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Hodge equations with change of type

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Abstract. A geometric interpretation is given for certain elliptic-hyperbolic systems in the plane. Among several examples, one which reduces to the equations for harmonic 1-forms on the projective disc is studied in detail. A boundary-value problem for this example is formulated and shown to possess weak solutions.

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1. Introduction

Harmonic forms u on a Riemannian manifold satisfy the *Hodge equations*

$$\delta u = du = 0, \tag{1}$$

where d is the exterior derivative and δ its adjoint. In the case of 1-forms, these equations have the local form

$$|G|^{-1/2}\,\partial_i\left(G^{ij}\sqrt{|G|}u_j\right) = 0, \tag{2}$$

$$\partial_i u_j dx^i \wedge dx^j = \frac{1}{2}\left(\partial_i u_j - \partial_j u_i\right) dx^i \wedge dx^j = 0, \tag{3}$$

where G_{ij} is the metric tensor on the manifold. If eqs. (2), (3) are defined on a singular 2-manifold, it may happen that the equations can be written as a system of mixed type in which the parabolic curve lies along the singularity of the manifold. This yields a geometric interpretation of certain elliptic-hyperbolic systems in the plane.

Perhaps the simplest example is a metric which changes from Euclidean to Minkowskian along the x-axis. In this case the system (2), (3) reduces to a potential equation on one side of the metric singularity and to a wave equation on the other side, leading to a first-order system of the form

$$u_{1x} + sgn(y)u_{2y} = 0,$$
$$u_{1y} - u_{2x} = 0.$$

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This corresponds, in the case $u_1 = u_x,\ u_2 = u_y$, to the Lavrent'ev–Bitsadze equation.

An example possessing more interesting geometry can be constructed by taking Beltrami's hyperbolic model [1] for the projective disc $\mathbb{P}^2$ as the underlying surface. The metric tensor in this model is the matrix

$$G_{ij} = \frac{1}{\left(1-x^2-y^2\right)^2}\begin{bmatrix} 1-y^2 & xy \\ xy & 1-x^2 \end{bmatrix}.$$

The matrix

$$G^{ij} = \left(1-x^2-y^2\right)\begin{bmatrix} 1-x^2 & -xy \\ -xy & 1-y^2 \end{bmatrix}$$

becomes indefinite, and the determinant

$$G = \frac{1}{1-x^2-y^2}$$

becomes singular, on the line at infinity of the model, which corresponds to the circle $x^2+y^2=1$.

But the equations can be redefined so that the metric singularity on the unit circle in $\mathbb{P}^2$ is replaced by a change of type on the unit circle in $\mathbb{R}^2$. Writing out eq. (2) in coordinates, we obtain

$$\begin{aligned}\left(1-x^2-y^2\right)\Big\{ \left[\left(1-x^2\right)u_1\right]_x - (xyu_1)_y - (xyu_2)_x \\ + \left[\left(1-y^2\right)u_2\right]_y - (xu_1+yu_2) \Big\} = 0.\end{aligned} \tag{4}$$

Equation (3) implies that

$$(xyu_1)_y + (xyu_2)_x = 2xyu_{1y} + xu_1 + yu_2. \tag{5}$$

Equations (4), (5) are well defined on so-called "ideal points" lying outside the unit circle, and possess wave-like solutions in which disturbances propagate along null geodesics of the distance element

$$ds^2 = \frac{\left(1-y^2\right)dx^2 + 2xydxdy + \left(1-x^2\right)dy^2}{\left(1-x^2-y^2\right)^2}.$$

Borrowing the terminology of fluid dynamics, we call an expression such as ds^2 the *flow metric* associated to a system such as (4), (5).

In order for a 1-form u to satisfy (4), (5), it is sufficient for u to satisfy a system of first-order equations on $\mathbb{R}^2$ having the form

$$Lu = g, \tag{6}$$

where

$$L = (L_1, L_2)\,,\ \ g = (g_1, g_2)\,,$$
$$u = (u_1(x,y), u_2(x,y))\,,\ \ (x,y) \in \Omega \subset\subset \mathbb{R}^2,$$
$$(Lu)_1 = \left[\left(1-x^2\right)u_1\right]_x - 2xyu_{1y} + \left[\left(1-y^2\right)u_2\right]_y - 2xu_1 - 2yu_2, \tag{7}$$

and

$$(Lu)_2 = u_{1y} - u_{2x}.$$

If $y^2 \neq 1$, we can replace the second component of L by the expression

$$(Lu)_2 = \left(1 - y^2\right)\left(u_{1y} - u_{2x}\right), \tag{8}$$

which has the same annihilator.

The second-order terms of eqs. (6)–(8) can be written in the form $Au_x + Bu_y$, where

$$A = \begin{bmatrix} 1 - x^2 & 0 \\ 0 & -\left(1 - y^2\right) \end{bmatrix}$$

and

$$B = \begin{bmatrix} -2xy & 1 - y^2 \\ 1 - y^2 & 0 \end{bmatrix}.$$

If $y^2 \neq 1$, the characteristic equation

$$|A - \lambda B| = -\left(1 - y^2\right)\left[\left(1 - y^2\right)\lambda^2 + 2xy\lambda + \left(1 - x^2\right)\right]$$

possesses two real roots λ_1, λ_2 on Ω precisely when $x^2 + y^2 > 1$. Thus the system is elliptic in the intersection of Ω with the open unit disc centered at $(0, 0)$ and hyperbolic in the intersection of Ω with the complement of the closure of this disc. The boundary of the unit disc, along which this change in type occurs, is the line at infinity in $\mathbb{P}^2$ and a line singularity of the tensor G_{ij}.

