Annali di Matematica 181, 463–500 (2002)

Digital Object Identifier (DOI) 10.1007/s10231-002-0051-3

M. Engliš* · J. Peetre

Green functions and eigenfunction expansions for the square of the Laplace–Beltrami operator on plane domains

Received: August 21, 2000
Published online: October 30, 2002 –

Abstract. For a certain class of domains $\Omega \subset \mathbb{C}$ with smooth boundary and $\widetilde{\Delta}_\Omega = w^2 \Delta$ the Laplace–Beltrami operator with respect to the Poincaré metric $ds^2 = w(z)^{-2} dz\, d\overline{z}$ on Ω, we (1) show that the Green function for the biharmonic operator $\widetilde{\Delta}^2_\Omega$, with Dirichlet boundary data, is positive on $\Omega \times \Omega$; and (2) obtain an eigenfunction expansion for the operator $\widetilde{\Delta}_\Omega$, which reduces to the ordinary non-Euclidean Fourier transform of Helgason for $\Omega = \mathbb{D}$ (the unit disc). In both cases the proofs go via uniformization, and in (1) we obtain a Myrberg-like formula for the corresponding Green function. Finally, the latter formula as well as the eigenfunction expansion are worked out more explicitly in the simplest case of Ω an annulus, and a result is established concerning the convergence of the series $\sum_{\omega \in G} (1 - |\omega 0|^2)^s$ for G the covering group of the uniformization map of Ω and $0 < s < 1$.

Mathematics Subject Classification (2000). Primary 58J32, 35P10; Secondary 33E30, 30F35

Key words. Green function – Laplace–Beltrami operator – eigenfunction expansion – Poincaré metric – biharmonic operator – Fuchsian group

0. Introduction

Let Ω be a domain in $\mathbb{C}$ with a smooth boundary, or, more generally, such that $\mathbb{C} \setminus \Omega$ consists of at least two points. By the uniformization theorem, the universal covering surface of Ω is then biholomorphic to the unit disc $\mathbb{D}$, and we denote by $\phi : \mathbb{D} \to \Omega$ the covering projection (the uniformization map). Projecting the invariant metric $(1 - |x|^2)^{-1} |dx|$ on the disc via ϕ, we obtain the Poincaré metric $w(z)^{-1} |dz|$ on Ω. Up to a constant multiple, it is the unique complete Riemannian metric on Ω of constant negative curvature, and is preserved by biholomorphic maps. Associated to this metric are the Poincaré measure $d\mu_\Omega(z) = w(z)^{-2}\, dz\, d\overline{z}$ and the Laplace–Beltrami operator $\widetilde{\Delta}_\Omega = w(z)^2 \Delta$, which are the projections to Ω of the invariant measure $d\mu_{\mathbb{D}}(x) = (1 - |x|^2)^{-2}\, dx\, d\overline{x}$ and the invariant Laplacian

M. Engliš: MÚ AV ČR, Žitná 25, 11567 Prague 1, Czech Republic,
e-mail: `englis@math.cas.cz`

J. Peetre: Centre of Mathematics, University of Lund, Box 118, S-22100 Sweden,
e-mail: `jaak@maths.lth.se`

* The first author was supported by GA AV ČR grants no. A1019701 and A1019005.

$\widetilde{\Delta}_{\mathbb{D}} = (1-|z|^2)^2\Delta$ on $\mathbb{D}$, respectively; $\widetilde{\Delta}_\Omega$ is a formally selfadjoint second-order differential operator on $L^2(\Omega, d\mu_\Omega)$. In this paper we want to pursue two themes from the spectral theory of the operator $\widetilde{\Delta}_\Omega$: namely, (1) the constancy of the sign of the Green function for its square, the biharmonic Laplace–Beltrami operator $\widetilde{\Delta}^2_\Omega$, with Dirichlet boundary conditions; and (2) the eigenfunction expansion for the operator $\widetilde{\Delta}_\Omega$.

The paper consists of several parts which below, for convenience, we refer to as I, II, III and IV.

I. *Positivity.* (Sections 1–2) Our original motivation for (1) stemmed from the recent renewal of interest, motivated in turn by applications to the function theory in the Bergman spaces, in the constancy of the sign of the Green function of the biharmonic operator Δ^2 (or, even more generally, of the operators $\Delta\rho^{-1}\Delta$, with a weight $\rho > 0$), with the Dirichlet boundary data; see [20, 13]. It is well known that the Green function for the Laplace operator on Ω, with Dirichlet boundary data, is negative on $\Omega \times \Omega$; in fact, this is just a disguised form of one of the fundamental results in analysis, the maximum principle. Passing from Δ to Δ^2, the situation changes drastically: in that case, the Green function turns out to still have a constant sign (this time, positive) e.g. for the disc, but fails to do so for many other domains, even very nice ones (sufficiently elongated ellipses or rectangles, annuli, etc.; see the discussion in [13] and the references therein). One source of this failure is the *non-invariance* of the operator Δ: under a biholomorphism $f : \Omega_1 \to \Omega_2$, the operator Δ on Ω_2 is not transformed into Δ again, but rather into $|f'|^{-2}\Delta$ on Ω_1. In the case of Δ itself, this causes no problem — quite generally, for any linear differential operator L and a weight function ρ, the Green functions of L and ρL are related by $G_{\rho L}(x, y) = \rho(x)^{-1}G_L(x, y)$; hence, in particular, the Green function for Δ on Ω_1 is of the same sign as that for $|f'|^{-2}\Delta$, i.e. as that for Δ on Ω_2. However, in the case of Δ^2, passing from one domain to another by a conformal map f as above shows only that the Green function for Δ^2 on the latter domain has the same sign as that for the operator $\Delta|f'|^{-2}\Delta$ on the former; and, obviously, on a given domain there is no connection between the Green functions for Δ^2 and $\Delta|f'|^{-2}\Delta$ in general. From this point of view it is therefore very natural to consider instead the biholomorphically invariant operators $\widetilde{\Delta}_\Omega$ and $\widetilde{\Delta}^2_\Omega$. (In this case, the operator $\widetilde{\Delta}_\Omega$ being singular at the boundary, some care must be exercised regarding the "Dirichlet boundary conditions", but we assure the reader that everything can be fixed with ease). For $\widetilde{\Delta}_\Omega$, as has been pointed out a few lines above, the Green function differs from that for Δ only by a factor of w^{-2} (as $\widetilde{\Delta}_\Omega = w^2\Delta$), and, consequently, is also of constant (negative) sign. For its square $\widetilde{\Delta}^2_\Omega$ and Ω the unit disc $\mathbb{D}$, the Green function has been computed explicitly by the present authors in [15] and shown to be of constant sign (positive). As, in contrast to Δ^2, the operators $\widetilde{\Delta}^2_\Omega$ are invariant under biholomorphic maps (since the Poincaré metric is), it follows immediately that the Green function for $\widetilde{\Delta}^2_\Omega$ is of constant (positive) sign for any simply connected plane domain Ω (other than $\mathbb{C}$ itself). A very natural question thus arises at this point, namely, can it be true that the Green function for $\widetilde{\Delta}^2_\Omega$ is in fact positive for any, say, smoothly bounded plane domain Ω?

The main result of the first part of this paper is that the answer is indeed in the affirmative: for any bounded domain $\Omega \subset \mathbb{C}$ with smooth boundary, the Green function for the operator $\widetilde{\Delta}_\Omega^2$ with the Dirichlet boundary data is positive. In more detail, we use uniformization to express the Green function in question as an (infinite) sum of translates of the same Green function on the disc; this formula (which resembles Myrberg's formula for the Green function of the ordinary Δ on Ω, cf. [32, Theorem XI.13] immediately implies the positivity. As a by-product, a rigorous proof of the existence of the Green function is also obtained.

One suspects that an analogous situation might also prevail in higher dimensions — for instance, for the square of the Laplace–Beltrami operator with respect to the invariant metric on a bounded symmetric domain, or with respect to the Kähler–Einstein metric on a strictly pseudoconvex domain in $\mathbb{C}^n$ (in both instances the metric is unique to within a constant multiple). We give some trifling evidence in support of this belief by establishing the positivity of the Green function for the square of the invariant Laplacian $\widetilde{\Delta}_\mathbb{B}$ on the unit ball $\mathbb{B}$ of $\mathbb{C}^d$.

II. *Spectral theory.* (Sections 3–4) Our second theme concerns the analogue for Ω of the non-Euclidean Fourier transform of Helgason on $\mathbb{D}$ [21]. Recall that the latter is given by the Fourier inversion formulas

$$f(x) = \int_\mathbb{R} \int_{\partial\mathbb{D}} \tilde{f}(\lambda, b) e_{-\lambda,b}(x) |c(\lambda)|^{-2} \, d\lambda \, db,$$

$$\tilde{f}(\lambda, b) = \int_\mathbb{D} f(x) e_{\lambda,b}(x) \, d\mu_\mathbb{D}(x),$$

and the Plancherel formula

$$\int_\mathbb{D} |f(x)|^2 \, d\mu_\mathbb{D}(x) = \int_\mathbb{R} \int_{\partial\mathbb{D}} |\tilde{f}(\lambda, b)|^2 |c(\lambda)|^{-2} \, d\lambda \, db,$$

where

$$e_{\lambda,b}(x) = \left(\frac{1 - |x|^2}{|1 - \overline{b}x|^2} \right)^{\frac{1}{2} + i\lambda}, \qquad (\lambda \in \mathbb{R}, b \in \partial\mathbb{D})$$

and

$$|c(\lambda)|^{-2} = \frac{\pi\lambda}{2} \tanh \frac{\pi\lambda}{2}.$$

These formulas in turn are a special case of a result of Harish–Chandra, valid in the context of any Riemannian symmetric space $\mathcal{M} = G/K$, with G a semi-simple Lie group and K its maximal compact subgroup. The proof uses the homogeneity of $\mathcal{M}$ in an essential way and thus cannot be adapted to other domains. The first two formulas are valid for $f \in \mathcal{D}(\mathbb{D})$ (the space of infinitely differentiable functions with compact support), but can be extended to $L^2(\mathbb{D}, d\mu_\mathbb{D})$ using the third formula; the correspondence $f \longleftrightarrow \tilde{f}$ then sets up a unitary isomorphism of the latter space onto the subspace in $L^2(\mathbb{R} \times \partial\mathbb{D}, |c(\lambda)|^{-2} \, d\lambda \, db)$ of functions satisfying a certain symmetry condition. (The image of $\mathcal{D}(\mathbb{D})$ under this correspondence can also be described explicitly, see [21].) Note that each function $e_{\lambda,b}$ is an eigenfunction of $\widetilde{\Delta}_\mathbb{D}$ (with eigenvalue $-\lambda^2 - \frac{1}{4}$) which vanishes everywhere on $\partial\mathbb{D}$ except at the point b, where it peaks into a singularity of a very specific nature: in fact, $|e_{\lambda,b}|^2$ is precisely the Poisson kernel at the point b. Using uniformization again as a substitute for homogeneity, we establish a complete analogue of the above eigenfunction decomposition for the domain Ω: namely, for any smoothly bounded $\Omega \subset \mathbb{C}$ which satisfies a certain condition, we show that

$$F(z) = \int_{\mathbb{R}} \int_{\partial\Omega} \tilde{F}(\lambda, \zeta)\, E_{-\lambda,\zeta}(z)\, d\zeta\, |c(\lambda)|^{-2}\, d\lambda,$$
$$\tilde{F}(\lambda, \zeta) = \int_{\Omega} F(z) E_{\lambda,\zeta}(z)\, d\mu(z), \qquad (\lambda \in \mathbb{R}, \zeta \in \partial\Omega),$$

and

$$\int_{\Omega} |F(z)|^2\, d\mu(z) = \int_{\mathbb{R}} \int_{\partial\Omega} |\tilde{F}(\lambda, \zeta)|^2\, d\zeta\, |c(\lambda)|^{-2}\, d\lambda,$$

where $E_{\lambda,\zeta}$ is an eigenfunction of $\widetilde{\Delta}_\Omega$ with the same eigenvalue $-\lambda^2 - \frac{1}{4}$ as for the disc which vanishes everywhere on $\partial\Omega$ except at ζ, where $|E_{\lambda,\zeta}|^2$ has a singularity of the same kind as the Poisson kernel of the domain. The condition on the domain Ω is the following: if $\phi : \mathbb{D} \to \Omega$ is the uniformization map, then it should hold that

$$\sum_{\phi(x)=\phi(0)} (1 - |x|^2)^{1/2} |\log(1 - |x|^2)| < \infty. \tag{0.1}$$

This result is not new, but has been established by Elstrodt [12, Part II] and Patterson [27, Part I], even in the more general case of the operator $\widetilde{\Delta}_\Omega$ replaced by the similar operators acting not on functions but on sections of certain line bundles; our derivation seems more straightforward. For domains Ω not satisfying the condition (0.1), the eigenfunction expansions have been obtained, using more sophisticated methods, by Fay [17] and Patterson [27, Part III], [30]. The fact that $|E_{\lambda,\zeta}|^2$ has the same kind of singularity as the Poisson kernel is a consequence of a certain uniqueness property of the eigenfunctions of $\widetilde{\Delta}_\Omega$ (Theorem 4.2); both these facts seem to have gone unnoticed in the literature.

III. *Annulus.* (Sections 5–6) In the third part of the paper we deal in more detail with the simplest case when Ω is an annulus. In this special situation a different approach can be used which makes it possible to give the results a somewhat more explicit form. Namely, any annulus $\mathbb{A} = \{z : 1 < |z| < R\}$ is invariant under the circle group $z \mapsto e^{i\theta} z$ ($\theta \in \mathbb{R}$), which together with the reflection $z \mapsto R/z$ makes up all biholomorphic automorphisms of $\mathbb{A}$. Both the Poincaré metric and the Laplace–Beltrami operator are invariant under this group action. Performing the Fourier decomposition with respect to θ (i.e. separating the variables),

$$f(e^{t+i\theta}) = \sum_{n\in\mathbb{Z}} f_n(t) e^{ni\theta} \qquad (\theta \in \mathbb{R},\ 0 < t < \log R),$$

the operator $\widetilde{\Delta}_{\mathbb{A}}$ thus splits into a family, indexed by n, of ordinary differential operators

$$\widetilde{\Delta}_{\mathbb{A}} f(e^{t+i\theta}) = \sum_{n\in\mathbb{Z}} L_n f_n(t)\, e^{ni\theta},$$

where $L_n F = \sin^2 t \cdot (F'' - n^2 F)$. Thus, for instance, the problem of obtaining an eigenfunction expansion for $\widetilde{\Delta}_{\mathbb{A}}$ reduces to that of obtaining it for the operators L_n, which can be handled by the standard theory for ordinary differential operators (the Kodaira–Titschmarsh theorem — see [25, Chapter VI, §21]; [31, Chapter III]; [8, Chapter 9]; [11, Chapter XIII, §5]). In this way we obtain explicitly, for each n, the n-th Fourier part of the sought eigenfunction expansion of $\widetilde{\Delta}_{\mathbb{A}}$. The resulting eigenfunctions come in the form of hypergeometric functions of $-\cot^2 t$, while the corresponding Plancherel measures are expressed in terms of gamma functions. Unfortunately, these expressions do not seem to be summable over all $n \in \mathbb{Z}$ to anything nice so as to yield the eigenfunction expansion in a closed form. The same approach also works for the other problem of our interest, viz., finding the Green function for $\widetilde{\Delta}^2_{\mathbb{A}}$. We first establish an Almansi-type theorem characterizing the functions on $\mathbb{A}$ annihilated by $\widetilde{\Delta}^2_{\mathbb{A}}$ as those for which, in the above decomposition,

$$f_n(t) = A_n e^{nt} + B_n e^{-nt} + C_n \Phi_n(t) + D_n \Psi_n(t),$$

where A_n, B_n, C_n, D_n are constants and Φ_n, Ψ_n are again certain hypergeometric functions of e^{it} (and some modification must be done for $n = 0$). This reduces the problem to solving, for each n, a system of linear equations for the coefficients $A_n, \ldots, D_n$. (For the Green function of the ordinary biharmonic operator Δ^2 on $\mathbb{A}$, this method was used in [14].) The resulting n-th Fourier components of the sought Green function again turn out to be too complicated to permit summation in a closed form, but they reveal at least the nature of the singularities and the analytic continuation of the transcendental functions that turn up; in particular, it transpires that the Green function for $\widetilde{\Delta}^2_{\mathbb{A}}$, unlike the one for $\widetilde{\Delta}_{\mathbb{A}}$, cannot be expressed in terms of the Jacobi theta functions.

