.4) shows that if condition (2.3) holds for some $\tau \in \mathbb{B}$ and all $x \in \mathbb{B}$, then τ must be a stationary point of $\mathcal{S} = \{F(t)\}_{t \geq 0}$, and hence a null point of f.

If f has no null point, then it can be shown exactly as in theorem 3.1 of [3] that there is a unique boundary point $\tau \in \partial\mathbb{B}$ such that (2.3) holds. This point τ is the sink point for the flow generated by f.

In order to classify the asymptotic behavior of flows we will consider a condition which is finer than (2.3). More precisely, for a point $\tau \in \overline{\mathbb{B}}$ and a mapping $f \in \mathcal{GN}_\rho(\mathbb{B})$, we consider the following two real non-negative functions on $(0, \infty)$:

$$\omega_\flat(s) := \inf_{d_\tau(x) \leq s} 2 \operatorname{Re} \left\langle f(x), x^* \right\rangle, \quad s > 0, \tag{2.5}$$

and

$$\omega^\sharp(s) := \inf_{d_\tau(x) = s} 2 \operatorname{Re} \left\langle f(x), x^* \right\rangle, \quad s > 0, \tag{2.6}$$

where x^* is defined by (2.2).

It is clear that

$$\omega^\sharp(s) \geq \omega_\flat(s) \geq 0,$$

and that $\omega_\flat(s)$ is decreasing on $(0, \infty)$.

Let $\mathcal{M}(0, \infty)$ denote the class of all positive functions ω on $(0, \infty)$ such that $\frac{1}{\omega}$ is Riemann integrable on each closed interval $[a, b] \subset (0, \infty)$ and

(*) $$\int_{0^+} \frac{ds}{\omega(s)s} \text{ is divergent.}$$

Note that for each $\omega \in \mathcal{M}(0,\infty)$, the function Ω defined by

$$\Omega(s) := \int_{s}^{d_\tau(x)} \frac{d\lambda}{\omega(\lambda)\lambda} \tag{2.7}$$

is a strictly decreasing positive function on $(0, d_\tau(x)]$ which maps this interval onto $[0,\infty)$. We denote its inverse function by $V : [0,\infty) \mapsto (0, d_\tau(x)]$.

Theorem 2.1. *Let $f \in \mathcal{G}\mathcal{N}_\rho(\mathbb{B})$ be continuous and let $\mathcal{S} = \{F(t)\}_{t\geq 0}$ be the flow generated by f. Given a point $\tau \in \overline{\mathbb{B}}$ and a function $\omega \in \mathcal{M}(0,\infty)$, the following conditions are equivalent:*

(i) for all $s \in (0,\infty)$,

$$\omega^\sharp(s) \geq \omega(s),$$

where $\omega^\sharp(s)$ is defined by (2.6)*;*

(ii) for any differentiable function W on $[0,\infty)$ such that $V(t) \leq W(t)$, $V(0) = W(0)$ and $V'(0) = W'(0)$,

$$d_\tau(F(t)x) \leq W(t), \quad x \in \mathbb{B},\ t \geq 0,$$

where $V = \Omega^{-1}$ and Ω is defined by (2.7).

In particular, $d_\tau(F(t)x) \leq V(t)$; hence τ is a globally attractive point for $\mathcal{S}$.

Proof. Consider the function $\Psi : \mathbb{R}^+ \times \mathbb{B} \mapsto \mathbb{R}^+$ defined by

$$\Psi(t,x) = d_\tau(F(t)x). \tag{2.8}$$

By direct calculations we have

$$\frac{\partial \Psi}{\partial t}\,|_{t=0^+} = -2\Psi(0,x)\,\mathrm{Re}\left\langle f(x), x^*\right\rangle. \tag{2.9}$$

Let us first assume that condition (ii) holds. Since $\Psi(0,x) = d_\tau(x) = W(0)$, we get, by (2.9) and (ii), that

$$\begin{aligned} 2\Psi(0,x)\,\mathrm{Re}\left\langle f(x), x^*\right\rangle &= -\tfrac{\partial \Psi}{\partial t}\,|_{t=0^+} \geq -\tfrac{d}{dt}\,[W(t)]_{t=0^+} \\ &= -\frac{d}{dt}\,[V(t)]_{t=0^+} = -\tfrac{1}{\Omega'(d_\tau(x))} = d_\tau(x)\omega\left(d_\tau(x)\right). \end{aligned}$$

Varying $x \in \partial E(\tau,s) = \{x \in \mathbb{B} : d_\tau(x) = s\}$, we see that this inequality immediately implies (i).

Conversely, let condition (i) hold. It follows by (2.8) and the semigroup property (1.3), that for all $x \in \mathbb{B}$ and $s, t \geq 0$,

$$\Psi(s+t,x) = \Psi(s, F(t)x).$$

Hence by (2.9) and the continuity of f, Ψ is differentiable at each $t \geq 0$ and we deduce from (i) and (2.9) that

$$\frac{\partial \Psi(t,x)}{\partial t} \leq -\Psi(t,x)\omega^{\sharp}(\Psi(t,x)) \leq -\Psi(t,x)\omega(\Psi(t,x)).$$

Separating variables we get

$$\int_{d_\tau(F(t)x)}^{d_\tau(x)} \frac{d\Psi}{\omega(\Psi)\Psi} = \Omega\left(d_\tau(F(t)x)\right) \geq t,$$

which is equivalent to condition (ii). Theorem 2.1 is proved. □

We will call a function $\omega \in \mathcal{M}(0,\infty)$ which satisfies condition (i), an appropriate lower bound for $f \in \mathcal{G}\mathcal{N}_\rho(\mathbb{B})$.

Remark 2.1. Of course, if the function $\omega_\flat$ defined by (2.5) belongs to $\mathcal{M}(0,\infty)$, then one can use it as an appropriate lower bound.

However, examples show that sometimes $\omega_\flat$ may be identically zero, while $\omega^\sharp$ itself belongs to the class $\mathcal{M}(0,\infty)$. Moreover, we will see below that for a semigroup of holomorphic mappings with a boundary sink point, $\omega_\flat$ is always a constant which determines the best rate of uniform exponential convergence of the flow.

To illustrate Theorem 2.1 and to motivate our next definition, we now present several one-dimensional examples.

Example 1. Let Δ be the open unit disk of the complex plane $\mathbb{C}$, let n be a positive integer and let $f : \Delta \mapsto \mathbb{C}$ be defined by

$$f(z) = -(1-z)^2\,\frac{1+z^n}{1-z^n}.$$

If we set $\tau = 1$ and

$$z^* = \frac{z}{1-|z|^2} - \frac{1}{1-\bar{z}},$$

then we get

$$\operatorname{Re} f(z)\bar{z^*} = \frac{|1-z|^2}{1-|z|^2}\operatorname{Re}\frac{1+z^n}{1-z^n} = d_1(z)\operatorname{Re}\frac{1+z^n}{1-z^n} > 0.$$

Since f is holomorphic on Δ, this inequality implies that f generates a flow $\mathcal{S} = \{F(t)\}_{t\geq 0}$ of holomorphic self-mappings of Δ (cf. [4]). In addition, it can be shown (see Theorem 3.1 below) that

$$\omega(s) = \omega_\flat(s) = \inf_{d_1(z)\leq s} 2\operatorname{Re} f(z)\bar{z^*} = const. = \frac{2}{n}.$$

Hence f satisfies the conditions of Theorem 2.1. In this case,

$$\Omega(s) = \frac{n}{2}\int_s^{d_1(x)} \frac{d\lambda}{\lambda} = -\frac{n}{2}\ln\frac{s}{d_1(z)}$$

and

$$V(t) = \Omega^{-1}(t) = \exp\left\{-\frac{2}{n}t\right\} d_1(z).$$

Thus we have an exponential rate of convergence of the flow $\mathcal{S}$ to the boundary point $\tau = 1$:

$$d_1(F_t(z)) = \frac{|1 - F(t)\,z|^2}{1 - |F(t)\,z|^2} \leq \exp\left\{-\frac{2}{n}t\right\} \frac{|1 - z|^2}{1 - |z|^2}.$$

Note also that although f has $n + 1$ null points $\{a_k : k = 1, 2, \ldots, n + 1\}$ on the unit circle, only $a_1 = 1$ is an attractive point of $\mathcal{S} = \{F(t)\}_{t \geq 0}$. The reason is that $\operatorname{Re} f'(a_1) > 0$, while $\operatorname{Re} f'(a_k) < 0$, $k = 2, 3, ..., n + 1$ (see Theorem 3.2 below). See Figure 1 for the case $n = 3$.