Hua used variable separation, the Poisson kernel, and D'Alembert methods to solve boundary-value problems for a scalar equation which resembles the system (6)–(8) [3]. Precisely, the scalar equation studied by Hua consists of the conserved quantities in an equation which can be obtained from (6)–(8) by choosing $u_1 = u_x$, $u_2 = u_y$, and $g_1 = g_2 = 0$. A form of the equation studied by Hua with $g_1 \neq 0$ was solved by Ji and Chen [4]. Inside the unit disc, these choices correspond to replacing the Hodge operator on 1-forms with the Laplace–Beltrami operator on scalars, modulo lower-order terms. We emphasize that eqs. (6)–(8), even without the lower-order terms, are not equivalent to an equation of the form studied by Hua if $g_2 \neq 0$ or if the vector (u_1, u_2) is not continuously differentiable. Beyond this, the form of the lower-order terms which do not appear in [3] affect our analysis of the equations in Sections 3–6. (In Section 6 we consider the equations in the absence of lower-order terms.) Finally, in [3] and [4] conditions are placed on characteristics, as in the classical Tricomi problem; in Sections 3–6 conditions are placed only on the non-characteristic part of the boundary, as in the classical Frankl' problem.

2. Other systems of mixed type

The analogy between the two systems (2), (3) and (6)–(8) can be extended to other equations of mixed type, although generally these systems will have less interesting geometry than the projective disc. For example, the system introduced by Morawetz [5] as a vehicle for studying the Chaplygin equations is of a broadly similar form, as is the system studied in [8].

2.1. Equations of fluid dynamics

The geometry of eqs. (6)–(8) is in some sense dual to that of a well-known transform of the velocity potential for transonic flow in the hodograph plane. Denote by $(u_1(x,y), u_2(x,y))$ the velocity components of a steady flow expressed in coordinates (x,y). The hodograph transformation introduces u_1, u_2 as independent coordinates. The continuity equations for the velocity potential under standard simplifying assumptions can now be written in the linear form ([2, eq. (3.6)])

$$\left(c^2 - u_1^2\right) y_{u_2} + u_1 u_2 \left[x_{u_2} + y_{u_1}\right] + \left(c^2 - u_2^2\right) x_{u_1} = 0,$$
$$x_{u_2} - y_{u_1} = 0.$$

Here

$$c^2 = 1 - \frac{\gamma_a - 1}{2}\left(u_1^2 + u_2^2\right),$$

where $\gamma_a > 1$ is the adiabatic constant of the medium. This system corresponds to eqs. (2), (3) where the parabolic curve is a circle of radius $\sqrt{2/(\gamma_a+1)}$ centered at the point $u_1 = 0$, $u_2 = 0$ and the metric tensor in eq. (2) is the matrix

$$\widetilde{G}_{ij} = \frac{1}{c^2\left(c^2 - u_2^2 - u_1^2\right)} \begin{bmatrix} c^2 - u_1^2 & -u_1 u_2 \\ -u_1 u_2 & c^2 - u_2^2 \end{bmatrix}.$$

Consider for simplicity the lower limit of the range of values for γ_a, in which c^2 is approximately normalized. In this artificially simple case, the change of type occurs on the boundary of the unit circle and the continuity equations in the hodograph plane reduce to a replacement of the metric tensor $G_{ij}(u_1, u_2)$ of eq. (2) by the tensor $\left(1 - u_1^2 - u_2^2\right)^{-2} G^{ij}(u_1, u_2)$ (ignoring lower-order terms). We obtain a second-order scalar equation if we introduce a function $\chi(u_1, u_2)$ satisfying

$$x = \chi_{u_1}, \ \ y = \chi_{u_2}$$

(*cf.* eq. (3.8) of [2]). The characteristic curves of the resulting equation are relatively complicated, as they are given by a family of epicycloids which intersect the parabolic curve in a family of cusps. This leads to complicated boundary-value problems for the equation. By contrast, the characteristic curves corresponding to the "dual" system (6)–(8) are exceedingly simple, as they are given by the set of all tangent lines to the unit disc. This leads, in our case, to relatively simple boundary-value problems. How much can be said a priori about relations between solutions of the two sets of boundary-value problems is not immediately clear, however.

