Annali di Matematica 181, 453–462 (2002)

Digital Object Identifier (DOI) 10.1007/s10231-002-0050-4

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# The existence of embeddings for which the Gauss map is an embedding

Received: March 11, 2001
Published online: October 30, 2002 – © Springer-Verlag 2002

**Mathematics Subject Classification (2000).** 14N05

## Introduction

**0.1.** Let $X \subset \mathbb{P}^N$ be a smooth $n$-dimensional projective variety. For $x \in X$ we have an embedded tangent space $T_x(X) \subset \mathbb{P}^N$ (see 0.3). It is an $n$-space in $\mathbb{P}^N$, hence it corresponds to a point $\gamma(x)$ in the Grassmannian $\mathbb{G}(n, N)$. We obtain the Gauss map

$$\gamma : X \to \mathbb{G}(n, N)$$

(see [2, Lecture 15]). In the case that the ground field has characteristic 0, this Gauss map is finite and birational on its image (see [4, I, Corollary 2.8]). Although a lot of work has been done concerning the inseparability of Gauss maps in positive characteristic (see [3]), not much work has been done concerning better properties for the Gauss map. In this paper we look for embeddings such that the Gauss map is an embedding too. In particular, we prove that each $n$-dimensional smooth projective variety $X$ has an embedding in $\mathbb{P}^{2n+1}$ such that the Gauss map is an embedding.

**0.2.** We start by fixing some notations, making some conventions and giving a more precise description of our main result.

In this paper we assume the ground field is the field $\mathbb{C}$ of the complex numbers (due to the Lefschetz principle the results hold for any algebraically closed ground field of characteristic 0). Nevertheless this restriction on the characteristic is not used in the injectivity part of the paper (Section 1).

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A variety is always assumed to be irreducible. If one considers a variety as a scheme then in this paper a point means a closed point. Writing $X$ we always mean a smooth $n$-dimensional projective variety.

**0.3.** Let $\mathcal{L}$ be an invertible sheaf on $X$ and let $V$ be an $(N+1)$-dimensional subvectorspace of $\Gamma(X, \mathcal{L})$.

For a non-zero element $s \in V$ we denote the associated divisor on $X$ by $D_s$. We obtain a linear system $\{D_s : s \in V\}$, which is denoted by $\mathbb{P}(V)$. In the case $V = \Gamma(X, \mathcal{L})$, we write $\mathbb{P}(\mathcal{L})$ (this convention applies to more notations and will not be mentioned explicitly any more).

For $x \in X$, let $I_x(V) = \{s \in V : s = 0 \text{ or } x \in D_s\}$ and $S_x(V) = \{s \in V : s = 0$ or $D_s$ is singular at $x\}$.

We know that $I_x(V)$ is a codimension 1 subvectorspace of $V$ unless $x$ is a fixed point of $\mathbb{P}(V)$. Assume $\dim(\mathbb{P}(V)) \geq 1$, then we obtain a rational map

$$i_V : X \dashrightarrow \mathbb{P}(V^*) = \mathbb{P}^N,$$

with $i_V(x) = \mathbb{P}((V/I_x(V))^*)$. The domain of $i_V$ is the complement of the set of fixed points of $\mathbb{P}(V)$. We say $i_V$ is a *local immersion* at $x$ if $S_x(V)$ has codimension $n+1$ in $V$ (this codimension cannot be more; in this case $i_V$ is defined at $x$ and the tangent map $d_x(i_V) : T_x(X) \to T_x(\mathbb{P}^N)$ is injective). If $i_V$ is a local immersion at $x$ then $\mathbb{P}((V/S_x(V))^*)$ is an $n$-plane in $\mathbb{P}(V^*)$; it is *the embedded tangent space* $T_x(i_V(X))$ of $X$ at $x$. In that case it defines an element $\gamma_V(x) \in \mathbb{G}(n; N)$. Let $U \subset X$ be the set of points $x$ such that $i_V$ is a local immersion at $x$. It is an open subset of $X$, in the case that $U$ is non-empty we obtain a rational map with domain $U$, called *the Gauss map* of $\mathbb{P}(V)$:

$$\gamma_V : X \dashrightarrow \mathbb{G}(n, N).$$

If $\mathbb{P}(V)$ is very ample then $i_V$ is an embedding of $X$ in $\mathbb{P}^N$ and $\gamma_V$ has domain $X$.

**0.4.** Let $\mathcal{L}$ be a very ample invertible sheaf on $X$ and consider the embedding $X \subset \mathbb{P}(\mathcal{L})$. Assume $\dim(\mathbb{P}(\mathcal{L})) \geq 2n+1$ and let $V \subset \Gamma(X, \mathcal{L})$ be a general linear subspace of dimension $k \geq 2n+2$, then $i_V$ is an embedding. Indeed, $i_V$ is obtained from $X \subset \mathbb{P}(\mathcal{L})$ using a projection with a general center $\Lambda = \mathbb{P}((\Gamma(X, \mathcal{L})/V)^*) \subset \mathbb{P}(\mathcal{L})$. Since the secant variety $S(X)$ has dimension at most $2n+1$, the claim follows. Similarly, since the tangential variety $T(X)$ has dimension at most $2n$, in the case that $V$ is a general linear subspace of dimension $k = 2n+1$, then $i_V$ is a local immersion and hence $\gamma_V$ is globally defined on $X$. We are now able to state the main result of this paper.