IV. *Zeta function.* (Section 7) The fourth and last part of the paper arose from our efforts to understand the condition (0.1) for a smoothly bounded domain Ω, but we hope that it is of interest on its own. We consider, quite generally, the function

$$\zeta(s) = \sum_{\phi(x)=\phi(0)} (1 - |x|^2)^s. \tag{0.2}$$

It is clear that (0.1) holds if $\zeta(s)$ converges for some $s < \frac{1}{2}$ (in fact, the left-hand side of (0.1) is just the derivative $-\zeta'(\frac{1}{2})$). Observe that the set $\{x : \phi(x) = \phi(0)\}$ is precisely the orbit of $0 \in \mathbb{D}$ under the covering group G of ϕ. Thus (0.2) may be transcribed as

$$\zeta(s) = \sum_{\omega \in G} (1 - |\omega(0)|^2)^s.$$

In this guise the definition makes sense for any Fuchsian group G acting on $\mathbb{D}$. It is then known that for a general group G, $\zeta(s)$ converges for all $s > 1$; for Ω a bounded plane domain (0.2) converges even for all $s \geq 1$ owing to the Blaschke condition, and for Ω an annulus it converges $\forall s > 0$. In general it is clear that $\zeta(s)$ converges if $\operatorname{Re} s > s_0$ and diverges if $\operatorname{Re} s < s_0$ for a certain critical value s_0, called the *exponent of convergence* of G (or of Ω). The exponent of convergence of Fuchsian groups has been studied by Akaza [1], Beardon [4,5], Nicholls [26], and

Patterson [28,29]. It is known that $s_0 > 0$ if G is not elementary (i.e. for $\mathbb{D}/G \cong \Omega$ not the disc, the punctured disc, or the annulus, for which one has $s_0 = -\infty, \frac{1}{2}$ and 0, respectively), $s_0 > \frac{1}{2}$ if G is not elementary and contains a parabolic element, $s_0 < 1$ if G is finitely generated and of the second kind, and for G of the second kind and containing no parabolic elements, s_0 equals the Hausdorff dimension of the limit set of G (the subset on $\partial\mathbb{D}$ of accumulation points of $\{\omega 0 : \omega \in G\}$). Further, for any positive ϵ there exist free groups generated by two hyperbolic elements such that $0 < s_0(G) < \epsilon$, free groups generated by two hyperbolic elements such that $s_0(G) > 0.67$, and Hecke groups (= groups generated by the reflection $z \mapsto -z$ and a parabolic element) for which $1 - \epsilon < s_0(G) < 1$. Observe that a Hecke group cannot be the covering group of the uniformization map of a plane domain, as the latter do not contain any elements of finite order ([18, Chapter VI, §1]). For the free groups G generated by two hyperbolic elements (Schottky groups), the quotient $\mathbb{D}/G$ is a bordered Riemann surface of connectivity 3. Our main result in the last part of this paper is that for smoothly bounded plane domains Ω of any given connectivity $k \geq 3$, the exponent of convergence may be arbitrarily close to 0 as well as arbitrarily close or bigger than $\frac{\log\sqrt{3}}{\log(1+\sqrt{2})} = 0.623\ldots$. It seems that the domains for which s_0 is small are those which are sufficiently "thick", i.e. whose bounded components of $\mathbb{C} \setminus \Omega$ ("holes") are small compared to Ω. It is known that, for smoothly bounded plane domains, s_0 cannot get arbitrarily close to 1 [10].

The paper is organized as follows. The formula for the Green function for $\widetilde{\Delta}^2_\Omega$ on a smoothly bounded plane domain Ω is derived in Section 1, after briefly recalling the pertinent preliminaries for the disc from [16]. The case of the unit ball $\mathbb{B}$ of $\mathbb{C}^d$ is treated in the short Section 2. The analogue of the Helgason–Fourier transform is established in Section 3, and the relation between $|E_{\lambda,\zeta}|^2$ and the Poisson kernel is discussed in Section 4. The special case of Ω an annulus, is the subject of the next two sections: the eigenfunction expansion for $\widetilde{\Delta}_\mathbb{A}$ is derived in Section 5, and the formula for the Fourier components of the Green function for $\widetilde{\Delta}^2_\mathbb{A}$ in Section 6. Finally, the question of the convergence of the series (0.2) is discussed in the last Section 7.

Throughout the paper, we sometimes drop the subscripts Ω and $\mathbb{D}$ in $\widetilde{\Delta}_\Omega$, $d\mu_\mathbb{D}$ etc. if there is no danger of confusion, and sometimes (notably in Sections 3–5) we switch from the usual Laplacian to the Laplacian $\Delta = \partial\overline{\partial}$, which differs by a factor of 4. The letter C is employed universally to denote a constant, whose value may vary from one occurrence to another, and $\phi(x)$, $\omega(x)$ etc. are frequently abbreviated to ϕx, ωx etc.

1. The disc and planar domains

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$, $\widetilde{\Delta}_\mathbb{D} g(x) = (1 - |x|^2)^2 \Delta g(x)$ the invariant Laplacian on $\mathbb{D}$. Then the Green function of the operator $\widetilde{\Delta}^2_\mathbb{D}$ on $\mathbb{D}$ with the "Dirichlet" boundary conditions (i.e. having the least possible growth at the boundary) and with the pole at a point $y \in \mathbb{D}$ is given by [16]

$$G^\mathbb{D}_y(x) = \Lambda\left(\left|\frac{x-y}{1-\overline{y}x}\right|^2\right),$$

where

$$\Lambda(t) = -\frac{1}{16\pi}\left(2\mathrm{Li}_2(t) + \log t \log(1-t) - \frac{\pi^2}{3}\right).$$

In particular, for each fixed y and $|x-y| > \delta$,

$$\begin{aligned} |G_y^{\mathbb{D}}(x)| &\le C(y)(1-|x|^2)\ \log_1(1-|x|^2), \\ \|\nabla G_y^{\mathbb{D}}(x)\| &\le C(y,\delta)\ \log_1(1-|x|^2), \\ |\widetilde{\Delta}_{\mathbb{D}} G_y^{\mathbb{D}}(x)| &\le C(y,\delta)(1-|x|^2), \end{aligned} \tag{1.1}$$

and

$$G_y^{\mathbb{D}}(x) > 0 \qquad \forall x \in \mathbb{D}.$$

Here we have introduced the notation

$$\log_1 s := \max\{1, 1 - \log s\}.$$

The function $G^{\mathbb{D}}$ has the following property: for any C^∞ function g with support in a compact subset of $\mathbb{D}$, $f(x) = \int_{\mathbb{D}} g(y) G_x^{\mathbb{D}}(y)\, d\mu_{\mathbb{D}}(y)$ satisfies

$$\widetilde{\Delta}_{\mathbb{D}}^2 f = g$$

and the boundary estimates (1.1). In particular, for any such g,

$$\int_{\mathbb{D}} \widetilde{\Delta}_{\mathbb{D}}^2 g(x) G_y^{\mathbb{D}}(x)\, d\mu_{\mathbb{D}}(x) = g(y), \qquad \forall y \in \mathbb{D}. \tag{1.2}$$

(Here $d\mu_{\mathbb{D}}(x) = (1-|x|^2)^{-2}\, dx$ stands for the invariant measure and dx for the Lebesgue area measure on $\mathbb{D}$). We are not going to go into the subtleties of extending this to more general functions g on $\mathbb{D}$. Using $d\mu_{\mathbb{D}}$ for identifying locally integrable functions on $\mathbb{D}$ with the corresponding distributions, one can interpret (1.2) as saying that $\widetilde{\Delta}_{\mathbb{D}}^2 G_y^{\mathbb{D}} = \delta_y$, the Dirac function at y (cf. [21, § II.5.1]).

For later reference we put down the following submultiplicativity property of the function $\log_1$, the proof of which is left to the reader.

Lemma 1.1. *For any $a, b > 0$,*

$$\log_1 ab \le \log_1 a \cdot \log_1 b$$

(*and, hence, also*

$$\log_1 ab \ge \frac{\log_1 a}{\log_1(1/b)}.)$$

Now let Ω be a smoothly bounded plane domain of hyperbolic type and $\phi : \mathbb{D} \to \Omega$ the uniformization map. Recall that the *Poincaré metric* on Ω is given by $ds^2 = w(z)^{-2}|dz|^2$, where

$$w(\phi(x)) = (1-|x|^2)|\phi'(x)|. \tag{1.3}$$

(It is easy to see that this definition is consistent). It is known that w vanishes on $\partial\Omega$ (in fact, $w(z) \leq \operatorname{dist}(z, \partial\Omega)$; cf. [23, p. 45]). The Laplace–Beltrami operator on Ω is given by

$$\widetilde{\Delta}_\Omega f(z) = w(z)^2 \Delta f(z),$$

where Δ is the ordinary Laplacian. One has

$$(\widetilde{\Delta}_\Omega f) \circ \phi = \widetilde{\Delta}_{\mathbb{D}}(f \circ \phi). \tag{1.4}$$

The *covering group* of ϕ is $G = \{\omega \in \operatorname{Aut}(\mathbb{D}) : \ \phi \circ \omega = \phi\}$. We will often write ωx instead of $\omega(x)$. The open set $\mathcal{O} = \{x \in \mathbb{D} : \ d(x, 0) < d(\omega x, 0) \ \forall \omega \in \ G, \omega \neq \ \mathrm{id}\}$, where d is the hyperbolic distance, is a *fundamental domain* for G. The operator $\widetilde{\Delta}_\Omega$ is formally selfadjoint with respect to the *Poincaré measure*

$$d\mu_\Omega(z) = w(z)^{-2}\, dz.$$

Theorem 1.2. *For each $t \in \Omega$, there exists a unique function G_t^Ω on Ω such that*

$$\widetilde{\Delta}_\Omega^2 G_t^\Omega = \delta_t$$

in the sense of distributions (that is, (1.2) *holds with G_t^Ω and $\widetilde{\Delta}_\Omega$ in the place of $G_y^{\mathbb{D}}$ and $\widetilde{\Delta}_{\mathbb{D}}$, respectively, for any compactly supported C^∞ function g on Ω), and near the boundary*

$$\left|G_t^\Omega\right| \leq C(t) w \log_1 w, \quad \left\|\nabla G_t^\Omega\right\| \leq C(t) \log_1 w, \quad \left|\widetilde{\Delta}_\Omega G_t^\Omega\right| \leq C(t) w. \tag{1.5}$$

Furthermore, if $y \in \mathbb{D}$ is such that $t = \phi(y)$, then

$$G_t^\Omega(z) = \sum_{x \in \phi^{-1}(z)} G_y^{\mathbb{D}}(x). \tag{1.6}$$

In particular, $G_t^\Omega > 0$ on Ω.

The assertion concerning uniqueness is immediate from the following lemma:

Lemma 1.3. *Let h be a function on Ω which is annihilated by $\widetilde{\Delta}_\Omega^2$ and such that*

$$|h| \leq C w \log_1 w, \qquad |\widetilde{\Delta}_\Omega h| \leq C w.$$

Then $h = 0$.

Proof of the Lemma. The function $w^2 \Delta h$ is harmonic and vanishes at the boundary, in view of the second condition; hence it vanishes identically. Thus h is itself harmonic, and vanishes at the boundary owing to the first condition; hence $h = 0$. □

Proof of the Theorem. Let $f : \Omega \to \mathbb{C}$ and $g : \mathbb{D} \to \mathbb{C}$ be two functions. Then $f \circ \phi$ is a function on $\mathbb{D}$ and, whenever the left-hand side exists,

$$\begin{aligned}\int_{\mathbb{D}} f(\phi(x))g(x)\,d\mu_{\mathbb{D}}(x) &= \sum_{\omega\in G}\int_{\mathcal{O}} f(\underbrace{\phi(\omega x)}_{=\phi(x)})g(\omega x)\,d\mu_{\mathbb{D}}(\omega x)\\ &= \int_{\mathcal{O}} f(\phi(x))\sum_{\omega\in G} g(\omega x)\,d\mu_{\mathbb{D}}(x)\\ &= \int_{\Omega} f(z)\,Tg(z)\,d\mu_{\Omega}(z),\end{aligned}\tag{1.7}$$

where Tg is the function on Ω given by

$$Tg(\phi(x)) = \sum_{\omega\in G} g(\omega x),$$

that is,

$$Tg(z) = \sum_{x\in\phi^{-1}(z)} g(x).\tag{1.8}$$

Thus T is the formal adjoint of the pullback operator $f \mapsto f\circ\phi$.

Set $G_t^{\Omega} = TG_y^{\mathbb{D}}$ (so that (1.6) holds). We claim that this function has the required properties.

Let f be a compactly supported C^∞ function on Ω. There exists $R \in (0,1)$ such that $\mathcal{O}\cap\operatorname{supp}(f\circ\phi)\subset R\mathbb{D}$. The image of $R\mathbb{D}$ under $\omega\in G$ is a disc (in the Euclidean metric) centered at $\frac{1-R^2}{1-R^2|\omega 0|^2}\omega 0$ and of radius $\frac{R(1-|\omega 0|^2)}{1-R^2|\omega 0|^2}$; therefore, by a simple calculation,

$$\frac{1-|\omega x|^2}{1-|\omega 0|^2} \le \frac{1+R}{1-R} \qquad \forall x\in R\mathbb{D},\ \omega\in G.$$

Thus by (1.1)

$$\begin{aligned}\int_{\mathbb{D}} \left|\widetilde{\Delta}_{\mathbb{D}}^2 f(\phi(x))G_y^{\mathbb{D}}(x)\right| d\mu_{\mathbb{D}}(x) &\le \sum_{\omega\in G}\int_{\mathcal{O}}\left|\widetilde{\Delta}_{\mathbb{D}}^2 f(\underbrace{\phi(\omega x)}_{=\phi(x)})\,G_y^{\mathbb{D}}(\omega x)\right| d\mu_{\mathbb{D}}(x)\\ &\le \sum_{\omega\in G}\sup_{\mathcal{O}}\left|\widetilde{\Delta}_{\mathbb{D}}^2(f\circ\phi)\right|\int_{R\mathbb{D}}\left|G_y^{\mathbb{D}}(\omega x)\right| d\mu_{\mathbb{D}}(x)\\ &\le \sum_{\omega\in G}\sup_{\Omega}\left|\widetilde{\Delta}_{\Omega}^2 f\right|\,\sup_{R\mathbb{D}}\left|G_y^{\mathbb{D}}(\omega x)\right|\int_{R\mathbb{D}} d\mu_{\mathbb{D}}(x)\\ &\le C\sum_{\omega\in G}\sup_{R\mathbb{D}}\left[(1-|\omega x|^2)\log_1(1-|\omega x|^2)\right]\\ &\le C\sum_{\omega\in G}(1-|\omega 0|^2)\log_1(1-|\omega 0|^2).\end{aligned}$$

We will show below that the last sum is finite. Hence an application of (1.7) to the integral

$$\int_{\mathbb{D}} \widetilde{\Delta}^2_{\mathbb{D}} f(\phi(x))\, G^{\mathbb{D}}_y(x)\, d\mu_{\mathbb{D}}(x) \tag{1.9}$$

is legitimate and shows that it equals

$$\int_{\Omega} \widetilde{\Delta}^2_{\Omega} f(z)\, TG^{\mathbb{D}}_y(z)\, d\mu_{\Omega}(z) = \int_{\Omega} \widetilde{\Delta}^2_{\Omega} f(z) G^{\Omega}_t(z)\, d\mu_{\Omega}(z).$$

On the other hand, the integral (1.9) equals $f(\phi(y)) = f(t)$, by (1.2). [Strictly speaking, the equality (1.2) is valid only for compactly supported C^∞ functions g on $\mathbb{D}$; however, the existence of the integral (1.9) also justifies rewriting the $\int_{\mathbb{D}} \dots dx$ there as $\sum_{\omega\in G} \int_{\mathbb{D}} \chi_{\omega\Theta}(x) \dots dx$ (χ stands for the characteristic function), and one can then apply (1.2) to the compactly supported C^∞ functions on $\mathbb{D}$ given by $g = \chi_{\omega\Theta} \cdot (f\circ\phi)$, with $\widetilde{\Delta}^2_{\mathbb{D}} g = \chi_{\omega\Theta} \widetilde{\Delta}^2_{\mathbb{D}} (f\circ\phi)$. This gives $\sum_{\omega\in G} \chi_{\omega\Theta}(y) f(\phi(y)) = f(\phi(y))$, as claimed.]