Without its lower-order terms and after a trivial relabelling of coordinates, system (6)–(8) can be interpreted as the hodograph image of a quasilinear system having the form

$$\left(1 - u_1^2 - u_2^2\right)^m \left[\left(1 - u_2^2\right) u_{1x} + u_1 u_2 \left(u_{1y} + u_{2x}\right) + \left(1 - u_1^2\right) u_{2y}\right] = 0, \tag{9}$$
$$u_{2x} - u_{1y} = 0, \tag{10}$$

for $m \in \mathbb{R}$. If the components $u_1(x, y)$ and $u_2(x, y)$ are continuously differentiable in x and y, then there is a potential function $\varphi(x, y)$ such that

$$d\varphi(x, y) = \varphi_x dx + \varphi_y dy = u_1 dx + u_2 dy$$

on any domain having trivial de Rham cohomology. If $m = -3/2$, then the resulting equation is the *Hodge dual* of the minimal surface equation, in the sense of [9, eqs. (2.23)–(2.29)]. If $\left(1 - u_1^2 - u_2^2\right)^m \neq 0$, then the flow metric for eqs. (9), (10) is conformally equivalent to the metric

$$ds^2 = dx^2 + dy^2 - (d\varphi)^2 .$$

By comparison, the flow metric for the gas dynamics equation

$$\left(1 - \frac{u_1^2}{c^2}\right) u_{1x} - \frac{u_1 u_2}{c^2}\left(u_{1y} + u_{2x}\right) + \left(1 - \frac{u_2^2}{c^2}\right) u_{2y} = 0$$

is conformally equivalent to the metric

$$ds'^2 = dx^2 + dy^2 - (*d\varphi)^2 ,$$

where, in this case, $\varphi(x, y)$ is the flow potential and $*$ is the Hodge isomorphism. We note that the difference between the metrics ds'^2 and ds^2 corresponds physically to a difference between a composite metric with non-Euclidean part conformally equivalent to a metric on streamlines, and a composite metric with non-Euclidean part conformally equivalent to a metric on potential lines. This correspondence arises from relating the differential of the *stream function* ψ to the differential of the flow potential φ by the equation

$$d\psi = c^{2/(\gamma_a - 1)} * d\varphi.$$

2.2. *Cauchy–Riemann equations*

An alternative to considering the functions u_1, u_2 to be components of a 1-form in $\mathbb{R}^2$ is to treat them as components of a function in $\mathbb{C}$. This is a standard approach in which, for example, the continuity equations in the hodograph plane are associated with a generalized Cauchy–Riemann operator. Among its many advantages, this approach has the disadvantage of giving special emphasis to dimension 2 and to the conformal group (or to quasiconformal mappings in the quasilinear case). In fact, the natural invariance group for eqs. (6)–(8) is the projective group rather than the conformal group, a circumstance which has some interesting consequences. For instance, whereas there are many conic sections in $\mathbb{R}^2$, every non-degenerate conic section is projectively equivalent to the unit circle; so the parabolic degeneracy at the point at infinity in the projective metric corresponds under projective mappings to a variety of parabolic curves in a Euclidean metric (*cf.* [7, Section 120, Theorem 21]; [3, p. 633]).

3. A boundary-value problem

The Dirichlet problem for the systems introduced in the preceding section involves prescribing the value of the 1-form $u_1dx + u_2dy$ on the boundary of a domain of $\mathbb{R}^2$. In the following we consider an analogue of the Dirichlet problem in which we show the existence of weak solutions to (6)–(8) which satisfy the boundary condition

$$u_1\frac{dx}{ds} + u_2\frac{dy}{ds} = 0, \tag{11}$$

where s denotes arc length, on the non-characteristic part of the domain boundary. The proof is based on methods introduced in [5] for boundary-value problems in the Chaplygin model.

Denote by R the region bounded by the rectangle $1/\sqrt{2} < x \le 1$, $-1/\sqrt{2} < y < 1/\sqrt{2}$. Let C be any smooth curve lying entirely in the interior of R except for two distinct points, which intersect the characteristic line $x = 1$ at $(1, y_0)$ and $(1, y_1)$, $-1/\sqrt{2} < y_0 < y_1 < 1/\sqrt{2}$. Define Ω to be the domain bounded by $C \cup \Gamma$, where Γ is the line segment $(1, y_0) \le (x, y) \le (1, y_1)$. Assume that $dy \le 0$ on C.

Define U to be the vector space consisting of all pairs of measurable functions $u = (u_1, u_2)$ for which the weighted L^2 norm

$$\|u\|_* = \left[\iint_\Omega \left(\left|2x^2-1\right| u_1^2 + \left|2y^2-1\right| u_2^2\right) dxdy\right]^{1/2}$$

is finite. Denote by W the linear space defined by pairs of functions $w = (w_1, w_2)$ having continuous derivatives and satisfying:

$$w_1dx + w_2dy = 0$$

on Γ;

$$w_1 = 0$$

on C;

$$\iint_\Omega \left[\left|2x^2-1\right|^{-1}\left(L^*w\right)_1^2 + \left|2y^2-1\right|^{-1}\left(L^*w\right)_2^2\right] dxdy < \infty.$$

Here

$$\left(L^*w\right)_1 = \left[\left(1-x^2\right)w_1\right]_x - 2xyw_{1y} + \left[\left(1-y^2\right)w_2\right]_y + 2xw_1,$$

and

$$\left(L^*w\right)_2 = \left(1-y^2\right)\left(w_{1y} - w_{2x}\right) + 2yw_1.$$

Define the Hilbert space H to consist of pairs of measurable functions $h = (h_1, h_2)$ for which the norm

$$\|h\|^* = \left[\iint_\Omega \left(\left|2x^2-1\right|^{-1} h_1^2 + \left|2y^2-1\right|^{-1} h_2^2\right) dxdy\right]^{1/2}$$

is finite.