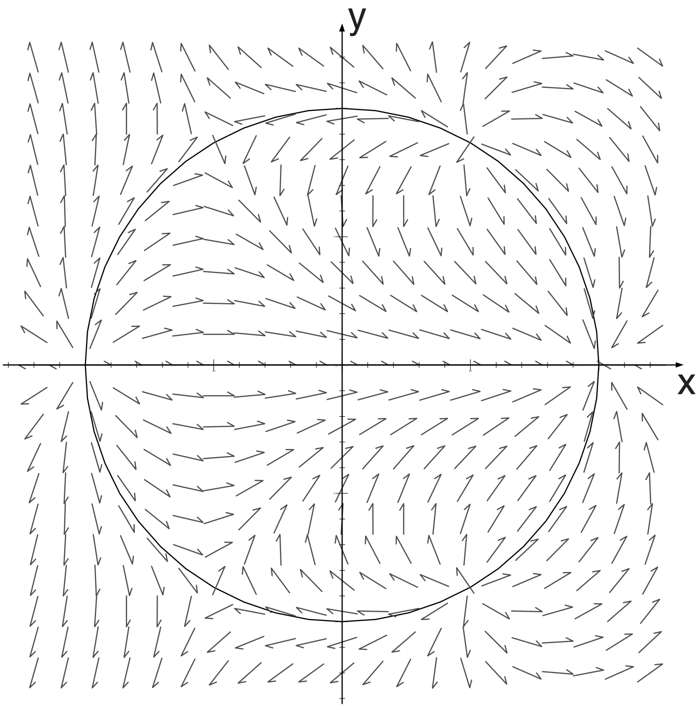

Fig. 1. The flow generated by $f(z) = -(1 - z)^2 \frac{1+z^3}{1-z^3}$

Example 2. Let Δ be as above and let $f : \Delta \mapsto \mathbb{C}$ be defined by

$$f(z) = -(1 - z)^2 \frac{1 + cz^n}{1 - cz^n},$$

with $|c| < 1$.

Once again, if we define z^* as in Example 1, then we have

$$\operatorname{Re} f(z)\overline{z^*} = \frac{|1-z|^2}{1-|z|^2} \operatorname{Re} \frac{1+cz^n}{1-cz^n} \geq d_1(z) \frac{1-|c|}{1+|c|} > 0.$$

In this case, $\omega_\flat(s) = 0$ for all $s \in (0, \infty)$ and we cannot use it as an appropriate lower bound. However, we can define $\omega(s) = as$, where $a = \frac{1-|c|}{1+|c|}$, and we find

$$\Omega(s) = \frac{1}{a} \int_s^{d_1(z)} \frac{d\lambda}{\lambda^2} = \frac{1}{a} \left(\frac{1}{s} - \frac{1}{d_1(z)} \right).$$

Thus we get, by Theorem 2.1, the following rate of convergence:

$$d_1(F_t(z)) = \frac{|1-F(t)z|^2}{1-|F(t)z|^2} \leq \frac{1}{1+atd_1(z)} \frac{|1-z|^2}{1-|z|^2}.$$

Example 3. Let Δ be as above and let $z = x + iy \in \Delta$. Define $f : \Delta \to \mathbb{C}$ by

$$f(z) = x^{\frac{7}{3}} + iy^{\frac{7}{3}}.$$

Since

$$\operatorname{Re} f(z)\bar{z} = x^{\frac{10}{3}} + y^{\frac{10}{3}} \geq 0,$$

f is ρ-monotone (see [13]) and the origin is the unique null point of f.

Hence, if we set $\tau = 0$, then we have

$$d_0(z) = \frac{|z|^2}{1-|z|^2}$$

and

$$\omega^\sharp(s) = 2 \inf_{d_0(z)=s} \frac{1}{|z|^2(1-|z|^2)} \operatorname{Re} f(z)\bar{z} =$$

$$= 2 \inf_{x^2+y^2=\frac{s}{s+1}} \frac{x^{\frac{10}{3}} + y^{\frac{10}{3}}}{(x^2+y^2)(1-x^2-y^2)} = 2^{\frac{1}{3}} s^{\frac{2}{3}} (1+s)^{\frac{1}{3}}.$$

Setting $\omega(s) = \omega^\sharp(s)$ we get

$$\Omega(s) = \int_s^{d_0(z)} \frac{d\lambda}{\omega(\lambda)\lambda} = \frac{1}{2^{\frac{1}{3}}} \int_s^{d_0(z)} \frac{d\lambda}{\lambda^{\frac{5}{3}}(\lambda+1)^{\frac{1}{3}}}.$$

Finally, we obtain the estimate

$$d_0(F(t)z) \leq V(t) = \frac{d_0(z)}{\left[\frac{2^{\frac{4}{3}}}{3} t d_0(z)^{\frac{2}{3}} + (d_0(z)+1)^{\frac{2}{3}} \right]^{\frac{3}{2}} - d_0(z)}.$$

This inequality is equivalent to the estimate

$$|F(t)z| \leq \frac{|z|}{\left[\frac{(2|z|)^{\frac{4}{3}}}{3}t + 1\right]^{\frac{3}{4}}}.$$

Note that one can calculate $F(t)$ directly by solving the Cauchy problem and get

$$|F(t)z|^2 = \frac{x^2}{\left(\frac{4}{3}x^{\frac{4}{3}}t + 1\right)^{\frac{3}{2}}} + \frac{y^2}{\left(\frac{4}{3}y^{\frac{4}{3}}t + 1\right)^{\frac{3}{2}}}.$$

Thus for $x = y$ we obtain

$$|F(t)z| = \frac{|z|}{\left[\frac{(2|z|)^{\frac{4}{3}}}{3}t + 1\right]^{\frac{3}{4}}}.$$

So, the rate of non-exponential convergence we have obtained is sharp.

Remark 2.2. We will see below that a similar phenomenon is impossible for holomorphic mappings.

Namely: *if a flow of holomorphic self-mappings converges locally uniformly to an interior stationary point, then the convergence must be of exponential type.*

These examples and Theorem 2.1 above motivate the following definitions:

Definition 2.1. *Let $\mathcal{S} = \{F(t)\}_{t\geq 0}$ be a flow with a stationary (or sink) point $\tau \in \overline{\mathbb{B}}$. We will say that the asymptotic behavior of $\mathcal{S}$ at τ is of order not less than $\alpha > 0$ if there is a function $\omega \in \mathcal{M}(0, \infty)$ such that*

$$\liminf_{s\to 0^+} \left\{ \frac{\omega(s)}{s^{\frac{1}{\alpha}}} \right\} > 0 \tag{2.10}$$

and

$$d_\tau(F(t)x) \leq \frac{1}{\left(1 + \frac{t}{\alpha}\omega(d_\tau(x))\right)^\alpha} d_\tau(x) \tag{2.11}$$

for all $x \in \mathbb{B}$ and $t \geq 0$.

Definition 2.2. *We will say that the asymptotic behavior of $\mathcal{S}$ at τ is of exponential type if there is a decreasing function $\omega \in \mathcal{M}(0, \infty)$ such that*

$$d_\tau(F(t)x) \leq \exp\left(-t\omega(d_\tau(x))\right) d_\tau(x), \tag{2.12}$$

for all $x \in \mathbb{B}$ and $t \geq 0$.