**Theorem I.** *Let $X$ be a smooth $n$-dimensional projective variety, let $\mathcal{N}$ be a very ample invertible sheaf on $X$ and let $\mathcal{L} = \mathcal{N}^{\otimes t}$ for some $t \geq 2$. Let $V$ be a general $k$-dimensional subvectorspace of* $\Gamma(X, \mathcal{L})$ *with* $k \geq 2n+1$. *Then the Gauss map $\gamma_V$ is an embedding.*

The proof of this theorem is divided into two parts. In the first part we investigate the injectivity of $\gamma_V$; in the second part we prove $\gamma_V$ is a local immersion at each point $x \in X$. In both parts, we make general considerations on the behavior of the Gauss map under projections and we apply them to the situation of the theorem.

## 1. Injectivity of the Gauss map

**1.1.** We start by stating some lemmas concerning projections and Gauss maps, using the following notations.

If $i : X \hookrightarrow \mathbb{P}^N$, we define, for $l = 1, 2, 3$, $Z_{n-l,i} \subset X \times X$ as $\{(q, q') : \dim(T_{i(q)}(i(X)) \cap T_{i(q')}(i(X))) = n - l\}$. Let $s$ be a general point of $\mathbb{P}^N$, then $i_s$ denotes the map (possibly embedding) $p_s \circ i : X \to \mathbb{P}^{N-1}$, where $p_s$ denotes the projection of $\mathbb{P}^N$ with center $s$. If $q$ is a point in $X$, to simplify notations, we write $q$ instead of $i(q)$ (or $i_s(q)$).

The notation genfib$(f)$, is short for "general fiber of the map $f$".

**1.2. Lemma.** *If $i : X \hookrightarrow \mathbb{P}^N$, with $N > 2n$, $\gamma_i$ injective and* $\dim(Z_{n-1,i}) < n$, *then the Gauss map $\gamma_{i_s}$ is injective.*

*Proof.* Assume that $\gamma_{i_s}$ is not injective, i.e. there exist two (distinct) points $q$ and $q' \in X$ for which $T_q(i_s(X)) = T_{q'}(i_s(X))$. We know this is only possible if $\langle T_q(i(X)), s\rangle = \langle T_{q'}(i(X)), s\rangle$. We also assume that $\gamma_i$ is injective, so $T_q(i(X)) \neq T_{q'}(i(X))$. This implies that $\dim(\langle T_q(i(X)) \cup T_{q'}(i(X))\rangle) = \dim(\langle T_q(i(X)), s\rangle) = n + 1$ and $s \in \langle T_q(i(X)) \cup T_{q'}(i(X))\rangle$. So $\dim(T_q(i(X)) \cap T_{q'}(i(X))) = 2\dim(T_q(i(X))) - \dim(\langle T_q(i(X)) \cup T_{q'}(i(X))\rangle) = 2n - (n + 1) = n - 1$, i.e. $(q, q') \in Z_{n-1,i}$.

Define $I \subset Z_{n-1,i} \times \mathbb{P}^N$, by $I := \{((r, r'), t) : t \in \langle T_r(i(X)) \cup T_{r'}(i(X))\rangle\}$. Let $p_1 : I \to Z_{n-1,i}$ (resp. $p_2 : I \to \mathbb{P}^N$) be the first (resp. second) projection map. Obviously $p_1$ is dominant, and $\dim(\text{genfib}(p_1)) = n + 1$. Then $((q, q'), s) \in I$; and, because $s$ is a general point of $\mathbb{P}^N$, we get that $p_2$ is dominant, i.e. $\dim(\text{Im}(p_2)) = N$. On the other hand, $\dim(\text{Im}(p_2)) \leq \dim(I)$, and $\dim(I) = \dim(Z_{n-1,i}) + \dim(\text{genfib}(p_1)) < n + (n + 1) = 2n + 1 \leq N$. So $\dim(\text{Im}(p_2)) < N$ and we get a contradiction. □

**1.3. Lemma.** *If $i : X \hookrightarrow \mathbb{P}^N$, with $N > 2n + 1$, $\gamma_i$ injective,* $\dim(Z_{n-2,i}) < 2n$ *and* $\dim(Z_{n-1,i}) < n$, *then* $\dim(Z_{n-1,i_s}) < n$.

*Proof.* Assume that $\dim(Z_{n-1,i_s}) \geq n$. Then, if $(q, q')$ is a general element of $Z_{n-1,i_s}$, $(q, q') \notin Z_{n-1,i}$. This implies that $\dim(T_q(i_s(X)) \cap T_{q'}(i_s(X))) = n - 1$ and $\dim(T_q(i(X)) \cap T_{q'}(i(X))) \neq n - 1$. Moreover, because $\gamma_i$ is injective, $\dim(T_q(i(X)) \cap T_{q'}(i(X))) \leq n - 1$. And, we know that by projecting from a point, the dimension of the intersection of two tangent spaces remains the same, unless the center of the projection belongs to the span of the two tangent spaces, in this case, the dimension of the intersection increases by one. So $\dim(T_q(i_s(X)) \cap T_{q'}(i_s(X))) \leq \dim(T_q(i(X)) \cap T_{q'}(i(X))) + 1$. Thus, $\dim(T_q(i(X)) \cap T_{q'}(i(X))) = n - 2$, i.e. $(q, q') \in Z_{n-2,i}$. And $s \in \langle T_q(i(X)) \cup T_{q'}(i(X))\rangle$. Let us now define $J \subset Z_{n-2,i} \times \mathbb{P}^N$ by $J := \{((r, r'), t) : t \in \langle T_r(i(X)) \cup T_{r'}(i(X))\rangle\}$. The first (resp. second) projection map we denote by $q_1$ (resp. $q_2$). Then $((q, q'), s) \in J$, and thus $\dim(\text{Im}(q_2)) = N$ (because $s$ is a general point of $\mathbb{P}^N$). Obviously $\dim(\text{Im}(q_1)) = \dim(Z_{n-2,i})$, and $\dim(\text{genfib}(q_1)) = n + 2$. And $\dim(J) = \dim(Z_{n-2,i}) + \dim(\text{genfib}(q_1)) = N + \dim(\text{genfib}(q_2))$. Because $Z_{n-1,i_s} \cap Z_{n-1,i}$ is a closed subset of $Z_{n-1,i_s}$, we have that $\dim(Z_{n-1,i_s}) = \dim(Z_{n-1,i_s} \setminus (Z_{n-1,i_s} \cap Z_{n-1,i}))$. We also showed that $Z_{n-1,i_s} \setminus (Z_{n-1,i_s} \cap Z_{n-1,i}) \subset q_2^{-1}(s)$. But, $s$ is a general point of $\mathbb{P}^N$, so $\dim(\text{genfib}(q_2)) = \dim(q_2^{-1}(s))$. Thus, $\dim(Z_{n-1,i_s}) \leq \dim(\text{genfib}(q_2)) = \dim(Z_{n-2,i}) - N + n + 2 < 2n - (2n+2) + n + 2 = n$. So $\dim(Z_{n-1,i_s}) < n$, which gives a contradiction with the assumption $\dim(Z_{n-1,i_s}) \geq n$, and this proves our claim. □