It follows that $\widetilde{\Delta}^2_{\Omega} G^{\Omega}_t = \delta_t$.

It remains to show (1.5). Note that by (1.3),

$$(1-|x|^2)|\partial(f\circ\phi)(x)| = |(w\,\partial f)(\phi(x))|. \tag{1.10}$$

(Here ∂, $\overline{\partial}$ are the Wirtinger operators.) Now in view of (1.1), we have

$$\left|G^{\Omega}_t(z)\right| \le C(t) \sum_{x\in\phi^{-1}(z)} (1-|x|^2)\, \log_1(1-|x|^2).$$

Similarly, by (1.10) and (1.4), for $|z-t| > \delta$,

$$\begin{aligned}
\left|w(z)\partial G^{\Omega}_t(z)\right| &\le \sum_{x\in\phi^{-1}(z)} (1-|x|^2)\left|\partial G^{\mathbb{D}}_y(x)\right| \\
&\le C(t,\delta) \sum_{x\in\phi^{-1}(z)} (1-|x|^2)\, \log_1(1-|x|^2), \\
\left|\widetilde{\Delta}_{\Omega} G^{\Omega}_t(z)\right| &\le \sum_{x\in\phi^{-1}(z)} \left|\widetilde{\Delta}_{\mathbb{D}} G^{\mathbb{D}}_y(x)\right| \\
&\le C(t,\delta) \sum_{x\in\phi^{-1}(z)} (1-|x|^2).
\end{aligned}$$

It therefore suffices to prove that

$$\sum_{\omega\in G} (1-|\omega x|^2)\, \log_1(1-|\omega x|^2) \le Cw(\phi(x))\, \log_1 w(\phi(x)), \qquad \forall x\in\mathbb{D}, \tag{1.11}$$

and

$$\sum_{\omega\in G} (1-|\omega x|^2) \le Cw(\phi(x)), \qquad \forall x\in\mathbb{D}.$$

We prove (1.11), the proof of the other inequality being completely analogous (in fact, even slightly simpler). As both sides are invariant under G, it is enough to show this for $x \in \overline{\mathcal{O}}$, the closure of the fundamental domain $\mathcal{O}$. As in our case (Ω smoothly bounded) ϕ' is bounded and bounded away from zero on $\overline{\mathcal{O}}$ (cf. [18, Chapter VI, §2]), we can then replace $w(\phi(x))$ on the right-hand side by $(1-|x|^2)$ (using (1.3) and the submultiplicativity of the function $\log_1$). Thus (1.11) reduces to

(1.12)
$$\sum_{\omega \in G}(1-|\omega x|^2)\log_1(1-|\omega x|^2) \leq C(1-|x|^2)\log_1(1-|x|^2), \qquad \forall x \in \overline{\mathcal{O}}.$$

The group G is at most countable; let us enumerate by $\omega_0 = \mathrm{id}, \omega_1, \omega_2, \dots$ the elements of G and set $a_n = \omega_n^{-1}(0)$. Recall that (loc. cit.) the cluster points of the sequence $\{a_n\}$ form a subset of the unit circle lying at a positive distance from $\overline{\mathcal{O}}$; thus there exists $\delta > 0$ such that

$$\delta < |1-\overline{a}_n x| < 2, \qquad \forall n,\ \forall x \in \overline{\mathcal{O}}. \tag{1.13}$$

Consequently, the quantity

$$1-|\omega_n x|^2 = \frac{(1-|x|^2)(1-|a_n|^2)}{|1-\overline{a}_n x|^2} \tag{1.14}$$

satisfies

(1.15)
$$\frac{1}{4}(1-|a_n|^2)(1-|x|^2) \leq 1-|\omega_n x|^2 \leq \frac{1}{\delta^2}(1-|a_n|^2)(1-|x|^2), \qquad \forall x \in \overline{\mathcal{O}}.$$

Substituting this into (1.12) and using once more the submultiplicativity of $\log_1$, the factors $(1-|x|^2)\log_1(1-|x|^2)$ can be cancelled, and we see that (1.12) will follow if we show that

$$\sum_{n>0}(1-|a_n|^2)\log_1(1-|a_n|^2) \leq C. \tag{1.16}$$

As was shown by Dalzell [9], this estimate is equivalent to the seemingly weaker inequality

$$\sum_{n>0}(1-|a_n|^2) < \infty. \tag{1.17}$$

However, the points a_n are precisely the zeros of the bounded analytic function $\phi - \phi(0)$ on $\mathbb{D}$, and thus satisfy the Blaschke condition $\sum(1-|a_n|) < \infty$. This completes the proof. $\square$

2. The unit ball

The square of the invariant Laplacian $\widetilde{\Delta}_{\mathbb{B}}$ on the unit ball $\mathbb{B}$ of $\mathbb{C}^d$ has been studied in [16], and in particular the corresponding Green function $G^{\mathbb{B}}$ has been found there: $G^{\mathbb{B}}_y(x) = \Lambda_d(|\phi_y(x)|^2)$, where ϕ_y is the holomorphic symmetry of $\mathbb{B}$ interchanging y and the origin and $\Lambda_d(t) = \frac{(d-1)!}{16\pi^d}(g_1(t) - C_d f_1(t) - A_d)$, where $C_d = g_0(1)$, A_d is chosen so that $\Lambda_d(1) = 0$, and f_1, g_0, g_1 are functions satisfying

$$f_1 = \frac{1}{d}\log\frac{1}{1-t}, \qquad g_0' = \frac{(1-t)^{d-1}}{t^d},$$

and

$$g_1' = \frac{1}{d}\frac{1}{1-t}\, g_0 - \frac{1}{d}\frac{(1-t)^{d-1}}{t^d}\log\frac{1}{1-t}.$$

Theorem 2.1. *$\Lambda_d(t) > 0$ for all t, $0 \le t < 1$. Hence, $G^{\mathbb{B}} > 0$ on $\mathbb{B}\times\mathbb{B}$.*

Proof. Since $\Lambda_d(1) = 0$ by the choice of A_d, it is enough to show that $\Lambda_d' < 0$, i.e. $g_1' - C_d f_1' < 0$ on $(0, 1)$. Substituting the formulas above and also writing $g_0(t)$ as $C_d - \int_t^1 g_0'(s)\, ds$, this becomes

$$\frac{1}{d}\frac{1}{1-t}\left(C_d - \int_t^1 \frac{(1-s)^{d-1}}{s^d}\, ds\right) - \frac{1}{d}\frac{(1-t)^{d-1}}{t^d}\log\frac{1}{1-t} - C_d\frac{1}{d}\frac{1}{1-t} < 0.$$

The C_d's cancel, yielding

$$\frac{(1-t)^d}{t^d}\log\frac{1}{1-t} + \int_t^1 \frac{(1-s)^{d-1}}{s^d}\, ds > 0.$$

However, by integrating by parts on the left-hand side this can be rewritten as

$$\int_t^1 \frac{(1-s)^{d-1}}{s^{d+1}}\log\frac{1}{1-s}\, ds > 0,$$

and the claim follows since the integrand is positive. □

The above formulas also show that the analogue of (1.5) and (1.1) remains in force for $G^{\mathbb{B}}$ (with $w(z) := 1 - \|z\|^2$). Namely, for $\|x - y\| > \delta$,

$$\begin{aligned} \left|G^{\mathbb{B}}_y(x)\right| &\le C(y)(1-\|x\|^2)^d \log_1(1-\|x\|^2), \\ \left\|\nabla G^{\mathbb{B}}_y(x)\right\| &\le C(y,\delta)(1-\|x\|^2)^{d-1}\log_1(1-\|x\|^2), \\ \left|\widetilde{\Delta}_{\mathbb{B}} G^{\mathbb{B}}_y(x)\right| &\le C(y,\delta)(1-\|x\|^2)^d. \end{aligned}$$

3. A Plancherel formula via uniformization

Recall that for a compactly supported C^∞ function f on $\mathbb{D}$, one has the Helgason–Fourier inversion formula

$$\begin{aligned} f(x) &= \int_{\mathbb{R}} \int_{\partial\mathbb{D}} \tilde{f}(\lambda, b) e_{-\lambda,b}(x) |c(\lambda)|^{-2}\, d\lambda\, db, \\ \tilde{f}(\lambda, b) &= \int_{\mathbb{D}} f(x) e_{\lambda,b}(x)\, d\mu_{\mathbb{D}}(x), \end{aligned} \tag{3.1}$$

and also the corresponding Plancherel formula

$$\int_{\mathbb{D}} |f(x)|^2\, d\mu_{\mathbb{D}}(x) = \int_{\mathbb{R}} \int_{\partial\mathbb{D}} |\tilde{f}(\lambda, b)|^2 |c(\lambda)|^{-2}\, d\lambda\, db. \tag{3.2}$$

Here $d\lambda$ is a suitably normalized Lebesgue measure on $\mathbb{R}$, db is the arc-length measure on $\partial\mathbb{D}$, $c(\lambda)$ is the Harish–Chandra c-function, and $e_{\lambda,b}$ are the "plane waves"

$$e_{\lambda,b}(x) = \left(\frac{1 - |x|^2}{|1 - \overline{b}x|^2} \right)^{\frac{1}{2}+i\lambda}, \qquad x \in \mathbb{D}, \lambda \in \mathbb{R}, b \in \partial\mathbb{D}. \tag{3.3}$$

They are eigenfunctions of $\widetilde{\Delta}_{\mathbb{D}}$ with eigenvalue $-\lambda^2 - \frac{1}{4}$. For $\omega \in \mathrm{Aut}(\mathbb{D})$, the action of $U_\omega : f \mapsto f \circ \omega$ on $e_{\lambda,b}$ is

$$U_\omega e_{\lambda,b} = e_{\lambda,b}(\omega 0)\, e_{\lambda,\omega^{-1}b}, \tag{3.4}$$

from which it follows that the action of U_ω on the level of $\tilde{f}$ is

$$\widetilde{U_\omega f}(\lambda, b) = e_{\lambda,b}(\omega^{-1}0)\, \tilde{f}(\lambda, \omega b). \tag{3.5}$$

For later use we note one more consequence of (3.4)

$$e_{\lambda,\omega b}(\omega 0)\, e_{\lambda,b}(\omega^{-1}0) = 1, \tag{3.6}$$

and also the formulas

$$e_{\lambda,b}(\omega^{-1}0) = |\omega'(b)|^{\frac{1}{2}+i\lambda}, \tag{3.7}$$

$$e_{\lambda,b}(\omega^{-1}0)\, e_{-\lambda,b}(\omega^{-1}0) = |\omega'(b)|, \tag{3.8}$$

which are easily proved by direct calculation from (3.3).

We now use the uniformization map ϕ to transfer the formulas (3.1), (3.2) to the smoothly bounded plane domain Ω of hyperbolic type. We retain the notations from Section 1.

Theorem 3.1. *Let Ω be a bounded and smoothly bounded domain in $\mathbb{C}$ for which*

$$\sum \left(1-|a_n|^2\right)^{1/2} \left|\log\left(1-|a_n|^2\right)\right| < \infty. \tag{3.9}$$

Then the recipe

$$E_{\lambda,\phi b}(\phi x) := |\phi'(b)|^{-\frac{1}{2}-i\lambda} \sum_{\omega\in G} e_{\lambda,b}(\omega x) \tag{3.10}$$

yields a well-defined function $E_{\lambda,\zeta}(z)$ of $z \in \Omega, \zeta \in \partial\Omega, \lambda \in \mathbb{R}$, and for any compactly supported C^∞ function F on Ω, the Fourier inversion formulas

$$\tilde{F}(\lambda,\zeta) = \int_\Omega F(z) E_{\lambda,\zeta}(z)\, d\mu_\Omega(z), \qquad \lambda \in \mathbb{R}, \zeta \in \partial\Omega, \tag{3.11}$$

$$F(z) = \int_{\mathbb{R}} \int_\Omega \tilde{F}(\lambda,\zeta)\, E_{-\lambda,\zeta}(z)\, d\zeta\, |c(\lambda)|^{-2}\, d\lambda, \tag{3.12}$$

and the Plancherel formula

$$\int_\Omega |F(z)|^2\, d\mu_\Omega(z) = \int_{\mathbb{R}} \int_{\partial\Omega} |\tilde{F}(\lambda,\zeta)|^2\, d\zeta\, |c(\lambda)|^{-2}\, d\lambda, \tag{3.13}$$

hold. Here $d\zeta$ is the arc-length measure on $\partial\Omega$.

Proof. Let F be a compactly supported function on Ω. Proceeding, for the time being, formally, let us apply (3.1) to $f = F \circ \phi$. We get, similarly as in (1.7),

$$\begin{aligned} \tilde{f}(\lambda,b) &= \sum_{\omega\in G} \int_{\mathcal{O}} f(\omega x) e_{\lambda,b}(\omega x)\, d\mu_{\mathbb{D}}(x) \\ &= \int_\Omega F(z) \mathcal{E}_{\lambda,b}(z)\, d\mu_\Omega(z), \end{aligned} \tag{3.14}$$

where

$$\mathcal{E}_{\lambda,b}(\phi x) := \sum_{\omega\in G} e_{\lambda,b}(\omega x) \tag{3.15}$$

(that is, $\mathcal{E}_{\lambda,b}(z) = \sum_{x\in\phi^{-1}(z)} e_{\lambda,b}(\omega x)$) obeys the transformation law (by (3.4))

$$\mathcal{E}_{\lambda,b}(z) = e_{\lambda,b}(\omega^{-1}0)\, \mathcal{E}_{\lambda,\omega b}(z) \qquad \forall \omega \in G. \tag{3.16}$$

Comparing this with (3.7), we thus see that (3.10) indeed gives a well-defined function $E_{\lambda,\zeta}(z)$ of $z \in \Omega, \zeta \in \partial\Omega, \lambda \in \mathbb{R}$.

