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The overdetermined Cauchy problem in some classes of ultradifferentiable functions

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Abstract. In this paper we study the Cauchy problem for overdetermined systems of linear partial differential operators with constant coefficients in some spaces of ultradifferentiable functions of class (M_p). We show that evolution is equivalent to the validity of a Phragmén-Lindelöf principle for entire and plurisubharmonic functions on some irreducible affine algebraic varieties, and make applications in different situations. We find necessary and sufficient conditions for well posedness, and relate the hyperbolicity of a given system to that of its principal part.

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1. Introduction

In this paper we continue the study of the Cauchy problem for overdetermined systems of linear partial differential operators, following [32], [34], [10], [11]. Here we are concerned with spaces of ultradifferentiable functions of class (M_p). The consideration of these classes is natural when we consider perturbations of evolution problems and well posedness for systems with a hyperbolic principal part.

As in [11], we formulate the Cauchy problem for general pairs (K_1, K_2) of closed subsets of $\mathbb{R}^N$ with $K_1 \subset K_2$:

(C.P.) *given Whitney sections f on K_2 and u on K_1, with $A(D)u = f|_{K_1}$, and knowing that f and u belong to classes $\mathfrak{F}(K_2)$, $\mathfrak{U}(K_1)$ of Whitney ultradifferentiable functions, find a Whitney section $\tilde{u}$ in K_2, belonging to a given class $\tilde{\mathfrak{U}}(K_2)$ of ultradifferentiable Whitney functions, satisfying $A(D)\tilde{u} = f$ in K_2 and $\tilde{u}|_{K_1} = u$.*

This formulation, involving Whitney functions (and hence *formal power series* in the transversal directions when K_1 is a submanifold of $\mathbb{R}^N$), allows us to bypass the question of the coherence of the initial data that could be especially intricate

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in the overdetermined case. In this case we need only to require that the f in (C.P.) satisfies the natural *integrability conditions*, which in the case of systems with constant coefficients can be expressed by a suitable system of homogeneous equations $B(D)f = 0$: the pair (u, f) of $f \in \mathfrak{F}(K_2)$, $u \in \mathfrak{U}(K_1)$ with $B(D)f = 0$ and $A(D)u = f|_{K_1}$ will be called an *admissible pair*.

As in [11], we shall say that the pair (K_1, K_2) is of *evolution* (and of *causality*, *hyperbolic*) in the class $(\mathfrak{U}(K_1), \mathfrak{F}(K_2); \tilde{\mathfrak{U}}(K_2))$ for $A(D)$ if (C.P.) has at least one solution (and at most one solution, one and only one solution respectively) for each pair of admissible data.

In general it is convenient to insert the transpose of the total symbol of an overdetermined system of partial differential operators with constant coefficients into a Hilbert resolution of a finitely generated unitary module $\mathfrak{M}$ over the ring $\mathbb{C}[\zeta_1, ..., \zeta_N]$ of polynomials in N indeterminates.

Then evolution, causality and hyperbolicity in the different classes of Whitney functions we consider translate into the vanishing of some Ext groups for the module $\mathfrak{M}$. This formulation enlightens the algebraic invariance of the problem. It also makes natural some algebraic reductions to simpler systems of partial differential equations. In particular in most cases (C.P.) for a *vector valued* $\tilde{u}$ may be reduced to a (C.P.) involving only a *scalar* unknown function.

In most of our applications, K_1 will be contained in an affine n-dimensional subspace of a Euclidean space $\mathbb{R}^N$, $N = n + k$, $k > 0$. With $\mathbb{R}^N \simeq \mathbb{R}^k_t \times \mathbb{R}^n_x$, we shall take $K_1 \subset \Sigma = \{t = 0\} \simeq \mathbb{R}^n_x$ and consider data (u, f) and solutions $\tilde{u}$ either in the class $W^{(M_p)}$ of Whitney functions which are ultradifferentiable of class (M_p) in both variables t and x, or, allowing different scales of regularity in t and x, in the topological tensor products $\widetilde{W}^{(M_p)}$ and $W^{(N_p),(M_p)}$, the first not requiring ultradifferentiability in the t-variables, the second, (N_p) ultradifferentiability in t. The t-regularity of the initial datum u actually imposes restrictions on the rate of growth of the coefficients of its formal Taylor series in the t-variables along K_1.

In the study of existence and uniqueness of the Cauchy problem we found convenient to introduce, in an intrinsic algebraic setting, the notion of a *formally non-characteristic*, *strongly non-characteristic*, *free* and *quasi-free* affine subspace Σ for a unitary finitely generated $\mathbb{C}[\zeta_1, \ldots, \zeta_N]$-module $\mathfrak{M}$.

A key step is the fact that evolution is equivalent to the validity of a Phragmén–Lindelöf principle for holomorphic functions or, equivalently, for plurisubharmonic functions, on some associated irreducible affine algebraic varieties. From this we obtain, for instance, the sufficiency of a Petrowski-type condition for evolution, when Σ is formally non-characteristic and quasi-free, in the class $\tilde{\gamma}^{(s)}$ of smooth functions which belong uniformly, with their t-derivatives, to the (small) Gevrey class of order $s > 1$ in x, or in the class $\gamma^{(r,s)}$ of (small) Gevrey functions of order $r > 1$ in t and $s > 1$ in x for $r \geq bs$, where b is the reduced order of the system (cf. [19]).

Then we investigate the well-posedness of the given Cauchy problem (C.P.) for a general pair of convex sets (K_1, K_2), obtaining necessary and sufficient geometric conditions for hyperbolicity.

In the special case in which $K_1 = \Sigma$ and K_2 is a closed wedge with edge equal to Σ, we obtain a Petrowski-type sufficient condition for hyperbolicity, when the reduced order of the system is less than or equal to 1 and Σ is formally non-characteristic and quasi-free.

The main reason to consider (C.P.) for classes of ultradifferentiable functions was for us to understand the behaviour of a system which is *close* to one for which the pair (K_1, K_2) is of evolution, or hyperbolic. It is known that the principal part of a hyperbolic system is always hyperbolic, while the reverse is, in general, false. Generalising a classical result (cf. [22]), (and precising the notion of principal part in the overdetermined case), we prove that there is a rational $\sigma_0 \in [0, 1)$ such that the hyperbolicity of its principal part implies that of the system in the classes $\widetilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$, for $1 < s \leq \frac{1}{\sigma_0}$, $r > 1$. Moreover, following [32], we give a necessary and sufficient condition for hyperbolicity in terms of a homogeneous cone associated to the characteristic variety of the given system, which generalises in the overdetermined case the propagation cones of classical hyperbolicity.

For non-evolution pairs, the question of finding the right necessary and sufficient conditions for causality for a Cauchy problem with initial data on a submanifold of co-dimension larger than one remains largely an open question. Here we showed that for the pair (Σ, H) of a *hypersurface* Σ and a half-space H, causality in the class of ultradifferentiable functions is equivalent to causality in the class of smooth functions and to Σ being strongly non-characteristic for the system.

2. Algebraic preliminaries and notation

A) *Reduction to the associated prime ideals*

We denote by $\mathscr{P}$ the unitary commutative ring $\mathbb{C}[\zeta_1, \ldots, \zeta_N]$ of polynomials with complex coefficients in N indeterminates. For a unitary finitely generated $\mathscr{P}$-module $\mathfrak{M}$ we denote by $\mathrm{Spec}(\mathfrak{M})$, $\mathrm{Ann}(\mathfrak{M})$ and $\mathrm{Supp}(\mathfrak{M})$, respectively, the spectrum, the annihilator and the support of $\mathfrak{M}$.

In most cases differential problems involving a $q \times p$ matrix $A(D)$ of partial differential operators with constant coefficients can be more invariantly formulated in terms of the unitary $\mathscr{P}$-module $\mathfrak{M} = \mathrm{coker}\, {}^t A(\zeta) : \mathscr{P}^q \longrightarrow \mathscr{P}^p$. Here $A(\zeta)$ is the matrix with polynomial entries obtained by the formal substitution $\partial/\partial x_j \longrightarrow i\zeta_j$. A further useful reduction is to substitute to the consideration of $\mathfrak{M}$ that of the $\mathscr{P}$-modules $\mathscr{P}/\mathfrak{p}$ for the prime ideals $\mathfrak{p}$ belonging to $\mathrm{Ass}(\mathfrak{M})$. In several interesting cases, and namely in investigations concerning the Cauchy problem, this can be achieved by employing the following propositions (see [32] and [11]):

Proposition 2.1. *Let $\mathfrak{M}$ and $\mathfrak{F}$ be unitary $\mathscr{P}$-modules, with $\mathfrak{M}$ of finite type, and let p be a nonnegative integer. Then the following statements are equivalent:*

$$
\begin{array}{llll}
(i) & \mathrm{Ext}^j_{\mathscr{P}}(\mathfrak{M}, \mathfrak{F}) = 0 & \forall j \leq p & \\
(ii) & \mathrm{Ext}^j_{\mathscr{P}}(\mathscr{P}/\mathfrak{p}, \mathfrak{F}) = 0 & \forall j \leq p, & \forall \mathfrak{p} \in \mathrm{Ass}(\mathfrak{M}) \\
(iii) & \mathrm{Ext}^j_{\mathscr{P}}(\mathscr{P}/\mathfrak{p}, \mathfrak{F}) = 0 & \forall j \leq p, & \forall \mathfrak{p} \in \mathrm{Supp}(\mathfrak{M}) \\
(iv) & \mathrm{Ext}^j_{\mathscr{P}}(\mathfrak{L}, \mathfrak{F}) = 0 & \forall j \leq p, & \forall \mathscr{P} - \textit{submoldule } \mathfrak{L} \subset \mathfrak{M}.
\end{array}
$$

Proposition 2.2. *Let $\mathfrak{M}$ and $\mathfrak{F}$ be unitary $\mathcal{P}$-modules, with $\mathfrak{M}$ of finite type, and let p be a nonnegative integer. Assume moreover that*

$$\mathrm{Ext}^j_{\mathcal{P}}(\mathcal{P}/\mathfrak{p}, \mathfrak{F}) = 0 \quad \forall j > p, \quad \forall \mathfrak{p} \in \mathrm{Supp}(\mathfrak{M}).$$

Then the following statements are equivalent:

$$\begin{array}{rlll} (i) & \mathrm{Ext}^j_{\mathcal{P}}(\mathfrak{M}, \mathfrak{F}) = 0 & \forall j \geq p & \\ (ii) & \mathrm{Ext}^j_{\mathcal{P}}(\mathcal{P}/\mathfrak{p}, \mathfrak{F}) = 0 & \forall j \geq p, & \forall \mathfrak{p} \in \mathrm{Ass}(\mathfrak{M}) \\ (iii) & \mathrm{Ext}^j_{\mathcal{P}}(\mathfrak{L}, \mathfrak{F}) = 0 & \forall j \geq p, & \forall \mathcal{P} - \textit{submoldule } \mathfrak{L} \subset \mathfrak{M}. \end{array}$$

B) *Differential $\mathcal{P}$-modules*

Let $\mathfrak{F}$ be a $\mathbb{C}$-linear space of (Whitney) functions or (ultra)distributions defined on a subset of $\mathbb{R}^N$ and such that $\partial f/\partial x_j \in \mathfrak{F}$ for every $f \in \mathfrak{F}$ and $j = 1, \dots, N$. To every polynomial $p(\zeta) = \sum_{|\alpha| \leq m} a_\alpha \zeta^\alpha$ in $\mathcal{P}$, we associate the differential operator with constant coefficients $p(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, where, as usual, for a multi-index $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$, we set $|\alpha| = \alpha_1 + ... + \alpha_N$, $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_N^{\alpha_N}$,$D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$ and $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$ for $j = 1, ..., N$. We consider $\mathfrak{F}$ as a unitary $\mathcal{P}$-module by letting $p(\zeta) \in \mathcal{P}$ act on $f \in \mathfrak{F}$ by:

$$p(\zeta) f = f p(\zeta) = p(D) f. \tag{2.1}$$

We call such an $\mathcal{F}$ a *differential $\mathcal{P}$-module*.

Let be given an $a_1 \times a_0$ matrix $A_0(D)$ of linear partial differential operators with constant coefficients in $\mathbb{R}^N$. Let $\mathfrak{F}$ be a differential $\mathcal{P}$-module. To solve the system

$$\begin{cases} u \in \mathfrak{F}^{a_0} \\ A_0(D) u = f \in \mathfrak{F}^{a_1} \end{cases} \tag{2.2}$$

it is necessary that the right-hand side f satisfies suitable integrability conditions. To take those into account, it is convenient to insert the $\mathcal{P}$-homomorphism ${}^t A_0(\zeta) : \mathcal{P}^{a_1} \longrightarrow \mathcal{P}^{a_0}$ into a Hilbert resolution

$$\begin{array}{l} 0 \longrightarrow \mathcal{P}^{a_d} \xrightarrow{{}^t A_{d-1}(\zeta)} \mathcal{P}^{a_{d-1}} \longrightarrow \cdots \\ \cdots \longrightarrow \mathcal{P}^{a_2} \xrightarrow{{}^t A_1(\zeta)} \mathcal{P}^{a_1} \xrightarrow{{}^t A_0(\zeta)} \mathcal{P}^{a_0} \longrightarrow \mathfrak{M} \longrightarrow 0 \end{array} \tag{2.3}$$

of the unitary finitely generated $\mathcal{P}$-module $\mathfrak{M} = \mathrm{coker}\left({}^t A_0(\zeta) : \mathcal{P}^{a_1} \longrightarrow \mathcal{P}^{a_0}\right)$. The rows of the matrix $A_1(D)$ give a system of generators for the module of all integrability conditions for f in (2.2) that can be expressed in terms of partial differential operators. By the natural isomorphisms $\mathrm{Hom}_{\mathcal{P}}(\mathcal{P}^{a_i}, \mathfrak{F}) \simeq \mathfrak{F}^{a_i}$ we obtain the isomorphisms:

$$\begin{cases} \mathrm{Ext}^0_{\mathcal{P}}(\mathfrak{M}, \mathfrak{F}) \simeq \left\{ f \in \mathfrak{F}^{a_0} \,\middle|\, A_0(D) f = 0 \right\}, \\ \\ \mathrm{Ext}^j_{\mathcal{P}}(\mathfrak{M}, \mathfrak{F}) \simeq \dfrac{\ker\left(A_j(D) : \mathfrak{F}^{a_j} \longrightarrow \mathfrak{F}^{a_{j+1}}\right)}{\mathrm{Image}\left(A_{j-1}(D) : \mathfrak{F}^{a_{j-1}} \longrightarrow \mathfrak{F}^{a_j}\right)}, \qquad \text{for} \quad j \geq 1. \end{cases} \tag{2.4}$$

Thus uniqueness for (2.2) is equivalent to $\mathrm{Ext}^0_{\mathcal{P}}(\mathfrak{M}, \mathfrak{F}) = 0$; its solvability for every right-hand side $f \in \mathfrak{F}^{a_1}$ satisfying the integrability conditions $A_1(D)f = 0$ translates into the equation $\mathrm{Ext}^1_{\mathcal{P}}(\mathfrak{M}, \mathfrak{F}) = 0$. This shows that these are properties of the $\mathcal{P}$-module $\mathfrak{M}$ rather than of the given matrix $A_0(D)$. Proposition 2.2 shows that, even when we are only interested in solving (2.2), the consideration of higher order Ext groups could be important.

C) *Some algebraic notions related to the Cauchy problem*

In most of our applications we shall be interested in the Cauchy problem with initial data on an affine subspace Σ of $\mathbb{R}^N$. After a suitable choice of coordinates, we shall assume that $\mathbb{R}^N = \mathbb{R}^k_t \times \mathbb{R}^n_x$, and that

$$\Sigma = \left\{(t, x) \in \mathbb{R}^N \,\middle|\, t = 0\right\} \simeq \mathbb{R}^n_x \subset \mathbb{R}^N. \tag{2.5}$$

We denote by $(t, x) = (t_1, ..., t_k, x_1, ..., x_n)$ the coordinates in $\mathbb{R}^N$ and by $(\tau, \zeta) = (\tau_1, ..., \tau_k, \zeta_1, ..., \zeta_n) \in \mathbb{C}^N = \mathbb{C}^k_\tau \times \mathbb{C}^n_\zeta$ their *dual* coordinates in $\mathbb{C}^N$.

We consider $\mathcal{P}_n = \mathbb{C}[\zeta_1, ..., \zeta_n]$ as a unitary subring of $\mathcal{P}$. Given a $\mathcal{P}$-module $\mathfrak{M}$ we denote by $(\mathfrak{M})_n$ the set $\mathfrak{M}$ considered as a $\mathcal{P}_n$-module by change of the base ring.

Let $\mathfrak{M}$ be a unitary finitely generated $\mathcal{P}$-module. We say that Σ is *formally noncharacteristic* for $\mathfrak{M}$ if $(\mathfrak{M})_n$ is a $\mathcal{P}_n$-module of finite type (see [3], [12], [9]). For a matrix $A_0(D)$ such that $\mathfrak{M} = \operatorname{coker} {}^tA_0(\zeta)$, this condition means that every formal power series solution $u = \sum_{\alpha \in \mathbb{N}^k} u_\alpha(x) t^\alpha$ in $t_1, ..., t_n$, with coefficients $u_\alpha(x)$ that are (vector valued) smooth functions of $x_1, ..., x_n$ in $\mathbb{R}^n$, of the equation $A_0(D)u = 0$, is completely determined by a finite number of its coefficients $u_\alpha(x)$.

We also note that this notion is an algebraically invariant formulation, valid for the case of constant coefficients, of the one introduced originally in [1] (cf. [5] for a further discussion of this topic).

We shall use the following characterization, involving the natural projection map:

$$\pi_n : \mathbb{C}^N \ni (\tau, \zeta) \longrightarrow \zeta \in \mathbb{C}^n. \tag{2.6}$$

Proposition 2.3. *The following conditions are equivalent, for a finitely generated unitary $\mathcal{P}$-module $\mathfrak{M}$:*

(1) *Σ is formally noncharacteristic for $\mathfrak{M}$;*
(2) *let $V(\mathfrak{p})$ be the irreducible affine algebraic variety in $\mathbb{C}^N$ associated to a prime $\mathfrak{p}$ in $\mathrm{Ass}(\mathfrak{M})$; then $V_n(\mathfrak{p}) = \pi_n(V(\mathfrak{p}))$ is an irreducible affine algebraic variety in $\mathbb{C}^n$ and the map*

$$V(\mathfrak{p}) \ni (\tau, \zeta) \longrightarrow \zeta \in V_n(\mathfrak{p}) \subset \mathbb{C}^n \tag{2.7}$$

is surjective, finite and dominant;
(3) *Σ is formally noncharacteristic for every unitary $\mathcal{P}$-submodule $\mathfrak{L}$ of $\mathfrak{M}$;*
(4) *Σ is formally noncharacteristic for $\mathcal{P}/\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$;*
(5) *Σ is formally noncharacteristic for $\mathcal{P}/\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Supp}(\mathfrak{M})$;*

(6) *there exist real constants* λ, b *such that, for every* $\mathfrak{p} \in \mathrm{Supp}(\mathfrak{M})$, *we have:*

$$|\tau| \le \lambda(1 + |\zeta|)^b \quad \forall(\tau, \zeta) \in V(\mathfrak{p}). \tag{2.8}$$

Under the assumption that Σ is formally noncharacteristic for $\mathfrak{M}$, the Tarski–Seidenberg theorem implies that there exists a smallest b, which is a rational number, such that (2.8) is valid. We call this number b the *reduced order of* Σ *for* M.

In the case that (2.8) holds true, for every $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$, with a reduced order $b \le 1$, we say that Σ is *strongly noncharacteristic* for $\mathfrak{M}$. This is the necessary and sufficient condition for the well-posedness of the analytic Cauchy problem with initial data on Σ (see for instance [3]).

The *initial data* for an overdetermined Cauchy problem must in general satisfy some tangential differential equations (see [3] for the case of constant coefficients, or [29] for the example of pluriharmonic functions). We now consider algebraic conditions that guarantee that initial data can be chosen arbitrarily.

We say that Σ is *free* for $\mathfrak{M}$ if $(\mathfrak{M})_n$ is a free $\mathcal{P}_n$-module. This notion is not stable for passing from the primary $\mathcal{P}$-submodules of $\mathfrak{M}$ to $\mathfrak{M}$. It is therefore more convenient to introduce a more general notion: we say that Σ is *quasi-free* for $\mathfrak{M}$ if $(\mathfrak{M})_n$ is a torsion free $\mathcal{P}_n$-module. We have:

Proposition 2.4. *The following conditions are equivalent, for a finitely generated unitary* $\mathcal{P}$*-module* $\mathfrak{M}$*:*

(1) Σ *is quasi-free for* $\mathfrak{M}$*;*
(2) Σ *is quasi-free for every unitary* $\mathcal{P}$*-submodule* $\mathfrak{L}$ *of* $\mathfrak{M}$*;*
(3) Σ *is quasi-free for* $\mathcal{P}/\mathfrak{p}$ *for every* $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$*;*
(4) *for every* $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$*, the image* $V_n(\mathfrak{p})$ *of the irreducible affine algebraic variety* $V(\mathfrak{p})$ *is dense in* $\mathbb{C}^n$*.*

We note that in (4) we can consider equally the Euclidean or the Zariski topology of $\mathbb{C}^n$. The notions of free and quasi-free do not require that Σ be formally noncharacteristic for $\mathfrak{M}$: if we add this assumption we obtain that $V_n(\mathfrak{p}) = \mathbb{C}^n$ for every $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$.

Example. Consider $\mathbb{R}^2 = \mathbb{R}_t \times \mathbb{R}_x$ and let $\mathfrak{p}$ be the principal ideal in $\mathcal{P} = \mathbb{C}[\tau, \zeta]$ generated by the polynomial $\tau\zeta + 1$. Then $\Sigma = \{(t, x) \in \mathbb{R}^2 \,|\, t = 0\}$ is quasi-free but not free for $\mathcal{P}/\mathfrak{p}$. Note that in this case $(\mathcal{P}/\mathfrak{p})_1$ is isomorphic to the field $\mathbb{C}(\zeta)$ of rational functions. Hence it is not a finitely generated $\mathbb{C}[\zeta]$-module and Σ is not formally noncharacteristic for $\mathcal{P}/\mathfrak{p}$.