If the curve C is chosen so that x is bounded below away from the value $1/\sqrt{2}$ and y is bounded above and below away from the values $\pm\sqrt{1/2}$, then the above weighted inner products can all be replaced by the L^2 inner product.

Definition. *We say that u is a* weak solution *of the system* (6)–(8), (11) *in Ω if $u \in U$ and for every $w \in W$,*

$$-(w, g) = \left(L^* w, u\right), \tag{12}$$

where

$$(w, g) = \iint_\Omega (w_1 g_1 + w_2 g_2)\, dxdy.$$

The following proposition shows that this notion of weak solution is well defined:

Proposition 1. *Any continuously differentiable weak solution of the boundary-value problem* (6)–(8), (11) *with $g \in H$ is a classical solution of the system* (6)–(8)*, with* (11) *satisfied on the noncharacteristic curve C.*

Proof. In the interest of generality, we prove the proposition by an argument that applies to any smooth domain having a characteristic line segment on the boundary; we do not use any of the special properties of the line $x = 1$ or of the first and fourth quadrants.

$$\begin{aligned}
\left(L^* w\right)_1 u_1 &= \left[\left(1 - x^2\right) w_1\right]_x u_1 - 2xyw_{1y}u_1 + \left[\left(1 - y^2\right) w_2\right]_y u_1 + 2xw_1u_1 \\
&= \left[\left(1 - x^2\right) w_1u_1\right]_x - \left(1 - x^2\right) w_1u_{1x} - [2xyw_1u_1]_y + 2xw_1u_1 \\
&\quad + 2xyw_1u_{1y} + \left[\left(1 - y^2\right) w_2u_1\right]_y - \left(1 - y^2\right) w_2u_{1y} + 2xw_1u_1,
\end{aligned}$$

and

$$\begin{aligned}
\left(L^* w\right)_2 u_2 &= \left(1 - y^2\right) \left(w_{1y} - w_{2x}\right) u_2 + 2yw_1u_2 \\
&= \left[\left(1 - y^2\right) w_1u_2\right]_y + 2yw_1u_2 - \left(1 - y^2\right) w_1u_{2y} \\
&\quad - \left[\left(1 - y^2\right) w_2u_2\right]_x + \left(1 - y^2\right) w_2u_{2x} + w_1 2yu_2.
\end{aligned}$$

Application of Green's theorem to the derivatives of products yields

$$\begin{aligned}
&\iint_\Omega \left[\left(1 - x^2\right) w_1u_1 - \left(1 - y^2\right) w_2u_2\right]_x dxdy \\
&\qquad - \iint_\Omega \left[2xyw_1u_1 - \left(1 - y^2\right)(w_2u_1 + w_1u_2)\right]_y dxdy \\
&= \int_{\partial\Omega} \left[\left(1 - x^2\right) w_1u_1 - \left(1 - y^2\right) w_2u_2\right] dy \\
&\qquad + \int_{\partial\Omega} \left[2xyw_1u_1 - \left(1 - y^2\right)(w_2u_1 + w_1u_2)\right] dx.
\end{aligned}$$

On the characteristic line segment Γ this integral splits into the sum $I_1 + I_2$, where

$$I_1 = \int_\Gamma \left(1 - x^2\right) w_1 u_1 dy + \left[2xyw_1u_1 - \left(1 - y^2\right) w_2 u_1\right] dx$$
$$= \int_\Gamma \left[2xyw_1u_1 + \left(1 - x^2\right) w_1 u_1 \left(\frac{dy}{dx}\right) - \left(1 - y^2\right) w_2 u_1\right] dx,$$

and

$$I_2 = -\int_\Gamma \left(1 - y^2\right) u_2 \left(w_1 dx + w_2 dy\right).$$

The integral I_2 vanishes by the boundary condition for elements of W, from which we also obtain

$$I_1 = \int_\Gamma \left\{2xyw_1u_1 - \left[\left(1 - x^2\right)\left(\frac{dy}{dx}\right)^2 + \left(1 - y^2\right)\right] w_2 u_1\right\} dx. \qquad (13)$$

On the characteristic curves,

$$\left(1 - y^2\right) dx^2 + 2xydxdy + \left(1 - x^2\right) dy^2 = 0,$$

so

$$\left(1 - x^2\right) \frac{dy^2}{dx^2} = -\left(1 - y^2\right) - 2xy\frac{dy}{dx}. \qquad (14)$$

Substituting (14) into (13) yields

$$I_1 = \int_\Gamma \left\{2xyw_1u_1 - \left[-\left(1 - y^2\right) - 2xy\frac{dy}{dx} + \left(1 - y^2\right)\right] w_2 u_1\right\} dx$$
$$= \int_\Gamma 2xyu_1 \left(w_1 + w_2\frac{dy}{dx}\right) dx = 0.$$