Now let $E = \overline{\mathcal{O}} \cap \partial\mathbb{D}$ be the inverse image under ϕ of $\partial\Omega$. It is known that $\partial\mathbb{D} \setminus (\bigcup_{\omega\in G} \omega E)$ is precisely the set of limit points of $\{\omega 0\}_{\omega\in G}$ and has measure zero ([18, Chapter VI, §2]). We can thus write

$$\begin{aligned} \int_{\partial\mathbb{D}} \tilde{f}(\lambda,b) e_{-\lambda,b}(x)\, db &= \sum_{\omega\in G} \int_E \tilde{f}(\lambda,\omega b) e_{-\lambda,\omega b}(x) |\omega'(b)|\, db \\ \text{(by (3.5))} \quad &= \sum_{\omega\in G} \int_E \frac{\tilde{f}(\lambda,b)}{e_{\lambda,b}(\omega^{-1}0)}\, e_{-\lambda,\omega b}(x)\, |\omega'(b)|\, db \\ &\equiv \int_E \tilde{f}(\lambda,b) \mathcal{E}^*_{-\lambda,b}(x)\, db, \end{aligned} \tag{3.17}$$

where

$$\mathcal{E}^*_{-\lambda,b}(x) = \sum_{\omega\in G} \frac{e_{-\lambda,\omega b}(x)\,|\omega'(b)|}{e_{\lambda,b}(\omega^{-1}0)}. \tag{3.18}$$

However, in view of (3.8) and (3.4), respectively, we have

$$\begin{aligned}\mathcal{E}^*_{-\lambda,b}(x) &= \sum_{\omega\in G} e_{-\lambda,\omega b}(x) e_{-\lambda,b}(\omega^{-1}0)\\ &= \sum_{\omega\in G} e_{-\lambda,b}(\omega x) = \mathcal{E}_{-\lambda,b}(\phi x),\end{aligned}$$

and it follows that

$$\int_{\partial\mathbb{D}} \tilde{f}(\lambda, b)e_{-\lambda,b}(x)\,db = \int_E \tilde{f}(\lambda, b)\mathcal{E}_{-\lambda,b}(\phi x)\,db.$$

Letting $\tilde{F}(\lambda,\zeta)$ be defined by (3.11), we thus arrive at

$$\begin{aligned} f(x) &= \int_{\mathbb{R}}\int_{\partial\mathbb{D}} \tilde{f}(\lambda, b)e_{-\lambda,b}(x)\,db\,|c(\lambda)|^{-2}\,d\lambda\\ &= \int_{\mathbb{R}}\int_E \tilde{f}(\lambda, b)\mathcal{E}_{-\lambda,b}(\phi x)\,db\,|c(\lambda)|^{-2}\,d\lambda\\ &= \int_{\mathbb{R}}\int_E |\phi'(b)|^{\frac{1}{2}+i\lambda}\tilde{F}(\lambda,\phi b)|\phi'(b)|^{\frac{1}{2}-i\lambda} E_{-\lambda,\phi b}(\phi x)\,db\,|c(\lambda)|^{-2}\,d\lambda,\end{aligned} \tag{3.19}$$

which is (3.12).

It remains to prove the Plancherel formula. To this end, define

$$g(x) = \begin{cases} f(x) = F(\phi(x)) & x \in \mathcal{O}\\ 0 & x \notin \mathcal{O}.\end{cases} \tag{3.20}$$

As F has compact support, so does g. Clearly, by the invariance of the Poincaré metric,

$$\int_\Omega |F(z)|^2\,d\mu_\Omega(z) = \int_{\mathcal{O}} |f(x)|^2\,d\mu_{\mathbb{D}}(x) = \int_{\mathbb{D}} \overline{f(x)}g(x)\,d\mu_{\mathbb{D}}(x).$$

However (postponing the technical matters for the moment — f does not have compact support), by the Plancherel formula (3.2) for the ordinary Helgason–Fourier transform on $\mathbb{D}$,

$$\int_{\mathbb{D}} \overline{f(x)}g(x)\,d\mu_{\mathbb{D}}(x) = \int_{\mathbb{R}}\int_{\partial\mathbb{D}} \overline{\tilde{f}(\lambda, b)}\tilde{g}(\lambda, b)\,db\,|c(\lambda)|^{-2}\,d\lambda. \tag{3.21}$$

Now we have, as in (3.17),

$$
\begin{aligned}
\int_{\partial\mathbb{D}} \overline{\tilde{f}(\lambda, b)}\tilde{g}(\lambda, b)\, db &= \sum_{\omega\in G} \int_E \tilde{g}(\lambda, \omega b)\,\overline{\tilde{f}(\lambda, \omega b)}|\omega'(b)|\, db \\
&= \sum_{\omega\in G} \int_E \tilde{g}(\lambda, \omega b)\frac{\overline{\tilde{f}(\lambda, b)}}{e_{-\lambda,b}(\omega^{-1}0)}\,|\omega'(b)|\, db, \qquad \text{by (3.5)} \\
&= \sum_{\omega\in G} \int_E \overline{\tilde{f}(\lambda, b)}\tilde{g}(\lambda, \omega b)e_{\lambda,b}(\omega^{-1}0)\, db, \qquad \text{by (3.8).}
\end{aligned}
\tag{3.22}
$$

However, $\sum_{\omega\in G} U_\omega g = f$, so by (3.4)

$$
\sum_{\omega\in G} e_{\lambda,b}(\omega^{-1}0)\tilde{g}(\lambda, \omega b) = \tilde{f}(\lambda, b).
$$

Consequently,

$$
\begin{aligned}
\int_\Omega |F(z)|^2\, d\mu_\Omega(z) &= \int_{\mathbb{R}} \int_E |\tilde{f}(\lambda, b)|^2\, db\, |c(\lambda)|^{-2}\, d\lambda \\
&= \int_{\mathbb{R}} \int_E \left| |\phi'(b)|^{\frac{1}{2}+i\lambda} \tilde{F}(\lambda, \phi b)\right|^2 db\, |c(\lambda)|^{-2}\, d\lambda, \quad \text{by (3.14) and (3.11)} \\
&= \int_{\mathbb{R}} \int_E |\tilde{F}(\lambda, \phi b)|^2\, d(\phi b)\, |c(\lambda)|^{-2}\, d\lambda,
\end{aligned}
$$

which is the desired assertion (3.13).

Let us now turn to technical matters. The first is the question of convergence of the series (3.15). The series, of course, need not converge for b, a limit point of the set $\{\omega_n 0\}$, that is, $b \in \Lambda := \partial\mathbb{D} \setminus (\bigcup_{\omega\in G} \omega E)$; however, this is immaterial as the latter set is of measure zero on $\partial\mathbb{D}$. For other b and any $x \in \mathbb{D}$ we have $\inf_n |b - \omega_n x| = \delta > 0$, whence $|e_{\lambda,b}(\omega_n x)| \le \delta^{-1/2}(1 - |\omega_n x|^2)^{1/2}$, while by (1.14), $(1 - |\omega_n x|^2) \asymp (1 - |a_n|^2)$; thus (3.15) converges (for all $\lambda \in \mathbb{R}$, $x \in \mathbb{D}$ and b not in the limit set) if and only if

$$
\sum (1 - |a_n|^2)^{1/2} < \infty.
\tag{3.23}
$$

The series (3.15) then converges absolutely and uniformly for $\lambda \in \mathbb{R}$ and (x, b) in compact subsets of $\mathbb{D} \times (\partial\mathbb{D} \setminus \Lambda)$. (Observe that $|e_{\lambda,b}(x)| = e_{0,b}(x)$ for any real λ.) For any compactly supported F on Ω and $f = F \circ \phi$, the integral

$$
\int_{\mathbb{D}} f(x)e_{\lambda,b}(x)\, d\mu_{\mathbb{D}}(x)
$$

thus converges absolutely and the interchange of integration and summation in (3.14) is legitimate. In particular, $\tilde{f}(\lambda, b)$ is defined and, in the notation (3.20),

$$
\tilde{f}(\lambda, b) = \sum_{\omega\in G} \widetilde{U_\omega} g(\lambda, b),
$$

the last series converging absolutely, and uniformly as $\lambda \in \mathbb{R}$, for each $b \in \partial\mathbb{D} \setminus \Lambda$. Now assume that in addition to (3.23),

$$\int_{\mathbb{R}\times\partial\mathbb{D}} \sum_{\omega\in G} |\widetilde{U_\omega g}(\lambda, b)|\, |e_{-\lambda,b}(x)|\, |c(\lambda)|^{-2}\, d\lambda\, db < \infty, \qquad \forall x \in \mathbb{D}. \tag{3.24}$$

Then also the integral

$$\int_{\mathbb{R}} \int_{\partial\mathbb{D}} \tilde{f}(\lambda, b) e_{-\lambda,b}(x) |c(\lambda)|^{-2}\, d\lambda\, db$$

converges absolutely for each $x \in \mathbb{D}$, and equals

$$\sum_{\omega\in G} \int_{\mathbb{R}\times\partial\mathbb{D}} \widetilde{U_\omega g}(\lambda, b)\, e_{-\lambda,b}(x)\, |c(\lambda)|^{-2}\, d\lambda\, db = \sum_{\omega\in G} U_\omega g(x) = f(x),$$

and it follows that the manipulations in (3.19) are legitimate (that is, the Fourier inversion formulas (3.1) hold for f even though $f \notin L^2(\mathbb{D}, d\mu)$), and so are the ones in (3.17). The proof of (3.12) is thus made rigorous.

Finally, if also the integral

$$\int_{\mathbb{R}\times\partial\mathbb{D}} \sum_{\omega\in G} |\widetilde{U_\omega g}(\lambda, b)|\, |\tilde{g}(\lambda, b)|\, |c(\lambda)|^{-2}\, d\lambda\, db \tag{3.25}$$

is finite, then

$$\int_{\mathbb{R}\times\partial\mathbb{D}} \overline{\tilde{f}}\, \tilde{g} = \sum_{\omega\in G} \int_{\mathbb{R}\times\partial\mathbb{D}} \overline{\widetilde{U_\omega g}}\, g = \sum_{\omega\in G} \int_{\mathbb{D}} \overline{U_\omega g}\, g = \int_{\mathbb{D}} |g|^2 = \int_{\mathbb{D}} \overline{f}\, g,$$

so (3.21) holds and, further, its right-hand side converges absolutely, so the manipulations (3.22) used in course of the derivation of the Plancherel formula are likewise legitimate. In conclusion, we see that all our arguments above become rigorous subject to the conditions (3.23), (3.24) and (3.25).

Using (3.4)–(3.8), the integral (3.24) can be rewritten as

$$\sum_{\omega\in G} \int_{\mathbb{R}\times\partial\mathbb{D}} |\tilde{g}(\lambda, b)|\, |e_{-\lambda,b}(\omega x)|\, |c(\lambda)|^{-2}\, d\lambda\, db.$$

As g is compactly supported, $\tilde{g}$ is an entire function of uniform exponential type ([21, Theorem 4.2 in the Introduction]), i.e.

$$\sup_{\mathbb{R}\times\partial\mathbb{D}} (1 + |\lambda|)^m |\tilde{g}(\lambda, b)| < \infty, \qquad \forall m \geq 0. \tag{3.26}$$

Thus the integration over $\mathbb{R}$ poses no problems, and as $|e_{\lambda,b}| = e_{0,b}$, we see that (3.24) is equivalent to

$$\sum_{\omega\in G} \int_{\partial\mathbb{D}} e_{0,b}(\omega x)\, db < \infty.$$

However,

$$\int_{\partial\mathbb{D}} e_{0,b}(x)\, db = (1 - |x|^2)^{1/2}\, {}_2F_1\big(\tfrac{1}{2}, \tfrac{1}{2}; 1; |x|^2\big)$$

and the hypergeometric function on the right-hand side is $\approx -\log(1-|x|^2)$ as $x \to 1$. By the same argument as for the series (3.15), we thus see that (3.24) is equivalent to (3.9).

Finally, taking $m = 0$ in (3.26) we see that

$$|\tilde{g}(\lambda, b)| < c_0 = c_0 |e_{-\lambda,b}(0)|, \qquad \forall \lambda, b,$$

for some constant $c_0 < \infty$, and thus (3.25) is a consequence of the special case $x = 0$ of (3.24). Since the condition (3.9) is clearly stronger than (3.23), we see that all three conditions (3.23), (3.24) and (3.25) are fulfilled whenever (3.9) is, and the proof is complete. □

For Ω the annulus, one can take $a_n = \tanh n\alpha$ for some α (depending on the thickness of the annulus), so $1 - |a_n|^2 = \cosh^{-2} n\alpha \approx e^{-2|n|\alpha}$ and condition (3.9) is fulfilled; for other domains, we discuss this condition in Section 7.

4. Powers of the Poisson kernel

We supplement the discussion of the Plancherel formula with an observation concerning the eigenfunctions $E_{\lambda,\zeta}$. On the unit disc, the eigenfunctions $e_{\lambda,b}$ are simply the $(\frac{1}{2} + i\lambda)$-th powers of the familiar Poisson kernel $(1 - |x|^2)/|x - b|^2$. It turns out that a similar characterization is available for $E_{\lambda,\zeta}$.

Theorem 4.1. *For a domain Ω satisfying* (3.9)*, $E_{\lambda,\zeta}$ is the unique eigenfunction of $\widetilde{\Delta}_\Omega$ with eigenvalue $-(\lambda^2 + \frac{1}{4})$ such that*

$$E_{\lambda,\zeta}(z) = P_\zeta(z)^{\frac{1}{2}+i\lambda} + w(z)^{\frac{1}{2}+i\lambda} f(z), \tag{4.1}$$

where f extends continuously to $\overline{\Omega}$ and its first derivatives are bounded on Ω. Here $P_\zeta(z)$ is the Poisson kernel of Ω.

Proof. Fix $b \in \partial\mathcal{O} \cap \partial\mathbb{D}$ and let $x \in \overline{\mathcal{O}}$. Using (3.4) we may rewrite the definition of $\mathcal{E}_{\lambda,b}$ as

$$\begin{aligned} \mathcal{E}_{\lambda,b}(\phi x) &= \sum_{\omega\in G} e_{\lambda,b}(\omega 0) e_{\lambda,\omega^{-1}b}(x) \\ &= \sum_{\omega\in G} \frac{(1-|\omega 0|^2)^s (1-|x|^2)^s}{|b-\omega 0|^{2s}|x-\omega^{-1}b|^{2s}}, \qquad \left(s = \tfrac{1}{2} + i\lambda\right). \end{aligned}$$

Now, $\sum_{\omega\in G}(1-|\omega 0|^2)^s$ converges absolutely by assumption, while $|b - \omega 0|$ is both bounded and bounded away from zero as $\omega \in G$, and so is $|x - \omega^{-1}b|$ — even uniformly in x — as $\omega \in G \setminus \{\mathrm{id}\}$. Thus

$$\mathcal{E}_{\lambda,b}(\phi x) = (1-|x|^2)^s \left[\frac{1}{|x-b|^{2s}} + g_1(x)\right], \qquad x \in \overline{\mathcal{O}},$$

where g_1 is C^ω (= real analytic) in a neighbourhood of $\overline{\mathcal{O}}$.