In particular, if ω can be chosen to be a positive constant a, then we will say that $\mathcal{S}$ has a global uniform rate of convergence:

$$d_\tau(F(t)x) \leq \exp\left(-ta\right) d_\tau(x). \tag{2.13}$$

The following assertion is a consequence of Theorem 2.1:

Theorem 2.2. *Let* $S = \{F(t)\}_{t\geq 0}$ *be a flow generated by* $f \in \mathcal{G}\mathcal{N}_\rho(\mathbb{B})$ *with a null (or sink) point* $\tau \in \overline{\mathbb{B}}$. *Then the asymptotic behavior of* $\mathcal{S}$ *at* τ *is of order not less than* $\alpha > 0$ *if and only if there exists an appropriate lower bound* $\omega \in \mathcal{M}(0,\infty)$ *for* f *such that*

$$(**) \qquad \frac{\omega(s)}{s^{\frac{1}{\alpha}}} \text{ is decreasing on } (0,\infty)$$

Proof. We first observe that condition (2.11) with some $\omega \in \mathcal{M}(0,\infty)$ satisfying (2.10) is equivalent to the same condition with a function $\omega_1 \in \mathcal{M}(0,\infty)$ which satisfies both (2.10) and $(**)$. Indeed, for a given $\omega \in \mathcal{M}(0,\infty)$, define a function $\mu : (0,\infty) \mapsto (0,\infty)$ by

$$\mu(s) = \inf\left\{\frac{\omega(l)}{l^{\frac{1}{\alpha}}} : l \in (0,s]\right\}, \; s > 0.$$

It is clear that $\mu(s)$ is decreasing. Now setting $\omega_1(s) = s^{\frac{1}{\alpha}} \cdot \mu(s)$ we clearly see that ω_1 satisfies (2.10) and that $\omega_1(s) \leq \omega(s)$. Hence

$$\int_{0^+} \frac{ds}{\omega_1(s)s}$$

is divergent and $\omega_1 \in \mathcal{M}(0,\infty)$. In addition, we have the inequality

$$\frac{1}{\left[1+\frac{t}{\alpha}\omega(s)\right]^\alpha} \leq \frac{1}{\left[1+\frac{t}{\alpha}\omega_1(s)\right]^\alpha}$$

which proves our claim.

Thus we can assume for the rest of the proof that ω satisfies $(**)$. It remains to be shown that ω is an appropriate lower bound for f.

Indeed, defining $\Omega : (0, d_\tau(x)] \to [0,\infty)$ by (1.7) and using $(**)$ we have

$$\begin{aligned}\Omega(s) &= \int_s^{d_\tau(x)} \frac{d\lambda}{\omega(\lambda)\lambda} = \int_s^{d_\tau(x)} \frac{\lambda^{\frac{1}{\alpha}} d\lambda}{\omega(\lambda)\lambda^{\frac{1}{\alpha}+1}} \\ &\leq \frac{[d_\tau(x)]^{\frac{1}{\alpha}}}{\omega(d_\tau(x))} \int_s^{d_\tau(x)} \frac{d\lambda}{\lambda^{\frac{1}{\alpha}+1}} = \frac{\alpha}{\omega(d_\tau(x))}\left[s^{-\frac{1}{\alpha}}(d_\tau(x))^{\frac{1}{\alpha}} - 1\right].\end{aligned}$$

Inverting this expression we get

$$V(t) := \Omega^{-1}(t) \leq \frac{1}{\left(1+\frac{t}{\alpha}\omega(d_\tau(x))\right)} d_\tau(x) := W(t).$$

It is clear that the function $W(t)$ satisfies all the conditions of Theorem 2.1. This completes the proof of Theorem 2.2 □

Our next assertion is a direct consequence of Theorems 2.1 and 2.2.

Corollary 2.1. *Let* $\mathcal{S} = \{F(t)\}_{t\geq 0}$ *be a flow generated by a mapping* f *with a null (or sink) point* $\tau \in \overline{\mathbb{B}}$*. Then:*

(i) the asymptotic behavior of $\mathcal{S}$ *at* τ *is of exponential type if and only if*

$$\inf\{\omega^{\sharp}(l) : l \in (0, s]\} > 0, \quad s > 0; \tag{2.14}$$

(ii) the flow $\mathcal{S}$ *has a global uniform rate of exponential convergence if and only if*

$$\omega^{\sharp}(s) \geq a \tag{2.15}$$

for some $a > 0$.

Indeed, in both cases (i) and (ii), there is one function $\omega \in \mathcal{M}(0, \infty)$ such that the asymptotic behavior of $\mathcal{S}$ at τ is of order not less than α for all positive α. In case (i), ω can be chosen to be

$$\omega(s) := \inf\{\omega^{\sharp}(l) : l \in (0, s]\} > 0, \quad s > 0,$$

while in case (ii), ω can be chosen to be the constant a.

Remark 2.3. However, we will see in the next section that for holomorphic mappings, condition (2.15) holds, in fact, for some $a > 0$ whenever condition (2.14) holds. In other words, *for holomorphic flows any convergence of exponential type implies global uniform exponential convergence.*

The following example shows that *for a semigroup of* ρ*-non-expansive (but not holomorphic!) mappings, an asymptotic behavior of an exponential type does not imply, in general, a global uniform exponential rate of convergence.*

Example 4. Define a continuous mapping $f : \Delta \mapsto \mathbb{C}$ by the following formula:

$$f(x + iy) = x(1 - x)^2 + iy(1 - y)^2.$$

Since $\operatorname{Re} f(z)\bar{z} \geq 0$ for all $z = x + iy \in \Delta$, it follows that f is a generator of a semigroup $\mathcal{S} = \{F(t)\}_{t\geq 0}$ of ρ-non-expansive mappings such that each disk $\Delta_r = \{z \in \mathbb{C} : |z| < r < 1\}$ is $F(t)$-invariant. Setting $\tau = 0$ and $z^* = \frac{z}{|z|^2(1-|z|^2)}$, we have

$$\omega^{\sharp}(s) = \inf_{d_0(z)=s} \operatorname{Re} f(z)\bar{z^*} = \inf_{x^2+y^2=\frac{s}{s+1}} \frac{x^2(1-x)^2 + y^2(1-y)^2}{(x^2+y^2)(1-x^2-y^2)}.$$

It is easy to see that $\lim_{s\to 0^+} \omega^{\sharp}(s) = 1$ while $\omega^{\sharp}(s) \to 0$ as $s \to \infty$ (take, for example, $y = 0$ and $x = \sqrt{\frac{s}{s+1}} \to 1$). In Figure 2 we see, for instance, that the exponential rate corresponding to the initial point $z = 0.5$ is not appropriate for the point $z = 0.78$.

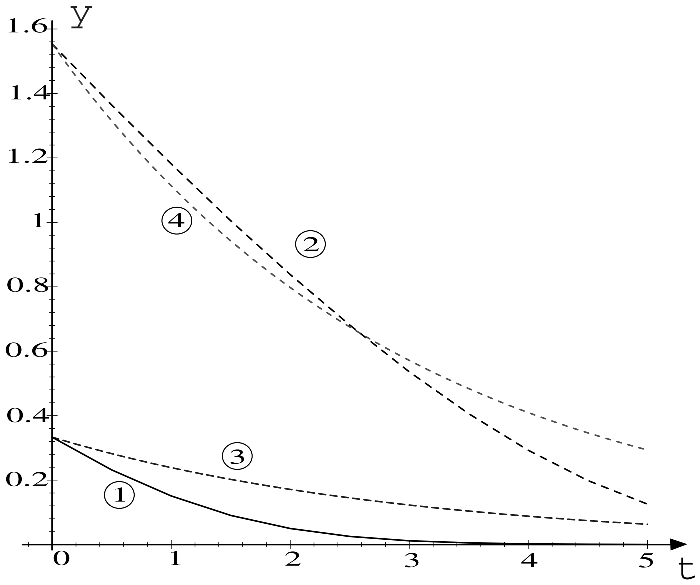

Fig. 2. 1. $y = d_0(F(t)0.5)$, 2. $y = d_0(F(t)0.78)$, 3. $y = d_0(0.5)\exp(-t/3)$, 4. $y = d_0(0.78)\exp(-t/3)$

3. Flows of holomorphic mappings

In this section we will study in more detail the flows $\mathcal{S} = \{F(t)\}_{t\geq 0}$ of self-mappings generated by holomorphic mappings $f \in \mathcal{G}\,\mathrm{Hol}(\mathbb{B})$ with stationary (or sink) points $\tau \in \overline{\mathbb{B}}$. We already know that the asymptotic behavior of a flow $\mathcal{S}$ at τ is of exponential type (Definition 2.2) if and only if the function

$$\omega^{\sharp}(s) = \inf_{d_\tau(x)=s} 2\mathrm{Re}\left\langle f(x), x^* \right\rangle, \quad s > 0, \tag{3.1}$$

satisfies (2.14). It turns out (see Theorem 3.1) that in this case this function and even the function

$$\omega_{\flat}(s) = \inf_{d_\tau(x)\leq s} 2\,\mathrm{Re}\left\langle f\,(x)\,, x^* \right\rangle \tag{3.2}$$

are bounded from below by a positive number. Moreover, for a boundary sink point the function $\omega_\flat$ is just a constant.