D) *The asymptotic variety*

Let $\mathcal{P} = \mathbb{C}[\zeta_1, \ldots, \zeta_N]$ as before. We introduce a new indeterminate ζ_0 and denote by $\tilde{\mathcal{P}}$ the ring $\mathbb{C}[\zeta_0, \zeta_1, \ldots, \zeta_N]$ of polynomials (with complex coefficients) in $N+1$ indeterminates. We define two maps:

$$\mathcal{P} \ni p \longrightarrow p^o \in \mathcal{P} \text{ (principal part) and } \mathcal{P} \ni p \longrightarrow \tilde{p} \in \tilde{\mathcal{P}} \text{ (homogenization)}$$

in the following way: both the principal part and the homogenization of the zero polynomial are the corresponding zero polynomial; if $p(\zeta) = \sum_{|\alpha|\le m} a_\alpha \zeta^\alpha \neq 0$ has degree $m \ge 0$, we set, respectively,

(2.9) $$p^o(\zeta) = \sum_{|\alpha|=m} a_\alpha \zeta^\alpha \qquad \text{and} \qquad \tilde{p}(\zeta_0, \zeta) = \sum_{|\alpha|\le m} a_\alpha \zeta_0^{m-|\alpha|} \zeta^\alpha .$$

Note that the two maps preserve multiplication and degree, but they do not in general preserve sums.

Lemma 2.5. *Let $\mathfrak{p}$ be an ideal in $\mathcal{P}$. Denote by $\mathfrak{p}^o$ the ideal in $\mathcal{P}$ which is generated by the principal parts of polynomials in $\mathfrak{p}$. Assume that $p_1, \ldots, p_r \in \mathfrak{p}$ and $p_1^o, \ldots, p_r^o$ generate $\mathfrak{p}^o$. Then:*

(i) $p_1, \ldots, p_r$ *generate* $\mathfrak{p}$*;*
(ii) $\tilde{p}_1, \ldots, \tilde{p}_r$ *generate an ideal* $\tilde{\mathfrak{p}}$ *in* $\tilde{\mathcal{P}}$ *which coincides with the ideal of* $\tilde{\mathcal{P}}$ *generated by the homogenizations of the polynomials of* $\mathfrak{p}$*;*
(iii) $\mathfrak{p}$ *is prime in* $\mathcal{P}$ *if and only if* $\tilde{\mathfrak{p}}$ *is prime in* $\tilde{\mathcal{P}}$*.*

Note that $\mathfrak{p}^o$ need not to be prime when $\mathfrak{p}$ is prime: for instance, when $N = 2$, the principal ideal generated by $p(\zeta_1, \zeta_2) = \zeta_1\zeta_2 + 1$ is prime, whereas $\mathfrak{p}^o$, which is the principal ideal generated by $p^o(\zeta_1, \zeta_2) = \zeta_1\zeta_2$, is not prime.

The easy proof of this lemma can be obtained by induction on the degree.

In the following we assume that $V = V(\mathfrak{p})$ is the irreducible affine algebraic variety associated to a prime ideal $\mathfrak{p} \subset \mathcal{P}$:

(2.10) $$V = V(\mathfrak{p}) = \left\{ \zeta \in \mathbb{C}^N \mid p(\zeta) = 0 \quad \forall p \in \mathfrak{p} \right\}.$$

The homogeneous affine algebraic variety,

(2.11) $$V^o = V^o(\mathfrak{p}) = \left\{ \zeta \in \mathbb{C}^N \mid p^o(\zeta) = 0 \quad \forall p \in \mathfrak{p} \right\},$$

of common zeros in $\mathbb{C}^N$ of the principal parts of polynomials in $\mathfrak{p}$ will be called the (reduced) *asymptotic variety* of V (or of $\mathfrak{p}$). We shall also consider the *projective completion* of V, defined by

(2.12) $$\tilde{V} = \tilde{V}(\mathfrak{p}) = \left\{ (\zeta_0, \zeta) \in \mathbb{CP}^N \mid \tilde{p}(\zeta_0, \zeta) = 0 \quad \forall p \in \mathfrak{p} \right\}.$$

Here we have used $(\zeta_0, \zeta) = (\zeta_0, \zeta_1, \ldots, \zeta_N)$ as homogeneous coordinates in $\mathbb{CP}^N$. The map $\iota : V \ni \zeta \longrightarrow (1, \zeta) \in \tilde{V}$ is one to one from V to the set of points of $\tilde{V}$ which do not belong to the hyperplane at infinity $\{\zeta_0 = 0\}$.

Lemma 2.6. *$\tilde{V}$ is the closure of $\iota(V)$ in $\mathbb{CP}^N$.*

Proof. Indeed this follows from the fact that $\tilde{V}$ is irreducible by Lemma 2.5. A point of $\tilde{V}$ not belonging to $\iota(V)$ must belong to the hyperplane $\zeta_0 = 0$. If it does not belong to the closure of $\iota(V)$ (either for the usual or for the Zariski topology), then it belongs to a component of $\tilde{V}$ which is all contained in $\zeta_0 = 0$. But there is no such a component because $\tilde{V}$ is irreducible.

Lemma 2.7. *We have:*

$$V^o = \left\{ \zeta \in \mathbb{C}^N \;\middle|\; \begin{array}{l} \exists\{\zeta^\nu\} \subset V,\ \exists\{\epsilon_\nu\} \subset \mathbb{R} \\ \textit{with } \epsilon_\nu > 0,\ \epsilon_\nu \longrightarrow 0 \end{array} \textit{ such that } \epsilon_\nu \zeta^\nu \longrightarrow \zeta \right\}. \tag{2.13}$$

Proof. First of all it is easy to show that the set defined by the right-hand side of (2.13) is contained in V^o: if $\zeta = \lim_{\nu\longrightarrow\infty} \epsilon_\nu \zeta^\nu$ with $\zeta^\nu \in V$, and $\epsilon_\nu \longrightarrow 0$ we have

$$0 = \epsilon_\nu^m p(\zeta^\nu) = p^0(\epsilon_\nu\zeta^\nu) + \epsilon_\nu p^{(1)}(\epsilon_\nu\zeta^\nu) + \cdots + \epsilon_\nu^{m-1} p^{(m-1)}(\epsilon_\nu\zeta^\nu) + \epsilon_\nu^m p^{(m)}$$

for every polynomial p of degree m in $\mathfrak{p}$, where we have used the notation $p^{(j)}$ for the homogeneous part of degree $m-j$ of p. Passing to the limit for $\nu \longrightarrow \infty$ we obtain $p^o(\zeta) = 0$. This proves the first inclusion.

To prove the opposite inclusion, we introduce the cone W in $\mathbb{C}^{N+1}$ corresponding to the projective variety $\tilde{V}$:

$$W = \left\{ (\zeta_0, \zeta) \in \mathbb{C}^{N+1} \;\middle|\; \tilde{p}(\zeta_0, \zeta) = 0 \quad \forall p \in \mathfrak{p} \right\}.$$

Using the preparation lemma (see, for instance, [4]), after a linear change of the coordinates $\zeta_1, \ldots, \zeta_N$ in $\mathbb{C}^N$, we can assume that the map

$$W \ni (\zeta_0, \zeta) \longrightarrow (\zeta_0, \zeta_1, \ldots, \zeta_d) \in \mathbb{C}^{d+1}$$

is finite, dominant and surjective. Since every neighbourhood of a point $(0, \theta)$ of W projects onto a full open neighbourhood of $(0, \theta_1, ..., \theta_d)$ in $\mathbb{C}^{d+1}$, we can choose a sequence $\{(\epsilon_\nu, \theta^\nu)\}$ such that ϵ_ν is real and positive and $\epsilon_\nu \longrightarrow 0$, $\theta^\nu \longrightarrow \theta$. Since every point θ of V^o corresponds to a point $(0, \theta)$ of W, and $\zeta^\nu = \epsilon_\nu^{-1}\theta^\nu \in V$, we have proved the opposite inclusion, completing the proof of the lemma.

The function

$$f(r) = \inf\left\{ |\zeta - \theta| \mid \zeta \in V,\ \theta \in V^o,\ |\zeta| \le r \right\} \tag{2.14}$$

is a semi-algebraic function of $r \in \mathbb{R}$. When V has positive dimension it is defined for all $r \gg 1$: thus we have, with an exponent $\sigma \in \mathbb{Q}$ and a constant $C \ge 0$:

$$f(r) = C\, r^\sigma\, (1 + o(1)) \qquad \text{for} \quad r \gg 1.$$

We obtain

Lemma 2.8. *Assume that V has positive dimension. Then there is a smallest nonnegative real number σ such that, for some constants $C_1, C_2 > 0$, we have:*

$$\operatorname{dist}(\zeta, V^o) \le C_1 |\zeta|^\sigma + C_2 \qquad \forall \zeta \in V. \tag{2.15}$$

The number σ is rational and strictly smaller than 1.

Proof. Indeed the characterization of the asymptotic variety given in the previous lemma shows that the exponent σ in (2.15), which is rational by the Tarski–Seidenberg theorem, must be less than one.

The exponent σ measures somehow how far $V = V(\mathfrak{p})$ is from being a homogeneous cone. It will enter into perturbation arguments related to the Cauchy problem. It is therefore useful to have a general a priori bound from above for σ. Assume that $\dim_{\mathbb{C}} V = d > 0$. The *degree* of V is defined as the number of points of the generic intersection of V with an $(N - d)$ affine subspace of $\mathbb{C}^N$. We have:

Lemma 2.9. *Let V be an irreducible affine algebraic variety in $\mathbb{C}^N$, of positive dimension and of degree m. Then the exponent σ in (2.15) can be taken as less or equal than $(m-1)/m$.*

Proof. We use the preparation lemma in [4], to arrange affine coordinates in $\mathbb{C}^N$ in such a way that the map

$$\pi : V \ni \zeta \longrightarrow \zeta' = (\zeta_1, \dots, \zeta_d) \in \mathbb{C}^d$$

induced by the projection is surjective and we have for some positive constant C:

$$|\zeta| \le C\,(1 + |\zeta'|) \qquad \forall \zeta \in V.$$

Moreover, we can assume that $\mathfrak{p}$ contains a monic polynomial Q of degree m of the form:

$$Q(\zeta', \zeta_N) = \zeta_N^m + q_1(\zeta')\zeta_N^{m-1} + \dots + q_{m-1}(\zeta')\zeta_N + q_m(\zeta')$$

with q_j of degree $\le j$ in ζ'; here m is the degree of $V = V(\mathfrak{p})$. Denote by $\Delta(\zeta')$ the discriminant of Q with respect to the ζ_N variable and by S the set of zeros in $\mathbb{C}^d$ of Δ. Then $V \setminus S$ is an m-fold covering of $\mathbb{C}^d \setminus S$; for each $j = d+1, \dots, N-1$ there is a polynomial $P_j \in \mathbb{C}[\zeta_1, \dots, \zeta_d, \zeta_N]$ such that

$$\Delta(\zeta')\zeta_j - P_j(\zeta', \zeta_N) \in \mathfrak{p} \qquad \text{for} \quad d+1 \le j \le N-1.$$

Moreover, the degree of P_j is less or equal to the degree of Δ plus 1. Since the statement of the lemma is known in the case where V is a hypersurface (i.e. $\mathfrak{p}$ is principal: see [19]), the estimate for σ is clear if we stay away from the set S of zeros of Δ and the set S^o of zeros of Δ^o. (Note that Δ^o is the discriminant of Q^o with respect to ζ_N.) This we can do by considering the coordinates $\zeta_{d+1}, \dots, \zeta_{N-1}$ on V and V^o as algebraic functions of ζ' and computing their value at the points over S and S^o by averaging on circles at a distance ≥ 1 from $S \cup S^o$.

Let us go back to the situation of point C), in particular using coordinates t, x in $\mathbb{R}^N$ and dual coordinates τ, ζ in $\mathbb{C}^N$. We fix a unitary finitely generated $\mathscr{P}$-module $\mathfrak{M}$ and consider the affine space Σ of (2.5). The condition of Σ being strongly noncharacteristic for $\mathfrak{M}$ is equivalent to the fact that the asymptotic varieties $V^o(\mathfrak{p})$, for $\mathfrak{p} \in \mathrm{Ass}(M)$, do not contain elements of the form $(\tau, 0)$ with $\tau \neq 0$.

There is a weaker notion of noncharacteristic which ensures uniqueness in the Cauchy problem (in the smooth category) for $\mathfrak{M}$ with initial data on Σ and solutions defined in a full neighbourhood of Σ: namely it suffices, according to a classical theorem of F. John (see also [9]), that no variety $V^o(\mathfrak{p})$, for $\mathfrak{p} \in \mathrm{Ass}(M)$, contains an element of the form $(\tau, 0)$ with $\tau \in \mathbb{R}^k \setminus \{0\}$. This notion coincides with that of strongly noncharacteristic in the case $k = 1$, but the two concepts are different when $k > 1$.

3. Some spaces of ultradifferentiable functions

A) *Ultradifferentiable functions of class* (M_p)

We revise in this section some definitions and properties of spaces of ultradifferentiable functions, for which we refer to [26] and [27].

Let $\{M_p\}_{p\in\mathbb{N}}$ be a sequence of positive numbers such that

$$\text{(M)}\qquad \begin{cases} \text{(M.0)} & M_0 = 1 \\ \text{(M.1)} & M_p^2 \leq M_{p-1}M_{p+1} \quad \forall p \geq 1 \\ & \qquad \text{(logarithmic convexity)} \\ \text{(M.2)} & \exists A,\ H\ :\ M_p \leq AH^p \min\limits_{0\leq q\leq p} M_q M_{p-q} \quad \forall p \geq 0 \\ & \qquad \text{(stability under ultradifferential operators)} \\ \text{(M.3)} & \exists A\ :\ \sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leq A\ p\frac{M_p}{M_{p+1}} \quad \forall p \geq 1 \\ & \qquad \text{(strong non-quasi-analyticity).} \end{cases}$$

We shall assume, as we can, that the constants A, H in (M) are > 1, and that the constant A in (M2) and (M3) is the same.

Let Ω be an open subset of $\mathbb{R}^N$. We denote by $\mathcal{E}(\Omega)$ the space of complex valued smooth functions on Ω, endowed by its usual Fréchet–Schwartz topology. It is a differential $\mathcal{P}$-module and is injective when Ω is convex (see [17], [35], [28], [2]).

The space $\mathcal{E}^{(M_p)}(\Omega)$ of *ultradifferentiable functions of class* (M_p) in Ω is defined by

$$(3.1)\qquad \mathcal{E}^{(M_p)}(\Omega) = \left\{ f \in \mathcal{E}(\Omega) \,\middle|\, \forall K \Subset \Omega,\ \sup_{\alpha} \|D^\alpha f\|_K \big/ (\epsilon^{|\alpha|} M_{|\alpha|}) < \infty \right\}.$$

Here we denote by $\|f\|_K$ the sup-norm $\|f\|_K = \sup_{x\in K} |f(x)|$ on the compact K. We consider on $\mathcal{E}^{(M_p)}(\Omega)$ the topology defined by the family of seminorms

$$(3.2)\qquad p_{\epsilon,K}(f) = \sup_K \sum_{\alpha\in\mathbb{N}^N} \frac{|D^\alpha f(x)|}{\epsilon^{|\alpha|} M_{|\alpha|}} \qquad \text{for} \quad \epsilon > 0 \quad \text{and} \quad K \Subset \Omega.$$

The space $\mathcal{E}^{(M_p)}(\Omega)$ is then Fréchet–Schwartz, a Fréchet algebra for the multiplication of functions, a differential $\mathcal{P}$-module by (M.2). We note that (M.3) guarantees the existence of partitions of unity subordinated to any open covering of Ω; hence we have the notion of support for the elements of its strong dual $\mathcal{E}'^{(M_p)}(\Omega)$. This dual space is called the *space of ultradistributions of Beurling type*.

The *associated function* to the sequence $\{M_p\}$ is defined for $t > 0$ by:

$$(3.3)\qquad m(t) = \sup_p \log \frac{t^p}{M_p}.$$

We note that m is nondecreasing and that, by (M2),

$$m(k_1 t) + m(k_2 t) \le \log A + m\left(H(k_1 + k_2)t\right) \qquad \forall t, k_1, k_2 > 0. \tag{3.4}$$

We shall use the associated function $m(t)$ to construct the function

$$M(\zeta) = m(1 + |\zeta|) \qquad (\zeta \in \mathbb{C}^N); \tag{3.5}$$

it is plurisubharmonic, being the supremum of a sequence of plurisubharmonic functions, is nonnegative and uniformly bounded on compact subsets of $\mathbb{C}^N$. Most of the properties of $M(\zeta)$ are a consequence of the following:

Lemma 3.1. *With the constants $A, H > 1$ in (M), for all $k > 0$ we have:*

$$m(k\rho) - m(\rho) \ge \frac{\log(\rho/A) \cdot \log k}{\log H} \qquad \forall \rho > 0. \tag{3.6}$$

Lemma 3.2. *There is a constant $k_0 > 0$ such that*

$$\left|M(\zeta + \zeta') - M(\zeta)\right| \le k_0 \qquad \textit{if} \quad \zeta, \zeta' \in \mathbb{C}^N \quad \textit{and} \quad |\zeta'| \le 1. \tag{3.7}$$

Proof. Using (3.6) we obtain, when $\zeta, \zeta' \in \mathbb{C}^N$ and $|\zeta + \zeta'| \ge |\zeta| \gg 1$,

$$\left|M(\zeta + \zeta') - M(\zeta)\right| \le (\log H)^{-1} \log\left(\frac{1 + |\zeta|}{A}\right) \cdot \log\left(\frac{1 + |\zeta| + |\zeta'|}{1 + |\zeta|}\right).$$

Analogously, when $\zeta, \zeta' \in \mathbb{C}^N$ and $1 \ll |\zeta + \zeta'| < |\zeta|$ we obtain:

$$\left|M(\zeta + \zeta') - M(\zeta)\right| \le (\log H)^{-1} \log\left(\frac{1 + |\zeta + \zeta'|}{A}\right) \cdot \log\left(\frac{1 + |\zeta|}{1 + |\zeta + \zeta'|}\right).$$

When $|\zeta'| \le 1$ and $|\zeta| \longrightarrow \infty$, the right-hand sides of these inequalities approach 0 and thus our statement follows.

The proof of Lemma 3.2 also yields:

Lemma 3.3. *Let $M(\zeta)$ be the plurisubharmonic function in $\mathbb{C}^N$ of (3.5), associated to a sequence of positive real numbers $\{M_p\}_{p \in \mathbb{N}}$ satisfying conditions (M). Then for every $\epsilon > 0$ there is a constant $C_\epsilon \ge 0$ such that*

$$M(\zeta) \le \epsilon|\zeta| + C_\epsilon \qquad \forall \zeta \in \mathbb{C}^N. \tag{3.8}$$

Given a real $s > 1$, the *Gevrey sequence* $M_0 = 1$, $M_p = p^{ps}$ for $p > 0$, satisfies conditions (M). The corresponding space $\mathcal{E}^{(p^{ps})}(\Omega)$ is called the *(small) Gevrey class of order s*, and is denoted by $\gamma^{(s)}(\Omega)$. The associated function is in this case $m(t) = m_s(t) = e^{-1}\, s\, t^{1/s}$.

Fix a sequence $\{M_p\}$ satisfying conditions (M). We note that all exponential functions $x \longrightarrow \exp(\langle x,\ \zeta\rangle)$, with $\zeta \in \mathbb{C}^N$, belong to $\mathcal{E}^{(M_p)}(\mathbb{R}^N)$. Thus, given any $f \in \mathcal{E}'^{(M_p)}(\mathbb{R}^N)$ we can consider its Fourier–Laplace transform

$$\hat{f}(\zeta) = \langle \exp(-i\langle \cdot, \zeta\rangle), f\rangle \qquad \text{for} \quad \zeta \in \mathbb{C}^N. \tag{3.9}$$

The following Paley–Wiener-type theorem for ultradistributions is proved in [27]:

Theorem 3.4. *Let K be a convex compact subset of $\mathbb{R}^N$. Then the following conditions are equivalent for an entire function $U \in \mathcal{O}(\mathbb{C}^N)$:*

(a) $U(\zeta)$ is the Fourier–Laplace transform of an ultradistribution of Beurling type $u \in \mathcal{E}'^{(M_p)}(\mathbb{R}^N)$ with support in K;
(b) there exist positive constants c and L such that

$$|U(\zeta)| \leq c \exp\{M(L\zeta) + H_K(Im\zeta)\} \qquad \forall \zeta \in \mathbb{C}^N. \tag{3.10}$$

Here we denote by $H_K(\xi)$ the supporting function of the compact convex subset K of $\mathbb{R}^N$:

$$H_K(\xi) = \sup_{x \in K} \langle x, \xi \rangle \qquad \text{for} \quad \xi \in \mathbb{R}^N. \tag{3.11}$$

In order to state the (overdetermined) Cauchy problem for ultradifferentiable functions of class (M_p), it is convenient to introduce the notion of ultradifferentiable Whitney function of class (M_p).

Let F be a locally closed subset of $\mathbb{R}^N$; we fix an open neighbourhood Ω of F such that $F = \Omega \cap \overline{F}$. Denote by $\mathcal{I}^{(M_p)}(F, \Omega)$ the ideal of all ultradifferentiable functions $f \in \mathcal{E}^{(M_p)}(\Omega)$ that vanish with all their derivatives on F. It is a closed subspace and a differential $\mathcal{P}$-submodule of $\mathcal{E}^{(M_p)}(\Omega)$. We define the space $W_F^{(M_p)}$ of *ultradifferentiable Whitney functions of class* (M_p) *in* F by the exact sequence:

$$0 \longrightarrow \mathcal{I}^{(M_p)}(F, \Omega) \longrightarrow \mathcal{E}^{(M_p)}(\Omega) \longrightarrow W_F^{(M_p)} \longrightarrow 0. \tag{3.12}$$

We endow $W_F^{(M_p)}$ with the quotient topology. Since $\mathcal{I}^{(M_p)}(F, \Omega)$ is a closed ideal and a differential $\mathcal{P}$-submodule of $\mathcal{E}^{(M_p)}(\Omega)$, the space $W_F^{(M_p)}$ is Fréchet–Schwartz, a Fréchet algebra, and a differential $\mathcal{P}$-submodule. These topological and algebraic structures are independent of the choice of the open neighbourhood Ω of F with $F = \Omega \cap \overline{F}$.