On the other hand, from Myrberg's Theorem [32, Theorem XI.18] — or by a similar argument as in Section 1 — it follows that the Poisson kernel on Ω is given by the formula

$$P_{\phi b}(\phi x) = \frac{1}{|\phi'(b)|} \sum_{\omega \in G} \frac{1 - |\omega x|^2}{|b - \omega x|^2}.$$

By the same reasoning as above we therefore obtain

$$|\phi'(b)| P_{\phi b}(\phi x) = (1 - |x|^2) \left[\frac{1}{|x - b|^2} + g_2(x) \right], \qquad x \in \overline{\mathcal{O}},$$

where g_2 is C^ω in a neighbourhood of $\overline{\mathcal{O}}$. Raising to the s-th power and subtracting we get

$$\frac{|\phi'(b)|^s \big(P_{\phi b}(\phi x)^s - E_{\lambda,\phi b}(\phi x)\big)}{(1 - |x|^2)^s} = |x - b|^{2-2s} \tilde{g}_2(x) + g_1(x),$$

with $\tilde{g}_2$ in C^ω in a neighbourhood of $\overline{\mathcal{O}}$. As $\mathrm{Re}(2-2s) = 1$, the function on the right-hand side is C^ω on $\mathcal{O}$, continuous on $\overline{\mathcal{O}}$, and its first derivatives are bounded on $\mathcal{O}$. Passing to Ω we thus obtain (4.1). It remains to prove uniqueness. Assume that (4.1) is satisfied (with, possibly, a different function f) for another eigenfunction F (with the same eigenvalue $-\lambda^2 - \frac{1}{4}$) in the place of $E_{\lambda,\zeta}$. Subtracting we see that the difference $g = F - E_{\lambda,\zeta}$ is of the form $g = w^s h$, where $h \in C(\overline{\Omega})$ and ∇h is bounded on Ω. Now, by Green's formula,

$$\begin{aligned}
\int_\Omega \big(\overline{g}\widetilde{\Delta}_\Omega g + w^2 \|\nabla g\|^2\big)\, d\mu_\Omega &= \int_\Omega (\overline{g}\Delta g + \|\nabla g\|^2)\, dx \\
&= \int_{\partial\Omega} \overline{g} \frac{\partial g}{\partial n}\, d\sigma \\
&= \int_{\partial\Omega} \left(w\overline{h}\frac{\partial h}{\partial n} + s|h|^2 \frac{\partial w}{\partial n} \right) d\sigma,
\end{aligned}$$

where dx is the Lebesgue area measure on Ω, $d\sigma$ is the arc-length measure on $\partial\Omega$, and we have made use of the fact $s + \overline{s} = 1$. Now the properties of h imply that the integral of the first term vanishes, and the assumption that $\partial\Omega$ is smooth implies that $\partial w/\partial n = -2$ on $\partial\Omega$. Since $\widetilde{\Delta}_\Omega g = -(\lambda^2 + \frac{1}{4})g$, we thus obtain

$$\int_\Omega \big[- \big(\lambda^2 + \tfrac{1}{4}\big)|g|^2 + w^2 \|\nabla g\|^2 \big]\, d\mu_\Omega = -2s \int_{\partial\Omega} |h|^2\, d\sigma.$$

However, the left-hand side is a real number, while the right-hand side is a real multiple of $s = \frac{1}{2} + i\lambda$. Hence if $\lambda \neq 0$, both sides must vanish, and so $h = 0$ on $\partial\Omega$; as $\|\nabla h\|$ is bounded, it follows that $h = O(w)$, whence $g = w^s h = O(w^{3/2})$. If $\lambda = 0$, then by the boundedness of h we at least have $g = O(w^{1/2})$. We claim that in either case, these O-conditions already imply that $g = 0$.

To see this, let $f = g \circ \phi$ be the lifting of f to $\mathbb{D}$, and for $y \in \mathbb{D}$ and $0 < r < 1$ set

$$F(r, y) = \frac{1}{2\pi} \int_0^{2\pi} f\left(\frac{re^{i\theta} + y}{1 + \overline{y}re^{i\theta}} \right) d\theta.$$

As f is an eigenfunction of $\widetilde{\Delta}_{\mathbb{D}}$ with eigenvalue $-(\lambda^2+\frac{1}{4})$, it is known ([21, §4 of the Introduction]) that

$$F(r,y)=\phi_s(r)f(y), \qquad \big(s=\tfrac{1}{2}+i\lambda\big),$$

where ϕ_s is the spherical function

$$\phi_s(r)=(1-r^2)^s\,{}_2F_1(s,s,1,r^2).$$

It is also known that as $r\to 1$, $\phi_s(r)\approx \operatorname{Re}[A(1-r^2)^s]$ for a certain non-zero constant A if $s\neq\frac{1}{2}$, and $\phi_s(r)\approx A(1-r^2)^{1/2}\log(1-r^2)$ if $s=1/2$ (see [3, formulas (1) and (12) in §2.10]). Thus if we show that

$$F(r,y)=\begin{cases} O((1-r^2)^{\frac{1}{2}}/\log(1-r^2)) & \text{if } s\neq\frac{1}{2}\\ O((1-r^2)^{\frac{1}{2}}) & \text{if } s=\frac{1}{2},\end{cases} \tag{4.2}$$

it will follow, upon letting $r\to 1$, that $f(y)=0$ for any $y\in\mathbb{D}$, and we are done.

It remains to prove (4.2). We do this for $y=0$, the general case being analogous. Let $G=\{\omega_0,\omega_1,\omega_2,\dots\}$ with $\omega_0=\mathrm{id}$, and denote again $a_n=\omega_n^{-1}(0)$. Consider first $s\neq\frac{1}{2}$. The fact that $g=O(w^{3/2})=O(w^{1/2}/\log w)$ implies that

$$\begin{aligned}|f(y)| &\le C\frac{w(\phi y)^{1/2}}{\max(1,-\log w(\phi y))}\\ &= C\frac{(1-|y|^2)^{1/2}|\phi'(y)|^{1/2}}{\max(1,-\log(1-|y|^2)-\log|\phi'(y)|)}.\end{aligned}$$

If $y=\omega_n x$ with $x\in\mathcal{O}$, then $|\phi'(y)|=|\phi'(x)/\omega_n'(x)|=|\phi'(x)||1-\overline{a}_n x|^2/(1-|a_n|^2)\asymp(1-|a_n|^2)^{-1}$, by (1.13) (remember that ϕ' and $1/\phi'$ are bounded on $\mathcal{O}$ as Ω is smoothly bounded). Further, since

$$\Big|\frac{\omega_n x-\omega_n y}{x-y}\Big|=\frac{1-|a_n|^2}{|1-\overline{a}_n x|\,|1-\overline{a}_n y|}, \qquad (x,y\in\mathbb{D}), \tag{4.3}$$

we see by (1.13) that for $x,y\in\mathcal{O}$ and any n, $|\omega_n x-\omega_n y|\le C(1-|a_n|^2)$, so, in particular, the length of the portion of the circle $|x|=r$ intersecting $\omega_n\mathcal{O}$ has arc-length $\le C(1-|a_n|^2)$. Thus

$$\begin{aligned}\int_{|x|=r}|f(x)|\,|dx| &\le \sum_n C\frac{(1-r^2)^{1/2}(1-|a_n|^2)^{1/2}}{\max(1,-\log(1-r^2)+\log(1-|a_n|^2)+C_1)}\\ &= C\,\frac{(1-r^2)^{1/2}}{-\log(1-r^2)}\sum_n\frac{-(1-|a_n|^2)^{1/2}\log(1-r^2)}{\max(1,-\log(1-r^2)+\log(1-|a_n|^2)+C_1)},\end{aligned}$$

with C and C_1 independent of r and n. When $r\to 1$, the last series is dominated by the series in (3.9), and hence its sum tends to $\sum_n(1-|a_n|^2)^{1/2}$ by the dominated convergence theorem. As the latter sum is finite by the same hypothesis (3.9), we thus arrive at

$$|F(r,0)|\le C\frac{(1-r^2)^{1/2}}{-\log(1-r^2)},$$

as asserted. For $s=\frac{1}{2}$, the argument is the same, only with the log-terms omitted. This completes the proof. □

In the last part of the proof above we have, in effect, established the following uniqueness assertion:

Theorem 4.2. *For Ω a domain satisfying* (3.9)*, if $g : \Omega \to \mathbb{C}$ is a solution to $\widetilde{\Delta}_\Omega g + (\lambda^2 + \frac{1}{4})g = 0$ ($\lambda \in \mathbb{R}$) and $g = O(w^{1/2})$ if $\lambda = 0$ or $g = O(w^{1/2}/\log_1 w)$ if $\lambda \neq 0$, then $g = 0$.*

As the simple counter-example involving the spherical functions $\phi_{\frac{1}{2}+i\lambda}$ on the disc shows, the conditions cannot be relaxed to $g = O(w^{1/2}\log w)$ for $\lambda = 0$ or $g = O(w^{1/2})$ for $\lambda \neq 0$.

5. The eigenfunction expansion for the annulus

In this section, we make the above inversion and Plancherel formula more explicit in the case of an annulus. For simplicity, we consider the annulus $\mathbb{A} = \{z : 1 < |z| < R\}$, where $R = e^\pi$, with the Poincaré metric

$$ds = \frac{|dz|}{|z|\sin(\log|z|)},$$

or, writing $z = e^{t+i\theta}$ $(0 < t < \pi)$,

$$ds^2 = \frac{dt^2 + d\theta^2}{\sin^2 t}.$$

(In the case of the general R, the only difference is that one must replace t by $\frac{\pi}{\log R}t$.) The corresponding Laplace–Beltrami operator reads

$$\widetilde{\Delta}_{\mathbb{A}} = \sin^2 t\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2}\right). \tag{5.1}$$

Taking the Friedrichs extension we may view $\mathcal{P} = -\widetilde{\Delta}_{\mathbb{A}}$ as a self-adjoint operator in the Hilbert space L^2 with respect to the Poincaré measure on the annulus; the spectrum is the interval $\lambda > \frac{1}{4}$. It admits the following symmetries: $\theta \mapsto \theta + h$; $t \mapsto \pi - t$. Thus, separating variables, we are led to the eigenvalue equation

$$\sin^2 t\left(\frac{d^2 f}{dt^2} - n^2 f\right) + \lambda f = 0 \tag{5.2}$$

on the interval $0 < t < \frac{\pi}{2}$ with $n = 0, \pm 1, \pm 2, \ldots$, the boundary conditions at the right endpoint $t = \frac{\pi}{2}$, where the equation is not singular, being

$$\text{either I.}\quad f\left(\frac{\pi}{2}\right) = 0, \qquad \text{or else II.}\quad f'\left(\frac{\pi}{2}\right) = 0.$$

By the general theory, the spectral multiplicity of the corresponding differential operator is one (see [25, Chapter VI, §21]).

Let us investigate the solutions to (5.2) letting (for a while) λ be complex. We define a parameter α, implicitly, by

$$\alpha^2 - \alpha + \lambda = 0, \tag{5.3}$$

or, explicitly, by

$$\alpha = \frac{1}{2} + \frac{\sqrt{1-4\lambda}}{2}, \qquad \operatorname{Re}\alpha \geq \frac{1}{2}. \tag{5.4}$$

At the left endpoint $t = 0$ we have a regular singular point. It turns out that the indices are given by (5.3). Thus one index is the number α that we defined in (5.4) and the other is $1 - \alpha$. Therefore, there exists a solution $\mathfrak{f} = \mathfrak{f}(t) = \mathfrak{f}(t, \alpha)$ which behaves as t^α near $t = 0$. Then $\mathfrak{f}_- = \mathfrak{f}_-(t) = \mathfrak{f}_-(t, \alpha) \stackrel{\text{def}}{=} \mathfrak{f}(t, 1-\alpha)$ is also a solution but behaves as $t^{1-\alpha}$ there. By virtue of our choice of the square root in (5.4), for $\lambda \notin [\frac{1}{4}, \infty)$ the solution $\mathfrak{f}$ belongs to $L^2(0, \frac{\pi}{2})$ while $\mathfrak{f}_-$ does not. One can show (the substitution $\zeta = \sin^2 t$ does it; cf. e.g. [22, p. 49]) that

$$\mathfrak{f}(t, \alpha) = \sin^\alpha t \; {}_2F_1\big(\tfrac{\alpha}{2} + i\tfrac{n}{2}, \tfrac{\alpha}{2} - i\tfrac{n}{2}; \tfrac{1}{2} + \alpha; \sin^2 t\big),$$

and, consequently,

$$\mathfrak{f}_-(t, \alpha) = \sin^{1-\alpha} t \; {}_2F_1\big(\tfrac{1-\alpha}{2} + i\tfrac{n}{2}, \tfrac{1-\alpha}{2} - i\tfrac{n}{2}; \tfrac{3}{2} - \alpha; \sin^2 t\big);$$

here, generally speaking, ${}_2F_1(a, b; c; x)$ is the Gauss hypergeometric function. Note that in our case $c = a + b + \frac{1}{2}$. Below we collect some relevant facts about this function.

For $c \neq 0, -1, -2, \ldots$ the hypergeometric function is given by the series expansion

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

where we have used the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2)\ldots(a+n-1).$$

For $0 < \operatorname{Re} b < \operatorname{Re} c$ one has the integral representation

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-t)^{c-b-1} t^{b-1} (1-tx)^{-a} dt.$$

Letting $x \to 1$ we obtain from it, for $0 < \operatorname{Re} b < \operatorname{Re} c$ and $\operatorname{Re}(c - a - b) > 0$, the limit

$$\lim_{x \to 1} {}_2F_1(a, b; c; x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

In the special case of interest to us, viz. $c = a + b + \frac{1}{2}$, it becomes

$$\lim_{x \to 1} {}_2F_1\big(a, b; a+b+\tfrac{1}{2}; x\big) = \frac{\sqrt{\pi}\,\Gamma\big(a+b+\frac{1}{2}\big)}{\Gamma\big(a+\frac{1}{2}\big)\Gamma\big(b+\frac{1}{2}\big)}. \tag{5.5}$$

We also need an analogous result involving the derivative, namely,

$$\lim_{x\to 1}(1-x)^{\frac{1}{2}}\,{}_2F_1'\big(a,b;a+b+\tfrac{1}{2};x\big)=\frac{\sqrt{\pi}\,\Gamma\big(a+b+\frac{1}{2}\big)}{\Gamma(a)\Gamma(b)}. \tag{5.6}$$

Beside the previous basis $\{\mathfrak{f},\mathfrak{f}_-\}$ we need another basis $\{\mathfrak{u},\mathfrak{v}\}$ for the solutions of our eigenvalue equation: $\mathfrak{u}=\mathfrak{u}(t)=\mathfrak{u}(t,\alpha)$ is a normalized solution satisfying the boundary conditions II

$$\mathfrak{u}\big(\tfrac{\pi}{2}\big)=1,\quad \mathfrak{u}'\big(\tfrac{\pi}{2}\big)=0,$$

while $\mathfrak{v}=\mathfrak{v}(t)=\mathfrak{v}(t,\alpha)$ is such a solution satisfying the boundary conditions I

$$\mathfrak{v}\big(\tfrac{\pi}{2}\big)=0,\quad \mathfrak{v}'\big(\tfrac{\pi}{2}\big)=1.$$

Then we can write

$$\mathfrak{f}=A\mathfrak{u}+B\mathfrak{v},\quad \mathfrak{f}_-=A_-\mathfrak{u}+B_-\mathfrak{v}, \tag{5.7}$$

where the constants are found from (5.5) and (5.6) upon putting $a=\frac{\alpha}{2}+i\frac{n}{2}$, $b=\frac{\alpha}{2}-i\frac{n}{2}$ and $a=\frac{1-\alpha}{2}+i\frac{n}{2}$, $b=\frac{1-\alpha}{2}-i\frac{n}{2}$ there, respectively,

$$A=\frac{\sqrt{\pi}\,\Gamma\big(\alpha+\frac{1}{2}\big)}{\Gamma\big(\frac{\alpha}{2}+i\frac{n}{2}+\frac{1}{2}\big)\Gamma\big(\frac{\alpha}{2}-i\frac{n}{2}+\frac{1}{2}\big)},\quad B=\frac{-2\sqrt{\pi}\,\Gamma\big(\alpha+\frac{1}{2}\big)}{\Gamma\big(\frac{\alpha}{2}+i\frac{n}{2}\big)\Gamma\big(\frac{\alpha}{2}-i\frac{n}{2}\big)}, \tag{5.8}$$

and (replace α by $1-\alpha$)

$$A_-=\frac{\sqrt{\pi}\,\Gamma\big(\frac{3}{2}-\alpha\big)}{\Gamma\big(\frac{1-\alpha}{2}+i\frac{n}{2}+\frac{1}{2}\big)\Gamma\big(\frac{1-\alpha}{2}-i\frac{n}{2}+\frac{1}{2}\big)},\quad B_-=\frac{-2\sqrt{\pi}\,\Gamma\big(\frac{3}{2}-\alpha\big)}{\Gamma\big(\frac{1-\alpha}{2}+i\frac{n}{2}\big)\Gamma\big(\frac{1-\alpha}{2}-i\frac{n}{2}\big)},$$

respectively.