**1.4. Lemma.** *If* $i : X \hookrightarrow \mathbb{P}^N$, *with* $N > 2n+2$, $\gamma_i$ *injective and* $\dim(Z_{n-2,i}) < 2n$, *then* $\dim(Z_{n-2,i_s}) < 2n$.

*Proof.* Obviously, $\dim(Z_{n-2,i_s}) \leq 2n$, because $Z_{n-2,i_s} \subset X \times X$. So assume $\dim(Z_{n-2,i_s}) = 2n$. If $(q, q')$ is a general element of $Z_{n-2,i_s}$, then $(q, q') \notin Z_{n-2,i}$. This implies $\dim(T_q(i_s(X)) \cap T_{q'}(i_s(X))) = n-2$ and $\dim(T_q(i(X)) \cap T_{q'}(i(X))) \neq n-2$. Moreover, we know that by projecting from a point, the dimension of the intersection of two tangent spaces remains the same, unless the center of the projection belongs to the span of the two tangent spaces, in this case, the dimension of the intersection increases by one. So, either $\dim(T_q(i_s(X)) \cap T_{q'}(i_s(X))) = \dim(T_q(i(X)) \cap T_{q'}(i(X)))$; or $\dim(T_q(i_s(X)) \cap T_{q'}(i_s(X))) = \dim(T_q(i(X)) \cap T_{q'}(i(X))) + 1$. This implies that $\dim(T_q(i(X)) \cap T_{q'}(i(X))) = n-3$, i.e. $(q, q') \in Z_{n-3,i}$; and $s \in \langle T_q(i(X)) \cup T_{q'}(i(X)) \rangle$. Let us now define $L \subset Z_{n-3,i} \times \mathbb{P}^N$ by $L := \{((r, r'), p) : p \in \langle T_r(Y_i) \cup T_{r'}(Y_i) \rangle\}$. Denote the first (resp. second) projection map by $r_1$ (resp. $r_2$). Then $((q, q'), s) \in L$, and thus $\dim(\text{Im}(r_2)) = N$ (because $s$ is a general point of $\mathbb{P}^N$). Obviously $\dim(\text{Im}(r_1)) = \dim(Z_{n-3,i})$, and $\dim(\text{genfib}(r_1)) = n+3$. Thus $\dim(L) = \dim(Z_{n-3,i}) + \dim(\text{genfib}(r_1)) = N + \dim(\text{genfib}(r_2))$. Because $Z_{n-2,i_s} \cap Z_{n-2,i}$ is a closed subset of $Z_{n-2,i_s}$, we have that $\dim(Z_{n-2,i_s}) = \dim(Z_{n-2,i_s} \setminus (Z_{n-2,i_s} \cap Z_{n-2,i}))$. We also showed that $Z_{n-2,i_s} \setminus (Z_{n-2,i_s} \cap Z_{n-2,i}) \subset r_2^{-1}(s)$. But, $s$ is a general point of $\mathbb{P}^N$, so $\dim(\text{genfib}(r_2)) = \dim(r_2^{-1}(s))$. Thus, $\dim(Z_{n-2,i_s}) \leq \dim(\text{genfib}(r_2)) = \dim(Z_{n-3,i}) - N + n + 3 < 2n - (2n+2) + n + 3 = n + 1 < 2n$. So $\dim(Z_{n-2,i_s}) < 2n$, which gives a contradiction with the assumption $\dim(Z_{n-2,i_s}) = 2n$, and this proves our claim. □

**1.5.** Now, let $X$ and $\mathcal{L}$ be as in the statement of Theorem I, and let $i = i_{\mathcal{L}} : X \to \mathbb{P}^N$. In view of Lemmas 1.2, 1.3 and 1.4, in order to prove that $\gamma_V$ is injective for a general $k$-dimensional linear subvectorspace $V$ of $\Gamma(X, \mathcal{L})$, it is sufficient to show that $\dim(Z_{n-1,i}) < n$ and $\dim(Z_{n-2,i}) < 2n$.