Given an element $f \in W_F^{(M_p)}$, all its partial derivatives $D^\alpha f$ are well defined at points $x_0 \in F$: thus we can consider, for each $x_0 \in F$, the *formal* Taylor series of f at x_0:

$$\tau_{x_0}(f) = \sum_{\alpha \in \mathbb{N}^N} \frac{1}{\alpha!} \frac{\partial^\alpha f(x_0)}{\partial x^\alpha} (x - x_0)^\alpha,$$

and the Taylor polynomial of degree m of f at x_0:

$$\tau_{x_0}^m(f) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \frac{\partial^\alpha f(x_0)}{\partial x^\alpha} (x - x_0)^\alpha.$$

We have (see [15]):

Lemma 3.5. *Let $\{f_\alpha\}_{\alpha \in \mathbb{R}^N}$ be a family of continuous functions on F. A necessary and sufficient condition in order that there exists an element $f \in W_F^{(M_p)}$ such that*

$D^\alpha f(x) = f_\alpha(x)$ for all $\alpha \in \mathbb{R}^N$ and $x \in F$ is that the following two conditions are satisfied:

$$
(3.13)\quad \begin{cases} \forall \epsilon > 0, \quad \forall \emptyset \neq K \Subset F, \qquad \exists C > 0 \quad \textit{such that} \\ (i) \quad \sup_{x \in K} \sum_{\alpha \in \mathbb{N}^N} |f_\alpha(x)|\, \epsilon^{-|\alpha|} M_{|\alpha|}^{-1} \le C; \\ (ii) \quad \textit{for every } h \in \mathbb{N} \\ \qquad \sup_{\substack{x,y \in K \\ x \neq y}} \sum_{|\alpha| \le h} \frac{\left| f_\alpha(y) - D_y^\alpha \left(\sum_{|\beta| \le h} f_\beta(x)(y-x)^\beta / \beta! \right) \right|}{|x-y|^{|\alpha|-h}\, \epsilon^{|\alpha|} M_{|\alpha|}} \le C. \end{cases}
$$

Note that condition (ii) is empty when K reduces to a single point. For $F = \{0\}$, the space $W_{\{0\}}^{(M_p)}$ can be identified to the space of all formal power series $\sum_\alpha a_\alpha x^\alpha$ whose coefficients a_α satisfy that for every $\epsilon > 0$ there is a constant $c_\epsilon > 0$ such that $|a_\alpha| \le c_\epsilon \epsilon^{|\alpha|} M_{|\alpha|} \big/ \alpha!$ for every $\alpha \in \mathbb{N}^N$.

Denote by $\mathrm{Aff}(F)$ the affine span of $F \subset \mathbb{R}^N$ and let $\mathrm{Int}_{\mathrm{aff}}(F)$ be the interior of F as a subspace of $\mathrm{Aff}(F)$.

A closed subset F of $\mathbb{R}^N$ has the *Whitney property* (P) if $\overline{\mathrm{Int}_{\mathrm{aff}}(F)} = F$ and for every compact subset K there is a constant $c_K > 0$ such that every pair x, y of points of K can be joined by a rectifiable curve in F of length nonexceeding $c_K|x - y|$. Closed convex subsets F of $\mathbb{R}^N$ are trivial examples of closed subsets enjoying the Whitney property (P).

For the proof of the following proposition we refer to [15]:

Proposition 3.6. *Let F be a closed subset of $\mathbb{R}^N$, having the Whitney property (P). Then the family of seminorms $p_{\epsilon,K}$ in (3.2), for $\epsilon > 0$ and $K \Subset F$, defines the topology of $W_F^{(M_p)}$.*

By dualizing the exact sequence (3.12), we deduce that the strong dual $\left(W_F^{(M_p)}\right)'$ of $W_F^{(M_p)}$ can be identified to a subspace of ultradistributions of the class (M_p), having compact support contained in F. The proposition above yields:

Proposition 3.7. *Let F be a closed subset of $\mathbb{R}^N$, having the Whitney property (P). Then the natural inclusion induces an isomorphism $\left(W_F^{(M_p)}\right)' \simeq \mathcal{E}'^{(M_p)}_F$, where $\mathcal{E}'^{(M_p)}_F = \left\{ f \in \mathcal{E}'^{(M_p)}(\mathbb{R}^N) \,\middle|\, \mathrm{supp} f \subset F \right\}$.*

For instance, $\left(W_{\{0\}}^{(M_p)}\right)'$ is identified to the space of ultradistributions of the form $\sum_\alpha c_\alpha \delta_0^{(\alpha)}$ where the c_α are complex coefficients such that, for some $R > 0$, we have $|c_\alpha| \le R^{|\alpha|}/M_{|\alpha|}$ for all $\alpha \in \mathbb{N}^N$. Here $\delta_0^{(\alpha)}$ denotes the D^α-derivative of the Dirac delta function with unit mass concentrated in $\{0\} \subset \mathbb{R}^N$.

B) *Topological tensor products*

In the study of evolution from an affine subspace $\Sigma = \{(t,x) \in \mathbb{R}^k_t \times \mathbb{R}^n_x \mid t = 0\}$ it is natural to consider spaces of (Whitney) functions satisfying different regularity requirements with respect to the t and the x variables. They can be defined as topological tensor products. In this section we collect the definitions and properties that will be relevant for our discussion.

A partially ordered set $(A, \prec)$ is a *directed set* if for every pair $\alpha,\ \beta$ of elements of A there is $\gamma \in A$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$. Let $\mathfrak{F}$ be a topological $\mathbb{C}$-vector space. A *generalized sequence* $\{x_\alpha\}_{\alpha\in A}$ in $\mathfrak{F}$ is a map $A \ni \alpha \longrightarrow x_\alpha \in \mathfrak{F}$ defined on a directed set A; it is called *fundamental* if for every open neighbourhood U of 0 in $\mathfrak{F}$ there exists $\alpha_0 \in A$ such that $x_\alpha - x_\beta \in U$ for $\alpha,\ \beta \succ \alpha_0$; an element x_∞ is a *limit* of the generalized sequence $\{x_\alpha\}_{\alpha\in A}$ if for every neighbourhood U of 0 in $\mathfrak{F}$ there exists $\alpha_0 \in A$ such that $x_\infty - x_\alpha \in U$ for $\alpha \succ \alpha_0$. A generalized sequence $\{x_\alpha\}_{\alpha\in A}$ admitting a limit x_∞ in $\mathfrak{F}$ is said to be *convergent* in $\mathfrak{F}$. The limit is unique if $\mathfrak{F}$ is Hausdorff.

The topological vector space $\mathfrak{F}$ is *complete* if every fundamental generalized sequence is convergent. We have (see [16]):

Proposition 3.8. *Every Hausdorff locally convex topological vector space $\mathfrak{F}$ can be embedded in a locally convex Hausdorff complete topological vector space $\hat{\mathfrak{F}}$ as a dense subspace. This space $\hat{\mathfrak{F}}$ is unique up to isomorphisms of topological vector spaces and is called the completion of $\mathfrak{F}$.*

Let $\mathfrak{E}$, $\mathfrak{F}$ be topological vector spaces (over $\mathbb{C}$). Denote by $\mathfrak{L}(\mathfrak{E},\mathfrak{F})$ the space of continuous linear maps from $\mathfrak{E}$ to $\mathfrak{F}$. This is a topological vector space with the topology of *bounded convergence*. A basis for this topology is defined by the open sets of the form $\mathcal{U}(B,U) = \{u \in \mathfrak{L}(\mathfrak{E},\mathfrak{F}) \mid u(B) \subset U\}$, where U is open in $\mathfrak{F}$ and B is bounded (i.e. absorbed by every open neighbourhood of 0) in $\mathfrak{E}$.

Let $\mathfrak{G}$ be a third topological $\mathbb{C}$-vector space and denote by $\mathfrak{B}(\mathfrak{E},\mathfrak{F};\mathfrak{G})$ the space of continuous bilinear maps $b : \mathfrak{E}\times\mathfrak{F} \longrightarrow \mathfrak{G}$. This is also a topological vector space with the topology of bounded convergence, defined by the basis of open sets $\mathcal{U}(B_1, B_2; U) = \{b \in \mathfrak{B}(\mathfrak{E},\mathfrak{F};\mathfrak{G}) \mid b(B_1 \times B_2) \subset U\}$ for U open in $\mathfrak{G}$, B_1 and B_2 bounded in $\mathfrak{E}$ and $\mathfrak{F}$ respectively. (Note that every bounded subset of the topological product $\mathfrak{E}\times\mathfrak{F}$ is contained in a bounded set of the form $B_1 \times B_2$ with B_1 bounded in $\mathfrak{E}$ and B_2 bounded in $\mathfrak{F}$).

The (algebraic) tensor product $\mathfrak{E}\otimes\mathfrak{F}$ is characterized by the universal property that for every $\mathbb{C}$-vector space $\mathfrak{G}$ there is a unique linear isomorphism between the space $\mathfrak{B}_{\text{alg}}(\mathfrak{E},\mathfrak{F};\mathfrak{G})$ of bilinear maps $b : \mathfrak{E}\times\mathfrak{F} \longrightarrow \mathfrak{G}$ and the space $\mathfrak{L}_{\text{alg}}(\mathfrak{E}\otimes\mathfrak{F},\mathfrak{G})$ of linear maps $u : \mathfrak{E}\otimes\mathfrak{F} \longrightarrow \mathfrak{G}$, say $b \longrightarrow \tilde{b}$, such that $\tilde{b}(x\otimes y) = b(x,y)$ for all $x \in \mathfrak{E}$ and $y \in \mathfrak{F}$. Here we used the subscript alg to stress the fact that no topologies are involved.

The *projective tensor product topology* is characterized by the following proposition, for which we refer to [20, Ch. I, Prop. 2]:

Proposition 3.9. *Let $\mathfrak{E}$, $\mathfrak{F}$ be two Hausdorff locally convex topological vector spaces. Then there is a unique Hausdorff locally convex topology on $\mathfrak{E}\otimes\mathfrak{F}$ such* *that, for every Hausdorff locally convex topological vector space* $\mathfrak{G}$*, the algebraic isomorphism* $\mathfrak{B}_{\text{alg}}(\mathfrak{E}, \mathfrak{F}; \mathfrak{G}) \simeq \mathcal{L}_{\text{alg}}(\mathfrak{E} \otimes \mathfrak{F}, \mathfrak{G})$ *induces by restriction an ismomorphism of locally convex topological vector spaces* $\mathfrak{B}(\mathfrak{E}, \mathfrak{F}; \mathfrak{G}) \simeq \mathcal{L}(\mathfrak{E} \otimes \mathfrak{F}, \mathfrak{G})$.

Moreover, if $\{\alpha_i\}_{i\in I}$, $\{\beta_j\}_{j\in J}$ *are fundamental systems of continuous seminorms on* $\mathfrak{E}$ *and* $\mathfrak{F}$*, respectively, this topology on* $\mathfrak{E}\otimes\mathfrak{F}$ *is defined by the family of seminorms* $\{\gamma_{i,j} = \alpha_i \otimes \beta_j\}_{(i,j)\in I\times J}$ *defined by*

$$\gamma_{i,j}(u) = \inf \left\{ \sum_{h=1}^{k} \alpha_i(x_h)\beta_j(y_h) \;\middle|\; u = \sum_{h=1}^{k} x_h \otimes y_h \right\}.$$

The completeness of $\mathfrak{E}$ and $\mathfrak{F}$ does not imply, in general, the completeness of the tensor product $\mathfrak{E} \otimes \mathfrak{F}$ for the projective tensor product topology. We define the *completed projective tensor product* $\mathfrak{E}\hat{\otimes}\mathfrak{F}$ as the completion of $\mathfrak{E} \otimes \mathfrak{F}$ for the projective tensor product topology. We refer to [36, Ch. III, Section 6] for a proof of the following:

Proposition 3.10. *Let* $\mathfrak{E}$, $\mathfrak{F}$ *be metrizable locally convex topological vector spaces. Then each element* u *of* $\mathfrak{E}\hat{\otimes}\mathfrak{F}$ *is the sum of an absolutely convergent series* $\sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$*, where* $\sum_{i=1}^{\infty} |\lambda_i| < \infty$ *and* $\{x_i\}$, $\{y_i\}$ *are sequences converging to* 0 *in* $\mathfrak{E}$, $\mathfrak{F}$*, respectively.*

Duality will play a fundamental role in our study of the overdetermined Cauchy problem. Thus we shall briefly discuss duality for topological tensor products.

Let $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}$ be topological $\mathbb{C}$-vector spaces. Denote by $\mathfrak{B}_{\text{sep}}(\mathfrak{E}, \mathfrak{F}; \mathfrak{G})$ the linear space of $b \in \mathfrak{B}_{\text{alg}}(\mathfrak{E}, \mathfrak{F}; \mathfrak{G})$ that are *separately continuous*, i.e. such that $b(x, y)$ is a continuous function of $x \in \mathfrak{E}$ for fixed $y \in \mathfrak{F}$ and of $y \in \mathfrak{F}$ for fixed $x \in \mathfrak{E}$. Denote by $\mathfrak{E}'_w$ and $\mathfrak{F}'_w$ the strong duals of $\mathfrak{E}$ and $\mathfrak{F}$, respectively, endowed with the weak* topologies $\sigma(\mathfrak{E}', \mathfrak{E})$ and $\sigma(\mathfrak{F}', \mathfrak{F})$. We consider the topology of the *bi-equicontinuous convergence* on $\mathfrak{B}_{\text{sep}}(\mathfrak{E}'_w, \mathfrak{F}'_w; \mathfrak{G})$: a basis for this topology is given by the open sets $\mathcal{U}(B_1, B_2; U) = \{b \in \mathfrak{B}_{\text{sep}}(\mathfrak{E}'_w, \mathfrak{F}'_w; \mathfrak{G}) \,|\, b(B_1 \times B_2) \subset U\}$ with U open in $\mathfrak{G}$ and B_1, B_2 *equicontinuous* in $\mathfrak{E}'_w$, $\mathfrak{F}'_w$, respectively. There is a natural inclusion $\mathfrak{E} \otimes \mathfrak{F} \hookrightarrow \mathfrak{B}_{\text{sep}}(\mathfrak{E}'_w, \mathfrak{F}'_w; \mathfrak{G})$ and thus we can consider the topology of bi-equicontinuous convergence on $\mathfrak{E} \otimes \mathfrak{F}$. We denote by $\mathfrak{E}\hat{\hat{\otimes}}\mathfrak{F}$ the completion of $\mathfrak{E} \otimes \mathfrak{F}$ with respect to this topology.

For instance, if $\mathfrak{E}$ is the space $\mathcal{E}(\Omega)$ of smooth complex-valued functions defined on an open subset Ω of $\mathbb{R}^N$ and $\mathfrak{F}$ is a Hausdorff locally convex complete topological vector space, then $\mathfrak{E}\hat{\hat{\otimes}}\mathfrak{F}$ can be identified to the space $\mathcal{E}(\Omega, \mathfrak{F})$ of $\mathfrak{F}$-valued smooth functions on Ω (cf. [21, Ch. III, n. 8]).

To obtain a good characterization of duals of topological tensor products, we need the following notion: a Hausdorff topological vector space $\mathfrak{E}$ is *nuclear* if for every Hausdorff locally convex topological vector space $\mathfrak{F}$ the natural map $\mathfrak{E}\hat{\otimes}\mathfrak{F} \longrightarrow \mathfrak{B}_{\text{sep}}(\mathfrak{E}'_w, \mathfrak{F}'_w; \mathbb{C})$ associating to a tensor of rank one $x \otimes y \in \mathfrak{E} \otimes \mathfrak{F} \subset \mathfrak{E}\hat{\otimes}\mathfrak{F}$ the bilinear map $b \in \mathfrak{B}_{\text{sep}}(\mathfrak{E}'_w, \mathfrak{F}'_w; \mathbb{C})$ defined by $b(f, g) = f(x)g(y)$, is an isomorphism.

The following (with their usual Schwartz topologies) are examples of nuclear spaces (cf. [20, Ch. II, Section 2, n. 3]):

the space $\mathcal{E}(\mathbb{R}^N)$ of smooth functions in $\mathbb{R}^N$;
the space $\mathcal{D}(\mathbb{R}^N)$ of smooth functions with compact support in $\mathbb{R}^N$;
the space $\mathcal{S}(\mathbb{R}^N)$ of smooth functions in $\mathbb{R}^N$ rapidly decreasing at infinity with all derivatives;

and their strong duals:

the space $\mathcal{E}'(\mathbb{R}^N)$ of distributions with compact support in $\mathbb{R}^N$;
the space $\mathcal{D}'(\mathbb{R}^N)$ of distributions in $\mathbb{R}^N$;
the space $\mathcal{S}'(\mathbb{R}^N)$ of tempered distributions in $\mathbb{R}^N$.

We collect in the following proposition (see [20, Ch. II, Th. 9]) the properties of nuclear spaces:

Proposition 3.11. *(a) Every linear subspace of a nuclear space is nuclear.*
(b) Every Hausdorff quotient of a nuclear space is nuclear.
(c) The topological vector product of a family of nuclear spaces is nuclear.
(d) The projective tensor product of two nuclear spaces is nuclear.
(e) The strong dual of a Fréchet nuclear space is nuclear.

Moreover, we shall use the following proposition from [20, Ch. I, Corollary 1 of Proposition 30]:

Proposition 3.12. *Let $\mathfrak{E}$, $\mathfrak{F}$ be two Fréchet spaces. Assume that $\mathfrak{E}$ is nuclear and $\mathfrak{F}$ is reflexive. Then the strong dual of $\mathfrak{E}\hat{\hat{\otimes}}\mathfrak{F}$ is isomorphic to $\mathfrak{E}'\hat{\otimes}\mathfrak{F}'$.*

C) *The spaces $\tilde{W}_F^{(M_p)}$*

Let Ω be an open subset of $\mathbb{R}^N = \mathbb{R}^k_t \times \mathbb{R}^n_x$. We fix a sequence $\{M_p\}_{p\in\mathbb{N}}$ satisfying conditions (M). If $\Omega = \Omega_1 \times \Omega_2$ with Ω_1 open in $\mathbb{R}^k_t$ and Ω_2 open in $\mathbb{R}^n_x$, we set

$$\tilde{\mathcal{E}}^{(M_p)}(\Omega) = \mathcal{E}\left(\Omega_1, \mathcal{E}^{(M_p)}(\Omega_2)\right) \simeq \mathcal{E}(\Omega_1)\hat{\hat{\otimes}}\mathcal{E}^{(M_p)}(\Omega_2), \tag{3.14}$$

with the topology described in B).

For a general Ω, we denote by $\tilde{\mathcal{E}}^{(M_p)}(\Omega)$ the space of smooth functions on Ω whose restrictions to all $\Omega_1\times\Omega_2 \subset \Omega$ belong to $\tilde{\mathcal{E}}^{(M_p)}(\Omega_1\times\Omega_2)$, with the projective limit topology defined by the restriction maps $\tilde{\mathcal{E}}^{(M_p)}(\Omega) \longrightarrow \tilde{\mathcal{E}}^{(M_p)}(\Omega_1\times\Omega_2)$. This is the space of smooth functions $f(t,x)$ in Ω, whose t-derivatives are of class (M_p) with respect to the x-variables, uniformly for t varying on compact subsets:

(3.15)
$$\tilde{\mathcal{E}}^{(M_p)}(\Omega) = \left\{ f \in \mathcal{E}(\Omega) \;\middle|\; \left\{ \begin{array}{l} \forall K \Subset \Omega \;\; \forall\epsilon > 0, \;\; \forall\beta \in \mathbb{N}^k, \;\; \exists c > 0 : \\ \sup_K \left| D_t^\beta D_x^\alpha f(x,t) \right| \leq c\epsilon^{|\alpha|} M_{|\alpha|} \quad \forall\alpha \in \mathbb{N}^n \end{array} \right. \right\}.$$

When $\Omega = \Omega_1 \times \Omega_2$, we can use Proposition 3.12 to characterize the strong dual $\left(\tilde{\mathcal{E}}^{(M_p)}(\Omega)\right)'$ of $\tilde{\mathcal{E}}^{(M_p)}(\Omega)$: we have

$$\left(\tilde{\mathcal{E}}^{(M_p)}(\Omega_1 \times \Omega_2)\right)' \simeq \mathcal{E}'(\Omega_1)\hat{\otimes}\mathcal{E}'^{(M_p)}(\Omega_2). \tag{3.16}$$

We note that all exponential functions $(t, x) \longrightarrow e^{\langle t,\tau\rangle+\langle x,\zeta\rangle}$ (with $\tau \in \mathbb{C}^k$, $\zeta \in \mathbb{C}^n$) belong to $\tilde{\mathcal{E}}^{(M_p)}(\mathbb{R}^N)$. Thus we can compute the Fourier–Laplace transform

$$(3.17) \qquad \hat{f}(\tau,\zeta) = \langle \exp(-i\langle\cdot,\tau\rangle - i\langle\cdot,\zeta\rangle),\ f\rangle, \qquad \text{for} \quad \tau \in \mathbb{C}^k,\ \zeta \in \mathbb{C}^n$$

of the elements f of $\left(\tilde{\mathcal{E}}^{(M_p)}(\mathbb{R}^N)\right)'$. We note that they are ultradistributions with compact support. Using the Paley–Wiener Theorem and Theorem 3.4, by the characterization of the topological tensor product we obtain the following:

Theorem 3.13. *Let K be a compact convex subset of $\mathbb{R}^N$. A necessary and sufficient condition for an entire function $U \in \mathcal{O}(\mathbb{C}^N)$ to be the Fourier–Laplace transform of an element $u \in \left(\tilde{\mathcal{E}}^{(M_p)}(\mathbb{R}^N)\right)'$ with support contained in K is that there exist constants $c, L, L' \geq 0$ such that:*

(3.18)
$$|U(\tau,\zeta)| \leq c\ (1+|\tau|)^{L'} \exp(M(L\zeta) + H_K(Im\tau, Im\zeta)) \qquad \forall(\tau,\zeta) \in \mathbb{C}^N.$$

The statement is indeed straightforward when $K = K_1 \times K_2$ for compact convex sets $K_1 \subset \mathbb{R}^k_t$ and $K_2 \subset \mathbb{R}^n_x$. We recover the general statement by approximating a general compact convex $K \subset \mathbb{R}^N$ by a finite union of compact convex sets of the form $K_1 \times K_2$ and using related partitions of unity by functions with compact support belonging to $\tilde{\mathcal{E}}^{(M_p)}(\mathbb{R}^N)$.