Because w_1 vanishes on C, the boundary integral there has the form

$$-\int_C \left(1 - y^2\right) w_2 \left(u_1 dx + u_2 dy\right).$$

We obtain

$$\left(L^* w, u\right) = -\left(w, Lu\right) - \int_C \left(1 - y^2\right) w_2 \left(u_1 dx + u_2 dy\right), \qquad (15)$$

where

$$\begin{aligned}(w, Lu) = &\iint_\Omega \left[\left(1 - x^2\right) w_1 u_{1x} - 2xw_1u_1 - 2xyw_1u_{1y}\right.\\ &\left.+ \left(1 - y^2\right) w_2 u_{1y} - 2xw_1u_1 - 2yw_1u_2\right] dxdy\\ &- \iint_\Omega \left[2yw_1u_2 - \left(1 - y^2\right) w_1 u_{2y} + \left(1 - y^2\right) w_2 u_{2x}\right] dxdy\end{aligned}$$

$$
\begin{aligned}
&= \iint_\Omega \Big\{ \left[\left(1-x^2\right) u_1\right]_x - 2xyu_{1y} \\
&\qquad + \left[\left(1-y^2\right) u_2\right]_y - 2xu_1 - 2yu_2 \Big\} w_1 dxdy \\
&\quad + \iint_\Omega \left(1-y^2\right)\left(u_{1y} - u_{2x}\right) w_2 dxdy \\
&= \iint_\Omega \left[(Lu)_1 w_1 + (Lu)_2 w_2\right] dxdy.
\end{aligned}
$$

Combining eqs. (12) and (15) yields

$$
-(w, g) = \left(L^* w, u\right) = -(w, Lu) - \int_C \left(1-y^2\right) w_2 \left(u_1 dx + u_2 dy\right).
$$

Because w is arbitrary in W, we conclude that (11) is satisfied on C and $Lu = g$, which completes the proof. □

In Sections 4 and 5 we prove:

Theorem 2. *There exists a weak solution of the boundary-value problem* (6)–(8), (11) *on* Ω *for every* $g \in H$.

Remark. Switching the sign of the term $2yu_2$ in eq. (7) has no effect on the proof of Theorem 2.

4. An a priori estimate

Lemma 3. $\exists\, K \in \mathbb{R}^+ \ni \forall w \in W,\ K \|w\|_* \le \|L^* w\|^*$.

Proof. We use an abbreviated version of the Friedrichs *abc* method. Fixing a sufficiently differentiable function $a(x, y)$, consider the L^2 inner product

$$
\begin{aligned}
\left(L^* w, aw\right) &= \iint_\Omega \Big\{ \left[\left(1-x^2\right) w_1\right]_x - 2xyw_{1y} \\
&\qquad + \left[\left(1-y^2\right) w_2\right]_y + 2xw_1 \Big\} aw_1 dxdy \\
&\quad + \iint_\Omega \left[\left(1-y^2\right)\left(w_{1y} - w_{2x}\right) + 2yw_1\right] aw_2 dxdy \\
&= \iint_\Omega \sum_{i=1}^{7} \tau_i dxdy,
\end{aligned}
$$

where

$$
\tau_1 = \frac{1}{2}\left[\left(1-x^2\right) aw_1^2\right]_x - \left[\frac{1}{2}\left(1-x^2\right) a_x + ax\right] w_1^2; \tag{16}
$$

$$
\tau_2 = -\left(xyaw_1^2\right)_y + \left(ax + xya_y\right) w_1^2; \tag{17}
$$

$$
\tau_3 = \left[\left(1-y^2\right) aw_1 w_2\right]_y - \left(1-y^2\right) a_y w_2 w_1 - \left(1-y^2\right) aw_2 w_{1y}; \tag{18}
$$

$$\tau_4 = 2xaw_1^2; \ \ \tau_5 = \left(1-y^2\right) a w_{1y} w_2; \tag{19}$$

$$\tau_6 = -\frac{1}{2}\left[\left(1-y^2\right) a w_2^2\right]_x + \frac{1}{2}\left(1-y^2\right) a_x w_2^2; \tag{20}$$

$$\tau_7 = 2yaw_1w_2. \tag{21}$$

We ignore for a moment derivatives of products, as these will be integrated and become boundary terms. The coefficients of $w_{1y}w_2$ sum to zero in (18) and (19). Denoting the coefficients of w_1^2 off the boundary by α, those of w_2^2 by γ and those of w_1w_2 by 2β and choosing $a = x^2$, we obtain

$$\alpha = x\left(3x^2-1\right);$$
$$\gamma = x\left(1-y^2\right);$$
$$2\beta = 2yx^2.$$

The region R is defined so that the discriminant

$$\alpha\gamma - \beta^2 = x^2\left[y^2\left(1-4x^2\right)+3x^2-1\right]$$

is positive on Ω. Thus we have the estimate

$$2\beta w_1 w_2 \geq -2\left|\beta\right|\left|w_1\right|\left|w_2\right| > -2\sqrt{\alpha}\left|w_1\right|\sqrt{\gamma}\left|w_2\right| \geq -\alpha w_1^2 - \gamma w_2^2.$$