In both cases (interior stationary point or boundary sink point), the asymptotic behavior of a flow is completely determined by the value of $\omega^{\sharp}(0) := \liminf_{s\to 0^+} \omega^{\sharp}(s)$ which is related to the value of the derivative of f at its null point (for the interior case) or the so-called angular derivative (for the boundary case).

We begin with the following general assertion:

Theorem 3.1 (Theorem on universal rates of convergence). *Let* $f \in \mathcal{G}\,\mathrm{Hol}(\mathbb{B})$ *and let* $\{F(t)\}_{t\geq 0}$ *be the flow generated by* f. *If for some point* $\tau \in \overline{\mathbb{B}}$ *there is a decreasing function* $\omega : (0,\infty) \mapsto (0,\infty)$ *such that*

$$d_\tau(F(t)x) \leq e^{-t\omega(d_\tau(x))} d_\tau(x), \quad x \in \mathbb{B},\ t \geq 0, \tag{3.3}$$

then there exists a number $\mu > 0$ *such that*

$$d_\tau(F(t)x) \leq e^{-\mu t} d_\tau(x), \quad x \in \mathbb{B},\ t \geq 0. \tag{3.4}$$

Moreover:

(i) if $\tau \in \mathbb{B}$, *then* μ *can be chosen as* $\mu = \frac{\omega_\flat(0)}{4}$, *but* μ *cannot be larger than* $\omega_\flat(0)$ *(*$= \lim_{s\to 0^+} \omega_\flat(s)$*);*

(ii) if $\tau \in \partial\mathbb{B}$, *then the maximal* μ *for which* (3.4) *holds is exactly* $\omega_\flat(0)$, *that is,* $0 < \mu \leq \omega_\flat(0)$.

We will prove and discuss this theorem separately for the case where $\tau \in \mathbb{B}$ is a null point of f and for the case where $\tau \in \partial\mathbb{B}$, that is, when f is null point free.

1. Interior stationary point

Lemma 3.1. *Let* $f \in \mathcal{G}\,\mathrm{Hol}(\mathbb{B})$ *with* $f(0) = 0$ *and let* $\omega^\sharp$ *and* $\omega_\flat$ *be defined by (3.1) and (3.2). Then:*

(i) $\omega^\sharp(0) = \omega_\flat(0) = 2 \inf_{\|x\|=1} \mathrm{Re}\langle f'(0)x, x\rangle$*;*

(ii) $\frac{\omega_\flat(0)}{4} \leq \omega_\flat(s) \leq \omega_\flat(0)$.

Proof. First we show that

$$\omega^\sharp(0) \leq 2\nu, \tag{3.5}$$

where

$$\nu = \inf_{\|x\|=1} \mathrm{Re}\left\langle f'(0)x, x\right\rangle. \tag{3.6}$$

Since in our case $\tau = 0$, we have

$$\mathrm{Re}\left\langle f(x), x^*\right\rangle = \frac{1}{\|x\|^2(1-\|x\|^2)} \mathrm{Re}\left\langle f(x), x\right\rangle.$$

Now fixing $u \in \partial\mathbb{B}$, we set $x = ru$, where $r \in (0,1)$. Then we get

$$\mathrm{Re}\left\langle f(x), x^*\right\rangle = \mathrm{Re}\, \frac{1}{1-r^2}\left\langle \frac{1}{r} f(ru), u\right\rangle.$$

Therefore,

$$\omega^\sharp(s) \leq 2\ \mathrm{Re}\, \frac{1}{1-r^2}\left\langle \frac{1}{r} f(ru), u\right\rangle, \text{ where } r^2 = \|x\|^2 = \frac{s}{s+1}.$$

Letting s (and hence, r) tend to zero we obtain

$$\omega^{\sharp}(0) \leq 2 \ \mathrm{Re}\left\langle f'(0)u, u\right\rangle.$$

Since u is arbitrary, (3.5) follows.

On the other hand, it follows by the generalized Harnack inequality (see, for example, [4]) that for all $x \in \mathbb{B}$,

$$\mathrm{Re}\langle f(x), x\rangle \geq \mathrm{Re}\langle f'(0)x, x\rangle \frac{1-\|x\|}{1+\|x\|} \geq \nu\|x\|^2 \frac{1-\|x\|}{1+\|x\|}.$$

This implies that

$$\begin{aligned} 2 \ \mathrm{Re}\left\langle f(x), x^*\right\rangle &= \frac{2}{\|x\|^2(1-\|x\|^2)} \mathrm{Re}\ \langle f(x), x\rangle \\ &\geq \frac{2\nu\|x\|^2}{\|x\|^2(1-\|x\|^2)} \cdot \frac{1-\|x\|}{1+\|x\|} = \frac{2\nu}{(1+\|x\|)^2}. \end{aligned}$$

Hence

$$\begin{aligned} \omega_{\flat}(s) &= \inf_{d_\tau(x)\leq s} 2 \ \mathrm{Re}\left\langle f(x), x^*\right\rangle \geq \inf_{d_\tau(x)\leq s} \tfrac{2\nu}{(1+\|x\|)^2} \\ &= \inf_{\|x\|^2 \leq \frac{s}{s+1}} \frac{2\nu}{(1+\|x\|)^2} = \frac{2\nu}{\left(1+\sqrt{\frac{s}{s+1}}\right)^2} \left(\geq \tfrac{\nu}{2}\right). \end{aligned} \tag{3.7}$$

Letting s tend to 0^+ in (3.7) we see that $\omega_\flat(0) \geq 2\nu$. Since obviously $\omega^\sharp(0) \geq \omega_\flat(0)$, comparing the latter inequality with (3.5) we obtain (i). On the other hand, now substituting $\nu = \frac{\omega_\flat(0)}{2}$ in (3.7), we get assertion (ii), which completes the proof. □

To proceed we denote by M_τ the Möbius transformation of $\mathbb{B}$ defined by

$$M_\tau(x) = \frac{1}{1-\langle x, \tau\rangle} \left(\tau - \frac{\langle x, \tau\rangle}{\|\tau\|^2} - \sqrt{1-\|\tau\|^2}\left(x - \frac{\langle x, \tau\rangle}{\|\tau\|^2}\right)\right).$$

Note that M_τ is an automorphism of $\mathbb{B}$ (see, for example, [16, 9]) which has the following properties:

(a) $M_\tau^{-1} = M_\tau$ (involution property) with $M_\tau(0) = \tau$ and $M_\tau(\tau) = 0$;

(b) $1 - \|M_\tau(x)\|^2 = \sigma(x, y)$;

(c) $1 - \langle M_\tau(x), \tau\rangle = \frac{1-\|\tau\|^2}{1-\langle x,\tau\rangle}$.

These properties imply the equality

$$d_\tau(M_\tau(x)) = (1-\|\tau\|^2)d_0(x). \tag{3.8}$$

Now let us consider the flow $\{G(t)\}_{t\geq 0} \subset \mathrm{Hol}(\mathbb{B})$ defined by

$$G(t) = M_\tau \circ F(t) \circ M_\tau, \tag{3.9}$$

and let $g \in \mathcal{G}\,\mathrm{Hol}(\mathbb{B})$ be its generator, i.e.,

$$g(x) = -\frac{\partial}{\partial t} G(t)x \mid_{t=0^+} = \left[(M_\tau)'(x)\right]^{-1} f\left(M_\tau(x)\right). \tag{3.10}$$

Then $G_t(0) = 0$ for all $t \geq 0$ and $g(0) = 0$.