For a closed $F \subset \mathbb{R}^N$ we define the space $\tilde{W}_F^{(M_p)}$ of Whitney functions, of class (M_p) in the x-variables, on F, by the exact sequence:

$$(3.19) \qquad 0 \longrightarrow \tilde{\mathcal{E}}^{(M_p)}(\mathbb{R}^N) \cap \mathcal{I}(F,\mathbb{R}^N) \longrightarrow \tilde{\mathcal{E}}^{(M_p)}(\mathbb{R}^N) \longrightarrow \tilde{W}_F^{(M_p)} \longrightarrow 0.$$

We consider on $\tilde{W}_F^{(M_p)}$ the quotient topology. When F is convex, the strong dual $\left(\tilde{W}_F^{(M_p)}\right)'$ of $\tilde{W}_F^{(M_p)}$ can be identified to the space of ultradistributions in $\left(\tilde{\mathcal{E}}^{(M_p)}(\mathbb{R}^N)\right)'$ having compact support contained in F. In particular we can compute the Fourier–Laplace transform of the elements of $\left(\tilde{W}_F^{(M_p)}\right)'$ and we have:

Theorem 3.14. *Let F be a closed convex subset of $\mathbb{R}^N$. Then an entire function $U \in \mathcal{O}(\mathbb{C}^N)$ is the Fourier–Laplace transform of an element $u \in \left(\tilde{W}_F^{(M_p)}\right)'$ if and only if it satisfies the inequality (3.18) for some convex compact $K \Subset F$ and some constants $c, L, L' \geq 0$.*

D) *The spaces $W_F^{(N_p),(M_p)}$*

Let us now fix two sequences $\{N_p\}_{p\in\mathbb{N}}$ and $\{M_p\}_{p\in\mathbb{N}}$ satisfying the conditions (M). We denote by $n(t)$ the function associated to the sequence $\{N_p\}_{p\in\mathbb{N}}$ and by $N(\tau) = n(1+|\tau|)$, for $\tau \in \mathbb{C}^k$, the corresponding plurisubharmonic function in $\mathbb{C}^k$. Given open subsets $\Omega_1 \subset \mathbb{R}^k_t$ and $\Omega_2 \subset \mathbb{R}^n_x$, we set

(3.20)
$$\mathcal{E}^{(N_p),(M_p)}(\Omega_1 \times \Omega_2) = \mathcal{E}^{(N_p)}\left(\Omega_1, \mathcal{E}^{(M_p)}(\Omega_2)\right) = \mathcal{E}^{(N_p)}(\Omega_1)\hat{\hat{\otimes}}\mathcal{E}^{(M_p)}(\Omega_2).$$

As in the previous subsection, we can define $\mathcal{E}^{(N_p),(M_p)}(\Omega)$ for every open subset $\Omega \subset \mathbb{R}^N = \mathbb{R}^k_t \times \mathbb{R}^n_x$ as the set of all complex-valued smooth functions in Ω whose restriction to every $\Omega_1 \times \Omega_2 \subset \Omega$ belongs to $\mathcal{E}^{(N_p),(M_p)}(\Omega_1 \times \Omega_2)$, with the coarsest topology for which these restriction maps are continuous. By the characterization of the topological tensor product of two Fréchet nuclear spaces we obtain:

(3.21)
$$\mathcal{E}^{(N_p),(M_p)}(\Omega) = \left\{ f \in \mathcal{E}(\Omega) \;\middle|\; \begin{cases} \forall K \Subset \Omega, \ \forall \epsilon > 0, \ \exists c > 0 : \\ \sup\limits_{(t,x)\in K} \left| D_t^\beta D_x^\alpha f(t,x) \right| \le c\, \epsilon^{|\alpha|+|\beta|} M_{|\alpha|} N_{|\beta|} \\ \forall \alpha \in \mathbb{N}^n, \ \forall \beta \in \mathbb{N}^k \end{cases} \right\}.$$

Using Proposition 3.12 we can identify the strong dual of $\mathcal{E}^{(N_p),(M_p)}(\Omega_1 \times \Omega_2)$ (where Ω_1 and Ω_2 are open subsets of $\mathbb{R}^k_t$ and $\mathbb{R}^n_x$ respectively) to the topological tensor product $\left(\mathcal{E}^{(N_p)}(\Omega_1)\right)' \hat{\otimes} \left(\mathcal{E}^{(M_p)}(\Omega_2)\right)'$. Since $\mathcal{E}^{(N_p),(M_p)}(\mathbb{R}^N)$ contains all exponential functions $(t,x) \longrightarrow e^{\langle t,\tau\rangle + \langle x,\zeta\rangle}$ for $\tau \in \mathbb{C}^k$ and $\zeta \in \mathbb{C}^n$, we can compute the Fourier–Laplace transform of elements of $\left(\mathcal{E}^{(N_p),(M_p)}(\mathbb{R}^N)\right)'$. Then we have:

Theorem 3.15. *A necessary and sufficient condition for an entire function* $U \in \mathcal{O}(\mathbb{C}^N)$ *to be the Fourier–Laplace transform of an element u of* $\left(\mathcal{E}^{(N_p),(M_p)}(\mathbb{R}^N)\right)'$, *is that there exist a compact convex K in* $\mathbb{R}^N$ *and constants* $c, L, L' \ge 0$ *such that*

$$(3.22)\quad |U(\tau,\zeta)| \le c\, \exp\{M(L\zeta) + N(L'\tau) + H_K(Im\tau, Im\zeta)\} \qquad \forall (\tau,\zeta) \in \mathbb{C}^N.$$

And, vice versa, if for some convex compact K and some constants $c, L, L' \ge 0$ *inequality (3.22) holds true for the Fourier–Laplace transform of an ultradistribution u, then* $u \in \left(\mathcal{E}^{(N_p),(M_p)}(\mathbb{R}^N)\right)'$ *and* $\operatorname{supp} u \subset K$.

For a closed $F \subset \mathbb{R}^N$ we define the space $W_F^{(N_p),(M_p)}$ of Whitney functions, of class (N_p) in the t-variables and of class (M_p) in the x-variables, on F, by the exact sequence:

(3.23)
$$0 \longrightarrow \mathcal{E}^{(N_p),(M_p)}(\mathbb{R}^N) \cap \mathcal{I}(F,\mathbb{R}^N) \longrightarrow \mathcal{E}^{(N_p),(M_p)}(\mathbb{R}^N) \longrightarrow W_F^{(N_p),(M_p)} \longrightarrow 0.$$

The space $W_F^{(N_p),(M_p)}$ is Fréchet–Schwartz and nuclear. When F is a closed convex subset of $\mathbb{R}^N$, the strong dual $\left(W_F^{(N_p),(M_p)}\right)'$ of $W_F^{(N_p),(M_p)}$ is the space of ultradistributions in $\left(\mathcal{E}^{(N_p),(M_p)}(\mathbb{R}^N)\right)'$ having compact support contained in F. In particular we can compute the Fourier–Laplace transform of elements of $\left(W_F^{(N_p),(M_p)}\right)'$ and we obtain:

Theorem 3.16. *Let F be a closed convex subset of* $\mathbb{R}^N$. *An entire function* $U \in \mathcal{O}(\mathbb{C}^N)$ *is the Fourier–Laplace transform of an ultradistribution* $u \in \left(W_F^{(N_p),(M_p)}\right)'$ *if and only if inequality (3.22) holds true for some convex compact* $K \Subset F$ *and some constants* $c, L, L' \ge 0$. *If* $u \in \left(W_F^{(N_p),(M_p)}\right)'$ *and its Fourier–Laplace transform* $U = \hat{u}$ *satisfies (3.22), then* $\operatorname{supp} u \subset K$.

4. The Cauchy problem for a pair of convex sets

A) *The abstract setting*

Let F be a locally closed subset of $\mathbb{R}^N$. The space W_F of Whitney functions on F is defined by the exact sequence:

$$(4.1) \qquad 0 \longrightarrow \mathcal{I}(F, \Omega) \longrightarrow \mathcal{E}(\Omega) \longrightarrow W_F \longrightarrow 0,$$

where Ω is any open neighbourhood of F in $\mathbb{R}^N$ such that $F = \Omega \cap \overline{F}$ and $\mathcal{I}(F, \Omega)$ is the subspace of $\mathcal{E}(\Omega)$ consisting of smooth functions vanishing with all derivatives on F.

Let (K_1, K_2) be a pair of subsets of $\mathbb{R}^N$, with K_2 locally closed in $\mathbb{R}^N$, $K_1 \subset K_2$ and K_1 closed in K_2. We have a surjective restriction homomorphism $W_{K_2} \ni f \longrightarrow f|_{K_1} \in W_{K_1}$. This is a continuous linear map of Fréchet spaces and a differential $\mathcal{P}$-modules homomorphism.

Let $\mathfrak{F}(K_2) \subset W_{K_2}$ and $\mathfrak{E}(K_1) \subset W_{K_1}$ be two spaces of Whitney functions on K_2 and K_1 respectively. Let $A(D)$ be a $q \times p$ matrix of linear partial differential operators with constant coefficients. We consider the problem:

$$(4.2) \qquad \begin{cases} \text{Given} \quad f \in \mathfrak{F}(K_2)^q \quad \text{and} \quad \phi \in \mathfrak{E}(K_1)^p \\ \text{find} \quad u \in \mathfrak{F}(K_2)^p \quad \text{such that} \\ A(D)u = f, \\ u|_{K_1} = \phi. \end{cases}$$

Set $A_0(D) = A(D)$ and insert the $\mathcal{P}$-homomorphism ${}^t A(\zeta) : \mathcal{P}^q \longrightarrow \mathcal{P}^p$ into a Hilbert resolution (2.3) (with $a_0 = p$ and $a_1 = q$). Then to solve (4.2) for the pair of data (f, ϕ) it is necessary that they satisfy the *compatibility conditions*:

$$(4.3) \qquad \begin{cases} f \in \mathfrak{F}(K_2)^q, \quad f|_{K_1} \in \mathfrak{E}(K_1)^q, \quad \phi \in \mathfrak{E}(K_1)^p, \\ A_1(D) f = 0, \\ A_0(D)\phi = f|_{K_1}. \end{cases}$$

We say that the pair $(\mathfrak{E}(K_1), \mathfrak{F}(K_2))$ is

of evolution for $A(D)$ if (4.2) admits at least one solution for all data (f, ϕ) satisfying (4.3);
of causality for $A(D)$ if (4.2) admits at most one solution;
hyperbolic for $A(D)$ if (4.2) admits a unique solution for each pair (f, ϕ) satisfying (4.3).

Clearly causality means that 0 is the only solution of (4.2) corresponding to the data $f = 0$, $\phi = 0$. Problem (4.2) with $\phi = 0$ is the Cauchy problem *with zero initial data*; (4.2) with $f = 0$ will be referred to as *the homogeneous Cauchy problem*.

The compatibility conditions for (4.2) with zero initial data ($\phi = 0$) are expressed by $f \in \mathfrak{F}(K_2)^q \cap [\mathcal{I}(K_1, K_2)]^q$ and $A_1(D) f = 0$. (Here $\mathcal{I}(K_1, K_2)$ is the space of Whitney functions on K_2 vanishing with all derivatives on K_1.) In particular, the solvability of (4.2) (evolution) with zero initial data is expressed by the equation involving the unitary finitely generated $\mathcal{P}$-module $\mathfrak{M} = \operatorname{coker} {}^t A(\zeta) : \mathcal{P}^q \longrightarrow \mathcal{P}^p$:

$$\operatorname{Ext}^1_{\mathcal{P}}(\mathfrak{M}, \mathfrak{F}(K_2) \cap \mathcal{I}(K_1, K_2)) = 0, \tag{4.4}$$

and causality by the equation

$$\operatorname{Ext}^0_{\mathcal{P}}(\mathfrak{M}, \mathfrak{F}(K_2) \cap \mathcal{I}(K_1, K_2)) = 0. \tag{4.5}$$

The compatibility conditions for the homogeneous Cauchy problem ($f = 0$), with initial datum $\phi \in \mathfrak{E}(K_1)^p$, reduce to $A_0(D)\phi = 0$. Introducing the $\mathcal{P}$-module $\mathfrak{G}(K_1, K_2) = \{u \in \mathfrak{F}(K_2) \,|\, u|_{K_1} \in \mathfrak{E}(K_1)\}$, evolution and causality translate into surjectivity and injectivity of the map

$$\operatorname{Ext}^0_{\mathcal{P}}(\mathfrak{M}, \mathfrak{G}(K_1, K_2)) \longrightarrow \operatorname{Ext}^0_{\mathcal{P}}(\mathfrak{M}, \mathfrak{E}(K_1)) \tag{4.6}$$

induced by the restriction $\mathfrak{G}(K_1, K_2) \ni u \longrightarrow u|_{K_1} \in \mathfrak{E}(K_1)$.

We have:

Lemma 4.1. *Assume that the restriction map $\mathfrak{G}(K_1, K_2) \longrightarrow \mathfrak{E}(K_1)$ is onto. Then the pair $(\mathfrak{E}(K_1), \mathfrak{F}(K_2))$ is:*

of evolution for $A(D)$ if and only if (4.4) holds true;
of causality for $A(D)$ if and only if (4.5) holds true;
hyperbolic for $A(D)$ if and only if both (4.4) and (4.5) hold true.

Proof. Let (f, ϕ) satisfy (4.3). By the assumption, there is $\tilde{\phi} \in \mathfrak{G}(K_1, K_2)$ such that $\tilde{\phi}|_{K_1} = \phi$. Then $(f - A(D)\tilde{\phi}, 0)$ satisfy (4.3) and $u \in \mathfrak{F}(K_2)^p$ is a solution of (4.2) for the data (f, ϕ) if and only if $u - \tilde{\phi}$ is a solution of (4.2) for the data $(f - A(D)\tilde{\phi}, 0)$.

Lemma 4.2. *The pair $(\mathfrak{E}(K_1), \mathfrak{F}(K_2))$ is of causality for $A(D)$ if and only if (4.6) is injective. If $\operatorname{Ext}^1_{\mathcal{P}}(\mathfrak{M}, \mathfrak{G}(K_1, K_2)) = 0$, then the pair $(\mathfrak{E}(K_1), \mathfrak{F}(K_2))$ is of evolution for $A(D)$ if and only if (4.6) is surjective.*

Proof. The first assertion of the lemma is clear, since causality translates into the fact that 0 is the only solution of (4.2) when $f = 0$ and $\phi = 0$. Now let (f, ϕ) be data satisfying (4.3). If we can solve the equation $A(D)\psi = f$ with $\psi \in \mathfrak{G}(K_1, K_2)^p$, then $u \in \mathfrak{F}(K_2)$ is a solution of (4.2) for the data (f, ϕ) if and only $u - \psi$ is a solution of (4.2) for the data $(0, \phi - \psi|_{K_1})$.

B) *The overdetermined Cauchy problem for ultradifferentiable functions*

Let $\{M_p\}_{p\in\mathbb{N}}$ and $\{N_p\}_{p\in\mathbb{N}}$ be two sequences of real numbers satisfying conditions (M). We keep the notation of Section 3. To apply the general results of the previous subsection to the overdetermined Cauchy problem in the spaces of Whitney functions described in Section 3, we first prove:

Proposition 4.3. *If F is a closed convex subset of $\mathbb{R}^N$, then the differential $\mathcal{P}$-modules $\tilde{W}_F^{(M_p)}$ and $W_F^{(N_p),(M_p)}$ are injective.*

Proof. We recall that a $\mathcal{P}$-module $\mathfrak{F}$ is injective iff

$\operatorname{Ext}^j_{\mathcal{P}}(\mathfrak{M}, \mathfrak{F}) = 0$ for all $j > 0$ and all unitary $\mathcal{P}$-modules of finite type $\mathfrak{M}$. This condition is equivalent to $\operatorname{Ext}^1_{\mathcal{P}}(\mathcal{P}/\mathfrak{p}, \mathfrak{F}) = 0$ for every prime ideal $\mathfrak{p}$ in $\mathcal{P}$.

Let $\mathfrak{F}$ be one of the spaces $\tilde{W}_F^{(M_p)}$, $W_F^{(N_p),(M_p)}$ for F closed and convex in $\mathbb{R}^N$. Let (2.3) be a Hilbert resolution of the $\mathcal{P}$-module $\mathfrak{M} = \mathcal{P}/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of $\mathcal{P}$. We need to show that the sequence

$$\mathfrak{F} \xrightarrow{A_0(D)} \mathfrak{F}^{a_1} \xrightarrow{A_1(D)} \mathfrak{F}^{a_2} \tag{4.7}$$

is exact. Note that in this case we can take $a_0 = 1$ and ${}^tA_0(\zeta)$ as a row vector of polynomials giving a finite set of generators of the ideal $\mathfrak{p}$.

Using duality it suffices to show that the sequence

$$\mathfrak{F}'^{a_2} \xrightarrow{{}^tA_1(-D)} \mathfrak{F}'^{a_1} \xrightarrow{{}^tA_0(-D)} \mathfrak{F}' \tag{4.8}$$

is exact and that the map ${}^tA_0(-D) : \mathfrak{F}'^{a_1} \longrightarrow \mathfrak{F}'$ has a closed image. To this aim we use the Fourier–Laplace transform and the characterizations of Theorems 3.13, 3.16. Since the functions

$$H_K(Im\tau, Im\zeta) + M(L\zeta) + L' \log(1 + |\tau|),$$

$$H_K(Im\tau, Im\zeta) + M(L\zeta) + N(L'\tau),$$

for every compact convex $K \subset F$ and $L, L' \geq 0$ are plurisubharmonic and satisfy (3.7) in $\mathbb{C}^N = \mathbb{C}^k_\tau \times \mathbb{C}^n_\zeta$, the desired result follows from the division theorem for entire functions (see Theorem 7.6.11 in [24]).

Thus we obtain:

Theorem 4.4. *Let (K_1, K_2) be a pair of closed convex subsets of $\mathbb{R}^N$ with $K_1 \subset K_2$. Then the pair $\left(\tilde{W}_{K_1}^{(M_p)}, \tilde{W}_{K_2}^{(M_p)}\right)$ is:*

- *of evolution for $A(D)$ iff* $\operatorname{Ext}^1_{\mathcal{P}}(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)} \cap \mathcal{I}(K_1, K_2)) = 0$;
- *of causality for $A(D)$ iff* $\operatorname{Ext}^0_{\mathcal{P}}(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)} \cap \mathcal{I}(K_1, K_2)) = 0$;
- *hyperbolic for $A(D)$ iff*

$$\operatorname{Ext}^0_{\mathcal{P}}(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)} \cap \mathcal{I}(K_1, K_2)) = \operatorname{Ext}^1_{\mathcal{P}}(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)} \cap \mathcal{I}(K_1, K_2)) = 0.$$

Analogously the pair $\left(W_{K_1}^{(N_p),(M_p)}, W_{K_2}^{(N_p),(M_p)}\right)$ is:

- *of evolution for $A(D)$ iff* $\operatorname{Ext}^1_{\mathcal{P}}(\mathfrak{M}, W_{K_2}^{(N_p),(M_p)} \cap \mathcal{I}(K_1, K_2)) = 0$;
- *of causality for $A(D)$ iff* $\operatorname{Ext}^0_{\mathcal{P}}(\mathfrak{M}, W_{K_2}^{(N_p),(M_p)} \cap \mathcal{I}(K_1, K_2)) = 0$;
- *hyperbolic for $A(D)$ iff*

$$\operatorname{Ext}^0_{\mathcal{P}}(\mathfrak{M}, W_{K_2}^{(N_p),(M_p)} \cap \mathcal{I}(K_1, K_2)) = \operatorname{Ext}^1_{\mathcal{P}}(\mathfrak{M}, W_{K_2}^{(M_p)} \cap \mathcal{I}(K_1, K_2)) = 0.$$

This theorem shows that evolution, causality and hyperbolicity of the pairs $\left(\tilde{W}_{K_1}^{(M_p)}, \tilde{W}_{K_2}^{(M_p)}\right)$ and $\left(W_{K_1}^{(N_p),(M_p)}, W_{K_2}^{(N_p),(M_p)}\right)$ are related to the $\mathcal{P}$-module $\mathfrak{M}$ rather than to the specific matrix $A(D)$ of linear partial differential operators with constant coefficients. Thus we shall say in the following that (K_1, K_2) is of evolution (and, of causality, hyperbolic) for $\mathfrak{M}$ in the class $\tilde{W}^{(M_p)}$ if the pair $\left(\tilde{W}_{K_1}^{(M_p)}, \tilde{W}_{K_2}^{(M_p)}\right)$ is of evolution (and of causality, hyperbolic respectively) for $A(D)$ according to the definition above; and analogously that (K_1, K_2) is of evolution (and of causality, hyperbolic) in the class $W^{(N_p),(M_p)}$ if the pair $\left(W_{K_1}^{(N_p),(M_p)}, W_{K_2}^{(N_p),(M_p)}\right)$ is of evolution (and, of causality, hyperbolic, respectively) for $A(D)$. In this way we stress the algebraic invariance of these notions.

Next we note that from the short exact sequences:

$$0 \longrightarrow \tilde{W}_{K_2}^{(M_p)} \cap \mathcal{I}(K_1, K_2) \longrightarrow \tilde{W}_{K_2}^{(M_p)} \longrightarrow \tilde{W}_{K_1}^{(M_p)} \longrightarrow 0, \tag{4.9}$$

$$0 \longrightarrow W_{K_2}^{(N_p),(M_p)} \cap \mathcal{I}(K_1, K_2) \longrightarrow W_{K_2}^{(N_p),(M_p)} \longrightarrow W_{K_1}^{(N_p),(M_p)} \longrightarrow 0, \tag{4.10}$$

we obtain corresponding long exact sequences for the Ext functor: for every unitary finitely generated $\mathcal{P}$-module $\mathfrak{M}$ we have the long exact sequences:

(4.11)
$$\begin{aligned} 0 &\to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)} \cap \mathcal{I}(K_1, K_2)\right) \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)}\right) \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathfrak{M}, \tilde{W}_{K_1}^{(M_p)}\right) \\ &\to \mathrm{Ext}^1_{\mathcal{P}}\left(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)} \cap \mathcal{I}(K_1, K_2)\right) \to \mathrm{Ext}^1_{\mathcal{P}}\left(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)}\right) \to \mathrm{Ext}^1_{\mathcal{P}}\left(\mathfrak{M}, \tilde{W}_{K_1}^{(M_p)}\right) \\ &\to \mathrm{Ext}^2_{\mathcal{P}}\left(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)} \cap \mathcal{I}(K_1, K_2)\right) \to \mathrm{Ext}^2_{\mathcal{P}}\left(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)}\right) \to \cdots \end{aligned}$$

(4.12)
$$\begin{aligned} 0 &\to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathfrak{M}, W_{K_2}^{(N_p),(M_p)} \cap \mathcal{I}(K_1, K_2)\right) \to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathfrak{M}, W_{K_2}^{(N_p),(M_p)}\right) \\ &\to \mathrm{Ext}^0_{\mathcal{P}}\left(\mathfrak{M}, W_{K_1}^{(N_p),(M_p)}\right) \to \mathrm{Ext}^1_{\mathcal{P}}\left(\mathfrak{M}, W_{K_2}^{(N_p),(M_p)} \cap \mathcal{I}(K_1, K_2)\right) \\ &\to \mathrm{Ext}^1_{\mathcal{P}}\left(\mathfrak{M}, W_{K_2}^{(N_p),(M_p)}\right) \to \mathrm{Ext}^1_{\mathcal{P}}\left(\mathfrak{M}, W_{K_1}^{(N_p),(M_p)}\right) \\ &\to \mathrm{Ext}^2_{\mathcal{P}}\left(\mathfrak{M}, W_{K_2}^{(N_p),(M_p)} \cap \mathcal{I}(K_1, K_2)\right) \to \mathrm{Ext}^2_{\mathcal{P}}\left(\mathfrak{M}, W_{K_2}^{(N_p),(M_p)}\right) \to \cdots \end{aligned}$$

From these we deduce that

$$\begin{cases} \mathrm{Ext}^j_{\mathcal{P}}(\mathfrak{M}, \tilde{W}_{K_2}^{(M_p)} \cap \mathcal{I}(K_1, K_2)) = 0, \\ \\ \mathrm{Ext}^j_{\mathcal{P}}(\mathfrak{M}, W_{K_2}^{(N_p),(M_p)} \cap \mathcal{I}(K_1, K_2)) = 0 \\ \\ \qquad \forall j > 1. \end{cases} \tag{4.13}$$

Therefore, by Propositions 2.1 and 2.2, we have

Proposition 4.5. *Let (K_1, K_2) be closed convex subsets of $\mathbb{R}^N$, with $K_1 \subset K_2$ and let $\mathfrak{M}$ be a unitary finitely generated $\mathcal{P}$-module. Then the pair (K_1, K_2) is of evolution (and of causality, hyperbolic) in the class $\tilde{W}^{(M_p)}$ for $\mathfrak{M}$ if and only if it is* *of evolution (and, of causality, hyperbolic respectively) in the class* $\tilde{W}^{(M_p)}$ *for* $\mathcal{P}/\mathfrak{p}$ *for all* $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$*; the pair* (K_1, K_2) *is of evolution (and of causality, hyperbolic) in the class* $W^{(N_p),(M_p)}$ *for* $\mathfrak{M}$ *if and only if it is of evolution (and of causality, hyperbolic respectively) in the class* $W^{(N_p),(M_p)}$ *for* $\mathcal{P}/\mathfrak{p}$ *for all* $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$.