This shows that the inequality of the lemma is satisfied in Ω, but the resulting constant depends on the choice of the curve C. Rather, we prefer to obtain the explicit estimate

$$2\beta w_1 w_2 \geq -2x\left|xw_1\right|\left|yw_2\right| \geq -\left(x^3w_1^2 + xy^2w_2^2\right).$$

Applying Green's theorem to derivatives of products in (Lw, aw) results in a boundary integral of the form

$$\int_{\partial\Omega}\frac{x^2}{2}\left[\left(1-x^2\right)w_1^2 - \left(1-y^2\right)w_2^2\right]dy + \int_{\partial\Omega} x^2\left[xyw_1^2 - \left(1-y^2\right)w_1w_2\right]dx.$$

The definition of W implies that on Γ,

$$-\left(1-y^2\right)w_1w_2dx = \left(1-y^2\right)w_2^2dy,$$

so the boundary integral on Γ reduces to

$$\int_\Gamma \frac{x^2}{2}\left[\left(1-x^2\right)w_1^2 + \left(1-y^2\right)w_2^2\right]dy + x^3yw_1^2dx = \int_\Gamma\frac{x^2}{2}\left(1-y^2\right)w_2^2dy = 0.$$

Because w_1 vanishes on C, the remaining boundary integral is of the form

$$-\int_C \frac{x^2}{2}\left(1-y^2\right)w_2^2dy,$$

which is non-negative under the given orientation by the hypotheses on C.

We find that on Ω,

$$\left(L^*w, aw\right) \geq \frac{1}{\sqrt{2}} \int\int_\Omega \left(\left|2x^2-1\right| w_1^2 + \left|2y^2-1\right| w_2^2\right) dxdy. \tag{22}$$

It remains to estimate (L^*w, aw) from above. We have for any positive constant λ,

$$\begin{aligned}\left(L^*w, aw\right) \leq & \int\int_\Omega \left|\sqrt{2x^2-1}w_1\right| \left|\left(\sqrt{2x^2-1}\right)^{-1} \left(L^*w\right)_1\right| dxdy \\ & + \int\int_\Omega \left|\sqrt{\left|2y^2-1\right|}w_2\right| \left|\left(\sqrt{\left|2y^2-1\right|}\right)^{-1} \left(L^*w\right)_2\right| dxdy \\ \leq & \frac{1}{\lambda} \left\|L^*w\right\|^{*2} + \lambda \left\|w\right\|_*^2 .\end{aligned} \tag{23}$$

Choosing $\lambda < 1/\sqrt{2}$, inequalities (22) and (23) imply the assertion of Lemma 3 with $K = \sqrt{\left[\left(1/\sqrt{2}\right) - \lambda\right]\lambda}$. □

5. Existence

The proof of existence is straightforward, given the a priori estimates of the preceding section. We briefly outline the argument, following [5].

Define the scaled 1-forms

$$\widetilde{w} = \sqrt{2x^2-1}w_1dx + \sqrt{\left|2y^2-1\right|}w_2dy$$

and

$$\widetilde{g} = \frac{1}{\sqrt{2x^2-1}}g_1dx + \frac{1}{\sqrt{\left|2y^2-1\right|}}g_2dy.$$

Arguing as in (23), but applying the Schwarz inequality in place of Young's inequality, we have

$$|(w,g)| = |(\widetilde{w}, \widetilde{g})| \leq \|\widetilde{w}\|_2 \|\widetilde{g}\|_2 ,$$

where $\| \ \|_2$ is the (unweighted) L^2 norm. The extreme left- and right-hand sides of this inequality can be written as

$$|(w,g)| \leq \|w\|_* \|g\|^* \leq K^{-1} \left\|L^*w\right\|^* \|g\|^* \leq \widetilde{K}(g) \left\|L^*w\right\|^* ,$$

using Lemma 3. Thus the functional ξ defined for fixed g and all $w \in W$ by the formula

$$\xi\left(L^*w\right) = (w,g)$$

can be extended to a bounded linear functional on H. The Riesz representation theorem implies that $\forall w \in W$ there is an $h \in H$ for which

$$\xi\left(L^*w\right) = \left(L^*w, h\right)^* .$$

Defining $u = (u_1, u_2)$ so that

$$u_1 = -\left(2x^2 - 1\right)^{-1} h_1$$

and

$$u_2 = -\left|2y^2 - 1\right|^{-1} h_2,$$

we have $u \in U$ as $h \in H$; that is,

$$\begin{aligned}&\iint_\Omega \left[\left(2x^2-1\right) u_1^2 + \left|2y^2-1\right| u_2^2\right] dxdy \\ &\quad = \iint_\Omega \left[\left(2x^2-1\right)^{-1} h_1^2 + \left|2y^2-1\right|^{-1} h_2^2\right] dxdy \\ &\quad < \infty.\end{aligned}$$

We conclude that

$$\begin{aligned}-(w, g) &= -\xi\left(L^* w\right) = -\left(L^* w, h\right)^* \\ &= -\iint_\Omega \left[\left(2x^2-1\right)^{-1}\left(L^* w\right)_1 h_1 + \left|2y^2-1\right|^{-1}\left(L^* w\right)_2 h_2\right] dxdy \\ &= \iint_\Omega \Big[\left(2x^2-1\right)^{-1}\left(L^* w\right)_1 \left(2x^2-1\right) u_1 \\ &\qquad + \left|2y^2-1\right|^{-1}\left(L^* w\right)_2 \left|2y^2-1\right| u_2\Big] dxdy \\ &= \left(L^* w, u\right).\end{aligned}$$

Comparing the extreme left-hand side of this expression with its extreme right-hand side completes the proof of Theorem 2.