Lemma 3.2. *The following equality holds true:*

$$\frac{1}{1-\sigma(x,\tau)}\operatorname{Re}\left\langle f(x), \frac{x}{1-\|x\|^2}-\frac{\tau}{1-\langle\tau,x\rangle}\right\rangle \tag{3.11}$$
$$=\frac{1}{\|y\|^2}\operatorname{Re}\left\langle g(y), \frac{y}{1-\|y\|^2}\right\rangle,$$

where $y = M_\tau(x)$. *Thus the functions* $\omega^\sharp(s)$ *and* $\omega_\flat(s)$ *are invariant under the transformations* (3.9) *and* (3.10).

Proof. We have already seen in (2.8) and (2.9) that

$$\frac{\partial}{\partial t}\Big[d_\tau(F(t)x)\Big]_{t=0^+} = -2d_\tau(x)\operatorname{Re}\langle f(x), x^*\rangle$$
$$= -2\frac{d_\tau(x)}{1-\sigma(x,\tau)}\operatorname{Re}\left\langle f(x), \frac{x}{1-\|x\|^2}-\frac{\tau}{1-\langle\tau,x\rangle}\right\rangle. \tag{3.12}$$

On the other hand, by (3.8), (3.9) and property (a) of M_τ we have

$$\frac{\partial}{\partial t}\Big[d_\tau(F(t)x)\Big]_{t=0^+} = \frac{\partial}{\partial t}\Big[d_\tau(M_\tau G(t)y)\Big]_{t=0^+}$$
$$= \frac{\partial}{\partial t}\Big[(1-\|\tau\|^2)d_0(G(t)y)\Big]_{t=0^+}. \tag{3.13}$$

Since

$$\frac{\partial}{\partial t}\Big[d_0(G(t)y)\Big]_{t=0^+} = -\frac{d_0(y)}{\|y\|^2}\,2\operatorname{Re}\left\langle g(y), \frac{y}{1-\|y\|^2}\right\rangle$$
$$= -\frac{d_\tau(x)}{(1-\|\tau\|^2)\|y\|^2}\,2\operatorname{Re}\left\langle g(y), \frac{y}{1-\|y\|^2}\right\rangle,$$

we obtain (3.11) from (3.13) and (3.12). □

Now we are able to complete the proof of Theorem 3.1 for the case of an interior stationary point $\tau \in \mathbb{B}$ of $\mathcal{S}$.

To this end, let us assume that condition (3.3) holds. Then it follows, by Corollary 2.1(i) and Lemma 3.1(i), that $\omega_\flat(0) = \omega^\sharp(0) > 0$.

Let the flow $\{G(t)\}_{t\ge 0} \subset \operatorname{Hol}(\mathbb{B})$ and its generator $g \in \mathcal{G}\operatorname{Hol}(\mathbb{B})$ be defined by (3.9) and (3.10).

By Lemmata 3.2 and 3.1 we have

$$\inf_{d_0(x)=s}\operatorname{Re}\langle g(y), y^*\rangle \ge \frac{\omega_\flat(0)}{4}.$$

Then by Corollary 2.1(ii) we have

$$d_0(G(t)y) \le d_0(y)e^{-t\frac{\omega_\flat(0)}{4}}, \quad \text{for all } y \in \mathbb{B}.$$

Finally, setting $y = M_\tau(x)$ and using (3.8) we conclude that

$$d_\tau(F(t)x) \le d_\tau(x)e^{-t\frac{\omega_\flat(0)}{4}}. \tag{3.14}$$

Thus the proof of Theorem 3.1 for the case where $\tau \in \mathbb{B}$ is an interior stationary point is complete. □

Corollary 3.1. *Let $f \in \mathcal{G}\,\mathrm{Hol}(\mathbb{B})$ with $f(\tau) = 0$, $\tau \in \mathbb{B}$, and let $\{F(t)\}_{t\geq 0}$ be the flow generated by f. Then $\{F(t)\}_{t\geq 0}$ has a global uniform rate of exponential convergence if and only if $\omega^\sharp(0) > 0$.*

Corollary 3.2. *Let $\{F(t)\}_{t\geq 0}$ be a flow generated by $f \in \mathcal{G}\,\mathrm{Hol}(\mathbb{B})$ and let $\tau \in \mathbb{B}$. Then the following estimates are equivalent:*

$$\begin{aligned}
&(i) \qquad d_\tau(F(t)x) \leq e^{-t\mu} d_\tau(x), \quad x \in \mathbb{B},\ t \geq 0;\\
&(ii) \qquad \|M_\tau(F(t)x)\| \leq \|M_\tau(x)\| \cdot e^{-\mu\frac{1-\|M_\tau(x)\|^2}{2}t}, \quad x \in \mathbb{B},\ t \geq 0;\\
&(iii) \qquad \|M_\tau(F(t)x)\| \leq \|M_\tau(x)\| \cdot e^{-\nu\frac{1-\|M_\tau(x)\|}{1+\|M_\tau(x)\|}t}, \quad x \in \mathbb{B},\ t \geq 0,
\end{aligned}$$

where the numbers μ in (i) and (ii) can be chosen to be one and the same such that $0 \leq \frac{\omega_\flat(0)}{4} \leq \mu \leq \omega_\flat(0)$ and ν in (iii) is defined by

$$\nu = \frac{1}{2}\omega_\flat(0) = \frac{1}{2}\omega^\sharp(0) = \inf_{\|x\|=1} \mathrm{Re}\left\langle Bf'(\tau)B^{-1}x, x\right\rangle. \tag{3.15}$$

Here B is the linear operator defined by $B = P_\tau + \sqrt{1-\|\tau\|^2}(I - P_\tau)$ if $\tau \neq 0$ and $B = I$ if $\tau = 0$.

Proof. First we note that inequalities (ii) and (iii) are equivalent to the following ones:

$$\begin{aligned}
&(ii^*) \qquad \|G(t)y\| \leq \|y\| \cdot e^{-\mu\frac{1-\|y\|^2}{2}t}, \quad t \geq 0;\\
&(iii^*) \qquad \|G(t)y\| \leq \|y\| \cdot e^{-\nu\frac{1-\|y\|}{1+\|y\|}t}, \quad t \geq 0,
\end{aligned}$$

where $y = M_\tau(x) \in \mathbb{B}$ and the flow $\{G(t)\}_{t\geq 0}$ is defined by (3.9). First let us suppose that estimate (i) holds. By using (3.8) for the flow G we have

$$d_0(G(t)y) \leq e^{-\mu t} d_0(y).$$

Rewriting this inequality in the form

$$\frac{\|G(t)y\|^2}{1-\|G(t)y\|^2} \leq \frac{\|y\|^2}{1-\|y\|^2} \cdot e^{-\mu t},$$

we get, by direct calculations,

$$\|G(t)y\|^2 \leq \|y\|^2 \frac{1}{\|y\|^2 + (1-\|y\|^2)e^{\mu t}} \leq \|y\|^2 \cdot e^{-\mu t(1-\|y\|^2)},$$

which coincides with (ii^*).

Once again, let us suppose that inequality (ii) (and hence (ii^*)) holds. Differentiating both sides of this inequality with respect to t at $t = 0^+$ we obtain

$$-\frac{1}{\|y\|}\mathrm{Re}\langle g(y), y\rangle \leq -\|y\|\mu\frac{1-\|y\|^2}{2}. \tag{3.16}$$

This implies that $\omega^\sharp(s) \geq \mu$. Thus the decreasing function $\omega(s) \equiv a$ is an appropriate lower bound and the implication $(ii) \Rightarrow (i)$ follows by Theorem 2.2. The claimed estimate for the number μ is contained in Theorem 3.1(i).

