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C^∞ regularity of solutions of an equation of Levi's type in R^{2n+1}

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Abstract. We study the regularity of the solutions u of a class of P.D.E., whose prototype is the prescribed Levi curvature equation in $\mathbb{R}^{2n+1}$. It is a second-order quasilinear equation whose characteristic matrix is positive semidefinite and has vanishing determinant at every point and for every function $u \in C^2$. If the Levi curvature never vanishes, we represent the operator $\mathcal{L}$ associated with the Levi equation as a sum of squares of non-linear vector fields which are linearly independent at every point. By using a freezing method we first study the regularity properties of the solutions of a linear operator, which has the same structure as $\mathcal{L}$. Then we apply these results to the classical solutions of the equation, and prove their C^∞ regularity.

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Key words. Levi equation – sum of squares of non-linear vector fields – control distance – freezing method – smoothness

1. Introduction

We will study the regularity of the solutions of a class of equations modeled on the mean Levi-curvature equation

$$\mathcal{L}u = k(\cdot, u)\frac{(1+|\nabla u|^2)^{3/2}}{1+u_t^2} \quad in\ \Omega \subset \mathbb{R}^{2n+1}, \tag{1}$$

where

$$\mathcal{L}u = \sum_{i=1}^{n}\left(u_{x_i x_i} + u_{y_i y_i} + 2\frac{u_{y_i} - u_{x_i}u_t}{1+u_t^2}u_{x_i t} - 2\frac{u_{x_i} + u_{y_i}u_t}{1+u_t^2}u_{y_i t} + \frac{u_{x_i}^2 + u_{y_i}^2}{1+u_t^2}u_{tt}\right).$$

Here we have denoted (x, y, t) a point of $\mathbb{R}^{2n+1}$, with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$, u_{x_i} is the first derivative with respect to x_i, and ∇u the Euclidean gradient of u in $\mathbb{R}^{2n+1}$ and $k : \Omega \times \mathbb{R} \to \mathbb{R}$ is of class C^∞. If we identify $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$, the graph of a function $u : \Omega \to \mathbb{R}$ can be considered as a real hypersurface, and equation (1) naturally arises in the study of its geometric properties: if $c = 16\inf_{\Omega\times\mathbb{R}} |k| > 0$,

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then the radius of Ω is bounded by $2n/c$ (see [2]). Hence $k(\cdot, u)$ can be considered as a sort of mean curvature of u with respect to the complex structure and we call it mean Levi-curvature.

The majority of the results known for equation (1) refer to the case $n = 1$. This equation is a second-order quasilinear differential equation whose characteristic form is positive semidefinite and has vanishing determinant at every point of Ω and for every function u. Hence it is not elliptic at any point. However it has been initially treated as a strongly degenerate elliptic equation. With this approach Debiard and Gaveau in [8] proved a weak maximum principle, and Tomassini showed the following strong maximum principle (see [14]): *if $\Omega \subseteq \mathbb{R}^3$ is open and connected, $k : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and non-positive, u is a solution of the equation* (1) *of class C^2 in Ω and attains its maximum value at $\xi_0 = (x_0, y_0, t_0) \in \Omega$ then u is a constant function on the set*

$$\{(x, y, t) \in \Omega : \ t = t_0\}.$$

Later on Slodkowsky and Tomassini introduced an elliptic regularization of the problem, and proved that, *if $\Omega \subset \mathbb{R}^3$ is strictly pseudoconvex, and k satisfies a geometric hypothesis related to the Levi curvature of $\partial\Omega \times \mathbb{R}$, then the Dirichlet problem associated to equation* (1) *has a viscosity solution $u \in Lip(\bar{\Omega})$* (see [12]). They also proved a similar result for a "Levi equation" in $\mathbb{R}^{2n+1}$, which however is different from the one considered here, since it involves a curvature of the graph of u analogous to the Gaussian curvature for the complex structure (see [13]).

When $k = 0$, an existence and regularity result was established by Bedford and Gaveau with a geometric approach whose starting point is Bishop's theorem on a family of analytic discs: *if $\Omega \subset \mathbb{R}^3$ is a strictly pseudoconvex set, $\phi \in C^{m+5}(\partial\Omega)$, the boundary of Ω and the graph of ϕ satisfy a technical geometric condition, then the Dirichlet problem*

$$\begin{cases} \mathcal{L}(u) = 0 & in\ \Omega \\ u = \phi & on\ \partial\Omega \end{cases}$$

has a solution in $C^{m+\alpha}(\Omega) \cap Lip(\bar{\Omega})$, with $0 < \alpha < 1$ (see [3]).