6. Modifications of the problem

The lower-order terms of equations of mixed type are frequently modified in order to simplify the analysis [see, for example, eqs. (7) and (23) of [6] or eqs. (1.11) and (2.1) of [8]]. In addition to solving the system (6)–(8), we can also prove the existence of weak solutions to systems which differ from (6)–(8) only in the form of their lower-order terms. Among many possible examples, we choose two obvious ones.

6.1. A different distribution of the lower-order terms

We can replace (6)–(8) with a system having the more symmetric form

$$\widetilde{L} u = g,$$

where

$$\widetilde{L} = \left(\widetilde{L}_1, \widetilde{L}_2\right), \;\; g = (g_1, g_2),$$

$$\left(\widetilde{L} u\right)_1 = \left[\left(1 - x^2\right) u_1\right]_x - 2xyu_{1y} + \left[\left(1 - y^2\right) u_2\right]_y - 2xu_1,$$

and

$$\left(\widetilde{L}u\right)_2 = \left(1-y^2\right)\left(u_{1y}-u_{2x}\right)+2yu_2.$$

In the special case $g_1 = g_2 = 0$ both this system and eqs. (6)–(8) satisfy the equation

$$\left[\left(1-x^2\right)u_1\right]_x - 2xyu_{1y} + \left[\left(1-y^2\right)u_2\right]_y - 2\left(xu_1+yu_2\right) = \left(1-y^2\right)\left(u_{1y}-u_{2x}\right),$$

although the equated quantities differ in the different systems. The analysis of the modified system is a little simpler and the conditions on the non-characteristic part of the boundary considerably more lenient. However, the proof for this system does not apply in an obvious way to a domain lying in two contiguous quadrants.

Denote by Ω_m the region bounded by the characteristic line tangent to the unit disc at the point $(1,0)$ and a smooth curve C_m which intersects that line at exactly two points on the line segment Γ given by the interval $(1,-1) < (1,y) < (1,0)$. Assume that C_m is bounded on the left by the line $x = 0$, on the right by Γ, below by the line $y = -1$, and above by the x-axis. Orient $\partial\Omega_m$ in the counterclockwise direction. We assume that, with this orientation, the line element dy is non-positive on C_m. (Small modifications of the problem will define an analogous boundary-value problem in the second quadrant, a fact which is reflected below in our notation for the spaces U_m, W_m, and H_m.)

Denote by U_m the vector space consisting of all pairs of measurable functions $u = (u_1, u_2)$ for which the weighted L^2 norm

$$\|u\|_{m*} = \left\{\int\int_{\Omega_m}\left(|x|\,u_1^2+|y|\,u_2^2\right)dxdy\right\}^{1/2}$$

is finite. This norm is induced by the weighted inner product

$$(u,w)_{m*} = \int\int_{\Omega_m}\left(|x|\,u_1w_1+|y|\,u_2w_2\right)dxdy.$$

Denote by W_m the linear space defined by pairs of functions $w = (w_1, w_2)$ having continuous derivatives and satisfying:

$$w_1dx + w_2dy = 0$$

on Γ;

$$w_1 = 0$$

on C_m;

$$\int\int_{\Omega_m}\left[|x|^{-1}\left(\widetilde{L}^*w\right)_1^2+|y|^{-1}\left(\widetilde{L}^*w\right)_2^2\right]dxdy < \infty.$$

Here

$$\left(\widetilde{L}^*w\right)_1 = \left[\left(1-x^2\right)w_1\right]_x - 2xyw_{1y} + \left[\left(1-y^2\right)w_2\right]_y + 2xw_1,$$

and

$$\left(\widetilde{L}^*w\right)_2 = \left(1-y^2\right)\left(w_{1y}-w_{2x}\right)-2yw_2.$$

The space W_m is contained in the Hilbert space H_m consisting of pairs of measurable functions $h = (h_1, h_2)$ for which the norm

$$\|h\|_m^* = \left\{ \int \int_{\Omega_m} \left(|x|^{-1} h_1^2 + |y|^{-1} h_2^2 \right) dxdy \right\}^{1/2}$$

is finite.

To prove the analogue of Lemma 3 for this system under an analogous boundary condition we estimate the L^2 inner product $\left(w, \widetilde{L}^* w\right)$ as in (16)–(21). We obtain a result analogous to (22) with

$$\alpha = 2x, \ \gamma = -2y,$$

and

$$2\beta = 0.$$

Arguing as in (23) with $\lambda < 2$, we find that

$$\exists \, K_m \in \mathbb{R}^+ \ni \forall w \in W_m, \, K_m \, \|w\|_{m*} \leq \left\| \widetilde{L}^* w \right\|_m^*$$

with $K_m = \sqrt{(2 - \lambda) \, \lambda}$. The remainder of the existence proof proceeds as in the case of eqs. (6)–(8).