Now let us again suppose that inequality (ii) (hence, (ii^*) and (3.16)) holds with some number $\mu > 0$. In (3.16) setting $y = ru,\ u \in \partial\mathbb{B},\ r \in (0,1)$, and letting r tend to zero (cf. the proof of Lemma 3.1), we get $\operatorname{Re}\left\langle g'(0)u, u\right\rangle \geq \frac{\mu}{2} > 0$. A direct calculation shows that

$$g'(0) = \left[(M_\tau)'(0)\right]^{-1} f'(\tau)\,(M_\tau)'(0) = Bf'(\tau)B^{-1},$$

and so $\nu > 0$. Therefore, again by Harnack's inequality, we have

$$\operatorname{Re}\langle g(y), y\rangle \geq \operatorname{Re}\langle g'(0)y, y\rangle \frac{1-\|y\|}{1+\|y\|} \geq \nu\,\|y\|^2\,\frac{1-\|y\|}{1+\|y\|}.$$

On the other hand,

$$\begin{aligned}\frac{\partial \ln\|G(t)y\|}{\partial t} = \frac{1}{2}\frac{\partial \ln\|G(t)y\|^2}{\partial t} &= \frac{1}{\|G(t)y\|^2}\operatorname{Re}\left\langle \frac{\partial G(t)y}{\partial t}, G(t)y\right\rangle = \\ &= -\frac{1}{\|G(t)y\|^2}\operatorname{Re}\big\langle g\big(G(t)y\big), G(t)y\big\rangle.\end{aligned}$$

It also follows by the Schwarz lemma that $\|G(t)y\| \leq \|y\|$. Thus we have

$$\frac{\partial \ln\|G(t)y\|}{\partial t} \leq -\nu\frac{1-\|y\|}{1+\|y\|}.$$

Integrating this inequality we obtain the following estimate:

$$\ln\|G(t)y\| - \ln\|y\| \leq -\nu\frac{1-\|y\|}{1+\|y\|}t,$$

which coincides with (iii^*).

Finally, if condition (iii^*) holds, then differentiating it with respect to t at $t = 0^+$ we get

$$\operatorname{Re}\langle g(y), y\rangle \geq \frac{\nu}{(1+\|x\|)^2} \geq \frac{\nu}{4} > 0.$$

Thus $\omega^\sharp(0) > 0$ and by Theorem 3.1 the result follows. □

Remark 3.1. The above corollary asserts that an exponential rate of convergence in the sense of the "distance" $d_\tau(\cdot,\cdot)$ is equivalent to the same rate of convergence in the norm of $\mathcal{H}$. We remark in passing that when $\tau = 0$ estimate (iii) can be extended to an arbitrary Banach space (see [10, 18]). It is known that the original topology and the topology generated by the hyperbolic metric on $\mathbb{B}$ are locally equivalent (see, for example, [9]). Although global equivalence does not hold, we will prove that an exponential rate of convergence in the norm coincides with a global rate of exponential convergence with respect to the hyperbolic metric.

Corollary 3.3. *Let $\{F(t)\}_{t\geq 0}$ be a flow generated by $f \in \mathcal{G}\,\mathrm{Hol}(\mathbb{B})$ with $f(\tau) = 0$, $\tau \in \mathbb{B}$. Then the flow $\{F(t)\}_{t\geq 0}$ has an exponential rate of convergence if and only if the following estimate of convergence in the hyperbolic metric holds:*

$$\text{there exists a number } \eta > 0 \text{ such that}$$
$$\rho\,(F(t)x, \tau) \leq \rho(x, \tau) \cdot e^{-\eta \Lambda(x)t}, \tag{3.17}$$
$$\text{where } \Lambda(x) = e^{-2\rho(x,\tau)}, \quad x \in \mathbb{B} \text{ and } t \geq 0.$$

Moreover, the maximal value of the number η for which this inequality holds is $\nu = \inf_{\|x\|=1} \mathrm{Re}\left\langle Bf'(\tau)B^{-1}x, x\right\rangle$, where $B = P_\tau + \sqrt{1 - \|\tau\|^2}(I - P_\tau)$ if $\tau \neq 0$ and $B = I$ if $\tau = 0$.

Proof. Let us suppose that $\{F(t)\}_{t\geq 0}$ has an exponential rate of convergence. By Theorem 3.1, Corollary 3.2, Lemma 3.1 and formula (3.14) we have

$$\|M_\tau\,(F(t)x)\| \leq \|M_\tau(x)\| \cdot e^{-\nu \frac{1-\|M_\tau(x)\|}{1+\|M_\tau(x)\|}t}, \quad x \in \mathbb{B},$$

where the number ν is defined in (3.15).

Therefore one can write the estimate

$$\rho\,(F(t)x, \tau) = \rho\,(0, \|M_\tau(F(t)x)\|) \leq \rho\,(0, \|y\|)\, e^{-\nu \frac{1-\|M_\tau(x)\|}{1+\|M_\tau(x)\|}t},$$

which coincides with (3.17) for $\eta = \nu$. Now suppose that inequality (3.17) holds with some number $\eta \geq 0$. Again, by using the flow $\{G(t)\}_{t\geq 0}$ defined by (3.9), we can rewrite (3.17) as

$$\rho\,(G(t)y, 0) \leq \rho(y, 0) \cdot e^{-\eta\Lambda(y)t}, \text{ where } \Lambda(y) = e^{-2\rho(y,0)}.$$

Differentiating this inequality with respect to t at $t = 0^+$, we obtain

$$\mathrm{Re}\langle g(y), y^*\rangle \geq \eta \frac{\rho(y, 0)\Lambda(y)}{\|y\|}.$$

Now setting $y = ru, u \in \partial\mathbb{B}, r \in (0, 1)$, and letting r tend to zero (cf. the proofs of Lemma 3.1 and Corollary 3.2), we get $\nu \geq \eta$, as claimed. □

2. Boundary sink point (Wolff point [7,2,17])

As a matter of fact, if $\mathcal{S} = \{F(t)\}_{t\geq 0}$ converges to a boundary sink point $\tau \in \partial\mathbb{B}$ with a rate of convergence of exponential type,

$$d_\tau(F(t)x) \leq \exp(-t\omega(d_\tau(x)))d_\tau(x),$$

where $\omega \in \mathcal{M}(0, \infty)$ is a decreasing function, then this estimate can be improved as follows:

$$d_\tau(F(t)x) \leq \exp(-t\omega(0))d_\tau(x),$$

where $\omega(0) := \lim_{s\to 0^+} \omega(s)$.

In other words, we claim that, if the inequality

$$\omega^{\sharp}(s) \geq \omega(s)$$

holds for a decreasing ω, then the stronger inequality

$$\omega^{\sharp}(s) \geq \omega(0)$$

also holds. In particular, this fact holds for the function $\omega = \omega_{\flat}$. This implies, in turn, that $\omega_{\flat}$ is actually constant: $\omega_{\flat}(s) = \omega_{\flat}(0) = \beta$ for all $s \in (0, \infty)$ and is equal to the so-called angular derivative of f (if it exists) at the point $\tau \in \partial\mathbb{B}$. Moreover, this number β gives the best rate of exponential convergence of $\mathcal{S} = \{F(t)\}_{t\geq 0}$.

These claims will follow from Theorem 3.2 below, which is a continuous analog of the classical Julia–Wolff–Carathéodory theorem.

We will need some additional notions which are well known in the finite-dimensional case (see, for example, [5,16]).

Definition 3.1. *A curve $\Lambda : [0, 1) \mapsto \mathbb{B}$ is said to be asymptotically normal at a point $\tau \in \partial\mathbb{B}$ if:*

$$(i)\ \lim_{s\to 1^-} \frac{|\Lambda(s) - \lambda(s)|^2}{1 - |\lambda(s)|^2} = 0;$$

$$(ii)\ \frac{|\lambda(s) - \tau|}{1 - |\lambda(s)|} \leq M \leq \infty, \quad 0 \leq s < 1,$$

where $\lambda(s)$ is the orthogonal projection of $\Lambda(s)$ onto the complex line through 0 and τ:

$$\lambda(s) = \langle \Lambda(s), \tau\rangle\tau. \tag{3.18}$$

Definition 3.2. *Let h be a holomorphic function on $\mathbb{B}$ with values in the complex plane $\mathbb{C}$. We say that h has a restricted limit L at $\tau \in \partial\mathbb{B}$ if h has limit L along every curve which is asymptotically normal at τ.*

Definition 3.3. *Let $f : \mathbb{B} \mapsto \mathcal{H}$ be a holomorphic mapping on $\mathbb{B}$ and let $\tau \in \partial\mathbb{B}$. We say that f has a finite angular derivative at τ if, for some element $y \in \mathcal{H}$, the function $h : \mathbb{B} \mapsto \mathbb{C}$ defined by*

$$h(x) = \frac{\langle y - f(x), \tau\rangle}{1 - \langle x, \tau\rangle},$$

has a finite restricted limit at τ. We denote this limit by $\angle f'(\tau)$.