**1.6.** In the case $t \geq 3$, we prove the following much stronger fact:

**Lemma.** *For all distinct* $q$ *and* $q' \in X$, $\dim(T_q(i(X)) \cap T_{q'}(i(X))) = -1$.

*Proof.* Let $q$ and $q'$ be two distinct points of $X$. Because the sheaf $\mathcal{N}$ is very ample on $X$, $\text{codim}_{\Gamma(X,\mathcal{N})}(S_q(\mathcal{N})) = n+1$. So we can take a subvectorspace $V \subset \Gamma(X, \mathcal{N})$, with $\dim(V) = n+1$ and $V \cap S_q(\mathcal{N}) = \{0\}$. As $\mathcal{N}$ is very ample, we can take two divisors $D_1$ and $D_2 \in \mathbb{P}(\mathcal{N})$, with $q \notin D_i$ and $q' \in D_i$ $(i = 1, 2)$. Let $D_3$ be an element of $\mathbb{P}(V)$, then $D_3$ is not singular at $q$ (because $V \cap S_q(\mathcal{N}) = \{0\}$). So $D := D_1 + D_2 + D_3$ is singular at $q'$ and not singular at $q$. Let $V' \subset \Gamma(X, \mathcal{M})$ be such that $\mathbb{P}(V') = D_1 + D_2 + \mathbb{P}(V)$, then $\dim(V') = \dim(V) =$

$n+1$, $V' \subset S_{q'}(\mathcal{M})$ and $V' \cap S_q(\mathcal{M}) = \{0\}$. And thus $(S_q(\mathcal{M}) \cap S_{q'}(\mathcal{M})) \cap V' = \{0\}$. Looking at $S_q(\mathcal{M}) \cap S_{q'}(\mathcal{M})$ and $V'$ as subsets of $S_{q'}(\mathcal{M})$, we obtain $\operatorname{codim}_{S_{q'}(\mathcal{M})}(S_q(\mathcal{M}) \cap S_{q'}(\mathcal{M})) \geq n+1$. On the other hand, because $\mathcal{M}$ is very ample, $\operatorname{codim}_{\Gamma(X,\mathcal{M})}(S_q(\mathcal{M})) = n+1$. And thus $\operatorname{codim}_{S_{q'}(\mathcal{M})}(S_q(\mathcal{M}) \cap S_{q'}(\mathcal{M})) \leq n+1$. So we get that $\operatorname{codim}_{S_{q'}(\mathcal{M})}(S_q(\mathcal{M}) \cap S_{q'}(\mathcal{M})) = n+1$. This implies that $\dim(S_q(\mathcal{M}) \cap S_{q'}(\mathcal{M})) = \dim(S_{q'}(\mathcal{M})) - (n+1) = M+1-2(n+1)$, with $M := \dim(\mathbb{P}(\mathcal{M}))$.

If $H$ is a hyperplane in $\mathbb{P}^M$, then this corresponds to a divisor $D_H \in \mathbb{P}(\mathcal{M})$; and vice versa, if $D$ is a divisor of $\mathbb{P}(\mathcal{M})$ this corresponds to a hyperplane $[D] \subset \mathbb{P}^M$. We then have the following property: $D \in \mathbb{P}(S_q(\mathcal{M}) \cap S_{q'}(\mathcal{M}))$ if and only if $\langle T_q(i(X)) \cup T_{q'}(i(X))\rangle \subset [D]$. This means that $\dim(\mathbb{P}(S_q(\mathcal{M}) \cap S_{q'}(\mathcal{M}))) = M - \dim(\langle T_q(i(X)) \cup T_{q'}(i(X))\rangle) - 1$. Thus, $\dim(T_q(i(X)) \cap T_{q'}(i(X))) = \dim(\mathbb{P}(S_q(\mathcal{M}) \cap S_{q'}(\mathcal{M}))) - M + 2n + 1 = M - 2n - 2 - M + 2n + 1 = -1$. And this is exactly what we wanted to prove. □

**1.7.** In the case that $t = 2$ consider the 2-Veronese map $\nu : \mathbb{P}^M \hookrightarrow \mathbb{P}^{m(M+3)/2}$. If $q$ is a point in $\mathbb{P}^M$, to simplify notations, we write $q$ instead of $\nu(q)$.

**1.8. Lemma.** *For $q, q'$ distinct points on $\mathbb{P}^M$, one has* $\dim(T_q(\nu(\mathbb{P}^M)) \cap T_{q'}(\nu(\mathbb{P}^M))) = 0$.

*Proof.* In this proof let $\mathcal{M} = \mathcal{O}_{\mathbb{P}^M}(2)$. Since $\mathbb{P}(S_q(\mathcal{M}))$ is the linear system of quadrics in $\mathbb{P}^M$ singular at $q$, one has that $\mathbb{P}(S_q(\mathcal{M}) \cap S_{q'}(\mathcal{M}))$ is the linear system of quadrics in $\mathbb{P}^M$ singular along the line $\langle q, q'\rangle$. Such a quadric is a cone with vertex $\langle q, q'\rangle$ on a quadric in some $\mathbb{P}^{M-2}$. Comparing $\dim(\mathbb{P}(\mathcal{M}))$ with $\dim(\mathcal{O}_{\mathbb{P}^{M-2}}(2))$ the lemma follows. □

**1.9.** For $q, q'$ distinct points of $\mathbb{P}^M$, the tangent spaces $T_q(\nu(\mathbb{P}^M))$ and $T_{q'}(\nu(\mathbb{P}^M))$ intersect at one point. So for $X$ and $\mathcal{L}$ as in Theorem I, with $t = 2$, we find that $\dim(T_q(i_{\mathcal{L}}(X)) \cap T_{q'}(i_{\mathcal{L}}(X))) \leq 0$. This proves that $\dim(Z_{n-1,i_{\mathcal{L}}}) < n$ for all $n \geq 2$ and $\dim(Z_{n-2,i_{\mathcal{L}}}) < 2n$ for all $n \geq 3$. This implies that $\gamma_V$ is injective for $V$ as in Theorem I in the case $t = 2$ and $n > 2$.