Using the long exact sequences (4.11) and (4.12) the contents of Proposition 4.5 can be reformulated in terms of the maps

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathfrak{M}, \tilde{W}^{(M_p)}_{K_2}\right) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}\left(\mathfrak{M}, \tilde{W}^{(M_p)}_{K_1}\right) \tag{4.14}$$

and

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathfrak{M}, W^{(N_p),(M_p)}_{K_2}\right) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}\left(\mathfrak{M}, W^{(N_p),(M_p)}_{K_2}\right). \tag{4.15}$$

We have indeed:

Proposition 4.6. *Let* (K_1, K_2)*, with* $K_1 \subset K_2$*, be a pair of closed convex subsets of* $\mathbb{R}^N$ *and let* $\mathfrak{M}$ *be a unitary finitely generated* $\mathcal{P}$*-module. Then:*
(K_1, K_2) *is of evolution for* $\mathfrak{M}$ *in the class* $\tilde{W}^{(M_p)}$ *if and only if (4.14) is onto;*
(K_1, K_2) *is of causality for* $\mathfrak{M}$ *in the class* $\tilde{W}^{(M_p)}$ *if and only if (4.14) is injective;*
(K_1, K_2) *is hyperbolic for* $\mathfrak{M}$ *in the class* $\tilde{W}^{(M_p)}$ *if and only if (4.14) is an isomorphism;*
(K_1, K_2) *is of evolution for* $\mathfrak{M}$ *in the class* $W^{(N_p),(M_p)}$ *if and only if (4.15) is onto;*
(K_1, K_2) *is of causality for* $\mathfrak{M}$ *in the class* $W^{(N_p),(M_p)}$ *if and only if (4.15) is injective;*
(K_1, K_2) *is hyperbolic for* $\mathfrak{M}$ *in the class* $W^{(N_p),(M_p)}$ *if and only if (4.15) is an isomorphism.*

C) *Duality*

It will also be convenient to use Propositions 2.1 and 2.2, and restrict our consideration to the maps (4.14) and (4.15) when $\mathfrak{M}$ is a $\mathcal{P}$-module of the form $\mathcal{P}/\mathfrak{p}$ for a prime ideal $\mathfrak{p}$. We denote by $V(\check{\mathfrak{p}})$ the irreducible affine algebraic variety in $\mathbb{C}^N$:

$$V(\check{\mathfrak{p}}) = \left\{(\tau, \zeta) \in \mathbb{C}^N \,\middle|\, p(-\tau, -\zeta) = 0 \quad \forall p \in \mathfrak{p}\right\}. \tag{4.16}$$

Let $\mathcal{O}(V(\check{\mathfrak{p}}))$ be the space of holomorphic functions on $V(\check{\mathfrak{p}})$: this is the space of complex-valued continuous functions on $V(\check{\mathfrak{p}})$ that are restrictions of entire functions in $\mathbb{C}^N$. From the Ehrenpreis fundamental principle we have:

Lemma 4.7. *Let* K *be a convex compact subset of* $\mathbb{R}^N$ *and* $L, L' \geq 0$*. Let* ϕ *denote either the function* $H_K(Im\tau, Im\zeta) + M(L\zeta) + L' \log(1 + |\tau|)$ *or the function* $H_K(Im\tau, Im\zeta) + M(L\zeta) + N(L'\tau)$*. Then there are constants* $c_1, c_2 \geq 0$ *such that every* $U \in \mathcal{O}(V(\check{\mathfrak{p}}))$*, which satisfy the inequality*

$$|U(\tau, \zeta)| \leq \exp \phi(\tau, \zeta), \qquad \forall (\tau, \zeta) \in V(\check{\mathfrak{p}}), \tag{4.17}$$

is the restriction to $V(\check{\mathfrak{p}})$ *of an entire function* $\tilde{U} \in \mathcal{O}(\mathbb{C}^N)$ *satisfying*

$$|\tilde{U}(\tau, \zeta)| \leq c_1 \exp\{\phi(\zeta) + c_2 \log(1 + |\tau| + |\zeta|)\} \qquad \forall (\tau, \zeta) \in \mathbb{C}^N. \tag{4.18}$$

This is indeed a consequence of the fact that ϕ is a plurisubharmonic function in $\mathbb{C}^N$ satisfying (3.7) (see, e.g., [24], [8]).

When $\mathfrak{p}$ is a prime ideal and $\mathfrak{F}$ a differential $\mathscr{P}$-module, we have the standard identification:

$$\mathrm{Ext}^0_{\mathscr{P}}(\mathscr{P}/\mathfrak{p}, \mathfrak{F}) \simeq \mathfrak{F}_{\mathfrak{p}} = \{f \in \mathfrak{F} \mid \; p(D)u = 0 \quad \forall p \in \mathfrak{p}\}. \tag{4.19}$$

If $\mathfrak{F}$ has a topology in which differential operators with constant coefficients define $\mathbb{C}$-linear continuous maps, then $\mathfrak{F}_{\mathfrak{p}}$ is a closed subspace of $\mathfrak{F}$. Moreover, assuming that $\mathfrak{F}$ is a space of Fréchet–Schwartz, we have a natural identification (as topological vector spaces) of the strong dual of $\mathfrak{F}_{\mathfrak{p}}$ with the quotient of the strong dual $\mathfrak{F}'$ of $\mathfrak{F}$ by the annihilator in $\mathfrak{F}'$ of $\mathfrak{F}_{\mathfrak{p}}$:

$$\left(\mathfrak{F}_{\mathfrak{p}}\right)' \simeq \mathfrak{F}' \big/ \left(\mathfrak{F}_{\mathfrak{p}}\right)^o , \tag{4.20}$$

where

$$\left(\mathfrak{F}_{\mathfrak{p}}\right)^o = \left\{u \in \mathfrak{F}' \;\middle|\; \langle f, u\rangle = 0 \quad \forall f \in \mathfrak{F}_{\mathfrak{p}} \right\}. \tag{4.21}$$

Let us go back to the abstract situation described in A). Let (K_1, K_2) be a pair of closed convex sets in $\mathbb{R}^N$ with $K_1 \subset K_2$, and $\mathfrak{E}(K_1) = \mathfrak{F}(K_1)$, $\mathfrak{F}(K_2)$ spaces of Whitney functions on K_1, K_2 respectively. We assume that $\mathfrak{F}(K_1)$ and $\mathfrak{F}(K_2)$ are differential $\mathscr{P}$-module and Fréchet–Schwartz spaces, in which partial differential operators with constant coefficients act as continuous linear maps. We also assume that we have a continuous and surjective restriction map (commuting with the action of partial differential operators with constant coefficients) $\mathfrak{F}(K_2) \longrightarrow \mathfrak{F}(K_1)$. (We set $\mathfrak{E} = \mathfrak{F}$ as a reminder for the existence of such a restriction map.) We have a natural map:

$$\mathfrak{F}_{\mathfrak{p}}(K_2) \longrightarrow \mathfrak{F}_{\mathfrak{p}}(K_1), \tag{4.22}$$

induced by the restriction; evolution (and causality) of the pair $(\mathfrak{F}(K_1), \mathfrak{F}(K_2))$ for the homogeneous Cauchy problem associated to $\mathscr{P}/\mathfrak{p}$ is equivalent to the surjectivity (and injectivity, respectively) of (4.22). We consider the dual map:

$$\mathfrak{F}'_{\mathfrak{p}}(K_1) \longrightarrow \mathfrak{F}'_{\mathfrak{p}}(K_2). \tag{4.23}$$

By duality, we know that (4.22) is onto if and only if (4.23) is injective and has a closed image; (4.22) is injective if and only if (4.23) has a dense image.

To investigate evolution, we shall use:

Proposition 4.8. *Let $\{M_p\}_{p\in\mathbb{N}}$ and $\{N_p\}_{p\in\mathbb{N}}$ be two sequences satisfying conditions (M). We assume that either $\mathfrak{F}(K_1) = \tilde{W}_{K_1}^{(M_p)}$ and $\mathfrak{F}(K_2) = \tilde{W}_{K_2}^{(M_p)}$, or $\mathfrak{F}(K_1) = W_{K_1}^{(N_p),(M_p)}$ and $\mathfrak{F}(K_2) = W_{K_2}^{(N_p),(M_p)}$.*

Then the map (4.22) has a dense range, and hence the map (4.23) is injective.

Proof. This follows from the fact that, for every closed convex set $K \subset \mathbb{R}^N$, the restrictions to K of the exponential functions $(t, x) \longrightarrow \exp(i\langle t, \tau\rangle + i\langle x, \zeta\rangle)$, for $(\tau, \zeta) \in V(\mathfrak{p})$, generate a dense subspace $\mathfrak{S}$ of $\mathfrak{F}_{\mathfrak{p}}(K)$. This can be proved using the Hahn–Banach theorem and the characterization of the strong dual $\mathfrak{F}'(K)$ given in Theorems 3.14 and 3.16. Indeed, the Fourier–Laplace transform U of an element $u \in \mathfrak{F}'(K)$ that annihilates all elements of $\mathfrak{S}$ is an entire function in $\mathbb{C}^N$, that satisfies either (3.18) or (3.22), and vanishes on $V(\check{\mathfrak{p}})$. By the Nullstellensatz, if $p_1, \dots, p_\ell \in \mathscr{P}$ are generators of $\mathfrak{p}$, there are entire functions $X_1, \dots, X_\ell$ such that

$$U(\tau, \zeta) = p_1(\tau, \zeta)X_1(\tau, \zeta) + \cdots + p_\ell(\tau, \zeta)X_\ell(\tau, \zeta) \qquad \text{in} \quad \mathbb{C}^N. \tag{4.24}$$

Then, by the Ehrenpreis fundamental principle (see [24] or [8]), we can take the entire functions $X_1, \dots, X_\ell$ to satisfy the same inequalities (3.18) or (3.22) with different constants c, L, L'. Thus they are Fourier–Laplace transforms of elements $v_1, \dots, v_\ell \in \mathfrak{F}'(K)$ and

$$u = p_1(-D)v_1 + \cdots + p_\ell(-D)v_\ell. \tag{4.25}$$

Hence

$$\langle f, u\rangle = \langle p_1(D)f, v_1\rangle + \cdots + \langle p_\ell(D)f, v_\ell\rangle = 0, \qquad \forall f \in \mathfrak{F}_{\mathfrak{p}}(K),$$

and this by Hahn–Banach implies that $\mathfrak{S}$ is dense in $\mathfrak{F}_{\mathfrak{p}}(K)$. The Lemma follows, as stated in the beginning of the proof.

In this way we obtain the *dual* formulation of Proposition 4.6:

Proposition 4.9. *Let $\{M_p\}_{p\in\mathbb{N}}$ and $\{N_p\}_{p\in\mathbb{N}}$ be two sequences satisfying conditions (M). Let (K_1, K_2) be a pair of closed convex subsets of $\mathbb{R}^N$ with $K_1 \subset K_2$. Then:*

(K_1, K_2) is of evolution for $\mathfrak{M}$ in the class $\tilde{W}^{(M_p)}$ if and only if the map

$$\left(\tilde{W}_{K_1}^{(M_p)}\right)'_{\mathfrak{p}} \longrightarrow \left(\tilde{W}_{K_2}^{(M_p)}\right)'_{\mathfrak{p}} \tag{4.26}$$

has a closed image for all $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$.
(K_1, K_2) is of causality for $\mathfrak{M}$ in the class $\tilde{W}^{(M_p)}$ if and only if the map (4.26) has a dense image for all $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$.
(K_1, K_2) is hyperbolic for $\mathfrak{M}$ in the class $\tilde{W}^{(M_p)}$ if and only if the map (4.26) is an isomorphism for all $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$.
(K_1, K_2) is of evolution for $\mathfrak{M}$ in the class $W^{(N_p),(M_p)}$ if and only if the map

$$\left(W_{K_1}^{(N_p),(M_p)}\right)'_{\mathfrak{p}} \longrightarrow \left(\tilde{W}_{K_2}^{(N_p),(M_p)}\right)'_{\mathfrak{p}} \tag{4.27}$$

has a closed image for all $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$.
(K_1, K_2) is of causality for $\mathfrak{M}$ in the class $W^{(N_p),(M_p)}$ if and only if the map (4.27) has a dense image for all $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$.
(K_1, K_2) is hyperbolic for $\mathfrak{M}$ in the class $W^{(N_p),(M_p)}$ if and only if the map (4.27) is an isomorphism for all $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$.

5. A Phragmén–Lindelöf principle for evolution

Using the results of the previous section, we reduce the investigation of the Cauchy problem (4.2), under the compatibility conditions (4.3), to the study of the map (4.23). The Fourier–Laplace transform, and the algebraic reduction in Propositions 2.1 and 2.2, translate this into questions concerning the growth of entire functions on irreducible affine algebraic varieties.

A) *The general setting*

Let V be an irreducible affine algebraic variety in $\mathbb{C}^N$. An increasing sequence $\Phi = \{\phi_p\}_{p\in\mathbb{N}}$ of upper semicontinuous real-valued functions on V is said to be *weakly admissible* if it is uniformly bounded from below on compact subsets of V, and for every $p \in \mathbb{N}$ there is $q \in \mathbb{N}$ such that

$$\lim_{\substack{\zeta\in V\\ |\zeta|\longrightarrow\infty}} \{\phi_p(\zeta) - \phi_q(\zeta)\} = -\infty. \tag{5.1}$$

For a weakly admissible sequence $\Phi = \{\phi_p\}_{p\in\mathbb{N}}$ we set

$$\mathcal{O}_{\phi_p}(V) = \left\{ f \in \mathcal{O}(V) \,\Big|\, \sup_{\zeta\in V} \left| f(\zeta)\cdot e^{-\phi_p(\zeta)}\right| < \infty \right\}, \tag{5.2}$$

with the Banach space topology defined by the norm

$$\|f\|_{\phi_p} = \sup_{\zeta\in V} \left| f(\zeta)\cdot e^{-\phi_p(\zeta)}\right| \tag{5.3}$$

and consider the inductive limit:

$$\mathcal{O}_\Phi(V) = \varinjlim_{p\nearrow\infty} \mathcal{O}_{\phi_p}(V). \tag{5.4}$$

This is a locally convex topological vector space which is $(\mathcal{LF})$, $(\mathcal{DF})$, barrelled, Schwartz, Montel, bornological, reflexive and complete (see [9], [11], [21], [16]). Let us consider, for two weakly admissible sequences $\Phi = \{\phi_p\}_{p\in\mathbb{N}}$ and $\Psi = \{\psi_p\}_{p\in\mathbb{N}}$, the spaces $\mathcal{O}_\Phi(V)$ and $\mathcal{O}_\Psi(V)$ and the inclusion map

$$\mathcal{O}_\Phi(V) \bigcap \mathcal{O}_\Psi(V) \longrightarrow \mathcal{O}_\Psi(V). \tag{5.5}$$

We note that the sequence $\Phi\wedge\Psi = \{\min(\phi_p, \psi_p)\}_{p\in\mathbb{N}}$ is still weakly admissible and $\mathcal{O}_\Phi(V) \bigcap \mathcal{O}_\Psi(V) = \mathcal{O}_{\Phi\wedge\Psi}(V)$. Thus we can assume for simplicity that

$$\phi_p(\zeta) \le \psi_p(\zeta) \qquad \forall p \in \mathbb{N}, \quad \forall \zeta \in V. \tag{5.6}$$

Then the inclusion map (5.5) becomes a map

$$\mathcal{O}_\Phi(V) \longrightarrow \mathcal{O}_\Psi(V). \tag{5.7}$$

Then we have the following Phragmén–Lindelöf principle (cf. [7], [11]):

Theorem 5.1. *Let $\Phi = \{\phi_p\}_{p\in\mathbb{N}}$ and $\Psi = \{\psi_p\}_{p\in\mathbb{N}}$ be two weakly admissible sequences, satisfying (5.6). Then the following are equivalent:*

(i) *the inclusion map (5.7) has a closed image;*

(ii) *(Ph-L I)* $\begin{cases} \forall p \in \mathbb{N}\ \exists q \in \mathbb{N}\ \exists c > 0, \quad \text{such that} \\ f \in \mathcal{O}_\Phi(V),\ \|f\|_{\psi_p} \leq 1 \Rightarrow \|f\|_{\phi_q} \leq c; \end{cases}$

(iii) *(Ph-L II)* $\begin{cases} \forall p \in \mathbb{N}\ \exists q \in \mathbb{N} \quad \text{such that} \\ \mathcal{O}_\Phi(V) \cap \mathcal{O}_{\psi_p}(V) \subset \mathcal{O}_{\phi_q}(V). \end{cases}$

Theorem 5.2. *Let $\Phi = \{\phi_p\}_{p\in\mathbb{N}}$ and $\Psi = \{\psi_p\}_{p\in\mathbb{N}}$ be two weakly admissible sequences, satisfying (5.6). Then the following are equivalent:*

(i) *the inclusion map (5.7) is an isomorphism;*

(ii) *(Ph-L I)′* $\begin{cases} \forall p \in \mathbb{N}\ \exists q \in \mathbb{N}\ \exists c > 0, \quad \text{such that} \\ \|f\|_{\phi_q} \leq c\|f\|_{\psi_p} \quad \forall f \in \mathcal{O}(V); \end{cases}$

(iii) *(Ph-L II)′* $\begin{cases} \forall p \in \mathbb{N}\ \exists q \in \mathbb{N} \quad \text{such that} \\ \mathcal{O}_{\psi_p}(V) \subset \mathcal{O}_{\phi_q}(V). \end{cases}$

It is useful to give a formulation of the Phragmén–Lindelöf principles of the two theorems above in terms of plurisubharmonic, rather than entire functions on V. This is possible under additional assumptions on the sequences Φ, Ψ. The precise formulation will be given in Theorem 5.3 below. This result is analogous to the corresponding ones in [30] and [18], and is suggested by the treatment of analytic convexity in [23] and [4].

We recall that a function $u : V \longrightarrow [-\infty, +\infty[$ is *weakly plurisubharmonic* if it is plurisubharmonic in the complement in V of the set $S(V)$ of singular points of V, and moreover $u(\zeta) = \limsup_{\zeta' \longrightarrow \zeta} u(\zeta')$ for all $\zeta \in V$. We denote by $\mathfrak{P}(V)$ the set of all weakly plurisubharmonic functions on V.

An increasing sequence $\Phi = \{\phi_p\}_{p\in\mathbb{N}}$ of real-valued functions on V will be called *admissible* if it is weakly admissible and, in addition, the following conditions are satisfied:

(i) for each $p \in \mathbb{N}$, the function ϕ_p is the restriction to V of a plurisubharmonic function $\tilde{\phi}_p$ in $\mathbb{C}^N$;

moreover we require that the plurisubharmonic extensions $\{\tilde{\phi}_p\}_{p\in\mathbb{N}} \subset \mathfrak{P}(\mathbb{C}^N)$ in (i) satisfy:

(ii) for each $p \in \mathbb{N}$ and $c > 0$ there are an integer $q \geq 0$ and a constant $c' > 0$ such that

$$\tilde{\phi}_p(\zeta) + c\log(e + |\zeta|) \leq \tilde{\phi}_q(\zeta) + c' \quad \forall \zeta \in \mathbb{C}^N; \tag{5.8}$$

(iii) for every nonnegative integer p there are positive constants a_p, b_p, c_p such that

(5.9)
$$|\tilde{\phi}_p(\zeta + \zeta') - \tilde{\phi}_p(\zeta)| \leq c_p, \text{ if } \zeta, \zeta' \in \mathbb{C}^N \text{ and } |\zeta'| \leq a_p(1 + |\zeta|)^{-b_p}.$$

We have (see [13]):

Theorem 5.3. *Let $\Phi = \{\phi_p\}_{p\in\mathbb{N}}$ and $\Psi = \{\psi_p\}_{p\in\mathbb{N}}$ be two admissible sequences on V, satisfying (5.6). Then (Ph-L I) and (Ph-L II) are equivalent to:*

$$\text{(Ph-L III)} \qquad \begin{cases} \forall p \in \mathbb{N},\ \exists q \in \mathbb{N},\ \exists c > 0 \quad \textit{such that} \\ \textit{if } u \in \mathfrak{P}(V) \textit{ satisfies, for some } p_u \in \mathbb{N},\ c_u > 0\textit{:} \\ \begin{cases} u(\zeta) \le \psi_p(\zeta) & \forall \zeta \in V, \\ u(\zeta) \le \phi_{p_u}(\zeta) + c_u & \forall \zeta \in V, \end{cases} \\ \qquad \textit{then it also satisfies} \\ u(\zeta) \le \phi_q(\zeta) + c \qquad \forall \zeta \in V. \end{cases}$$

B) *Applications to the overdetermined Cauchy problem in classes of ultradifferentiable functions*

We fix a prime ideal $\mathfrak{p}$ in $\mathscr{P}$ and consider the irreducible affine algebraic variety $V = V(\check{\mathfrak{p}})$ defined in (4.16).