If the term $2yu_2$ in the component $\left(\widetilde{L}u\right)_2$ is multiplied by -1, then the domain of the solution switches from the fourth quadrant to the first quadrant. If the term $2xu_1$ in the component $\left(\widetilde{L}u\right)_1$ is multiplied by -1, then the domain switches from the fourth quadrant to the third quadrant. If both lower-order terms are multiplied by -1, then the domain switches from the fourth quadrant to the second quadrant. In the last two cases, Γ lies along the line $x = -1$.

6.2. *Neglected lower-order terms*

Finally, we consider a form of the system (6)–(8) in which no terms of order zero appear. This system consists of equations having the form

$$L_o u = g,$$

where

$$L_o = (L_{o1}, L_{o2}), \ g = (g_1, g_2),$$
$$(L_o u)_1 = \left[\left(1 - x^2\right) u_1 \right]_x - 2xyu_{1y} + \left[\left(1 - y^2\right) u_2 \right]_y,$$

and

$$(L_o u)_2 = \left(1 - y^2\right) \left(u_{1y} - u_{2x}\right).$$

In this case the boundary-value problem is simplified somewhat by the fact that $L_o = L_o^*$. For example, Lemma 3 implies the uniqueness in W of weak solutions, which are defined by direct analogy to the other two cases.

To prove the existence of weak solutions to the system $L_o u = g$, we fix positive numbers $\delta \ll 1/2$ and $\varepsilon \ll 1/2$ and denote by R_o the rectangle

$$\frac{1}{\sqrt{2}} < x \leq 1, \; \frac{1}{\sqrt{2-\delta}} < y \leq \sqrt{1-\varepsilon}.$$

Let C_o be a smooth curve lying in the interior of R_o with the exception of two distinct points, $(1, y_0)$ and $(1, y_1)$, $1/\sqrt{2-\delta} < y_0 < y_1 \leq \sqrt{1-\varepsilon}$, at which the curve intersects the characteristic line $x = 1$. Assume that $dy \leq 0$ on C_o. Define Γ to be the line segment $(1, y_0) \leq (x, y) \leq (1, y_1)$ and Ω_o to be the domain having boundary $C_o \cup \Gamma$. In this case we can take the associated Hilbert spaces, U_o and H_o, to be L^2, bearing in mind that our estimates will depend in a predictable way on the sizes of ε and δ. As in the preceding cases, we place the boundary condition (11) on the non-characteristic part of the boundary.

In order to prove the analogue of Lemma 3 for this system, we estimate the L^2 inner product

$$(L_o w, xyw) = \int\int_{\Omega_o} \sum_{i=1}^{5} \tau_i \, dxdy$$

where

$$\tau_1 = \frac{1}{2}\left[\left(1-x^2\right) xyw_1^2\right]_x - x^2 y w_1^2 - \frac{1}{2}\left(1-x^2\right) y w_1^2;$$
$$\tau_2 = -\left[x^2 y^2 w_1^2\right]_y + x^2 y w_1^2 + x^2 y w_1^2;$$
$$\tau_3 = \left[\left(1-y^2\right) xyw_1 w_2\right]_y - \left(1-y^2\right) x w_1 w_2 - \left(1-y^2\right) xy w_{1y} w_2;$$
$$\tau_4 = \left(1-y^2\right) w_{1y} xy w_2;$$
$$\tau_5 = -\frac{1}{2}\left[\left(1-y^2\right) xyw_2^2\right]_x + \frac{1}{2}\left(1-y^2\right) y w_2^2.$$

We have

$$\alpha = \frac{y}{2}\left(3x^2-1\right), \; \gamma = \frac{y}{2}\left(1-y^2\right),$$

and

$$2\beta = -\left(1-y^2\right) x,$$

yielding

$$\alpha\gamma - \beta^2 = \frac{y^2\left(1-y^2\right)}{4}\left(3x^2 - \frac{1-y^2}{y^2} x^2 - 1\right).$$

Because R_o is constructed so that

$$\frac{1-y^2}{y^2} < 1 - \delta < 1, \tag{24}$$

it is sufficient to show that

$$2x^2 - 1 > 0,$$

which also follows from the construction of R_o. Now

$$-2\beta w_1 w_2 \geq -\frac{\sqrt{1-y^2}}{2}\left[x^2 w_1^2 + \left(1-y^2\right) w_2^2\right]. \tag{25}$$

Taking square roots in (24), and applying the result to (25) yields

$$-2\beta w_1 w_2 > -\frac{\sqrt{1-\delta}}{2}\left[yx^2 w_1^2 + y\left(1-y^2\right) w_2^2\right].$$

Thus we have

$$(L_o w, xyw) \geq \frac{\varepsilon\left(1-\sqrt{1-\delta}\right)}{2\sqrt{2-\delta}} \|w\|_2^2 .$$

The remainder of the existence proof is exactly analogous to the arguments for the preceding cases.

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