**1.10. Lemma.** *Let $X$, $\mathcal{N}$ and $\mathcal{L}$ be as in Theorem I with $t = 2$, $n = 2$ and* $(X, \mathcal{N}) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. *Let $i = i_{\mathcal{L}} : X \hookrightarrow \mathbb{P}(\mathcal{L}^*)$, then* $\dim(Z_{n-2,i}) < 2n$.

*Proof.* For two distinct points $q, q' \in \mathbb{P}^M = \mathbb{P}(\mathcal{N}^*)$, let $L$ be the line $\langle q, q'\rangle$ in $\mathbb{P}^M$, then $\Gamma = \nu(L)$ is a conic. The tangent lines $T_q(\Gamma)$ and $T_{q'}(\Gamma)$ intersect, hence, from Lemma 1.8 it follows that

$$T_q(\nu(\mathbb{P}^M)) \cap T_{q'}(\nu(\mathbb{P}^M)) = T_q(\Gamma) \cap T_{q'}(\Gamma).$$

Now let $q, q'$ be distinct points on $X \subset \mathbb{P}^M$. One has that $T_q(\Gamma) \cap T_{q'}(\Gamma) \in T_q(\nu(X)) \cap T_{q'}(\nu(X))$ if and only if $\Gamma$ is tangent to $\nu(X)$ at both $q$ and $q'$; hence if and only if $L$ is tangent to $X$ at both $q$ and $q'$.

Now $Z_{0,i} = X \times X$ would imply $Z_0 = X \times X$ for the embedding $\nu(X) \subset \mathbb{P}^{M(M+3)/2}$, which is a projection of $i : X \hookrightarrow \mathbb{P}(\mathcal{L}^*)$. So $Z_{0,i} = X \times X$ would imply that for all $q, q' \in X$, the line $\langle q, q'\rangle$ is tangent to $X$ at $q$. Since $X$ is non-degenerate in $\mathbb{P}^M$ this would imply $T_q(X) = \mathbb{P}^M$. Because $X$ is smooth this would mean that $X = \mathbb{P}^M$. As $\dim(X) = 2$, we find that, in this case, $(X, \mathcal{N}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. □

**1.11.** Lemma 1.10 implies the injectivity of $\gamma_V$ in Theorem I for the case $\dim(X) = 2$ and $t = 2$ but $(X, \mathcal{N}) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. For this final case where $(X, \mathcal{N}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, we only need to use Lemma 1.2.

## 2. The Gauss map as a local immersion

**2.1.** First we mention some generalities on the tangent map of a Gauss map.

**2.1.1.** Assume $X \subset \mathbb{P}^M$ is a smooth $n$-dimensional projective variety and let $\gamma : X \to \mathbb{G}(n, M)$ be the corresponding Gauss map. For $x \in X$, we have a tangent map

$$d_x\gamma : T_x(X) \to T_{\gamma(x)}(\mathbb{G}(n, M)).$$

Let $N_X$ be the normal bundle of $X$ in $\mathbb{P}^M$ and, for $x \in X$, let $N_X(x)$ be its fiber at $x$. There is a well-known identification $T_{\gamma(x)}(\mathbb{G}(n, M)) \cong \mathrm{Hom}_{\mathbb{C}}(T_x(X), N_X(x))$ (see e.g. [2, Lecture 16]), so we obtain a bilinear form

$$d_x\gamma : T_x(X) \otimes T_x(X) \to N_X(x).$$

This is the so-called second fundamental form $\mathrm{II}_{X,x}$ of $X$ at $x$ (see [1] for the definition and in particular [1, 1.6.2] for the proof of the claim). Since $\mathrm{II}_{X,x}$ is symmetric we write it as a pairing on $T_x(X)$ with values in $N_X(x)$. If no confusion seems likely we omit some of the subscripts.

**2.1.2.** We are going to use the following description of the second fundamental form from [1, p. 369–370]:

Let $X$ and $x$ be as in 2.1.1. Choose homogeneous coordinates $(X_0 : X_1 : \ldots : X_M)$ on $\mathbb{P}^M$ such that $x = (1 : 0 : \ldots : 0)$ and $T_x(X) = Z(X_{n+1}; X_{n+2}; \ldots ; X_M)$. Let $z_1, \ldots, z_n$ be a system of regular parameters for $X$ at $x$ and let $x_1 = X_1/X_0, \ldots, x_M = X_M/X_0$ be the system of regular parameters for $\mathbb{P}^M$ at $x$. Then, for $n + 1 \leq \mu \leq M$, $x_\mu|_X$ is a regular function of $X$ at $x$ and it has the Taylor series expansion in $z_1, \ldots, z_n$:

$$x_\mu|_X = \sum_{\alpha,\beta=1}^{n} q_{\alpha,\beta,\mu} z_\alpha z_\beta + \text{(higher order terms)}.$$

We have a base $\left(\frac{\delta}{\delta z_1}\right)_x, \ldots, \left(\frac{\delta}{\delta z_n}\right)_x$ for $T_x(X)$, a base $\overline{\left(\frac{\delta}{\delta x_{n+1}}\right)_x}, \ldots, \overline{\left(\frac{\delta}{\delta x_M}\right)_x}$ for $N_X(x)$; and with respect to those bases we have

$$\mathrm{II}_{X,x}((a_1, \ldots, a_n), (b_1, \ldots, b_n))_\mu = \sum_{\alpha,\beta=1}^{n} q_{\alpha,\beta,\mu} \frac{a_\alpha b_\beta + a_\beta b_\alpha}{2},$$

for $n + 1 \leq \mu \leq M$.