This is clearly a global result, since the regularity of the solution depends on the boundary data. On the other hand E. Lanconelli conjectured that the interior regularity problem could be afforded with a vector fields method. A new approach was then introduced by one of the authors if k never vanishes. The first result in this direction is a strong comparison principle for the solutions of (1) (see [4]). If the dimension of the space is 3, a regularity result for classical solutions was proved:

If the Levi curvature k is smooth and always different from zero, any solution u of (1) *of class $C^{2,\alpha}$, with $\alpha > \frac{1}{2}$, is of class $C^{\infty}(\Omega)$* (see [6]).

In this paper we extend this result to the mean Levi-curvature equation in $\mathbb{R}^{2n+1}$.

In order to state our main theorem we need to recall some notations. Let $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ denote the coefficients which appear in the definition of the operator:

$$a_i = a_i(u) = \frac{u_{y_i} - u_{x_i} u_t}{1 + u_t^2}, \quad b_i = b_i(u) = -\frac{u_{x_i} + u_{y_i} u_t}{1 + u_t^2}, \tag{2}$$

and let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ be the first-order differential operators

$$X_i = \partial_{x_i} + a_i\partial_t, \quad Y_i = \partial_{x_i} + b_i\partial_t. \tag{3}$$

Then $\mathcal{L}$ can be formally represented as a sum of squares of these non-linear vector fields, plus a first-order term:

$$\mathcal{L}u = \sum_{i=1}^{n} \left(X_i^2 u + Y_i^2 u - (X_i a_i + Y_i b_i)\partial_t u\right)$$

(see (8)). Moreover, a direct computation (see (11)) shows that

$$\sum_{i=1}^{n}[X_i, Y_i] = -\frac{\mathcal{L}u}{1+(\partial_t u)^2}\partial_t.$$

Hence, if u is a solution of (1) and k is always different from zero in Ω, then

$$X_i, \; Y_i, \; \sum_{i=1}^{n}[X_i, Y_i] \tag{4}$$

are linearly independent at every point. This condition, formally analogous to the Hörmander condition for hypoellipticity, is also crucial in this non-linear situation. In particular it is possible to introduce a control distance d, and Lipschitz classes $C_{\mathcal{L}}^{m,\alpha}$ associated to the vector fields X_i and Y_i.

Our main theorem is the following:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^{2n+1}$ and $q \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R})$, with $q(\xi, s, p, \tau) \neq 0$ for every $(\xi, s, p, \tau) \in \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}$. If u is a solution of class $C^{2,\alpha}(\Omega)$, with $\alpha > \frac{1}{2}$, of*

$$\mathcal{L}u = q(\xi, u, a, b, \partial_t u) \quad \text{in } \Omega, \tag{5}$$

where a and b are defined in (2), then u is of class $C^\infty(\Omega)$.

Let us explicitly note that

$$\sum_{i=1}^{n}\left(a_i^2 + b_i^2\right) = \sum_{i=1}^{n} \frac{u_{y_i}^2 + u_{x_i}^2 u_t^2 + u_{x_i}^2 + u_{y_i}^2 u_t^2}{\left(1+u_t^2\right)^2} = \sum_{i=1}^{n}\frac{u_{y_i}^2 + u_{x_i}^2}{1+u_t^2}, \tag{6}$$

hence

$$\sum_{i=1}^{n}\left(a_i^2 + b_i^2\right) + 1 = \sum_{i=1}^{n}\frac{u_{y_i}^2 + u_{x_i}^2}{1+u_t^2} + 1 = \frac{1+|\nabla u|^2}{1+u_t^2},$$

and equation (5) simply becomes equation (1) if we choose

$$q(\xi, u, a, b, \partial_t u) = k(\xi, u)(|a|^2 + |b|^2 + 1)^{3/2}\left(1+u_t^2\right)^{1/2}.$$

Let us briefly sketch the proof. In order to study the regularity of the solutions of equation (1), we fix a solution $u \in C^{2,\alpha}$. The coefficients $a_i = a_i(u)$ and $b_i = b_i(u)$

defined in (2) are obviously of class $C^{1,\alpha}$, and the correspondent vector fields X_i and Y_i are well defined. Hence it is natural to call the linearized operator of $\mathcal{L}$ the linear operator L_u, formally defined as $\mathcal{L}$, in terms of these fixed vectors fields:

$$L_u v = \sum_{i=1}^{n} \left(X_i^2 v + Y_i^2 v - (X_i a_i + Y_i b_i)\partial_t v \right).$$