Remark. Using the formula for analytic continuation of the hypergeometric series ${}_2F_1(\frac{\alpha+in}{2},\frac{\alpha-in}{2};\alpha+\frac{1}{2};z)$ (cf. [3, formula (2.10.4)]), it transpires that $\mathfrak{u}$ and $\mathfrak{v}$ can even be expressed in terms of a single hypergeometric function:

$$\begin{aligned}\mathfrak{u}(t,\alpha)&=(\sin t)^{in}\,{}_2F_1\big(\tfrac{\alpha+in}{2},\tfrac{1-\alpha+in}{2};\tfrac{1}{2};-\cot^2 t\big),\\ \mathfrak{v}(t,\alpha)&=\cos t\,(\sin t)^{-1+in}\,{}_2F_1\big(\tfrac{1+\alpha-in}{2},\tfrac{2-\alpha-in}{2};\tfrac{3}{2};-\cot^2 t\big).\end{aligned}$$

We also note that, although this is not immediate neither from the last two formulas nor from (5.7), by virtue of (5.2) the functions $\mathfrak{u}$ and $\mathfrak{v}$ are in fact real-valued. □

We recall now some general facts from spectral theory. Let, for a moment, $\mathscr{P}$ denote any self-adjoint operator in a separable Hilbert space $\mathfrak{H}$.

Assume that the spectral multiplicity is one. Take an element $f_0\neq 0$ in $\mathfrak{H}$. Let f be another element in $\mathfrak{H}$ in the orbit of f_0, that is, $f=\phi(\mathscr{P})f_0$ for some scalar continuous function ϕ on the spectrum σ of $\mathscr{P}$. Then by the spectral theorem we have

$$f=\int_\sigma \phi(\lambda)\,d\mathcal{E}(\lambda)f_0,$$

where $\mathcal{E}(\lambda)$ is the family of spectral projections belonging to the operator $\mathcal{P}$. Moreover, by the same token, the norm of f is given by

$$\|f\|^2 = \int_\sigma |\phi(\lambda)|^2 \, d\langle \mathcal{E}(\lambda) f_0, f_0 \rangle.$$

By our assumption, the correspondence $\phi \mapsto f$ extends to a unitary transformation of $\mathcal{H}$ to itself. The inverse transformation is given by

$$g \mapsto \frac{d\langle g, \mathcal{E}(\lambda) f_0 \rangle}{d\langle f_0, \mathcal{E}(\lambda) f_0 \rangle}. \tag{5.9}$$

Assuming also that the spectrum is absolutely continuous and denoting the spectral density by $\mathcal{D}(\lambda)$,

$$\mathcal{D}(\lambda) = \frac{d\mathcal{E}(\lambda)}{d\lambda},$$

we let $\mathcal{R}(\lambda)$ denote the resolvent of $\mathcal{P}$,

$$\mathcal{R}(\lambda) = (\mathcal{P} - \lambda)^{-1};$$

it is well-defined for λ off σ. Then one has, for λ in σ, the formula

$$\mathcal{D}(\lambda) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} (\mathcal{R}(\lambda + i\epsilon) - \mathcal{R}(\lambda - i\epsilon)). \tag{5.10}$$

After these preliminaries we can begin to write down the sought eigenfunction expansion. But first we define for differentiable functions f, g on an interval the following bilinear differential operator ("transvectant" or "Wronskian"):

$$[f, g] = f'g - fg'.$$

It is well known that if f and g are solutions of one and the same ordinary second-order differential equation without the first-order term, then $[f, g]$ is a constant. We also note that our transvectant is alternating:

$$[f, f] = 0. \tag{5.11}$$

Consider, in particular, the previous functions $\mathfrak{f}, \mathfrak{f}_-, \mathfrak{u}, \mathfrak{v}$. Then $[\mathfrak{v}, \mathfrak{u}] = 1$ and so from the first equation (5.7) and from (5.11) we find

$$[\mathfrak{v}, \mathfrak{f}] = [\mathfrak{v}, A\mathfrak{u} + B\mathfrak{v}] = A. \tag{5.12}$$

In the same way, the second equation (5.7) yields

$$[\mathfrak{v}, \mathfrak{f}_-] = A_-.$$

Let now $\mathcal{P}$ be a selfadjoint extension of the ordinary differential operator (5.2) on $(0, \frac{\pi}{2})$, to fix the ideas taking the boundary condition I at $t = \frac{\pi}{2}$. For λ off σ, let $G(t, s, \lambda)$ be the kernel of the resolvent $\mathcal{R}(\lambda)$ with respect to the "Poincaré measure" $dt/\sin^2 t$; it satisfies the *inhomogeneous* differential equation (5.2) with the right-hand side $\delta(t-s)$ (the Dirac function). Then we have the formula

$$G(t,s,\lambda) = \frac{1}{A(\alpha)} \begin{cases} \mathfrak{f}(t,\alpha)\mathfrak{v}(s,\alpha) & \text{if } t < s, \\ \mathfrak{v}(t,\alpha)\mathfrak{f}(s,\alpha) & \text{if } t > s. \end{cases} \tag{5.13}$$

Here the parameter α, as before, is related to λ by equation (5.4) and we have taken care of the jump conditions at $t = s$, also using formula (5.12). (We also write $A(\alpha)$ etc. to indicate the dependence of A etc. on the parameter α.)

Similarly, let $S(t, s, \lambda)$ stand for the kernel of the operator $\mathcal{D}(\lambda)$ for $\lambda \in [\frac{1}{4}, \infty)$. Then (5.13) in conjunction with (5.10) yields, e.g. for $t < s$,

$$\begin{aligned} S(t,s,\lambda) = \frac{1}{2\pi i}\Bigg[\left(\mathfrak{u}(t,\alpha) + \frac{B(\alpha)}{A(\alpha)}\mathfrak{v}(t,\alpha)\right)\mathfrak{v}(s,\alpha) \\ - \left(\mathfrak{u}(t,\alpha) + \frac{B_-(\alpha)}{A_-(\alpha)}\mathfrak{v}(t,\alpha)\right)\mathfrak{v}(s,\alpha)\Bigg] \\ = \frac{1}{2\pi i}\left[\frac{B(\alpha)}{A(\alpha)} - \frac{B_-(\alpha)}{A_-(\alpha)}\right]\mathfrak{v}(t,\alpha)\mathfrak{v}(s,\alpha). \end{aligned}$$

The expression within the brackets on the last line can be written as

$$2i\,\mathrm{Im}\,\frac{B(\alpha)}{A(\alpha)} \stackrel{\text{def}}{=} 2iM^I(\alpha) \tag{5.14}$$

(the superscript being a reminder that we have fixed the boundary condition I). So we can write

$$S(t,s,\lambda) = \frac{1}{\pi}M^I(\alpha)\mathfrak{v}(t,\alpha)\mathfrak{v}(s,\alpha);$$

this formula is also valid for all t and s, not only for $t < s$, as is readily seen.

Feeding this back we see that

$$\begin{aligned} \langle g, \mathcal{D}(\lambda) f_0\rangle &= \int_0^{\pi/2} g(s)\overline{\left(\int_0^{\pi/2} S(t,s,\lambda)\,f_0(t)\,\frac{dt}{\sin^2 t}\right)}\,\frac{ds}{\sin^2 s} \\ &= \frac{1}{\pi}M^I(\alpha)\tilde{g}^I(\lambda)\overline{\tilde{f}_0^I(\lambda)}, \end{aligned}$$

where we have introduced the notation

$$\tilde{g}^I(\lambda) = \int_0^{\pi/2} g(s)\mathfrak{v}(s,\alpha)\,\frac{ds}{\sin^2 s}.$$

The isomorphism (5.9) thus becomes simply

$$g \mapsto \phi, \qquad \phi(\lambda) = \frac{\tilde{g}^I(\lambda)}{\tilde{f}_0^I(\lambda)},$$

and we obtain the inversion formula

$$\begin{aligned}
g(s) &= \int_\sigma \phi(\lambda)\mathcal{D}(\lambda) f_0(s)\, d\lambda \\
&= \int_\sigma \phi(\lambda)\left(\int_0^{\pi/2} S(t,s,\lambda) f_0(t)\, \frac{dt}{\sin^2 t}\right) d\lambda \\
&= \frac{1}{\pi}\int_\sigma \phi(\lambda) M^I(\alpha)\mathfrak{v}(s,\alpha)\left(\int_0^{\pi/2} \mathfrak{v}(t,\alpha) f_0(t)\, \frac{dt}{\sin^2 t}\right) d\lambda \\
&= \frac{1}{\pi}\int_\sigma \phi(\lambda) M^I(\alpha)\mathfrak{v}(s,\alpha)\tilde{f}_0^I(\lambda)\, d\lambda \\
&= \frac{1}{\pi}\int_\sigma \tilde{g}^I(\lambda)\mathfrak{v}(s,\alpha) M^I(\alpha)\, d\lambda,
\end{aligned}$$

and the Plancherel theorem

$$\|g\|^2 = \frac{1}{\pi}\int_\sigma |\tilde{g}^I(\lambda)|^2 M^I(\alpha)\, d\lambda,$$

both being valid for $g \in L^2((0,\frac{\pi}{2}), \sin^{-2} t\, dt)$.

Of course, the case of boundary condition II is susceptible to the same treatment, the only difference being that now $\mathfrak{u}$ takes over the role of $\mathfrak{v}$, and in place of $M^I(\alpha)$ we get

$$M^{II}(\alpha) = -\operatorname{Im}\frac{A(\alpha)}{B(\alpha)}. \tag{5.15}$$

The two formulas over $(0,\frac{\pi}{2})$ can be combined into a single formula for the whole interval $(0,\pi)$ (a manipulation reminiscent of combining Fourier sine and cosine transforms on $(0,\infty)$ into the ordinary Fourier transform on the whole line). Observe that in view of the symmetry $t \leftrightarrow \pi - t$ the solutions $\mathfrak{u}$, $\mathfrak{v}$ satisfy

$$\mathfrak{u}(\pi - t) = \mathfrak{u}(t), \qquad \mathfrak{v}(\pi - t) = -\mathfrak{v}(t).$$

Now, given any $f \in L^2((0,\pi), \sin^{-2} t\, dt)$, we can split it into its odd part f_{odd} and its even part f_{even} with respect to this symmetry. Then obviously

$$\langle f_{\text{even}}, \mathfrak{v}\rangle_{(0,\pi)} = 0, \qquad \langle f_{\text{even}}, \mathfrak{u}\rangle_{(0,\pi)} = 2\langle f_{\text{even}}, \mathfrak{u}\rangle_{(0,\pi/2)},$$

and similarly for f_{odd} (with $\mathfrak{u}$ and $\mathfrak{v}$ interchanged). We thus arrive at the following result:

Theorem 5.1. *The following inversion formulas and Plancherel theorem hold:*

$$\tilde{f}^I(\lambda) = \frac{1}{2}\int_0^\pi f(s)\mathfrak{v}(s,\alpha)\, \frac{ds}{\sin^2 s}, \qquad \tilde{f}^{II}(\lambda) = \frac{1}{2}\int_0^\pi f(s)\mathfrak{u}(s,\alpha)\, \frac{ds}{\sin^2 s},$$

$$f(s) = \frac{1}{\pi}\int_\sigma \tilde{f}^I(\lambda)\mathfrak{v}(s,\alpha)\, M^I(\alpha)\, d\lambda + \frac{1}{\pi}\int_\sigma \tilde{f}^{II}(\lambda)\mathfrak{u}(s,\alpha)\, M^{II}(\alpha)\, d\lambda, \tag{5.16}$$

$$\int_0^\pi |f(s)|^2\, \frac{ds}{\sin^2 s} = \frac{1}{\pi}\int_\sigma \Big[|\tilde{f}^I(\lambda)|^2 M^I(\alpha) + |\tilde{f}^{II}(\lambda)|^2 M^{II}(\alpha)\Big]\, d\lambda.$$

Here M^I and M^{II} are as in (5.14) *and* (5.15)*, respectively, while A and B are defined in* (5.8)*; and α is related to λ by* (5.4).

Finally, we can obtain the eigenfunction expansion on the annulus by first performing the Fourier decomposition with respect to θ,

$$f(z) = \sum_{n\in\mathbb{Z}} f_n(t)e^{in\theta}, \qquad f_n(t) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{t+i\theta})e^{-in\theta}\,d\theta,$$

and thereupon applying the formulas (5.16) for each n. In this way we finally arrive at

$$\hat{f}^I(\lambda,\theta) = \frac{1}{2}\int_{\mathbb{A}} f(z)V(\alpha, e^{i\theta}\overline{z})\,d\mu(z), \quad \hat{f}^{II}(\lambda,\theta) = \frac{1}{2}\int_{\mathbb{A}} f(z)U(\alpha, e^{i\theta}\overline{z})\,d\mu(z),$$

$$f(z) = \frac{1}{\pi}\int_\sigma\int_0^{2\pi}\Big[\hat{f}^I(\lambda,\theta)V(\alpha, e^{-i\theta}z) + \hat{f}^{II}(\lambda,\theta)U(\alpha, e^{-i\theta}z)\Big]\frac{d\theta}{2\pi}\,d\lambda,$$

$$\int_{\mathbb{A}} |f(z)|^2\,d\mu(z) = \int_\sigma\int_0^{2\pi}\Big[|\hat{f}^I(\lambda,\theta)|^2 + |\hat{f}^{II}(\lambda,\theta)|^2\Big]\frac{d\theta}{2\pi}\,d\lambda,$$

where U and V are given by

$$V(\alpha, e^{t+i\theta}) = \sum_{n\in\mathbb{Z}} \mathfrak{v}_n(t,\alpha)\sqrt{M_n^I(\alpha)}\,e^{in\theta},$$

$$U(\alpha, e^{t+i\theta}) = \sum_{n\in\mathbb{Z}} \mathfrak{u}_n(t,\alpha)\sqrt{M_n^{II}(\alpha)}\,e^{in\theta}$$

(here we wrote $\mathfrak{u}_n$, M_n^I etc. to indicate the dependence on n). We omit the details.

6. The Green function on the annulus

It is known that the Green function for the unsquared $\widetilde{\Delta}_{\mathbb{A}}$ (or, equivalently, Δ) on the annulus can be expressed in terms of the logarithm of a theta-function, cf. [7, p. 335–337]. The aim of this section is to compute "explicitly" (in the form of an infinite series) the corresponding Green function for $\widetilde{\Delta}^2_{\mathbb{A}}$, to see what kind of transcendental functions turn up in that case.