**2.2.** Next we make some general remarks on the behavior of the tangent map of a Gauss map under projections.

**2.2.1.** Consider an embedding $i : X \hookrightarrow \mathbb{P}^M$ and let $\gamma : X \to \mathbb{G}(n, M)$ be the Gauss map. For $x \in X$ and $P \in \mathbb{P}^M$ let $T_x(X; P)$ be the linear span of $T_x(X)$ and $P$ in $\mathbb{P}^M$. If $P \notin T_x(X)$ then the fiber at $x$ of the normal bundle of $T_x(X)$ in $T_x(X; P)$ corresponds to a one-dimensional subvectorspace of $N_X(x)$; this is denoted by $N_X(x; P) \subset N_X(x)$.

**2.2.2.** Take $P \in \mathbb{P}^M$ and assume for all $x \in X$ one has $P \notin T_x(X)$. Projecting with center $P$ we obtain $i' : X \to \mathbb{P}^{M-1}$; it is a local immersion at each point $x \in X$. Let $\gamma' : X \to \mathbb{G}(n, M-1)$ be the Gauss map associated to $i'$. The tangent map is the second fundamental form $\mathrm{II}'_X$ associated to $i'$. Let $N'_X$ be the normal bundle of $i'$, i.e. $N'_X = i'^*(T_{\mathbb{P}^{M-1}})/T_X$. We have a projection $N_X \to N'_X$ and for all $x \in X$ one has

$$\ker(N_X(x) \to N'_X(x)) = N_X(x; P).$$

From [1, (1.2.2)] it follows that, for $v, w \in T_x(X)$, the image of $\mathrm{II}_X(v, w)$ in $N'_X(x)$ is equal to $\mathrm{II}'_X(v, w)$. Fixing $v \in T_x(X)$ we obtain:

**Lemma.** $\mathrm{rk}((d_x\gamma')(v)) < \mathrm{rk}((d_x\gamma)(v))$ *if and only if* $N_X(x; P) \subset \mathrm{Im}((d_x\gamma)(v))$.
*In this case* $\mathrm{rk}((d_x\gamma')(v)) = \mathrm{rk}((d_x\gamma)(v)) - 1$. □

**2.2.3.** For $(x; v) \in \mathbb{P}(T_X)$ (hence $x \in X$; $v$ a non-zero element of $T_x(X)$) let $N_X(x; v)$ be the union of the spaces $T_x(X; Q)$ such that $N_X(x; Q) \subset \mathrm{Im}((d_x\gamma)(v))$. For $0 \leq k \leq n$, define

$$U_k := \{(x; v) \in \mathbb{P}(T_X) : \mathrm{rk}((d_x\gamma)(v)) \leq n - k\}.$$

In the case $(x; v) \in U_k$ we have $\dim([N_X(x; v)]) \leq 2n - k$. Define $U'_k$ in a similar way, using $i'$ and $\gamma'$. From the lemma in 2.2.2 it follows that $U_k \subset U'_k \subset U_{k-1}$. Let $U''_k$ be the closure of $U'_k \setminus U_k$. Again, from the lemma in 2.2.2 it follows that $(x; v) \in U'_k \setminus U_k$ if and only if $N_X(x; P) \subset \mathrm{Im}((d_x\gamma)(v))$, hence if and only if $P \in N_X(x; v)$. Define $Z_k \subset \mathbb{P}^M \times U_{k-1}$ by $(Q; (x; v)) \in Z_k$ if and only if $Q \in N_X(x; v)$. One has $\dim(Z_k) \leq \dim(U_{k-1}) + 2n - k + 1$. Moreover $U''_k$ is the closure of the projection on $\mathbb{P}(T_X)$ of the fiber of $Z_k$ over $P$. These observations imply the following:

**Lemma.** *Assume* $P \in \mathbb{P}^M$ *is general. Then* $U''_k$ *is empty if* $M > \dim(U_{k-1}) + 2n - k + 1$. *In the case that* $M \leq \dim(U_{k-1}) + 2n - k + 1$ *then* $\dim(U''_k) \leq \dim(U_{k-1}) + 2n - k + 1 - M$. □

**2.3.** For $X$, $\mathcal{N}$ and $\mathcal{L}$ as in Theorem I we are now going to investigate the tangent map of $\gamma_V$. First we consider the case $X = \mathbb{P}^M$; $\mathcal{N} = \mathcal{O}_{\mathbb{P}^M}(1)$.

**2.3.1.** Consider the Veronese embedding

$$\nu_t : \mathbb{P}^M \hookrightarrow \mathbb{P}^N$$

with $t \geq 2$; $N = \binom{N+t}{t} - 1$. Let $X_{M,t} = \nu_t(\mathbb{P}^M)$ and denote this, in short, by $X$.

Let $x \in X_{M,t}$ and choose homogeneous coordinates in $\mathbb{P}^M$ and $\mathbb{P}^N$ such that

$$\nu_t((1:0:\ldots:0)) = x$$
$$\nu_t((x_0:x_1:\ldots:x_M)) = \left(x_0^t : x_0^{t-1}x_1 : \ldots x_0^{t-1}x_M : x_0^{t-2}x_1^2 : x_0^{t-2}x_1x_2 : \right.$$
$$\left. x_0^{t-2}x_M^2 : x_0^{t-3}x_1^3 : \ldots : x_M^t\right).$$

Let $\mathbb{C}[X_0; \ldots ; X_M]$ be the homogeneous coordinate ring of $\mathbb{P}^M$ and let $\mathbb{C}[Y_I]$ be the homogeneous coordinate ring of $\mathbb{P}^N$ with subscripts $I = (i_0; \ldots ; i_M) \in \mathbb{Z}^{M+1}$; $i_j \geq 0$ for $0 \leq j \leq M$ and $\sum_{j=0}^{M} i_j = t$, i.e. $Y_I$ corresponds to the monomial $X_0^{i_0} X_1^{i_1} \ldots X_M^{i_M}$ of degree $t$.