Theorem 3.2. *Let the mapping $f \in \mathrm{Hol}(\mathbb{B}, \mathcal{H})$ be a generator of a semigroup $\mathcal{S} = \{F(t)\}_{t\geq 0}$ of holomorphic self-mappings of $\mathbb{B}$. Suppose that f has no null point in $\mathbb{B}$ and that $\tau \in \partial\mathbb{B}$ is the boundary sink point for $\mathcal{S}$. Then the following are equivalent:*

(i) the asymptotic behavior of $\mathcal{S}$ at τ is of exponential type;
(ii) there is a positive number γ such that

$$d_\tau(F(t)x) \leq e^{-\gamma t} d_\tau(x),\ \ x \in \mathbb{B} \text{ and } t \geq 0.$$

Moreover, if the angular derivative $\beta = \angle f'(\tau)$ of f at τ exists, then:
(a) β is a positive real number with $\beta = 2\inf\left\{\operatorname{Re}\langle f(x), x^*\rangle,\ x \in \mathbb{B}\right\}$;
(b) the maximal γ which satisfies condition (ii) is exactly β.

To prove Theorem 3.2 we will need the following assertion:

Lemma 3.3. *Let $F \in \operatorname{Hol}(\mathbb{B})$ be a holomorphic self-mapping of $\mathbb{B}$ with no fixed point in $\mathbb{B}$ and let τ be its boundary sink point. Then the curve $\Lambda : [0, 1) \mapsto \mathbb{B}$ defined by*

$$\Lambda(s) = F(s\tau)$$

is asymptotically normal at τ.

Proof. Since τ is a sink point of F, it follows by Julia's lemma that there is a number $0 < \delta(F) \leq 1$ (sometimes called the dilation coefficient or Julia's number [5, 16, 17]) such that

$$d_\tau(F(x)) \leq \delta(F) \cdot d_\tau(x), \quad x \in \mathbb{B}.$$

For $0 < s < 1$ we have, by (3.18),

$$\begin{aligned}
\frac{1 - \|\Lambda(s)\|}{1 - s}\frac{1 + s}{1 + \|\Lambda(s)\|} &\leq \frac{1 - \|\lambda(s)\|}{1 - s}\frac{1 + s}{1 + \|\lambda(s)\|} \\
&\leq \frac{\|\tau - \lambda\|^2}{1 - \|\lambda(s)\|^2}\frac{1 - s^2}{(1 - s)^2} \leq \frac{\|\tau - \lambda\|^2}{1 - \|\Lambda(s)\|^2}\frac{1 - s^2}{(1 - s)^2} \\
&= \frac{d_\tau(F(s\tau))}{d_\tau(s\tau)} \leq \delta(F).
\end{aligned} \tag{3.19}$$

Hence,

$$\limsup_{s \to 1^-} \frac{1 - \|\Lambda(s)\|}{1 - s} \leq \limsup_{s \to 1^-} \frac{1 - \|\lambda(s)\|}{1 - s} \leq \delta(F). \tag{3.20}$$

On the other hand, the Julia–Wolff–Carathéodory theorem asserts that

$$\delta(F) = \liminf_{x \to \tau} \frac{1 - \|F(x)\|}{1 - \|x\|} = \angle F'(\tau). \tag{3.21}$$

Thus by (3.19)–(3.21) we get

$$\begin{aligned}
&\lim_{s \to 1^-} \frac{1 - \|\Lambda(s)\|}{1 - s} \lim_{s \to 1^-} \frac{1 - \|\lambda(s)\|}{1 - s} \\
&= \lim_{s \to 1^-} \frac{\|\tau - \lambda(s)\|}{1 - s} = \delta(F).
\end{aligned} \tag{3.22}$$

This equality implies that

$$\lim_{s \to 1^-} \frac{\|\tau - \lambda(s)\|}{1 - \|\lambda(s)\|} = 1,$$

which proves condition (ii) of Definition 3.1.

To prove condition (i), we calculate as follows:

$$\lim_{s\to 1^-} \frac{\|\Lambda(s)-\lambda(s)\|^2}{1-\|\lambda(s)\|^2} = \lim_{s\to 1^-} \frac{\|\Lambda(s)\|^2+\|\lambda(s)\|^2-2\operatorname{Re}\langle\Lambda(s),\lambda(s)\rangle}{1-\|\lambda(s)\|^2}$$
$$= \lim_{s\to 1^-} \frac{\|\Lambda(s)\|^2+\|\lambda(s)\|^2-2\operatorname{Re}\overline{\langle\Lambda(s),\tau\rangle}\langle\Lambda(s),\tau\rangle}{1-\|\lambda(s)\|^2}$$
$$= \lim_{s\to 1^-} \frac{\|\Lambda(s)\|^2-\|\lambda(s)\|^2}{1-\|\lambda(s)\|^2} = 1-\lim_{s\to 1^-}\frac{1-\|\Lambda(s)\|^2}{1-\|\lambda(s)\|^2}\cdot\frac{1-s}{1-s} = 1-\frac{\delta(F)}{\delta(F)} = 0,$$

by (3.22). This proves condition (i) and completes the proof. □

Proof of Theorem 3.2. Let condition (i) of the theorem hold, i.e., for some decreasing function $\omega \in \mathcal{M}(0,\infty)$,

$$d_\tau(F(t)x) \le e^{-t\omega(d_\tau(x))} d_\tau(x)$$

or explicitly,

$$\frac{|1-\langle F(t)x,\tau\rangle|^2}{1-\|F(t)x\|^2} \le e^{-2t\omega(d_\tau(x))}\frac{|1-\langle x,\tau\rangle|^2}{1-\|x\|^2}.$$

This is equivalent to the inequality

$$\frac{|1-\langle F(t)x,\tau\rangle|^2}{|1-\langle x,\tau\rangle|^2} \le e^{-2t\omega(d_\tau(x))}\frac{1-\|F(t)x\|^2}{1-\|x\|^2}. \tag{3.23}$$

Once again, it follows from the Julia–Wolff–Carathéodory theorem that for a fixed $t \ge 0$,

$$\begin{aligned}\delta(F(t)) &:= \liminf_{x\to\tau} \frac{1-\|F(t)x\|}{1-\|x\|}\\ &= \angle[F(t)]'(\tau) := \lim_{x\to\tau}\frac{1-\langle F(t)x,\tau\rangle}{1-\langle x,\tau\rangle},\end{aligned} \tag{3.24}$$

where that last limit is taken along an asymptotically normal curve at τ.

Let us denote

$$\omega(0) = \lim_{s\to 1^-}\omega(d_\tau(s\tau)). \tag{3.25}$$

Thus setting $x = s\tau$ in (3.23) and letting s tend to 1^-, we get

$$\delta^2(F(t)) \le e^{-2t\omega(0)}\delta(F(t)),$$

or

$$\delta(F(t)) \le e^{-t\gamma},$$

where we set $\gamma = 2\omega(0)$.

Now by using Julia's lemma we obtain the implication (i)⇒(ii). The converse implication can be established by differentiating the inequality in (ii) at $t = 0^+$. Namely, we get

$$\operatorname{Re}\langle f(x), x^*\rangle \ge \frac{\gamma}{2} > 0. \tag{3.26}$$

So, one can set $\omega(s) \equiv \frac{\gamma}{2}$ and the asymptotic behavior of $\mathcal{S}$ at τ is seen to be of exponential type.

To prove the second part of the theorem we first observe that f is a generator if and only if the equation

$$x + tf(x) = y \tag{3.27}$$

is solvable for all $t \geq 0$ and $y \in \mathbb{B}$ (see, for example, [12]).