We again restrict attention to the case of the annulus $1 < |z| < R = e^\pi$, and use the same method as in [14], i.e. essentially the separation of variables $z = e^{(x+i\theta)}$ $(0 < x < \pi)$. We are looking for the Green function $G(z,w)$ with the pole at a point $w = e^t$ on the positive real axis $(0 < t < \pi)$, i.e. a solution to

$$\begin{cases} \widetilde{\Delta}^2_{\mathbb{A}} G(\cdot, w) = \delta_w \\ G(\cdot, w) \text{ has the least possible growth at the boundary.} \end{cases}$$

Our starting point is the following Almansi-type theorem (cf. [2], [14, p. 364]):

Proposition 6.1. *A function $f(z) = F(x)e^{ni\theta}$ satisfies $\widetilde{\Delta}^2_{\mathbb{A}} f = 0$ if and only if*

$$F(x) = Ae^{nx} + Be^{-nx} + C\Phi_n(x) + D\Psi_n(x) \qquad \text{if } n \neq 0,$$

$$F(x) = A + Bx + C\Phi_0(x) + D\Psi_0(x) \qquad \text{if } n = 0,$$

where

$$\begin{aligned}
\Phi_n(x) &= \frac{e^{nx}}{-in}\,{}_2F_1(1, -in; 1-in; e^{2ix}), \\
\Psi_n(x) &= \Phi_{-n}(x), \\
\Phi_0(x) &= \log\sin x, \\
\Psi_0(x) &= -2x\log(1-e^{2ix}) + x\log\sin x + ix^2 + i\mathrm{Li}_2(e^{2ix}).
\end{aligned} \tag{6.1}$$

Here ${}_2F_1$ *stands for the hypergeometric function.*

Proof. One has, by formula (5.1) of the preceding section,

$$\widetilde{\Delta}_{\mathbb{A}} f = \sin^2 x \cdot (F''(x) - n^2 F(x))\, e^{ni\theta}. \tag{6.2}$$

Thus $\widetilde{\Delta}_{\mathbb{A}} f = 0$ if and only if F is in the span of e^{nx} and e^{-nx}, and it suffices to check that Φ_n and Ψ_n are particular solutions to the equations

$$F'' - n^2 F = \frac{e^{nx}}{\sin^2 x}, \quad \text{respectively} \quad F'' - n^2 F = \frac{e^{-nx}}{\sin^2 x}. \tag{6.3}$$

By symmetry, it is enough to deal with the first equation. We look for a solution in the form $F(x) = e^{nx}G(x)$; this gives an equation for G

$$G'' + 2nG' = \frac{1}{\sin^2 x}.$$

Let us temporarily work in the half-plane $\operatorname{Im} x > 0$, i.e. $|e^{ix}| < 1$. Since the right-hand side has period π, we also expect G to have this period, i.e.

$$G(x) = \sum_{k\ge 0} g_k e^{2kix}, \qquad G'' + 2nG' = \sum_{k\ge 0}(4kni - 4k^2) g_k e^{2kix}.$$

Since, for $|e^{ix}| < 1$,

$$\frac{1}{\sin^2 x} = \frac{-4e^{2ix}}{(e^{2ix}-1)^2} = \sum_{k>0} -4ke^{2kix},$$

we get $g_k = 1/(k - ni)$, and the desired result follows.

For $n = 0$ the same argument applies with obvious modifications. □

We are thus lead to search for the Green function $G(z, w)$ in the form

$$G(z,w) = \begin{cases} \sum_{-\infty}^{\infty} \left(A_n^* e^{nx} + B_n^* e^{-nx} + C_n^* \Phi_n(x) + D_n^* \Psi_n(x)\right) e^{ni\theta} & , \ 0 < x < t \\ \sum_{-\infty}^{\infty} \left(A_n^{**} e^{nx} + B_n^{**} e^{-nx} + C_n^{**} \Phi_n(x) + D_n^{**} \Psi_n(x)\right) e^{ni\theta}, & t < x < \pi \end{cases} \tag{6.4}$$

($z = e^{x+i\theta}$, $w = e^t > 0$), where $A_n^*, \dots, D_n^{**}$ are unknown coefficients (depending on t), and for $n = 0$ the corresponding term has to be modified accordingly.

Let us first deal with $n \neq 0$.

To implement the boundary conditions, observe that

$$\frac{1}{a}\,{}_2F_1(1,a;1+a;y)=\sum_{k=0}^{\infty}\frac{y^k}{k+a}=\sum_{k=0}^{\infty}\frac{y^k}{k+1}+\sum_{k=0}^{\infty}y^k\left(\frac{1}{k+a}-\frac{1}{k+1}\right)$$
$$=\frac{1}{y}\log\frac{1}{1-y}-C-\psi(a)+o(1)\qquad\text{as } y\to 1,$$

where

$$\psi(a)=-C+\sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+a}\right)=\frac{\Gamma'(a)}{\Gamma(a)}.$$

Thus as $x\to 0$ ($x\in\mathbb{R}$),

$$\frac{1}{-in}\,{}_2F_1(1,-in;1-in;e^{2ix})=-\log|2x|-C-\psi(-in)+\frac{\pi i}{2}\operatorname{sign}x+o(1),$$

so as $x\searrow 0$,

$$\Phi_n(x)=-\log|2x|-C-\psi(-in)+\frac{\pi i}{2}+o(1)$$

(6.5) $$\Phi_n(\pi-x)=e^{n\pi}\left[-\log|2x|-C-\psi(-in)-\frac{\pi i}{2}\right]+o(1).$$

To get the least possible growth at $x=0$ we must therefore take, in view of the first equality in (6.5),

(6.6) $$C_n^*+D_n^*=0$$

(so that the log-terms vanish), and

(6.7) $$A_n^*+B_n^*+C_n^*\gamma_n=0$$

(so that the constant terms vanish), where

$$\gamma_n=\psi(in)-\psi(-in).$$

Taking the logarithmic derivative of the functional equations for the gamma function

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s},\qquad \Gamma(1-s)=(-s)\Gamma(-s),$$

shows that

$$\psi(s)-\psi(1-s)=-\pi\cot\pi s,\qquad \psi(1-s)=\psi(-s)-\frac{1}{s}.$$

It follows that

$$\gamma_n=\pi i\frac{e^{2n\pi}+1}{e^{2n\pi}-1}-\frac{1}{in}.$$

Similarly, from the second equation (6.5) we get, at $x=\pi$,

(6.8) $$C_n^{**}e^{n\pi}+D_n^{**}e^{-n\pi}=0,$$

(6.9) $$A_n^{**}e^{n\pi}+B_n^{**}e^{-n\pi}+C_n^{**}e^{n\pi}\gamma_n=0.$$

Now consider the conditions for the jump at $x = t$. Setting $\Delta A_n = A_n^{**} - A_n^*$ etc. these are:

$$\text{(6.10a)} \qquad e^{nt}\Delta A_n + e^{-nt}\Delta B_n + \Phi_n(t)\Delta C_n + \Psi_n(t)\Delta D_n = 0,$$

$$\text{(6.10b)} \qquad ne^{nt}\Delta A_n - ne^{-nt}\Delta B_n + \Phi_n'(t)\Delta C_n + \Psi_n'(t)\Delta D_n = 0,$$

$$\text{(6.10c)} \qquad n^2e^{nt}\Delta A_n + n^2e^{-nt}\Delta B_n + \Phi_n''(t)\Delta C_n + \Psi_n''(t)\Delta D_n = 0,$$

$$\text{(6.10d)} \qquad n^3e^{nt}\Delta A_n - n^3e^{-nt}\Delta B_n + \Phi_n'''(t)\Delta C_n + \Psi_n'''(t)\Delta D_n = 1/(2\pi \sin^2 t).$$

We now observe that

$$\left(\frac{d}{dx} + n\right)\Phi_n(x) = \frac{2ie^{nx}}{1 - e^{2ix}}.$$

(To see this, just use the series expansion of the function G in the proof of the last proposition.) This implies that:

$$\text{(6.11)} \qquad \begin{aligned} \Phi_n'(t) &= -n\ \Phi_n(t) + c_1e^{nt}, & \Psi_n'(t) &= n\ \Psi_n(t) + c_1e^{-nt}, \\ \Phi_n''(t) &= \ \ n^2\Phi_n(t) - c_2e^{nt}, & \Psi_n''(t) &= n^2\Psi_n(t) - c_2e^{-nt}, \\ \Phi_n'''(t) &= -n^3\Phi_n(t) + c_3e^{nt}, & \Psi_n'''(t) &= n^3\Psi_n(t) + \tilde{c}_3e^{-nt}, \end{aligned}$$

where

$$c_1 = \frac{2i}{1 - e^{2it}}, \qquad c_2 = \frac{4e^{2it}}{(1 - e^{2it})^2},$$

$$c_3 = \frac{2\left[n^2i(1 - e^{2it})^2 - 2ne^{2it}(1 - e^{2it}) - 4ie^{2it}(1 + e^{2it})\right]}{(1 - e^{2it})^3},$$

and $\tilde{c}_3$ is obtained from c_3 by replacing n by $-n$.

Inserting (6.11) into (6.10a) and forming the differences (c)$-n^2$(a) and (d)$-n^2$(b) gives two equations

$$-c_2e^{nt}\Delta C_n - c_2e^{-nt}\Delta D_n = 0,$$

$$\left(c_3 - n^2c_1\right)e^{nt}\Delta C_n + \left(\tilde{c}_3 - n^2c_1\right)e^{-nt}\Delta D_n = \frac{1}{2\pi \sin^2 t},$$

or

$$e^{-nt}\Delta D_n = -e^{nt}\Delta C_n,$$

$$(c_3 - \tilde{c}_3)e^{nt}\Delta C_n = \frac{1}{2\pi \sin^2 t}.$$

As $c_3 - \tilde{c}_3 = \dfrac{-8ne^{2it}}{(1 - e^{2it})^2} = \dfrac{2n}{\sin^2 t}$, the solution is

$$\Delta C_n = \frac{e^{-nt}}{4\pi n}, \qquad \Delta D_n = -\frac{e^{nt}}{4\pi n}.$$

Using these values in the first two equations (a), (b) in (6.10a) and forming (b)$+n$(a) and (b)$-n$(a), we obtain similarly

$$\Delta A_n = \frac{\Psi_n(t)}{4\pi n}, \qquad \Delta B_n = -\frac{\Phi_n(t)}{4\pi n}.$$

Now from (6.6) and (6.8) we have:

$$\begin{aligned}\Delta C_n &= -e^{-2n\pi} D_n^{**} + D_n^* = \frac{e^{-nt}}{4\pi n},\\ \Delta D_n &= \qquad\quad D_n^{**} - D_n^* = -\frac{e^{nt}}{4\pi n},\end{aligned}$$

and, hence, we can solve for D_n^{**}, D_n^* (and C_n^{**}, C_n^*).

Finally from (6.9) and (6.7) we can compute A_n^*, A_n^{**}, B_n^* and B_n^{**}. Switching to the notation $R = e^{\pi}$, we thus arrive at the following result:

$$\begin{aligned}C_n^* &= \frac{R^{2n}e^{-nt} - e^{nt}}{4\pi n(1 - R^{2n})},\\ D_n^* &= -C_n^*,\\ C_n^{**} &= \frac{e^{-nt} - e^{nt}}{4\pi n(1 - R^{2n})},\\ D_n^{**} &= \frac{R^{2n}(e^{nt} - e^{-nt})}{4\pi n(1 - R^{2n})},\\ A_n^{**} &= \frac{\Psi_n(t) - \Phi_n(t)}{4\pi n(1 - R^{2n})} - i\,\frac{[n\pi(1 + R^{2n}) - (1 - R^{2n})]\, e^{nt}}{4\pi n^2(1 - R^{2n})^2},\\ B_n^{**} &= R^{2n}\frac{\Phi_n(t) - \Psi_n(t)}{4\pi n(1 - R^{2n})} + i\,\frac{[n\pi(1 + R^{2n}) - (1 - R^{2n})]\, R^{2n}e^{-nt}}{4\pi n^2(1 - R^{2n})^2},\\ B_n^* &= \frac{\Phi_n(t) - R^{2n}\Psi_n(t)}{4\pi n(1 - R^{2n})} + \text{ the same second part as in } B_n^{**},\\ A_n^* &= \frac{R^{2n}\Psi_n(t) - \Phi_n(t)}{4\pi n(1 - R^{2n})} - \text{ the same second part as in } A_n^{**}.\end{aligned} \tag{6.12}$$

Remark. It can be checked that after plugging this into (6.4) one gets a function symmetric in t, x (i.e. $G(z, w) = G(w, z)$); on the coefficient level this amounts to

$$\begin{aligned}&A_n^*(t)e^{nx} + B_n^*(t)e^{-nx} + C_n^*(t)\Phi_n(x) + D_n^*(t)\Psi_n(x)\\ &\qquad = A_n^{**}(x)e^{nt} + B_n^{**}(x)e^{-nt} + C_n^{**}(x)\Phi_n(t) + D_n^{**}(x)\Psi_n(t).\end{aligned}$$

□

In a similar way we can deal with the exceptional case $n = 0$: one has

$$\lim_{x\searrow 0}\Psi_0(x) = i\frac{\pi^2}{6} =: \gamma_0^*, \qquad \lim_{x\searrow 0}[\Psi_0(\pi - x) + \pi\log x] = i\frac{\pi^2}{6} - 2\pi\log 2 =: \gamma_0^{**},$$

the boundary condition at $x = 0$ gives equations

$$C_0^* = 0, \qquad A_0^* + \gamma_0^* D_0^* = 0,$$

the condition at $x = \pi$ gives

$$C_0^{**} - \pi D_0^{**} = 0, \qquad A_0^{**} + \pi B_0^{**} + \gamma_0^{**} D_0^{**} = 0,$$

and the conditions at $x = t$ give

$$\begin{aligned}
\varDelta A_0 + t\,\varDelta B_0 + \log\sin t\,\varDelta C_0 + {} & \Psi_0(t)\,\varDelta D_0 = 0,\\
\varDelta B_0 + \cot t\,\varDelta C_0 + {} & (\log\sin t - t\cot t)\,\varDelta D_0 = 0,\\
-\frac{1}{\sin^2 t}\varDelta C_0 + {} & \frac{t}{\sin^2 t}\varDelta D_0 = 0,\\
\frac{2\cot t}{\sin^2 t}\varDelta C_0 + {} & \frac{1-2t\cot t}{\sin^2 t}\varDelta D_0 = \frac{1}{2\pi\sin^2 t}.
\end{aligned}$$

Solving out yields

$$\begin{aligned}
A_0^* &= \frac{i(\pi - t)}{12},\\
B_0^* &= \frac{6\Psi_0(t) - \pi^2 i + 12t\log 2 + 6\pi\log\sin t}{12\pi^2},\\
C_0^* &= 0,\\
D_0^* &= \frac{t-\pi}{2\pi^2},\\
A_0^{**} &= \frac{\pi^2 i - 6\Psi_0(t) - \pi i t}{12\pi},\\
B_0^{**} &= \frac{6\Psi_0(t) - \pi^2 i + 12t\log 2}{12\pi^2},\\
C_0^{**} &= \frac{t}{2\pi},\\
D_0^{**} &= \frac{t}{2\pi^2}.
\end{aligned} \tag{6.13}$$

Inserting all these values into (6.4) we thus get an infinite series representation for $G(z, w)$.

Theorem 6.2. *The Green function $G(z, w)$ is given by the series* (6.4)*, with Φ_n, Ψ_n as in* (6.1) *and the coefficients $A_n^*, A_n^{**}, \dots, D_n^*, D_n^{**}$ given by* (6.12) *and* (6.13) *for $n \neq 0$ and $n = 0$, respectively.*

Remark. In the Green function for the unsquared $\widetilde{\varDelta}_{\mathbb{A}}$, the logarithm of the Jacobi theta function

$$\vartheta_0(v|\tau) = \text{const}\cdot\prod_{j>0}(1 - \lambda R^{-2j})(1 - R^{2-2j}/\lambda) \tag{6.14}$$

$$(R = e^{-\pi i\tau},\ \operatorname{Im}\tau > 0,\ \lambda = Re^{2\pi i v}),$$

arises from the series

$$\sum_{n\neq 0}\frac{\lambda^n}{n(R^{2n}-1)}$$

upon expanding $1/(R^{2n} - 1)$ into a geometric series and interchanging the order of summation; see [14]. From this point of view, the series entering into our Green function are of the forms:

$$\sum_{n\neq 0} \frac{\lambda^n \Phi_n(t)}{n(R^{2n}-1)}, \qquad \sum_{n\neq 0} \frac{\lambda^n \Psi_n(t)}{n(R^{2n}-1)}, \tag{6.15}$$

$$\sum_{n\neq 0} \frac{\lambda^n}{n(R^{2n}-1)^2}, \qquad \sum_{n\neq 0} \frac{\lambda^n}{n^2(R^{2n}-1)}. \tag{6.16}$$

The first two series (6.15) do not seem to relate to any of the standard transcendental functions. Regarding the series in (6.16), using a similar manipulation as for the theta functions, the first of them can be rewritten as

$$-\log \prod_{j>0} (1-\lambda R^{-2j})^j (1-R^{2-2j}/\lambda)^j,$$

which is reminiscent of the logarithm of an infinite product of theta functions, while the second one admits an expression similar to (6.14) but with the log replaced by the dilogarithm

$$\sum_{j>0} \mathrm{Li}_2(R^{-2j}\lambda) + \mathrm{Li}_2(R^{2-2j}/\lambda).$$

We remark that using formulas (6.2), (6.3) one easily sees that at least $\widetilde{\Delta}_z G(z,w) = \sum_n (C_n e^{nx} + D_n e^{-nx}) e^{ni\theta}$ comes in the form of a logarithm of a theta function, as it should. □

7. The convergence of $\sum_{\omega\in G}(1-|\omega 0|^2)^s$

We have seen that an important role in the analysis on Ω is played by the "zeta function"

$$\zeta(s) = \sum_n (1-|a_n|^2)^s, \tag{7.1}$$

where, as before, $a_n = \omega_n^{-1}(0)$ and $G = \{\omega_0, \omega_1, \omega_2, \dots\}$. We have encountered $\zeta(\frac{1}{2})$ in (3.23) and $\zeta'(\frac{1}{2})$ in (3.9), and $\zeta'(1)$ and $\zeta(1)$ in (1.16) and (1.17).