We denote $I$ by 0 in the case $i_0 = t$; we denote $I$ by $j$ $(1 \leq j \leq M)$ in the case $i_0 = t-1$ and $i_j = 1$. We denote $I$ by $kj$ $(1 \leq k \leq j \leq M)$ in the case $i_0 = t-2$ and $i_k = 2$ if $k = j$; and $i_j = i_k = 1$ if $k > j$. We use local coordinates $y_I = Y_I/Y_0$ for $\mathbb{P}^N$ at $x$ with $I \neq 0$. Then $y_1, \ldots , y_M$ are local coordinates for $X_{M,t} = X$ at $x$ and

$$y_{kj}|_X = y_k y_j$$
$$y_I|_X \text{ has multiplicity at least 3 in } y_1, \ldots , y_n \text{ in the case } i_0 \leq t-3.$$

Let $e_I = \dfrac{\delta}{\delta y_I}$ be the base for $T_x(\mathbb{P}^N)$, then $e_1, \ldots , e_M$ is a base for $T_x(X_{M,t}) \subset T_x(\mathbb{P}^N)$ and we consider $\{e_I : i_0 \leq t-2\}$ as a base for $N_X(x)$. Let $v = \sum_{i=1}^{M} v_i e_i$ and $w = \sum_{i=1}^{M} w_i e_i$ be elements of $T_x(X)$. From 2.1.2 we find

$$\mathrm{II}_X(v, w) = \sum_{1 \leq i \leq j \leq n} \frac{v_i w_j + v_j w_i}{2} e_{ij}.$$

**Lemma.** *For all $v \in T_x(X)$ non-zero, the linear map $(d_x\gamma)(v)$ is injective.*

*Proof.* Assume $w \in T_x(X)$ is such that $(d_x\gamma(v))(w) = 0$, hence $\mathrm{II}(v, w) = 0$. Take $1 \leq m \leq M$ such that $v_m \neq 0$ and $v_i = 0$ for all $m < i \leq M$. For $m < i \leq M$ the coefficient of $e_{mi}$ in $\mathrm{II}(v; w)$ is equal to $(v_m w_i + v_i w_m)/2 = (v_m w_i)/2$. Hence $\mathrm{II}(v; w) = 0$ implies $w_{m+1} = \cdots = w_M = 0$. The coefficient of $e_{mm}$ in $\mathrm{II}(v; w)$ is equal to $v_m w_m$, hence $\mathrm{II}(v; w) = 0$ also implies $w_m = 0$. This implies that, for $1 \leq i < m$, the coefficient of $e_{im}$ in $\mathrm{II}(v, w)$ is equal to $(v_i w_m + v_m w_i)/2 = (v_m w_i)/2$. Hence $\mathrm{II}(v; w) = 0$ also implies $w_1 = \cdots = w_{m-1} = 0$, hence $w = 0$. □

**2.3.2.** Now let $X$; $\mathcal{N}$ and $\mathcal{L}$ be as in Theorem I and consider the embedding $i_{\mathcal{N}} : X \hookrightarrow \mathbb{P}^M$, defined by $\mathcal{N}$. Let $\phi : X \to \mathbb{P}^N$ be the composition of $i_{\mathcal{N}}$ with the Veronese embedding $i_t : \mathbb{P}^M \hookrightarrow \mathbb{P}^N$. Let $\gamma : X \to \mathbb{G}(n; N)$ be the Gauss map associated to $\phi$ and let $\mathrm{II}_X$ be its associated second fundamental form.

**Lemma.** *For $v, w \in T_x(X)$, one has that $\mathrm{II}_X(v; w) = 0$ implies $v = 0$ or $w = 0$.*

*Proof.* Let $y_1, \ldots, y_N$ be as in 2.3.1. We can assume that $y_1, \ldots, y_n$ are local coordinates for $X$ at $x$ and that, for $n+1 \le i \le N$, $y_i|_X$ vanish with order at least 2 in $y_1, \ldots, y_n$. Since $\{e_1, \ldots, e_n\}$ is a base for $T_x(X)$, we can consider $\{e_I : I \not\subseteq \{1, 2, \ldots, n\}\}$ as a base for $N_X(x)$. For $v = \sum_{i=1}^n v_i e_i$, $w = \sum_{i=1}^n w_i e_i$ in $T_x(X)$, we have $\mathrm{II}_{X,x}(v; w) \in N_X(x)$. The image under the projection $N_X(x) \to N_{X_{M,t}}(x)$ is equal to $\sum_{1 \le i \le j \le M} \frac{v_i w_j + v_j w_i}{2} e_{ij} = \mathrm{II}_{X_{M,t},x}(v; w)$. This proves the following diagram is commutative for all $v \in T_x(X)$:

$$\begin{array}{ccc} T_x(X) & \xrightarrow{(d_x\gamma)(v)} & N_X(x) \\ \big\downarrow & & \big\downarrow \\ T_x(X_{M,t}) & \xrightarrow{(d_x\gamma_{X_{M,t}})(v)} & N_{X_{M,t}}(x)\,. \end{array}$$

Since $(d_x\gamma_{X_{M,t}})(v)$ is injective (see 2.3.1) we get that $(d_x\gamma)(v)$ is injective. □