Let F be a closed convex set in $\mathbb{R}^N$. We fix a sequence $\{Q_p\}_{p\in\mathbb{N}}$ of compact convex subsets of F with $Q_p \subset Q_{p+1}$ for all p and $\cup_{p\in\mathbb{N}} Q_p = F$. Let $\{M_p\}$ be a sequence of real numbers satisfying conditions (M). Set

$$(5.10) \qquad \begin{cases} \phi_p^{(F)}(\tau,\zeta) = H_{Q_p}(Im\tau, Im\zeta) + M(p\zeta) + p\log(1+|\tau|) + p \\ \quad \text{for} \quad (\tau,\zeta) \in \mathbb{C}^N = \mathbb{C}^k_\tau \times \mathbb{C}^n_\zeta. \end{cases}$$

Then $\Phi^{(F)} = \{\phi_p^{(F)}\}_{p\in\mathbb{N}}$ defines an admissible sequence of plurisubharmonic functions in V. We have:

Proposition 5.4. *The commutative diagram:*

$$(5.11) \qquad \begin{array}{ccc} \left(\tilde{W}_F^{(M_p)}\right)' & \longrightarrow & \mathscr{O}_{\Phi^{(F)}}(\mathbb{C}^N) \\ \downarrow & & \downarrow \\ \left(\tilde{W}_F^{(M_p)}\right)'_{\mathfrak{p}} & \longrightarrow & \mathscr{O}_{\Phi^{(F)}}(V(\check{\mathfrak{p}})) \end{array}$$

where the top horizontal arrow is the Fourier–Laplace transform, the left vertical arrow is projection onto the quotient and the right vertical arrow the restriction map, defines an isomorphism between $\left(\tilde{W}_F^{(M_p)}\right)'_{\mathfrak{p}}$ and $\mathscr{O}_{\Phi^{(F)}}(V(\check{\mathfrak{p}}))$.

Proof. The top horizontal arrow is an isomorphism by Theorem 3.14. The right vertical arrow is onto because of Lemma 4.7. The proof that the bottom horizontal arrow is well defined and is an isomorphism can then be obtained by repeating the argument in the proof of Lemma 4.8 : it shows indeed that the difference of two elements of $\left(\tilde{W}_F^{(M_p)}\right)'$ whose Fourier–Laplace transforms agree on $V(\check{\mathfrak{p}})$ annihilates all elements of $\left(\tilde{W}_F^{(M_p)}\right)_{\mathfrak{p}}$.

Let (K_1, K_2) be a pair of closed convex subsets of $\mathbb{R}^N$ with $K_1 \subset K_2$. We fix two increasing sequences $\{Q'_p\}_{p\in\mathbb{N}}$ and $\{Q''_p\}_{p\in\mathbb{N}}$ of convex compact subsets of $\mathbb{R}^N$ with $K_1 = \bigcup_{p\in\mathbb{N}} Q'_p$ and $K_2 = \bigcup_{p\in\mathbb{N}} Q''_p$, to construct the two admissible sequences $\Phi^{(K_1)} = \{\phi^{(K_1)}\}_{p\in\mathbb{N}}$ and $\Phi^{(K_2)} = \{\phi^{(K_2)}\}_{p\in\mathbb{N}}$ by formula (5.10) (where $Q_p = Q'_p$ for K_1 and $Q_p = Q''_p$ for K_2).

Then, using Theorem 5.1, we obtain the following criterion:

Theorem 5.5. *Let $\{M_p\}_{p\in\mathbb{N}}$ be a sequence of real numbers satisfying conditions (M); let $\mathfrak{M}$ be a finitely generated unitary $\mathcal{P}$-module. Let (K_1, K_2) be a pair of convex closed subsets of $\mathbb{R}^N$ with $K_1 \subset K_2$. Then a necessary and sufficient condition for the pair (K_1, K_2) to be of evolution in the class $\tilde{W}^{(M_p)}$ for $\mathfrak{M}$ is that either of the following equivalent conditions* (5.12)*,* (5.13) *holds true for all* $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$*:*

(5.12)

$$\begin{cases} \text{For every } p \in \mathbb{N} \text{ there exists } p' \in \mathbb{N} \text{ such that:} \\ \text{if } U \in \mathcal{O}(V(\check{\mathfrak{p}})) \text{ satisfies:} \\ (i)\quad |U(\tau,\zeta)| \le e^p(1+|\tau|)^p e^{H_{Q''_p}(Im\tau, Im\zeta)+M(p\zeta)} \qquad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}}) \\ (ii)\quad |U(\tau,\zeta)| \le e^q(1+|\tau|)^q e^{H_{Q'_q}(Im\tau, Im\zeta)+M(q\zeta)} \qquad \text{for some } q\in\mathbb{N} \\ \qquad\qquad \text{and}\quad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}}) \\ \qquad \text{then it also satisfies} \\ (iii)\quad |U(\tau,\zeta)| \le e^{p'}(1+|\tau|)^{p'} e^{H_{Q'_{p'}}(Im\tau, Im\zeta)+M(p'\zeta)} \qquad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}}); \end{cases}$$

$$(5.13)\quad \begin{cases} \text{For every } p \in \mathbb{N} \text{ there exists } p' \in \mathbb{N} \text{ such that:} \\ \text{if } u \in \mathfrak{P}(V(\check{\mathfrak{p}})) \text{ satisfies:} \\ (i)\quad u(\tau,\zeta) \le p + p\log(1+|\tau|) \\ \qquad\qquad +H_{Q''_p}(Im\tau, Im\zeta) + M(p\zeta) \qquad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}}) \\ (ii)\quad u(\tau,\zeta) \le q + q\log(1+|\tau|) \qquad \text{for some } q\in\mathbb{N} \\ \qquad\qquad +H_{Q'_q}(Im\tau, Im\zeta) + M(q\zeta) \qquad \text{and}\,\forall(\tau,\zeta)\in V(\check{\mathfrak{p}}) \\ \qquad \text{then it also satisfies:} \\ (iii)\quad u(\tau,\zeta) \le p' + p'\log(1+|\tau|) \\ \qquad\qquad +H_{Q'_{p'}}(Im\tau, Im\zeta) + M(p'\zeta) \qquad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}}). \end{cases}$$

Evolution in the class $W^{(N_p),(M_p)}$, for two sequences $\{N_p\}_{p\in\mathbb{N}}$ and $\{M_p\}_{p\in\mathbb{N}}$ satisfying conditions (M), can be discussed in the same way. We construct our spaces of entire functions using sequences

$$(5.14)\qquad \phi_p^{(F)} = H_{Q_p}(Im\tau, Im\zeta) + M(p\zeta) + N(p\tau) + p \quad \text{for} \quad (\tau,\zeta)\in\mathbb{C}^N,$$

where F is a closed convex subset of $\mathbb{R}^N$ and $\{Q_p\}_{p\in\mathbb{N}}$ an increasing sequence of compact convex sets with $\bigcup_{p\in\mathbb{N}} Q_p = F$, and where N, M are the plurisubharmonic functions associated to the two sequences $\{N_p\}_{p\in\mathbb{N}}$ and $\{M_p\}_{p\in\mathbb{N}}$ as in Section 3. We obtain the following statement:

Theorem 5.6. *Let $\{M_p\}_{p\in\mathbb{N}}$, $\{N_p\}_{p\in\mathbb{N}}$ be sequences of real numbers satisfying conditions (M); let $\mathfrak{M}$ be a finitely generated unitary $\mathcal{P}$-module. Let (K_1, K_2) be a pair of convex closed subsets of $\mathbb{R}^N$ with $K_1 \subset K_2$. Then a necessary and sufficient condition for the pair (K_1, K_2) to be of evolution in the class $W^{(N_p),(M_p)}$ for $\mathfrak{M}$ is that either of the following equivalent conditions* (5.15), (5.16) *holds true for all $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$:*

(5.15)

For every $p \in \mathbb{N}$ there exists $p' \in \mathbb{N}$ such that:
if $U \in \mathcal{O}(V(\check{\mathfrak{p}}))$ satisfies:

$$(i)\quad |U(\tau,\zeta)| \le e^p e^{H_{Q''_p}(Im\tau, Im\zeta)+M(p\zeta)+N(p\tau)} \qquad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}})$$

$$(ii)\quad |U(\tau,\zeta)| \le e^q e^{H_{Q'_q}(Im\tau, Im\zeta)+M(q\zeta)+N(q\tau)} \qquad \textit{for some } q\in\mathbb{N} \textit{ and } \forall(\tau,\zeta)\in V(\check{\mathfrak{p}})$$

then it also satisfies

$$(iii)\quad |U(\tau,\zeta)| \le e^{p'} e^{H_{Q'_{p'}}(Im\tau, Im\zeta)+M(p'\zeta)+N(p'\tau)} \qquad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}});$$

(5.16)

For every $p \in \mathbb{N}$ there exists $p' \in \mathbb{N}$ such that:
if $u \in \mathfrak{P}(V(\check{\mathfrak{p}}))$ satisfies:

$$(i)\quad u(\tau,\zeta) \le p + H_{Q''_p}(Im\tau, Im\zeta) + M(p\zeta) + N(p\tau) \qquad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}})$$

$$(ii)\quad u(\tau,\zeta) \le q + H_{Q'_q}(Im\tau, Im\zeta) + M(q\zeta) + N(q\tau) \qquad \textit{for some } q\in\mathbb{N} \textit{ and } \forall(\tau,\zeta)\in V(\check{\mathfrak{p}})$$

then it also satisfies:

$$(iii)\quad u(\tau,\zeta) \le p' + H_{Q'_{p'}}(Im\tau, Im\zeta) + M(p'\zeta) + N(p'\tau) \qquad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}}).$$

6. The hypersurface case

In this section we consider the special case where $N = n+1$, $\mathbb{R}^N = \mathbb{R}_t \times \mathbb{R}^n_x$, and the initial data of our Cauchy problem are prescribed on

$$\Sigma = \left\{(t,x) \in \mathbb{R}^N \mid t = 0\right\} \simeq \mathbb{R}^n.$$

We also set

$$H = \left\{(t, x) \in \mathbb{R}^N \mid t \geq 0\right\}.$$

Denote by $\tilde{\mathscr{A}}(H)$ the space of Whitney functions $f(t, x)$ on H such that each t-derivative $D^\ell f(t, x)$ of $f(t, x)$ extends to an entire function of $x \in \mathbb{C}^n$, uniformly on compact subsets of $\mathbb{R}_t$. Analogously, define $\tilde{\mathscr{A}}(\Sigma)$ by the exact sequence:

$$0 \longrightarrow \tilde{\mathscr{A}}(H) \cap \mathcal{I}(\Sigma, H) \longrightarrow \tilde{\mathscr{A}}(H) \longrightarrow \tilde{\mathscr{A}}(\Sigma) \longrightarrow 0.$$

Let $\mathfrak{M}$ be a unitary finitely generated $\mathscr{P}$-module. In [32] the following version of a classical Volterra's theorem was proved:

Theorem 6.1. *Assume that Σ is formally noncharacteristic for $\mathfrak{M}$. Then the pair $(\tilde{\mathscr{A}}(\Sigma), \tilde{\mathscr{A}}(H))$ is of evolution for $\mathfrak{M}$.*

We can use this result to derive a criterion for causality:

Theorem 6.2. *Let $\mathfrak{M}$ be a finitely generated unitary $\mathscr{P}$-module. Let $(M_p)_{p\in\mathbb{N}}$ be a sequence satisfying conditions (M). Assume that the hypersurface Σ is formally noncharacteristic for $\mathfrak{M}$. Then a necessary and sufficient condition for the pair (Σ, H) to be of causality for $\mathfrak{M}$ in the class $\tilde{W}^{(M_p)}$ is that Σ is strongly noncharacteristic for $\mathfrak{M}$.*

Proof. Let $\mathfrak{F}$ denote the space

$$\mathfrak{F} = \left\{ f \in \tilde{W}_H^{(M_p)} \mid f|_\Sigma \in \tilde{\mathscr{A}}(\Sigma) \right\}.$$

From Theorem 6.1, it follows that under our assumptions the restriction map

$$\mathrm{Ext}^0_{\mathscr{P}}(\mathscr{P}/\mathfrak{p}, \mathfrak{F}) \longrightarrow \mathrm{Ext}^0_{\mathscr{P}}(\mathscr{P}/\mathfrak{p}, \tilde{\mathscr{A}}(\Sigma))$$

is an isomorphism for every $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$. We consider on these spaces the natural Fréchet–Schwartz topologies. Then the open mapping theorem yields, for each $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$, an a priori esimate of the form:

$$(6.1) \qquad |f(1,0)| \leq c \sup_{\ell \leq q} \sup_{\substack{x\in\mathbb{C}^n \\ |x|\leq A}} \left|D_t^\ell f(0, x)\right| \qquad \forall f \in \mathrm{Ext}^0_{\mathscr{P}}(\mathscr{P}/\mathfrak{p}, \mathfrak{F}),$$

for suitable constants c, $A > 0$ and $q \in \mathbb{N}$. Inserting the exponential solutions $e^{i\tau t + i\langle x, \zeta\rangle}$ for $(\tau, \zeta) \in V(\mathfrak{p})$ in (6.1), we obtain

$$(6.2) \qquad -Im\tau \leq A|\zeta| + c' \log(1 + |\tau|) + c'' \qquad \forall(\tau, \zeta) \in V(\mathfrak{p}),$$

with suitable constants c', $c'' > 0$. Passing to the limit for $|\tau| + |\zeta| \longrightarrow \infty$ on $V(\mathfrak{p})$, we obtain on the asymptotic cone $V^0(\mathfrak{p})$ the estimate

$$(6.3) \qquad -Im\tau \leq A|\zeta| \qquad \forall(\tau, \zeta) \in V^0(\mathfrak{p}),$$

from which we derive

$$(6.4) \qquad |\tau| \leq A|\zeta| \qquad \forall(\tau, \zeta) \in V^0(\mathfrak{p}),$$

because the asymptotic cone is homogeneous with respect to multiplication by complex numbers. This establishes necessity. Sufficiency follows from the classical Holmgren uniqueness theorem.

7. Evolution for quasi-free initial data

In this section we consider generalizations of the classical Petrowski conditions for evolution. They are sufficient to guarantee the validity of a Phragmèn–Lindelöf principle implying evolution in some of the classes of ultradifferentiable functions considered above. We need to assume that the initial affine manifold Σ carryinig the initial data is *quasi-free* for our given $\mathcal{P}$-module $\mathfrak{M}$, as explained in Section 1.

Let $\mathbb{R}^N = \mathbb{R}^k_t \times \mathbb{R}^n_x$ and let $\Sigma = \{0\} \times \mathbb{R}^n = \{(t,x) \in \mathbb{R}^N \,\big|\, t = 0\}$. We identify $\mathbb{R}^n_x$ to $\Sigma \subset \mathbb{R}^N$. Let K_1 be a compact convex subset of $\mathbb{R}^n_x$, with a nonempty interior. We can assume without loss of generality that $0 \in \overset{o}{K}_1$. Next we consider a closed proper convex cone Γ in $\mathbb{R}^k_t$, with vertex in 0, set $K_2 = \Gamma \times K_1$. We define

$$\kappa_\Gamma(\tau) = \sup_{\substack{t \in \Gamma \\ |t_i| \le 1\ i=1,\dots,k}} Im\ \langle t, \tau\rangle \qquad \forall \tau \in \mathbb{C}^k. \tag{7.1}$$

We recall from [11] that we have the following:

Proposition 7.1. *Let $\mathfrak{M}$ be a $\mathcal{P}$-module of finite type and assume that Σ is formally noncharacteristic and quasi-free for Σ. Assume that there is a constant c such that, for every $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$, we have*

$$\kappa_\Gamma(\tau) \le c \qquad \forall (\tau,\xi) \in V(\check{\mathfrak{p}}) \quad with \quad \xi \in \mathbb{R}^n. \tag{7.2}$$

Then the pair (K_1, K_2) is of evolution for $\mathfrak{M}$ in the class of Whitney functions.

Actually in [11] this was stated for the special case where K_1 is the closed ball of radius A centred at 0 in $\mathbb{R}^n_x$ and $\Gamma = \{t \in \mathbb{R}^k \,\big|\, t_i \ge 0 \quad \text{for} \quad i = 1, \dots, k\}$, but the same proof yields the slight, more general proposition we gave here.

We will show that a weaker condition than (7.2) implies evolution for the same pairs of closed sets in some spaces of ultradifferentiable functions.

A) *Evolution in the class $\tilde{\gamma}^{(s)}$*

We have the following:

Theorem 7.2. *Let $\mathfrak{M}$ be a $\mathcal{P}$ module of finite type, such that Σ is formally noncharacteristic and quasi-free for $\mathfrak{M}$. Let $s > 1$ and assume that there exist constants c_1, c_2 such that for every $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$*

$$\kappa_\Gamma(\tau) \le c_1 |\xi|^{1/s} + c_2 \qquad \forall (\tau,\xi) \in V(\check{\mathfrak{p}}), \quad with \quad \xi \in \mathbb{R}^n. \tag{7.3}$$

Then the pair (K_1, K_2) is of evolution for $\mathfrak{M}$ in the class $\tilde{\gamma}^{(s)}$.

Proof. The convex set $K_1 \subset \mathbb{R}^n$ is compact and thus is contained in a closed ball of radius $A > 0$, centred at 0, of $\mathbb{R}^n$. By the assumption that Σ is formally noncharacteristic and quasi-free for $\mathfrak{M}$, all compatible data on K_1 and K_2 can be extended to compatible data on $\{|x| \le A\}$ and $\Gamma \times \{|x| \le A\}$. Thus we will assume in the following, without loss of generality, that

$$K_1 = \left\{x \in \mathbb{R}^n \,\Big|\, |x| \le A\right\}.$$

We note that the Phragmèn–Lindelöf principle (5.12) of Theorem 5.5 reads, for the space $\tilde{\gamma}^{(s)}$ and the pair of convex sets (K_1, K_2):

for every $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$ and for every $m,\ \alpha,\ L \in \mathbb{N}$ there are $m',\ L' \in \mathbb{N}$ and $c > 0$ such that, if $F \in \mathcal{O}(\mathbb{C}^N)$ satisfies, for suitable constants $m_F,\ L_F \in \mathbb{N}$ and $c_F > 0$:

$$(7.4)\qquad \begin{cases} |F(\tau,\zeta)| \le (1+|\tau|)^m e^{\alpha\kappa_\Gamma(\tau)+A|Im\zeta|+L|\zeta|^{1/s}} & \forall(\tau,\zeta) \in V(\check{\mathfrak{p}}) \\ |F(\tau,\zeta)| \le c_F(1+|\tau|)^{m_F} e^{A|Im\zeta|+L_F|\zeta|^{1/s}} & \forall(\tau,\zeta) \in V(\check{\mathfrak{p}}) \end{cases}$$

it also satisfies

$$(7.5)\qquad |F(\tau,\zeta)| \le c(1+|\tau|)^{m'} e^{A|Im\zeta|+L'|\zeta|^{1/s}} \quad \forall(\tau,\zeta) \in V(\check{\mathfrak{p}}).$$

We fix a prime ideal $\mathfrak{p}$ in $\mathrm{Ass}(\mathfrak{M})$ and argue on the irreducible affine algebraic variety $V = V(\check{\mathfrak{p}})$. The map $\pi_n : V(\check{\mathfrak{p}}) \longrightarrow \mathbb{C}^n$ is surjective, finite and dominant by our assumptions that Σ is formally noncharacteristic and quasi-free for $\mathfrak{M}$. In particular, there are real constants λ and b such that

$$(7.6)\qquad |\tau| \le \lambda\,(1+|\zeta|)^b \quad \forall(\tau,\zeta) \in V.$$

Starting with $F \in \mathcal{O}(\mathbb{C}^N)$ satisfying (7.4) we obtain a plurisubharmonic function u in $\mathbb{C}^n$ by setting:

$$(7.7)\qquad u(\zeta) = \sup_{(\tau,\zeta)\in V} \log|F(\tau,\zeta)| \quad \text{for} \quad \zeta \in \mathbb{C}^n.$$

This plurisubharmonic function satisfies, in view of (7.6) and (7.4) :

$$(7.8)\qquad \begin{cases} \forall\delta > 0\ \exists C_\delta > 0 \quad \text{such that} \\ u(\zeta) \le (A+\delta)|\zeta| + C_\delta \qquad \forall\zeta \in \mathbb{C}^n \\ \\ u(\xi) \le L_1|\xi|^{1/s} + B \qquad \forall\xi \in \mathbb{R}^n. \end{cases}$$

The constants $C_\delta > 0$ depend on the entire function F, while the constants L_1 and B only depend on $m,\ \alpha,\ L$ and the constants c_1 and c_2 in (7.3). Then (7.5) is a consequence of the following:

Lemma 7.3. *Let $s > 1$ and $A,\ B,\ L_1 \ge 0$ be fixed. Then every plurisubharmonic function u in $\mathbb{C}^n$ that satisfies (7.8) also satisfies:*

$$(7.9)\qquad u(\zeta) \le A|Im\zeta| + L_1'|\zeta|^{1/s} + B \qquad \forall\zeta \in \mathbb{C}^n,$$

where $L_1' = 2^{2s/(2s-1)} L_1 \big/ (\cos(\pi/4s)\cos(\pi/2s))$.