The solution $x = \mathcal{J}_t(y) = (I + tf)^{-1}(y)$ is called the nonlinear resolvent of f. It has the following properties:

(1) for each $t \geq 0$, $\mathcal{J}_t : \mathbb{B} \mapsto \mathbb{B}$ is a holomorphic self-mapping of $\mathbb{B}$;

(2) for each $x \in \mathbb{B}$, the vector function $\mathcal{J}_t : \mathbb{R}^+ \mapsto \mathbb{B}$ is continuons and the semigroup $\mathcal{S} = \{F(t)\}_{t \geq 0}$ generated by f can be represented by the exponential formula

$$\lim_{n \to \infty} \left[\mathcal{J}_{\frac{t}{n}}\right]^n (x) = F(t)x, \quad x \in \mathbb{B},$$

where the limit is taken with respect to the locally uniform topology of $\mathbb{B}$;

(3) if f has no null point in $\mathbb{B}$ and $\tau \in \partial\mathbb{B}$ is the sink point of $\mathcal{S}$, then for each $t > 0$, τ is also the sink point of $\mathcal{J}_t$ and moreover, the following approximations hold:

$$\lim_{t \to \infty} \mathcal{J}_t(x) = \tau, \quad x \in \mathbb{B}$$

and

$$\lim_{t \to \infty} f(\mathcal{J}_t(x)) = 0, \quad x \in \mathbb{B}.$$

Now let us suppose that $\beta = \angle f'(\tau)$ exists (finitely). Then it follows, by properties (1), (3) above and Lemma 3.3, that for each $t > 0$, the curve $\Lambda_t(s) := \mathcal{J}_t(s\tau) : [0, 1) \mapsto \mathbb{B}$ is an asymptotically normal curve at $\tau \in \partial\mathbb{B}$. In addition, by equation (3.27) we have the identity

$$\Lambda_t(s) + tf(\Lambda_t(s)) = s\tau, \tag{3.28}$$

for all $s \in [0, 1)$ and $t \geq 0$.

Denote the angular derivative $\angle[\mathcal{J}_t]'(\tau)$ of $\mathcal{J}_t : \mathbb{B} \mapsto \mathbb{B}$ at the point τ by c_t.

Again by the Julia–Wolff–Carathéodory theorem,

$$c_t = \lim_{s \to 1^-} \frac{1 - \langle \mathcal{J}_t(s\tau), \tau \rangle}{1 - s}. \tag{3.29}$$

By (3.28) and (3.29) we get $\lim\limits_{s \to 1^-} f(\Lambda_t(s)) = 0$ and

$$\begin{aligned}
\beta &= \lim_{s \to 1^-} \frac{\langle f(\Lambda_t(s)), \tau \rangle}{\langle \Lambda_t(s), \tau \rangle - 1} = \lim_{s \to 1^-} \frac{1}{t} \cdot \frac{\langle (s\tau - \Lambda_t(s)), \tau \rangle}{\langle \Lambda_t(s), \tau \rangle - 1} \\
&= \lim_{s \to 1^-} \frac{1}{t} \cdot \frac{-s + \langle \Lambda_t(s), \tau \rangle}{1 - \langle \Lambda_t(s), \tau \rangle} \\
&= \lim_{s \to 1^-} \frac{1}{t} \cdot \left(\frac{\langle \Lambda_t(s), \tau \rangle - 1}{1 - \langle \Lambda_t(s), \tau \rangle} + \frac{1 - s}{1 - \langle \Lambda_t(s), \tau \rangle} \right) \\
&= \frac{1}{t}\left(-1 + \frac{1}{c_t}\right).
\end{aligned}$$

Thus we obtain that β is a non-negative real number and

$$c_t = \frac{1}{1+t\beta}.$$

Hence,

$$d_\tau(\mathcal{J}_t(x)) \le \frac{1}{1+t\beta} d_\tau(x), \tag{3.30}$$

by Julia's lemma.

Now applying the exponential formula (see property (2) above) we get

$$d_\tau(F(t)x) \le e^{-t\beta} d_\tau(x).$$

To conclude the proof of Theorem 3.2 it remains to be shown that if condition (ii), or equivalenty, inequality (3.26) holds for some $\gamma > 0$, then $\gamma \le \beta$.

Indeed, setting $x = s\tau$ in (3.26) we get

$$\begin{aligned}\operatorname{Re}\langle f(s\tau), (s\tau)^*\rangle &= \operatorname{Re}\left\langle f(s\tau), \frac{s\tau}{1-s^2} - \frac{\tau}{1-s}\right\rangle = \\ &= \frac{\operatorname{Re}\langle f(s\tau), \tau\rangle}{s-1} \cdot \frac{1}{1+s} \ge \gamma/2.\end{aligned}$$

Now letting s tend to 1^- we get

$$\frac{\operatorname{Re}\angle f'(\tau)}{2} = \frac{\beta}{2} \ge \frac{\gamma}{2},$$

i.e., $\beta \ge \gamma$. This completes the proof of Theorem 3.2 as well as that of Theorem 3.1. □

Acknowledgements. The work of M. Elin was partially supported by the Rashi–Sakta Fund. The work of S. Reich was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund – E. and M. Mendelson Research Fund. All the authors thank Alexander Goldvard for his technical help in producing the paper and the referee for several useful comments.

References

1. Abate, M.: Converging semigroups of holomorphic maps. Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. **82**, 223–227 (1988)
2. Abate, M.: The infinitesimal generators of semigroups of holomorphic maps. Ann. Mat. Pura Appl., IV. Ser. **161**, 167–180 (1992)
3. Aharonov, D., Elin, M., Reich, S., Shoikhet, D.: Parametric representations of semi-complete vector fields on the unit balls in $\mathbb{C}^n$ and in Hilbert space. Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. **10**, 229–253 (1999)
4. Aharonov, D., Reich, S., Shoikhet, D.: Flow invariance conditions for holomorphic mappings in Banach spaces. Math. Proc. R. Ir. Acad. **99A**, 93–104 (1999)
5. Cowen, C.C., MacCluer, B.D.: Composition Operators on Spaces of Analytic Functions. CRC Press, Boca Raton, FL 1995

6. Elin, M., Shoikhet, D.: Dynamic extension of the Julia–Wolff–Carathéodory theorem. Dyn. Syst. Appl. **10**, 421–437 (2001)
7. de Fabritiis, C.: On the linearization of a class of semigroups on the unit ball of $\mathbb{C}^n$. Ann. Mat. Pura Appl., IV. Ser. **166**, 363–379 (1994)
8. Franzoni, T., Vesentini, E.: Holomorphic Maps and Invariant Distances. North-Holland, Amsterdam 1980
9. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings. Marcel Dekker, New York, Basel 1984
10. Poreda, T.: On generalized differential equations in Banach space. Diss. Math. **310**, Warsaw 1991
11. Reich, S.: The asymptotic behavior of a class of nonlinear semigroups in the Hilbert ball. J. Math. Anal. Appl. **157**, 237–242 (1991)
12. Reich, S., Shoikhet, D.: Generation theory for semigroups of holomorphic mappings in Banach spaces. Abstr. Appl. Anal. **1**, 1–44 (1996)
13. Reich, S., Shoikhet, D.: Semigroups and generators on convex domains with the hyperbolic metric. Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. **8**, 231–250 (1997)
14. Reich, S., Shoikhet, D.: The Denjoy–Wolff theorem. Ann. Univ. Mariae Curie-Skłodowska, Sect. A **51**, 219–240 (1997)
15. Reich, S., Shoikhet, D.: Metric domains, holomorphic mappings and nonlinear semigroups. Abstr. Appl. Anal. **3**, 203–228 (1998)
16. Rudin, W.: Function Theory on the Unit Ball in $\mathbb{C}^n$. Springer, Berlin 1980
17. Shapiro, J.H.: Composition Operators and Classical Function Theory. Springer, Berlin 1993
18. Suffridge, T.J.: Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions. Complex Analysis (Proc. Conf. Univ. Kentucky, Lexington, KY, 1976), Lecture Notes in Math. **599**, 146–159 (1977)