For a general Riemann surface of hyperbolic type, (7.1) is known to converge for $\operatorname{Re} s > 1$ ([32, Theorem XI.8]; for $\operatorname{Re} s \geq 2$ see also [23, Lemma III.5.2]). For bounded plane domains this can be pushed to $\operatorname{Re} s \geq 1$, the points a_n are precisely the zeros of the bounded analytic function $\phi - \phi(0)$ on $\mathbb{D}$, and thus satisfy the Blaschke condition $\sum(1-|a_n|) < \infty$. By an easy application of the Riemann mapping theorem, this even remains in force for any domain whose complement contains a continuum; in fact, it suffices that the complement have positive capacity ([32, Theorem XI.13 and III.35]). An ingenious argument of Dalzell [9] shows that if $\zeta(1)$ is finite, then so are all its derivatives $\zeta^{(k)}(1)$, $k = 1, 2, \dots$. For the annulus, using the explicit expression for a_n mentioned at the end of Section 3, one easily shows that (7.1) converges for all $\operatorname{Re} s > 0$ and extends by analytic continuation to a meromorphic function in the whole complex plane, with simple poles at $s = -2m + \frac{\pi i}{\alpha} n$ ($m \geq 0$, $m, n \in \mathbb{Z}$). In this section, we present two results on the convergence of the series (7.1) for $0 < s < 1$ and domains of connectivity more than two.

It is clear that there exists $s_0 = s_0(\Omega) \geq 0$ such that (7.1) converges for $\operatorname{Re} s > s_0$ and diverges for $\operatorname{Re} s < s_0$; this s_0 is called the *exponent of convergence.* Obviously, condition (1.16) is fulfilled if $s_0 < 1$, and (3.9) holds if $s_0 < 1/2$. As mentioned in the last paragraph, $s_0 \leq 1$ for bounded plane domains, and $s_0 = 0$ for the annulus. Here is what can happen for plane domains of connectivity bigger than two:

Theorem 7.1. *(a) For any $k \geq 2$ and $\delta > 0$, there exists a smoothly bounded domain Ω of connectivity $k+1$ such that $s_0(\Omega) \leq \delta$.*
(b) For any $k \geq 2$ and $\delta < \frac{\log\sqrt{3}}{\log(1+\sqrt{2})} = 0.6232\ldots$, there exists a smoothly bounded domain Ω of connectivity $k+1$ such that $s_0(\Omega) \geq \delta$.

For various facts on Fuchsian groups and automorphic functions mentioned below we refer, for instance, to the beautiful exposition by Hadamard [19] or the books [23, especially §2 of Chapter I], [32, Chapter XI] and [6].

Proof. Let $K_1, K_2, \ldots, K_{2k}$ be $2k$ discs in the plane such that their closures are pairwise disjoint, their boundary circles are orthogonal to $\partial\mathbb{D}$ and meet it in the order indicated (i.e. the neighbours of K_j are K_{j-1} and K_{j+1}), and, for each $j = 1, 2, \ldots, k$, K_j is congruent to K_{2k+1-j}. For each j there exists then a unique hyperbolic self-map ω_j of $\mathbb{D}$ sending K_j into K_{2k+1-j}; clearly

$$\omega_{2k+1-j} = \omega_j^{-1} \qquad (j = 1, 2, \ldots, 2k), \tag{7.2}$$

and also

$$\omega_j(\mathbb{D} \setminus \overline{K}_j) \subset K_{2k+1-j} \qquad (j = 1, 2, \ldots, 2k). \tag{7.3}$$

Let G be the group generated by $\omega_1, \ldots, \omega_{2k}$ and let $\mathcal{O}$ be the complement in $\mathbb{D}$ of $K_1 \cup \cdots \cup K_{2k}$. Then G is a Schottky group, hence, it is a free group generated by the elements $\omega_1, \ldots, \omega_k$, all its elements are hyperbolic transformations, and $\mathcal{O}$ is a fundamental domain for G. The Riemann surface $\mathbb{D}/G$ obtained upon gluing together the sides of $\mathcal{O}$ by the maps $\omega_1, \ldots, \omega_k$ is clearly homeomorphic to a disc with k holes, hence, is of planar character (schlichtartig). Thus by Koebe's Fundamental Theorem ([32, Theorem IX.32]), it is biholomorphic to a planar domain Ω. As the closures of the K_j are disjoint, the fundamental domain $\mathcal{O}$ has no cusps, hence $\partial\Omega$ has no isolated points. Thus applying another conformal mapping if necessary, we can achieve that Ω is smoothly bounded (even by $k+1$ circles, for instance). Since G contains no elliptic elements, the canonical projection $\phi : \mathbb{D} \to \Omega \cong \mathbb{D}/G$ is a uniformization map for Ω. To summarize, we see that there is a smoothly bounded, $(k+1)$-connected plane domain Ω such that G is the covering group of the uniformization map $\phi : \mathbb{D} \to \Omega$.

Let us term a sequence $\boldsymbol{j} = j_1 \ldots j_m$ of numbers $1, 2, \ldots, 2k$ a *word* if $j_{i+1} \neq 2k+1-j_i$, $\forall i = 1, 2, \ldots, m-1$. The number m is the *length* of the word, denoted by $|\boldsymbol{j}|$, and we denote by W the set of all words. As G is the free group generated by $\omega_1, \ldots, \omega_k$, we have

$$G = \{\omega_{\boldsymbol{j}} \,|\, \boldsymbol{j} \in W\},$$

where $\omega_{\boldsymbol{j}} := \omega_{j_1} \dots \omega_{j_m}$ (for $\boldsymbol{j}$ the empty word, we define $\omega_{\boldsymbol{j}}$ to be the identity). Let us now estimate $1 - |\omega_{\boldsymbol{j}} 0|^2$. For each j the arc $\partial K_j \cap \mathbb{D}$ is the locus of all points $x \in \mathbb{D}$ satisfying $d(x, 0) = d(\omega_j x, 0)$ (or $d(x, 0) = d(x, \omega_j^{-1} 0)$, because $d(x, y) = d(\omega x, \omega y)$ for any holomorphic self-map ω of the disc), where $d(x, y) = \left|\frac{x-y}{1-\overline{y}x}\right|$ is the hyperbolic distance. Denoting $\omega_j^{-1} 0 = a_j$, this condition becomes $|x| = \left|\frac{x-a_j}{1-\overline{a}_j x}\right|$, or

$$1 - |x|^2 = 1 - \left|\frac{x - a_j}{1 - \overline{a}_j x}\right|^2 = \frac{(1 - |x|^2)(1 - |a_j|^2)}{|1 - \overline{a}_j x|^2},$$

that is,

$$\left|\frac{1}{\overline{a}_j} - x\right|^2 = \frac{1}{|a_j|^2} - 1.$$

Thus K_j has center $1/\overline{a}_j$ and radius $R_j = \sqrt{|a_j|^{-2} - 1}$. Now let $\xi = \omega_{j_2} \dots \omega_{j_m} 0$, with $m \geq 2$. From (7.3) it follows by induction that $\xi \in K_{2k+1-j_2}$. Also as $\boldsymbol{j}$ is a word, $2k + 1 - j_2 \neq j_1$. Thus

$$\left|\frac{1}{\overline{a}_{j_1}} - \xi\right| \geq R_{j_1} + d_{j_1, 2k+1-j_2},$$

where $d_{i,j} = \left|\frac{1}{\overline{a}_i} - \frac{1}{\overline{a}_j}\right| - R_i - R_j > 0$ is the distance between the discs K_i and K_j $(i \neq j)$, and

$$\left|\frac{1}{\overline{a}_{j_1}} - \xi\right| \leq D_{j_1, 2k+1-j_2},$$

where $D_{i,j} := \sup_{x \in K_j \cap \mathbb{D}} \left|\frac{1}{\overline{a}_i} - x\right|$ $(i \neq j)$. Hence the ratio

$$\begin{aligned}
\frac{1 - |\omega_{j_1 j_2 \dots j_m} 0|^2}{1 - |\omega_{j_2 \dots j_m} 0|^2} &= \frac{1 - |\omega_{j_1} \xi|^2}{1 - |\xi|^2} \\
&= \frac{(1 - |a_{j_1}|^2)(1 - |\xi|^2)}{|1 - \overline{a}_{j_1} \xi|^2 (1 - |\xi|^2)} \\
&= \frac{R_{j_1}^2}{\left|\frac{1}{\overline{a}_{j_1}} - \xi\right|^2}
\end{aligned}$$

satisfies

$$\left(\frac{R_{j_1}}{D_{j_1, 2k+1-j_2}}\right)^2 \leq \frac{1 - |\omega_{j_1 j_2 \dots j_m} 0|^2}{1 - |\omega_{j_2 \dots j_m} 0|^2} \leq \left(\frac{R_{j_1}}{R_{j_1} + d_{j_1, 2k+1-j_2}}\right)^2.$$

Setting $r = \min_j R_j$, $R = \max_j R_j$, $d = \min_{i \neq j} d_{i,j}$, $D = \max_{i \neq j} D_{i,j}$ and iterating the last formula, we thus obtain

$$\left(\frac{r}{D}\right)^{2(m-1)} \leq \frac{1 - |\omega_{j_1 j_2 \dots j_m} 0|^2}{1 - |\omega_{j_m} 0|^2} \leq \left(\frac{R}{r + d}\right)^{2(m-1)},$$

whence

$$C_1\Big(\frac{r}{D}\Big)^{2|\boldsymbol{j}|} \le 1 - |\omega_{\boldsymbol{j}}0|^2 \le C_2\Big(\frac{R}{r+d}\Big)^{2|\boldsymbol{j}|},$$

for some constants C_1, C_2 independent of $\boldsymbol{j}$. The number of words of a given length m is 1 if $m = 0$ and $2k(2k-1)^{m-1}$ if $m \ge 1$. Thus

$$C_1 \sum_{m=0}^{\infty} (2k-1)^m \Big(\frac{r}{D}\Big)^{2ms} \le \sum_{\boldsymbol{j}} (1 - |\omega_{\boldsymbol{j}}0|^2)^s \le C_2 \sum_{m=0}^{\infty} (2k)^m \Big(\frac{R}{r+d}\Big)^{2ms},$$

and we see that

$$\textit{if} \quad 2k\Big(\frac{R}{r+d}\Big)^{2s} < 1, \quad \textit{then } (7.1) \textit{ converges,} \quad \text{and}$$

$$\textit{if} \quad (2k-1)\Big(\frac{r}{D}\Big)^{2s} \ge 1, \quad \textit{then } (7.1) \textit{ diverges.} \tag{7.4}$$

Consequently, to prove the theorem it is enough to find $2k$ circles $K_1, \dots, K_{2k}$ with the above properties such that the corresponding condition in (7.4) is satisfied (the numbers r, R, d, D there depend only on the sizes of the discs K_j and on their mutual position). We now do this separately for both cases (a) and (b).

(a) Fix any $2k$ distinct points $\epsilon_1, \dots, \epsilon_{2k}$ on $\partial\mathbb{D}$ and let K_j be the disc of radius ρ and center $\epsilon_j\sqrt{1+\rho^2}$, where $\rho > 0$ is very small. Then $R = r = \rho$ and $d = \min_{i \ne j} \sqrt{1+\rho^2}|\epsilon_i - \epsilon_j| - 2\rho$. Thus as $\rho \to 0$, the number $\frac{R}{r+d}$ may be made arbitrarily small — in particular, smaller than $(2k)^{-1/2s}$ for any given $s > 0$ and $k \ge 1$. Then the first condition in (7.4) is fulfilled, and thus $\zeta(s) < \infty$.

(b) First let $k = 2$ and consider the discs K_1, K_2, K_3, K_4 from part (a) with the choice $\epsilon_1 = 1, \epsilon_2 = i, \epsilon_3 = -1, \epsilon_4 = -i$, and $0 < \rho < 1$. Then $r = R = \rho$ and $D = 1 + \sqrt{1+\rho^2}$, so as $\rho \to 1$, $\frac{r}{D} \to \frac{1}{1+\sqrt{2}}$. Consequently, if

$$\frac{3}{(1+\sqrt{2})^{2s}} > 1, \qquad \text{or} \qquad s < \frac{\log\sqrt{3}}{\log(1+\sqrt{2})},$$

then the second condition in (7.4) will be satisfied if ρ is sufficiently close to 1. This proves the assertion for $k = 2$.

For $k > 2$, consider the discs that solve the problem for $k = 2$, but relabel them as $K_1, K_2, K_{2k-1}, K_{2k}$ and add to them arbitrary $2k - 4$ discs $K_3, \dots, K_{2k-2}$ so that the required properties are not violated (i.e. $K_1, \dots, K_{2k}$ have pairwise disjoint closures, are orthogonal to $\partial\mathbb{D}$ and $K_j \cong K_{2k+1-j}$). The group G corresponding to the resulting fundamental domain $\mathcal{O}$ will clearly be larger than the one corresponding to $\mathbb{D} \setminus (K_1 \cup K_2 \cup K_3 \cup K_4)$ (the former is generated by $\omega_1, \dots, \omega_k$, the latter by ω_1 and ω_2 only). Hence, $\sum_{\omega \in G}(1 - |\omega 0|^2)^s$ will always be larger for the former domain than for the latter, and, in particular, will diverge for the former whenever it diverges for the latter. Thus the desired result for $k > 2$ follows from the one for $k = 2$. $\square$

Remark. The argument in the last paragraph shows, in particular, that for any $(k+1)$-connected smoothly bounded plane domain Ω and $k' > k$ there exists a $(k'+1)$-connected smoothly bounded domain Ω' such that $\zeta_{\Omega'}(s) \geq \zeta_\Omega(s)$ $\forall s$ and $s_0(\Omega') \geq s_0(\Omega)$. It follows that the sequence

$$\sigma_k := \sup\{s_0(\Omega) : \ \Omega \text{ is smoothly bounded of connectivity } k+1\}$$

is non-decreasing. For any finitely generated free group G without elliptic or parabolic elements (all covering groups of the uniformization maps of smoothly bounded plane domains are of this type), it follows from a theorem of Maskit [24] and a result of Doyle [10] that $s_0(G) \leq c < 1$ for some universal constant c (not depending on G). Thus, in particular, $\sup_k \sigma_k =: \sigma_\infty$ is strictly smaller than 1. The numerical values of $\sigma_2, \sigma_3, \ldots, \sigma_\infty$ or c seem to be unknown.

For plane domains Ω whose boundary contains an isolated point, G contains a parabolic element, and hence $s_0(\Omega) \geq 1/2$ (with equality only for Ω biholomorphic to the punctured disc $\mathbb{D} \setminus \{0\}$).

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