Proof. By substituting $u - B$ for u, we can assume that $B = 0$. Consider first the case $n = 1$. Set $Q = \{\zeta \in \mathbb{C} \,\big|\, Re\zeta > 0,\ \ Im\zeta > 0\}$. In polar coordinates $\zeta = re^{i\theta}$, with $r \ge 0$, $-\pi < \theta \le \pi$ we consider the functions

$$\phi(\zeta) = r^{1/s}\cos\left[\left(\theta - \frac{\pi}{4}\right)\Big/ s\right] \quad \text{and} \quad \psi(\zeta) = r^{3/2}\cos\left[\frac{3}{2}\left(\theta - \frac{\pi}{4}\right)\right].$$

They are harmonic in Q and continuous in $\overline{Q}$. Hence the functions

$$u_{\delta,\epsilon}(\zeta) = u(\zeta) - \left[L_1 \Big/ \cos\left(\frac{\pi}{4s}\right)\right]\phi(\zeta) - (A+\delta)Im\zeta - \epsilon\psi(\zeta)$$

are, for every $\delta, \epsilon > 0$, subharmonic in Q and uppersemicontinuous in $\overline{Q}$. For each fixed positive δ and ϵ the function $u_{\delta,\epsilon}$ is bounded, on the boundary of $Q_R = \{\zeta \in Q \,\big|\, |\zeta| < R\}$, by the constant C_δ, provided R is sufficiently large. By the maximum principle we deduce that $u_{\delta,\epsilon} \leq C_\delta$ in $\overline{Q}$. Letting $\epsilon \searrow 0$, we obtain that $u(\zeta) \leq (A+\delta)Im\zeta + \left[L_1 \big/ \cos\left(\frac{\pi}{4s}\right)\right] |\zeta|^{1/s} + C_\delta$ in $\overline{Q}$. Repeating the same argument in the other quadrants, we obtain:

$$u(\zeta) \leq (A+\delta)\,|Im\zeta| + \left[L_1 \Big/ \cos\left(\frac{\pi}{4s}\right)\right] |\zeta|^{1/s} + C_\delta \qquad \text{in} \quad \mathbb{C}. \tag{7.10}$$

Next we fix σ with $1 < \sigma < s$ and consider the functions:

$$\tilde{\phi}(\zeta) = r^{1/s}\cos\left[\frac{\theta - (\pi/2)}{s}\right] \quad \text{and} \quad \tilde{\psi}(\zeta) = r^{1/\sigma}\cos\left[\frac{\theta - (\pi/2)}{\sigma}\right].$$

They are harmonic in the upper half-plane $H = \{\zeta \in \mathbb{C} \,\big|\, Im\zeta > 0\}$ and continuous in $\overline{H}$. Moreover they satisfy:

$$r^{1/s}\cos\left(\frac{\pi}{2s}\right) \leq \tilde{\phi}(\zeta) \leq r^{1/s} \quad \text{and} \quad r^{1/\sigma}\cos\left(\frac{\pi}{2\sigma}\right) \leq \tilde{\psi}(\zeta) \leq r^{1/\sigma} \qquad \text{in} \quad \overline{H}.$$

Therefore the functions

$$w_{\delta,\epsilon}(\zeta) = u(\zeta) - (A+2\delta)Im\zeta - \left[\frac{L_1}{\cos(\pi/4s)\cos(\pi/2s)}\right]\tilde{\phi}(\zeta) - \epsilon\tilde{\psi}(\zeta),$$

for every $\delta, \epsilon > 0$ are subharmonic in H, upper semicontinuous in $\overline{H}$ and less or equal to zero on the boundary of $H_R = \{\zeta \in H \,\big|\, |\zeta| < R\}$, provided $R > 0$ is sufficiently large. By the maximum principle we obtain that $w_{\delta,\epsilon} \leq 0$ in $\overline{H}$. Letting $\epsilon \searrow 0$, we obtain that $u(\zeta) \leq (A+2\delta)Im\zeta + \left[L_1 \big/ (\cos(\pi/4s)\cos(\pi/2s))\right] |\zeta|^{1/s}$ in $\overline{H}$. Arguing in a similar way for the lower half-plane $\{Im\zeta < 0\}$ we get

$$u(\zeta) \leq (A+2\delta)\,|Im\zeta| + \left[L_1 \big/ (\cos(\pi/4s)\cos(\pi/2s))\right] |\zeta|^{1/s} \quad \text{in} \quad \mathbb{C}. \tag{7.11}$$

Letting $\delta \searrow 0$, we obtain:

$$u(\zeta) \leq A\,|Im\zeta| + \left[L_1 \big/ (\cos(\pi/4s)\cos(\pi/2s))\right] |\zeta|^{1/s} \quad \text{in} \quad \mathbb{C}, \tag{7.12}$$

which establishes the Lemma (with a better constant L_1') in the case $n = 1$. When $n > 1$, using a standard recursive argument, we obtain

$$u(\zeta) \;\leq\; A\sum_{i=1}^{n}|Im\zeta_i| + \frac{L_1}{\cos(\pi/4s)\cos(\pi/2s)}\sum_{i=1}^{n}|\zeta_i|^{1/s} \quad \text{in} \quad \mathbb{C}^n. \tag{7.13}$$

To complete the proof, we use the invariance of our assumptions with respect to *real* rotations in $\mathbb{C}^n$. Having fixed any point ζ^0 in $\mathbb{C}^n$, we can assume, after a linear change of coordinates by a matrix in $\mathbf{SO}(n)$, that $\zeta^0 = (\zeta_1^0, \zeta_2^0, 0, \ldots, 0)$ with $\zeta_2^0 \in \mathbb{R}$. Then $\sum_{i=1}^n \left|Im\zeta_i^0\right| = \left|Im\zeta^0\right|$ and $\sum_{i=1}^n \left|\zeta_i^0\right|^{1/s} = \left|\zeta_1^0\right|^{1/s} + \left|\zeta_2^0\right|^{1/s} \leq 2^{2s/(2s-1)}\left|\zeta^0\right|^{1/s}$ by the Hölder inequality This establishes the lemma, and hence completes the proof of the theorem.

Remark. Note that when (7.3) is valid for some fixed $s = \sigma > 1$, it is also valid for all $1 < s \leq \sigma$. Moreover, (7.2) implies (7.3) for all $s > 1$, so that when (7.2) is valid we have evolution in $\tilde{\gamma}^{(s)}$ for all $s > 1$. This already gives a fair amount of examples of systems for which the pair (K_1, K_2) is of evolution in the classes $\tilde{\gamma}^{(s)}$ for all $s > 1$.

When the cone Γ is semialgebraic, we note that by (7.6) the semialgebraic function

$$(7.14) \qquad g(r) = \sup\left\{\kappa_\Gamma(\tau) \,\middle|\, (\tau, \xi) \in V(\check{\mathfrak{p}}),\ \xi \in \mathbb{R}^n,\ |\xi| \leq r\right\}$$

is defined and finite for all $r \geq 0$. By the Tarski–Seidenberg theorem we obtain that there are a real $c \geq 0$ and a rational $q \in \mathbb{Q}$ such that

$$(7.15) \qquad g(r) = cr^q(1 + o(1)) \qquad \text{for} \quad r \gg 1.$$

The condition $q < 1$ is then necessary for the validity of (7.3). In this case we obtain evolution in $\tilde{\gamma}^{(s)}$ provided $1 < s \leq 1/q$.

Example 1. Let us fix integers $m > h \geq 1$ and consider the scalar differential operator

$$(7.16) \qquad P(D) = \left(i\frac{\partial}{\partial t} + a\frac{\partial^2}{\partial x^2}\right)^m - b\frac{\partial^h}{\partial y^h}$$

with $a, b \in \mathbb{C}$, $Im\, a \leq 0$, $b \neq 0$, in $\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}_x \times \mathbb{R}_y$. The associated affine algebraic variety is $V = \left\{(\tau, \zeta, \eta) \in \mathbb{C}^3 \,\middle|\, (-\tau - a\zeta^2)^m = b(i\eta)^h\right\}$. We note that $\Sigma = \{t = 0\}$ is formally noncharacteristic and free for the corresponding $\mathbb{C}[\tau, \zeta, \eta]$-module $\mathfrak{M} = \mathcal{P}/\mathfrak{p}$ (here $\mathfrak{p}$ is the principal ideal generated by $(-\tau - \zeta^2)^m - (i\eta)^h$). For real ζ and η we have

$$(7.17) \qquad (Im\tau)^+ \leq |b|^{1/m} \cdot |\eta|^{h/m} \quad \text{for} \quad (\tau, \zeta, \eta) \in V(\check{\mathfrak{p}}),\ \zeta, \eta \in \mathbb{R}.$$

Hence we have evolution for the pair consisting of a compact convex subset K_1 of $\Sigma \simeq \mathbb{R}_x \times \mathbb{R}_y$ and $K_2 = \left\{(t, x, y) \,\middle|\, (x, y) \in K_1,\ t \geq 0\right\}$ in the class $\tilde{\gamma}^{(s)}$, provided $1 < s \leq m/h$. We note that by [11], the pair (K_1, K_2) is not of evolution in the class of all Whitney functions.

Example 2. We consider in $\mathbb{R}^4$, with coordinates t_1, t_2, x, y, the matrix of partial differential operators:

$$(7.18) \qquad \begin{bmatrix} \frac{\partial^2}{\partial t_1^2} - \frac{\partial}{\partial x} \\ \frac{\partial^2}{\partial t_2^2} + \frac{\partial^4}{\partial x^4} - \frac{\partial}{\partial y} \end{bmatrix}.$$

Let us denote by $\tau_1, \tau_2, \zeta, \eta$, the corresponding dual coordinates in $\mathbb{C}^4$ and consider the prime ideal $\mathfrak{p}$ in $\mathcal{P} = \mathbb{C}[\tau_1, \tau_2, \zeta, \eta]$ generated by the polynomials $p_1 = \tau_1^2 + i\zeta$ and $p_2 = \tau_2^2 - \zeta^4 + i\eta$. Our matrix of partial differential operators corresponds to the $\mathscr{P}$-module $\mathfrak{M} = \mathscr{P}/\mathfrak{p}$ and $\Sigma = \{t_1 = t_2 = 0\}$ is formally noncharacteristic and free for $\mathfrak{M}$. We have:

$$\text{(7.19)} \qquad \begin{cases} |Im\tau_1| \le |\zeta|^{1/2}, \\ |Im\tau_2| \le |\eta|^{1/2}, \\ \text{for } (\tau_1, \tau_2, \zeta, \eta) \in V(\check{\mathfrak{p}}) \text{ with } \zeta, \eta \in \mathbb{R}. \end{cases}$$

Thus the pair consisting of any compact convex $K_1 \subset \Sigma$ and $K_2 = \Gamma \times K_1$ for any proper closed convex cone Γ of $\mathbb{R}^2_{t_1,t_2}$ is of evolution for $\mathfrak{M}$ in the classes $\tilde{\gamma}^{(s)}$ for all $1 < s \le 2$. One can show, using [11], that this is not true in the class of all Whitney functions.

B) *Evolution in the class* $\gamma^{(r,s)}$

We keep the notation introduced above. Let $\mathfrak{M}$ be a $\mathscr{P}$-module of finite type. We recall that, when Σ is formally noncharacteristic for $\mathfrak{M}$, there is a smallest rational b for which (7.6) holds true (for suitable constants $\lambda \ge 0$) on each irreducible affine algebraic variety $V(\check{\mathfrak{p}})$ for $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$. This exponent b is called the *reduced order* of Σ with respect to $\mathfrak{M}$. We obtain the following result on Gevrey regularity in t, complementing Theorem 7.2:

Theorem 7.4. *Let $\mathfrak{M}$ be a $\mathscr{P}$ module of finite type, such that Σ is formally non-characteristic and quasi-free for $\mathfrak{M}$. Let $s > 1$ and assume that (7.3) holds true on all irreducible affine algebraic varieties $V(\check{\mathfrak{p}})$ for $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$. Then the pair (K_1, K_2) is of evolution for $\mathfrak{M}$ in the class $\gamma^{(r,s)}$ if $s > 1$ and $r \ge bs$.*

Proof. We can assume as before that $K_1 = \{x \in \mathbb{R}^n \mid |x| \le A\}$ for $A > 0$. Then the pair (K_1, K_2) is of evolution for $\mathfrak{M}$ in the class $\gamma^{(r,s)}$ if and only if the following Phragmèn–Lindelöf principle holds true for all $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$:

for every $\alpha,\ L \in \mathbb{N}$ there are $L' \in \mathbb{N}$ and $c > 0$ such that if $F \in \mathscr{O}(\mathbb{C}^N)$ and satisfies, with constants $c_F,\ L_F > 0$:

$$\text{(7.20)} \qquad \begin{cases} |F(\tau, \zeta)| \le e^{\alpha\kappa_\Gamma(\tau) + A|Im\zeta| + L|\tau|^{1/r} + L|\zeta|^{1/s}} & \forall (\tau, \zeta) \in V(\check{\mathfrak{p}}) \\ |F(\tau, \zeta)| \le c_F\, e^{A|Im\zeta| + L_F|\tau|^{1/r} + L_F|\zeta|^{1/s}} & \forall (\tau, \zeta) \in V(\check{\mathfrak{p}}) \end{cases}$$

then it also satisfies

$$\text{(7.21)} \qquad |F(\tau, \zeta)| \le c\, e^{A|Im\zeta| + L'|\tau|^{1/r} + L'|\zeta|^{1/s}} \quad \forall (\tau, \zeta) \in V(\check{\mathfrak{p}}).$$

Using (7.6) and the fact that $b/r \le 1/s$ we reduce the proof of this implication to the one in Theorem 7.2.

Note that in both the Examples, 1 and 2, above, the reduced order b of Σ with respect to the $\mathscr{P}$-modules that we have considered is two: in Example 1 we obtain evolution in $\gamma^{(r,s)}$ provided $1 < s \le m/h$ and $r \ge 2s$; in Example 2 we get evolution in $\gamma^{(r,s)}$ when $1 < s \le 2$ and $r \ge 2s$.

C) *Some remarks on hyperbolicity*

Now we consider the case where condition (7.3) assures the hyperbolicity of the pair $(\Sigma, \Gamma \times \Sigma)$. We have:

Theorem 7.5. *Let $\mathfrak{M}$ be a $\mathcal{P}$ module of finite type, such that Σ is formally non-characteristic and quasi-free for $\mathfrak{M}$. Assume that the reduced order of Σ with respect to $\mathfrak{M}$ is less or equal than* 1 *and that, for some $s > 1$ the condition* (7.3) *holds true on all irreducible affine algebraic varieties $V(\check{\mathfrak{p}})$ for $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$. Then the pair $(\Sigma, \Gamma \times \Sigma)$ is hyperbolic for $\mathfrak{M}$ in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ if $r \geq s$.*

Proof. To prove hyperbolicity in the class $\tilde{\gamma}^{(s)}$ we use the two admissible sequences:

$$\phi_\alpha(\tau, \zeta) = \alpha\left(|Im\zeta| + |\zeta|^{1/s} + \log(1 + |\tau|) + 1\right)$$

and

$$\psi_\alpha(\tau, \zeta) = \alpha\left(\kappa_\Gamma(\tau) + |Im\zeta| + |\zeta|^{1/s} + \log(1 + |\tau|) + 1\right)$$

for $\alpha \in \mathbb{N}$.

Fix $\mathfrak{p}$ in $\mathrm{Ass}(\mathfrak{M})$ and consider the plurisubharmonic function u in $\mathbb{C}^n$ defined by

$$u(\zeta) = \sup_{(\tau,\zeta) \in V(\check{\mathfrak{p}})} \kappa_\Gamma(\tau). \tag{7.22}$$

In view of (7.6), valid with $b = 1$, and of (7.3), we can apply Lemma 7.3 to u: we obtain, with constants $c_3,\ c_4 \geq 0$:

$$\kappa_\Gamma(\tau) \leq \lambda|Im\zeta| + c_3|\zeta|^{1/s} + c_4 \qquad \forall(\tau, \zeta) \in V(\check{\mathfrak{p}}). \tag{7.23}$$

Thus, an entire function $F \in \mathcal{O}(\mathbb{C}^N)$ satisfying the first inequality in (7.4) also satisfies

$$|F(\tau, \zeta)| \leq (1 + |\tau|)^m e^{(A+\alpha\lambda)|Im\zeta| + (L+\alpha c_3)|\zeta|^{1/s} + c_4} \quad \forall(\tau, \zeta) \in V(\check{\mathfrak{p}}), \tag{7.24}$$

and thus we obtain the hyperbolicity of the pair $(\Sigma, \Gamma \times \Sigma)$ with respect to $\mathfrak{M}$ in the class $\tilde{\gamma}^{(s)}$ by Theorem 5.2. In a similar way we obtain the result for the class $\gamma^{(r,s)}$ with $r \geq s$.

Example 3. Consider the $\mathcal{P} = \mathbb{C}[\tau_1, \tau_2, \zeta, \eta]$-module $\mathfrak{M}$ corresponding to the matrix of partial differential operators:

$$\begin{bmatrix} \frac{\partial^2}{\partial t_1^2} - 2\frac{\partial^2}{\partial t_1 \partial x} + \frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial y} + i\frac{\partial}{\partial t_2} \\ \frac{\partial^2}{\partial t_2^2} - 2\frac{\partial^2}{\partial t_2 \partial y} + \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial x} + i\frac{\partial}{\partial t_1} \end{bmatrix} \tag{7.25}$$

in $\mathbb{R}^4$. We have $\mathfrak{M} = \mathcal{P}/\mathfrak{p}$ for the prime ideal $\mathfrak{p}$ generated by the polynomials

$$p_1 = (\tau_1 - \zeta)^2 + \eta + \tau_2 \quad \text{and} \quad p_2 = (\tau_2 - \eta)^2 + i\zeta + \tau_1.$$

We note that $\Sigma = \{t_1 = 0,\ t_2 = 0\} \simeq \mathbb{R}^2_{x,y}$ is formally noncharacteristic and free for $\mathfrak{M}$. Moreover, the reduced order of Σ with respect to Σ is 1 and from

$$\tau_1 = \zeta \pm \sqrt{\eta + \tau_2}, \qquad \tau_2 = \eta \pm \sqrt{i\zeta + \tau_1}$$

for $(\tau, \zeta) \in V(\check{\mathfrak{p}})$ we easily obtain (7.3) with $s = 2$. Therefore we have hyperbolicity for the pair $(\Sigma, \Gamma \times \Sigma)$ with respect to $\mathfrak{M}$ for every proper convex cone $\Gamma \subset \mathbb{R}^2_{t_1,t_2}$ in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(r,s)}$ provided $1 < s \leq 2$ and $r \geq s$.

8. Well-posedness

We keep the notation of Section 7.

A) *Hyperbolicity in the classes* $\tilde{W}^{(M_p)}$ *and* $W^{(N_p),(M_p)}$

Theorem 8.1. *Let* $\{M_p\}_{p\in\mathbb{N}}$, $\{N_p\}_{p\in\mathbb{N}}$ *be two sequences of positive real numbers satisfying conditions (M). Let* (K_1, K_2)*, be a pair of closed convex subsets of* $\mathbb{R}^N$*, with* $K_1 \subset K_2$*, and let* $\mathfrak{M}$ *be a finitely generated unitary* $\mathcal{P}$ *module. Then:*

a) A necessary and sufficient condition for the pair (K_1, K_2) *to be hyperbolic for* $\mathfrak{M}$ *in the class* $\tilde{W}^{(M_p)}$ *is that for every compact convex subset* Q_2 *of* K_2 *there exist a compact convex subset* Q_1 *of* K_1 *and a constant* $L > 0$ *such that for every prime ideal* $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$ *the following holds true:*

(8.1)
$$\begin{cases} H_{Q_2}(Im\tau, Im\zeta) \leq H_{Q_1}(Im\tau, Im\zeta) + M(L\zeta) + L\log(1+|\tau|) + L \\ \qquad \forall(\tau,\zeta) \in V(\check{\mathfrak{p}}). \end{cases}$$

b) A necessary and sufficient condition for the pair (K_1, K_2) *to be hyperbolic for* $\mathfrak{M}$ *in the class* $W^{(N_p),(M_p)}$ *is that for every compact convex subset* Q_2 *of* K_2 *there exist a compact convex subset* Q_1 *of* K_1 *and a constant* $L > 0$ *such that for every prime ideal* $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$ *the following holds true:*

$$\begin{cases} H_{Q_2}(Im\tau, Im\zeta) \leq H_{Q_1}(Im\tau, Im\zeta) + M(L\zeta) + N(L\tau) + L \\ \qquad \forall(\tau,\zeta) \in V(\check{\mathfrak{p}}). \end{cases} \tag{8.2}$$

Proof. We prove a); the proof of b) is similar and therefore will be omitted.

By Propositions 2.1 and 4.6, the hyperbolicity of the pair (K_1, K_2) for $\mathfrak{M}$ in the class $\tilde{W}^{(M_p)}$ is equivalent to the fact that for every prime ideal $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$ the map:

$$\mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\mathfrak{p}, \tilde{W}^{(M_p)}_{K_2}\right) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}\left(\mathcal{P}/\mathfrak{p}, \tilde{W}^{(M_p)}_{K_1}\right) \tag{8.3}$$

induced by the restriction is an isomorphism. In particular, by the Banach open mapping theorem, there are a positive integer m, and constants $\epsilon,\ C > 0$ such that

$$\begin{cases} \sup_{(t,x)\in K_2} |f(t,x)| \leq C \sup_{(t,x)\in K_1} \sum_{|\beta|\leq m} \sum_{\alpha\in\mathbb{N}^n} |D_t^\beta D_x^\alpha f(t,x)| \epsilon^{-|\alpha|} M_{|\alpha|}^{-1} \\ \qquad \forall f \in \mathrm{Ext}^0_{\mathcal{P}}(\mathcal{P}/\mathfrak{p}, \tilde{W}^{(M_p)}_{K_2}). \end{cases}$$

Then (8.1) follows by substituting to f in the a priori estimate above the (restrictions of the) exponential solutions $e^{i\langle t,\tau\rangle+i\langle x,\zeta\rangle}$ for $(\tau,\zeta)\in V(\mathfrak{p})$. The sufficiency of the condition (8.1) follows from Theorem 5.2, because of (3.4).

B) *Principal parts of hyperbolic systems*

We note that the functions $M(\zeta)$ and $N(\tau)$ are, respectively, $o(|\zeta|)$ and $o(|\tau|)$ for $|\zeta|\longrightarrow\infty$, $|\tau|\longrightarrow\infty$. Hence, when either (8.1) or (8.2) holds, we obtain

(8.4)
$$H_{Q_2}(Im\tau, Im\zeta)\leq H_{Q_1}(Im\tau, Im\zeta)\quad \text{for } (\tau,\zeta)\in V^o(\mathfrak{p}), \text{for every } \mathfrak{p}\in \mathrm{Ass}(\mathfrak{M}).$$

This, in turn, is equivalent to the hyperbolicity of (K_1, K_2) for $\mathcal{P}/\mathfrak{p}^o$ in the class of all Whitney functions and implies hyperbolicity in the classes $\tilde{W}^{(M_p)}$ and $W^{(N_p),(M_p)}$ when $N_p\geq M_p$ for all p.