**2.3.3.** Now consider the embedding $i : X \hookrightarrow \mathbb{P}(\mathcal{L}^*)$. The embedding considered in 2.3.2 can be obtained from a projection of $i$ together with some embedding $\mathbb{P}^{N'} \subset \mathbb{P}^N$ as a linear subspace. From [1, 1.22] and 2.3.2 we obtain:

**Lemma.** *For $x \in X$ and $v, w \in T_x(X)$ one has that $\mathrm{II}_{i(X),x}(v; w) = 0$ implies $v = 0$ or $w = 0$.*

*In particular, for $x \in X$ and $v \in T_x(X)$ with $v \ne 0$, one has $d_x\gamma_{\mathcal{L}}(v) : T_x(X) \to N_{i_{\mathcal{L}}(X)/\mathbb{P}(\mathcal{L}^*)}(x)$ is injective.* □

**2.4.** From now on we assume $X \subset \mathbb{P}^N$ is a smooth $n$-dimensional projective variety such that, for $x \in X$ and $v \in T_x(X)$, the linear map

$$(d_x\gamma)(v) : T_x(X) \to N_X(x)$$

is injective. Let $\Lambda$ be a general $k$-dimensional linear subspace of $\mathbb{P}^N$ for some $0 \le k \le N - 2n + 1$, let $i_\Lambda : X \to \mathbb{P}^{N-k-1}$ be the local embedding using the projection with center $\Lambda$; and let $\gamma_\Lambda : X \to \mathbb{G}(n; N-k-1)$ be the corresponding Gauss map. We are going to prove that $\gamma_\Lambda$ is an embedding. In combination with 2.3 this completes the proof of Theorem I.

We consider $\Lambda$ as the linear span of $k+1$ general points $P_1; \ldots; P_{k+1}$ in $\mathbb{P}^N$. For $0 \le j \le k+1$, let $\Lambda_j$ be the span of $P_1; \ldots; P_j$ and let $i_j$ be the embedding $X \subset \mathbb{P}^{N-j}$ obtained from the projection with center $\Lambda_j$. For $0 \le j \le k$ we obtain $i_{j+1}$ from $i_j$ using the projection with center $P_{j+1}$, considered as a general point of $\mathbb{P}^{N-j}$. Hence we can use the observations from 2.2. We have $U_{x,j} \subset \mathbb{P}(T_X)$ and $U''_{x,j}$ as in 2.2.3 for the embedding $i_j$. We know $U_{0,j} = \mathbb{P}(T_X)$, $U_{x,0} = \emptyset$ for all $x \ge 1$ and we need to prove $U_{n,k} = \emptyset$.

**2.4.1. Claim.** *Let $1 \le x \le n$:*
*if $xj < xN - 2(x+1)n + x^2 + 1$ then $U_{x,j} = \emptyset$;*
*if $xj \ge xN - 2(x+1)n + x^2 + 1$ then* $\dim(U_{x,j}) \le xj - xN + 2(x+1)n - x^2 - 1$.

*Proof.* Assume for some $j$ one has $U_{x,j} = \emptyset$ for all $x \ge 1$. From 2.2.2 we know $U_{x,j+1} = \emptyset$ for all $x > 1$ and if $U_{1,j+1} \ne \emptyset$ then $M - j \le \dim(U_{0,j}) + 2n = 4n - 1$

(see 2.2.3); hence $j \geq N-4n+1$. Since $U_{x,0}=\emptyset$ for $x \geq 1$ this implies $U_{x,j}=\emptyset$ for $x \geq 1$ in the case $j-1 < N-4n+1$, i.e. $j < N-4n+2$. This implies the first statement in the case $x=1$. For $j=N-4n+2$ we obtain $\dim(U_{1,j}) \leq 0$ (see 2.2.3). This is the second statement in the case $x=1$ and $j=N-4n+2$. Now assume $j > N-4n+2$ and assume the second statement holds for $x=1$ and smaller values of $j$. Then $U_{1,j}=U_{1,j-1} \cup U''_{1,j}$ and, from 2.2.3, it follows that $\dim(U''_{1,j}) \leq \dim(U_{0,j-1})+2n-(N-j+1)=j-N+4n-2$. This proves that $\dim(U_{1,j}) \leq j-N+4n-2$, hence the second statement in the case $x=1$. Now assume $x>1$ and the claim holds for smaller values of $x$. Assume for some value $j$ with $(x-1)j \geq (x-1)N-2xn+(x-1)^2+1$ we have $U_{x,j}=\emptyset$. From 2.2.2 we have $U_{y,j+1}=\emptyset$ for $y \geq x+1$; and if $N-j > \dim(U_{x-1,j})+2n-x+1$ then $U_{x,j+1}=\emptyset$. Since $\dim(U_{x-1,j})+2n-x+1 \leq (x-1)j-(x-1)N+2xn$ $-(x-1)^2-1+2n-x+1=(x-1)j-(x-1)N+2(x+1)n-(x-1)^2-x$, we have that $U_{x,j+1}=\emptyset$ if $xj < xN-2(x+1)n+(x-1)^2+x$. This proves that $U_{x,j}=\emptyset$ if $xj < xN-2(x+1)n+(x-1)^2+2x=xN-2(x+1)n+x^2+1$. This is the first statement for $x$. The proof of the second statement for $x$ can be done as the proof of the second statement for $x=1$. □

**2.4.2.** Now, take $x=n$. We find $U_{n,j}=\emptyset$ if $nj < nN-2(n+1)n+n^2+1$, hence if $j < N-n-2+\frac{1}{n}$; i.e. if $j \leq N-n-2$. Since $j \leq k \leq N-2n-1$ this condition is always satisfied.

This concludes the proof of Theorem I. □

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