In general, there is no unique way to associate to a general $\mathcal{P}$ module of finite type $\mathfrak{M}$ a homogeneous $\mathcal{P}$ module $\mathfrak{M}^o$. The construction of a *good* filtration of $\mathfrak{M}$, leading to a reasonable notion of *principal part* is done in the following way (see [6]): given any presentation

$$\mathcal{P}^b \xrightarrow{{}^tA(\tau,\zeta)} \mathcal{P}^a \longrightarrow \mathfrak{M} \longrightarrow 0$$

we can arbitrarily select integers $s_1, ..., s_a\leq 0$, and consider the filtration

$$\left(\mathcal{P}^a\right)_0\subset\left(\mathcal{P}^a\right)_1\subset\left(\mathcal{P}^a\right)_2\subset\cdots$$

of $\mathcal{P}^a$ given by:

$$\left(\mathcal{P}^a\right)_m=\left\{{}^t(p_1, ..., p_a)\in\mathcal{P}^a \,\middle|\, \deg(p_j)\leq m+s_j\right\}.$$

This induces a filtration

$$\mathfrak{M}_0\subset\mathfrak{M}_1\subset\mathfrak{M}_2\subset\cdots$$

of $\mathfrak{M}$ and we can consider the associated graded $\mathcal{P}$ module $\mathfrak{M}^o$. It turns out however that the set $\mathrm{Ass}(\mathfrak{M}^o)$ of associated prime ideals of $\mathfrak{M}^o$ is equal to

$$\bigcap\left\{\mathrm{Ass}(\mathcal{P}/\mathfrak{p}^o)\,\middle|\,\mathfrak{p}\in\mathrm{Ass}(\mathfrak{M})\right\}$$

and hence is independent of the choice of a *good* filtration of $\mathfrak{M}$. Thus, referring to any homogeneous $\mathcal{P}$ module $\mathfrak{M}^o$ obtained as above as to a *principal part* of $\mathfrak{M}$, we obtain:

Proposition 8.2. *Assume that the pair (K_1, K_2) is hyperbolic for $\mathfrak{M}$ either in the class $\tilde{W}^{(M_p)}$ or in the class $W^{(N_p),(M_p)}$. Let $\mathfrak{M}^o$ be a principal part of $\mathfrak{M}$. Then the pair (K_1, K_2) is also hyperbolic for $\mathfrak{M}^o$ in the classes of smooth Whitney functions and in the classes $\tilde{W}^{(M_p)}$, $W^{(N_p),(M_p)}$ if $N_p\geq M_p$ for all p.*

Next we note that, in the special case of a pair (K_1, K_2) in which K_1 and K_2 are compact, if the pair is hyperbolic for $\mathfrak{M}$ in one of the classes of ultradifferentiable functions considered above, we obtain

$$(8.5)\quad H_{K_1}(Im\tau, Im\zeta) = H_{K_2}(Im\tau, Im\zeta) \qquad \forall(\tau,\zeta) \in V^o(\mathfrak{p}), \quad \forall\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M}).$$

This suggests to consider the convex function:

(8.6)
$$\psi(t,x) = \sup_{(\tau,\zeta)\in\cup V^o(\mathfrak{p})} \langle t, Im\tau\rangle + \langle x, Im\zeta\rangle - H_{K_1}(Im\tau, Im\zeta), \quad \text{for} \quad (t,x) \in \mathbb{R}^N.$$

Since the varieties $V^o(\mathfrak{p})$ and the function H_{K_1} are homogeneous, $\psi(t,x)$ is the indicator function of a convex $\tilde{K}_1 \supset K_1$:

$$\psi(t,x) = \begin{cases} 0 & \text{if} \quad (t,x) \in \tilde{K}_1 \\ \infty & \text{if} \quad (t,x) \notin \tilde{K}_1. \end{cases}$$

Clearly the condition $K_1 \subset K_2 \subset \tilde{K}_1$ is then necessary for the pair (K_1, K_2) to be of hyperbolicity for $\mathfrak{M}$ either in the class of smooth or of the ultradifferentiable Whitney functions considered above and is necessary and sufficient (with the usual restriction for the two sequences (M_p), (N_p)) in the case of a principal part $\mathfrak{M}^o$ of $\mathfrak{M}$.

C) *Systems with a hyperbolic principal part*

Next we turn to investigate the opposite question. Namely, assuming that a pair (K_1, K_2) is hyperbolic for a principal part $\mathfrak{M}^o$ of a unitary finitely generated $\mathscr{P}$ module $\mathfrak{M}$, in the class W of smooth Whitney functions, we want to show that this condition implies hyperbolicity of the same pair for $\mathfrak{M}$, in some classes of ultradifferentiable functions. Considering here very general pairs of closed convex sets in $\mathbb{R}^N$, it will be convenient to denote by θ the general point (τ,ζ) of $\mathbb{C}^N$.

Lemma 8.3. *Let $\mathfrak{M}$ be a unitary finitely generated $\mathscr{P}$ module. Then there exists a smallest nonnegative real number σ_0 such that, for a positive constant C, we have:*

$$(8.7)\qquad \inf_{\theta'\in V^o(\mathfrak{p})} \left|Im\theta - Im\theta'\right| \leq C\,|\theta|^{\sigma_0} + C \quad \forall\theta \in V(\check{\mathfrak{p}}),\ \forall\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M}).$$

This number σ_0 is rational and smaller than $(m-1)/m$, where m is the largest degree of $V(\mathfrak{p})$ for $\mathfrak{p} \in \mathrm{Ass}(M)$.

We note that $0 \leq \sigma_0 \leq \sigma$, where σ is the exponent of Lemma 2.8. The fact that σ_0 is rational follows from the Tarski–Seidenberg theorem. This lemma and the characterization in Theorem 8.1, yield the following:

Theorem 8.4. *Let $\mathfrak{M}^o$ be a principal part of a unitary finitely generated $\mathcal{P}$ module $\mathfrak{M}$. Let (K_1, K_2) be a pair of convex closed sets with $K_1 \subset K_2$, hyperbolic for $\mathfrak{M}^o$ in the class W of smooth Whitney functions. If σ_0 is the exponent in Lemma 8.3, then the pair (K_1, K_2) is hyperbolic for $\mathfrak{M}$ in $\gamma^{(s)}$, provided $1 < s \leq 1/\sigma_0$. If $\sigma_0 = 0$, then (K_1, K_2) is hyperbolic for $\mathfrak{M}$ also in the class of smooth Whitney functions.*

The question of characterizing the perturbations of hyperbolic operators that preserve hyperbolicity in classes of ultradifferentiable functions has been considered by many authors (see for instance [31]). The problem is especially interesting in the case of the overdetermined Cauchy problem, as it was shown in [32] that hyperbolicity may not be time-symmetric in this case.

Example 4. Consider in $\mathbb{C}^3$, with coordinates τ, ζ_1, ζ_2, the prime ideal $\mathfrak{p}$ generated by the polynomials $p_1 = \sqrt{-1}\tau + \zeta_1$ and $p_2 = \zeta_1^3 - \zeta_2^2$. Then, with $\mathfrak{M} = \mathcal{P}/\mathfrak{p}$, we have $\mathfrak{M}^o = \mathcal{P}/\mathfrak{p}^o$, where $\mathfrak{p}^o$ is the ideal generated by p_1 and $p_2^0 = \zeta_1^3$. For $K_1 = \{(0, x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$, $K_2 = \{(t, x_1, x_2) \in \mathbb{R}^3 \mid t \geq 0\}$, $\check{K}_2 = \{(t, x_1, x_2) \in \mathbb{R}^3 \mid t \leq 0\}$, both the pairs (K_1, K_2) and $(K_1, \check{K}_2)$ are hyperbolic for $\mathfrak{M}^o$ in the class of smooth Whitney functions; (K_1, K_2) is hyperbolic for $\mathfrak{M}$ both in the class of Whitney functions and in the classes $\gamma^{(s)}$ for all $s > 1$. But $(K_1, \check{K}_2)$ is not hyperbolic for $\mathfrak{M}$ in the class of Whitney functions, while it is hyperbolic in the Gevrey classes $\gamma^{(s)}$ for $1 < s \leq 3/2$ by the previous theorem.

To investigate hyperbolicity in the class of Whitney functions for the pair consisting of an affine hypersurface Σ and one of the two closed half-spaces H^+, H^- of $\mathbb{R}^N$ intersecting in Σ, it was introduced in [32] the family of closed real algebraic sets $\tilde{V}^{\mathbb{R}}(\check{\mathfrak{p}})$, consisting, for each prime ideal $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$, of all points of $\mathbb{R}^N$ which are limits of sequences $\{\epsilon_\nu Im\theta^\nu\}$ for positive real ϵ_ν converging to zero and $\{\zeta^\nu\} \subset V(\check{\mathfrak{p}})$.

From [32] we have:

Proposition 8.5. *Let $\xi \in \mathbb{R}^N$ and set $\Sigma_\xi = \{\langle y, \xi\rangle = 0\}$, $H_\xi = \{\langle y, \xi\rangle \geq 0\}$. Then a necessary and sufficient condition in order that the pair (Σ_ξ, H_ξ) be hyperbolic for the unitary finitely generated $\mathcal{P}$ module $\mathfrak{M}$ in the class of smooth Whitney functions is that $\xi \notin \tilde{V}^{\mathbb{R}}(\check{\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M})$.*

We can generalize this statement to the following:

Proposition 8.6. *Let $K_1 \subset K_2$ be two closed convex sets in $\mathbb{R}^N$ and let $\mathfrak{M}$ be a finitely generated unitary $\mathcal{P}$ module. For each $y \in K_2 \setminus K_1$ let*

$$\Gamma(y, K_1) = \left\{t(y - y') \mid y' \in K_1,\ t \geq 0\right\} \tag{8.8}$$

and denote by $\Gamma^o(y, K_1)$ the polar cone:

$$\Gamma^o(y, K_1) = \{\xi \in \mathbb{R}^N \mid \langle y', \xi\rangle \geq 0 \quad \forall y' \in \Gamma(y, K_1)\}. \tag{8.9}$$

Then a necessary and sufficient condition for the pair (K_1, K_2) to be hyperbolic for $\mathfrak{M}$ in the class of Whitney functions is that

$$\Gamma^o(y, K_1) \cap \tilde{V}^{\mathbb{R}}(\check{\mathfrak{p}}) = \{0\} \qquad \forall y \in K_2 \setminus K_1,\ \forall \mathfrak{p} \in \mathrm{Ass}(\mathfrak{M}). \tag{8.10}$$

Proof. Let us prove sufficiency. Let $y \in K_2$. With $y' \in Q_1 \subset K_1$, we write:

$$\begin{aligned}\langle y, Im\theta\rangle &= \langle y', Im\theta\rangle + \langle (y-y'), Im\theta\rangle \\ &\leq H_{Q_1}(Im\theta) + \langle (y-y'), Im\theta\rangle.\end{aligned}$$

By our assumption, we can select y' in K_1 in such a way that

$$\langle (y-y'), Im\theta\rangle \leq -\epsilon \cdot |y-y'| \cdot |Im\theta|$$

with some $\epsilon > 0$, for all sufficiently large $\theta \in V(\check{\mathfrak{p}})$. This can be done uniformly for y varying in a convex compact subset Q_2 of K_2, obtaining a convex compact subset Q_1 of K_1 for the y', so that we finally obtain:

$$H_{Q_2}(Im\theta) \leq H_{Q_1}(Im\theta) + C \qquad \forall\theta \in V(\check{\mathfrak{p}}),\ \mathfrak{p} \in \mathrm{Ass}(\mathfrak{M}). \tag{8.11}$$

To prove necessity, we remark that the hyperbolicity for $\mathfrak{M}$ in the space of smooth Whitney functions of the pair $(H_{-\xi}, \mathbb{R}^N)$ is equivalent to that of the pair $(H_{-\xi}, H_{-\xi} \cup B(0,r))$, where $B(0,r)$ is a ball of arbitrarily small radius $r > 0$, implies that of the pair (Σ_ξ, H_ξ). This follows indeed from the translation invariance of partial differential operators with constant coefficients. In turn, the hyperbolicity of (Σ_ξ, H_ξ) is equivalent to that of $(H_{-\xi}, \mathbb{R}^N)$: this follows indeed from the exact sequence:

$$\begin{aligned}0 &\longrightarrow \mathrm{Ext}^0_{\mathcal{P}}(\mathfrak{M}, \mathcal{E}_{H_\xi}) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}(\mathfrak{M}, \mathcal{E}(\mathbb{R}^N)) \longrightarrow \mathrm{Ext}^0_{\mathcal{P}}(\mathfrak{M}, \mathcal{E}(H_{-\xi})) \\ &\longrightarrow \mathrm{Ext}^1_{\mathcal{P}}(\mathfrak{M}, \mathcal{E}_{H_\xi}) \longrightarrow \qquad 0.\end{aligned}$$

So, when (8.10) is not true, we can separate K_1 from a point y of K_2 by a hypersurface not forming a hyperbolic pair with the half-space containing y (for $\mathfrak{M}$ in the class W). Due to the previous remark, this prevents the pair (K_1, K_2) from being hyperbolic.

To extend the result above for hyperbolicity in the classes $\gamma^{(s)}$, we first introduce some notation. If $\mathfrak{p}$ is a prime ideal in $\mathcal{P}$, and $0 < \sigma < 1$, we define the homogeneous cone $\tilde{V}^{\sigma,\mathbb{R}}(\mathfrak{p})$ to be the set of all limits $\xi = \lim_{m\longrightarrow\infty} \epsilon_m(1+|\theta|)^{-\sigma} Im\theta_m$ where $\{\epsilon_m\}$ is an infinitesimal sequence of positive real numbers and $\{\theta_m\} \subset V(\mathfrak{p})$. Then with the same proofs, we obtain the statement for the case of Gevrey hyperbolicity:

Theorem 8.7. *Let $K_1 \subset K_2$ be two closed convex sets in $\mathbb{R}^N$ and let $\mathfrak{M}$ be a finitely generated unitary $\mathcal{P}$ module. A necessary and sufficient condition for the pair (K_1, K_2) to be hyperbolic for $\mathfrak{M}$ in the class $\gamma^{(s)}$ $(1 < s < \infty)$ is that*

$$\Gamma^o(y, K_1) \cap \tilde{V}^{1/s,\mathbb{R}}(\check{\mathfrak{p}}) = \{0\} \qquad \forall y \in K_2 \setminus K_1,\ \forall\mathfrak{p} \in \mathrm{Ass}(\mathfrak{M}). \tag{8.12}$$

We note that, again, hyperbolicity in the class $\gamma^{(s)}$ for $\mathfrak{M}$ implies hyperbolicity in the class W for a principal part $\mathfrak{M}^o$ of the module $\mathfrak{M}$. Also, vice versa, if the pair (K_1, K_2) of closed convex sets is hyperbolic in the class W for $\mathfrak{M}^o$, we have:

Theorem 8.8. *Let $\mathfrak{M}^o$ be a principal part of a unitary finitely generated $\mathcal{P}$ module $\mathfrak{M}$. Assume that the pair of closed convex sets (K_1, K_2) is hyperbolic for $\mathfrak{M}^o$ in the class W. Then, for each $y \in K_2 \setminus K_1$, there exists a smallest real σ_y with $0 \leq \sigma_y < 1$ such that, for every $\tau > \sigma_y$ there are a compact convex $Q_1 \subset K_1$, and a constant $C > 0$ such that*

$$(8.13) \qquad \langle y, Im\theta\rangle \leq H_{Q_1}(Im\theta) + C(1 + |\theta|)^\tau \qquad \forall\theta \in V(\check{\mathfrak{p}}),\ \mathfrak{p} \in \operatorname{Ass}(\mathfrak{M}).$$

Then

$$(8.14) \qquad \sigma(K_1, K_2) = \sup_{y\in K_2\setminus K_1} \sigma_y$$

is a real number with $0 \leq \sigma(K_1, K_2) < 1$. The pair (K_1, K_2) is hyperbolic for $\mathfrak{M}$ in the class $\gamma^{(s)}$ if $1 < s < 1/\sigma(K_1, K_2)$ and is not in case $s > 1/\sigma(K_1, K_2)$.

Proof. Let $y \in K_2 \setminus K_1$. Since $\{y\}$ is a compact convex subset of K_2, we can find a compact convex $Q_1' \subset K_1$ such that:

$$\langle y, Im\theta\rangle = H_{\{y\}}(Im\theta) \leq H_{Q_1'}(Im\theta) \qquad \forall\theta \in V^o(\mathfrak{p}),\ \mathfrak{p} \in \operatorname{Ass}(\mathfrak{M}).$$

Substituting to Q_1' a larger compact convex subset which is semialgebraic, we apply the Tarski–Seidenberg theorem to the semialgebraic function

$$g(r) = \sup\{\langle y, Im\theta\rangle - H_{Q_1}(Im\theta) \mid \theta \in \cup_{\mathfrak{p}\in\operatorname{Ass}(\mathfrak{M})} V(\check{\mathfrak{p}}),\ |\theta| \leq r\}.$$

We conclude that there is some rational λ, with $0 \leq \lambda < 1$ such that $g(r) = C\,r^\lambda(1 + o(1))$. Now, for each choice of such a Q_1, we already know that the corresponding λ is smaller than $(m-1)/m$, where m is the largest degree of the varieties $V(\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{M})$. Then, when we approximate Q_1' by a decreasing sequence of semialgebraic compact convex sets, the λ's that we find using the argument above are uniformly bounded by $(m-1)/m$ and hence their supremum $\sigma(Q_1')$ is strictly less than 1. Next we let Q_1' vary and we take for σ_y the infimum of the different numbers $\sigma(Q_1')$ obtained by the previous construction. Using the fact that, given a compact convex subset Q in $\mathbb{R}^N$, a point y_0 of its boundary, and a supporting hyperplane Σ for Q through y_0, we can always find a semialgebraic compact convex $\tilde{Q}$ containing Q and having Σ as a supporting hyperplane, we can conclude that for $r > \sigma_y$ the condition $\tilde{V}^{r,\mathbb{R}}(\check{\mathfrak{p}}) \cap \Gamma^o(y, K_1) = \{0\}$, if $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{M})$, is satisfied. This proves our first claim. Then, the remaining part of the statement follows by the previous theorem.

D) *Hyperbolic Cauchy problems with initial data on an affine submanifold*

Let $\mathbb{R}^N = \mathbb{R}^k_t \times \mathbb{R}^n_x$. Set $\Sigma = \{0\} \times \mathbb{R}^n \subset \mathbb{R}^N$ and consider a proper closed convex cone $\Gamma \subset \mathbb{R}^k$, having a nonempty interior. In this section we will be interested in the study of the hyperbolicity of the pair $(\Sigma, \Gamma \times \mathbb{R}^n)$ with respect to a finitely generated unitary $\mathcal{P}$ module $\mathfrak{M}$. In this case Theorem 8.1 yields:

Theorem 8.9. *Let $\{M_p\}_{p\in\mathbb{N}}$ and $\{N_p\}_{p\in\mathbb{N}}$ be two sequences of positive real numbers satisfying conditions (M). Let $\mathfrak{M}$ be a finitely generated unitary $\mathcal{P}$ module.*

a) A necessary and sufficient condition for the pair $(\Sigma, \Gamma\times\mathbb{R}^n)$ to be hyperbolic for $\mathfrak{M}$ in the class $\tilde{W}^{(M_p)}$ is that there exists a constant $L>0$ such that

(8.15)
$$\kappa_\Gamma(\tau) \le L|Im\zeta| + M(L\zeta) + L\log(2+|\tau|) \quad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}}),\ \forall\mathfrak{p}\in \mathrm{Ass}(\mathfrak{M}).$$

b) A necessary and sufficient condition for the pair $(\Sigma, \Gamma\times\mathbb{R}^n)$ to be hyperbolic for $\mathfrak{M}$ in the class $W^{(N_p),(M_p)}$ is that there exists a constant $L>0$ such that

(8.16)
$$\kappa_\Gamma(\tau) \le L|Im\zeta| + M(L\zeta) + N(L\tau) + L \quad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}}),\ \forall\mathfrak{p}\in \mathrm{Ass}(\mathfrak{M}).$$

Here κ_Γ is the function introduced at the beginning of Section 7. Since the functions $M(\zeta)$ and $N(\tau)$ have a sublinear rate of growth, we obtain:

Theorem 8.10. *Let $\mathfrak{M}^o$ be a principal part of a finitely generated $\mathcal{P}$ module $\mathfrak{M}$. If the pair $(\Sigma, \Gamma\times\mathbb{R}^n)$ is hyperbolic for $\mathfrak{M}$ either in the class $\tilde{W}^{(M_p)}$ or in the class $W^{(N_p),(M_p)}$, then there is a constant $L>0$ such that:*

$$\kappa_\Gamma(\tau) \le L|Im\zeta| \quad \forall(\tau,\zeta)\in V^o(\mathfrak{p}), \quad \forall\mathfrak{p}\in \mathrm{Ass}(\mathfrak{M}). \tag{8.17}$$

Moreover, $(\Sigma, \Gamma\times\mathbb{R}^n)$ is hyperbolic for $\mathfrak{M}^0$ in the class W and in the classes $\tilde{W}^{(A_p)}$, $W^{(B_p),(A_p)}$, for all sequences $\{A_p\}_{p\in\mathbb{N}}$ $\{B_p\}_{p\in\mathbb{N}}$, with $B_p \ge A_p$, satisfying conditions (M).

The proof is straightforward. We only need to notice that, from the condition that Γ has a nonempty interior in $\mathbb{R}^k$ and the estimate (8.17) we deduce that, for some constant $L'>0$:

$$|\tau| \le L'|\zeta| \qquad \forall(\tau,\zeta)\in V^o(\mathfrak{p}), \quad \forall\mathfrak{p}\in \mathrm{Ass}(\mathfrak{M}). \tag{8.18}$$

Then we can repeat the construction in the previous section to obtain:

Theorem 8.11. *Let $\mathfrak{M}$ be a unitary finitely generated $\mathcal{P}$ module. Assume that $\mathfrak{M}^o$ is a principal part of $\mathfrak{M}$ and that $(\Sigma, \Gamma\times\mathbb{R}^n)$ is hyperbolic for $\mathfrak{M}^o$ in the class W. Then for each $t\in\Gamma\setminus\{0\}$ there is a smallest real σ_t, with $0\le\sigma_t<1$, with the property: for every $r>\sigma_t$, there is a constant L_r such that*

$$\langle t, Im\tau\rangle \le L_r(|Im\zeta| + (1+|\zeta|)^r) \qquad \forall(\tau,\zeta)\in V(\check{\mathfrak{p}}), \quad \forall\mathfrak{p}\in \mathrm{Ass}(\mathfrak{M}). \tag{8.19}$$

Then $0\le\sigma_\Gamma = \sup_{t\in\Gamma\setminus\{0\}}\sigma_t < 1$ and $(\Sigma, \Gamma\times\mathbb{R}^n)$ is hyperbolic for $\mathfrak{M}$ in the classes $\tilde{\gamma}^{(s)}$ and $\gamma^{(s',s)}$ if $1<s<1/\sigma_\Gamma$ and $s'\ge s$ and is not hyperbolic for $\mathfrak{M}$ in the class $\tilde{\gamma}^{(s)}$ if $s>1/\sigma_\Gamma$.

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