Linear operators written in this way have been intensively studied by Hörmander, Stein, Folland and Stein, Rothschild and Stein, in suitable spaces $C_{\mathcal{L}}^{k,\alpha}$ of Lipchitz continuous functions defined in terms of the vector fields X_i and Y_i. However in order to show, for example, that the solution of the equation

$$L_u v = f \in C_{\mathcal{L}}^{1,\alpha}$$

is of class $C_{\mathcal{L}}^{3,\alpha}$, their technique requires that the coefficients of the vector fields X_i and Y_i are of class $C_{\mathcal{L}}^{4,\alpha}$, so that it is impossible to apply their technique to our situation. On the contrary, for our challenge we will adapt a freezing method introduced in [5] for the Levi equation in R^3. We will call frozen vector fields of X_i and Y_i at a fixed point ξ_0 the linear vector fields X_{i,ξ_0} and Y_{i,ξ_0}, whose coefficients are the first-order Taylor developments of the coefficients of X_i and Y_i. Then we define the frozen operator the linear operator L_{ξ_0} formally represented as $\mathcal{L}$ but in terms of these new vector fields

$$L_{\xi_0} = \sum_{i=1}^{n} \left(X_{i,\xi_0}^2 + Y_{i,\xi_0}^2 - (X_i a_i + Y_i b_i)(\xi_0)\partial_t \right). \tag{7}$$

Since X_{i,ξ_0} and Y_{i,ξ_0} are linear, nilpotent and C^∞ vector fields, which satisfy the Hörmander condition for hypoellipticity, there exists a fundamental solution Γ_{ξ_0} of L_{ξ_0}. For every fixed ξ_0 an explicit estimate of it is known in terms of the control distance d_{ξ_0} associated to L_{ξ_0} (see [11]). However the dependence of Γ_{ξ_0} and its derivatives on the variable ξ_0 was not known, and we study it here by means of suitable singular integrals, always dependent on the variable ξ_0. Then we write a representation formula for functions $v \in C_{\mathcal{L}}^{2,\alpha}$ in terms of Γ_{ξ_0} and $L_u v$. Some of the terms of this formula are singular integrals, and they can not be differentiated in a standard way. Hence we introduce some third-order difference quotients in the direction of the vector fields, and we use them to prove that, if $L_u v \in C_{\mathcal{L}}^{1,\alpha}$, then $v \in C_{\mathcal{L}}^{3,\alpha}$. Applying this result to the fixed solution u of the Levi equation we deduce that $u \in C_{\mathcal{L}}^{3,\alpha}$ and $\partial_t u \in C_{\mathcal{L}}^{2,\alpha}$. To study the higher regularity of u we iterate this technique: surprisingly, even though the operator is written in terms of X_i and Y_i, we have to show that if $u \in C_{\mathcal{L}}^{m,\alpha}$, its ordinary derivatives $\partial_{x_i} u$, $\partial_{y_i} u$ and $\partial_t u$ are of class $C_{\mathcal{L}}^{m,\beta}$, in order to conclude that $u \in C_{\mathcal{L}}^{m+1,\beta}$ for all $\beta < \alpha$.

The paper is organized as follows: in Sect. 2 we give the definition of derivatives in the directions X_i and Y_i and we describe the structure of $\mathcal{L}$. In Sect. 3 we study the frozen operator L_{ξ_0}, and in Sect. 4 we prove Theorem 1.1. The representation formula is stated without proof in Appendix A.

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2. Structure of the Levi operator

In this section we will always denote by u a fixed function of class C^2, and we will prove in detail some properties of the vector fields X and Y, introduced in (3) and of the operator $\mathcal{L}$. Then we will give the definition of some Lipschitz classes $C^{2,\alpha}_{\mathcal{L}}$, naturally associated to $\mathcal{L}$.

First note that, by (1), (2), and (6),

$$\mathcal{L}u = \sum_{i=1}^{n} \left(u_{x_i x_i} + u_{y_i y_i} + 2a_i u_{x_i t} + 2b_i u_{y_i t} + \left(a_i^2 + b_i^2\right) u_{tt}\right)$$

(since $u_{x_i x_i} + 2a_i u_{x_i t} + a_i^2 u_{tt} = (\partial_{x_i} + a_i \partial_t)^2 u - (\partial_{x_i} + a_i \partial_t) a_i \partial_t u = X_i^2 u - X_i a_i \partial_t u$)

$$= \sum_{i=1}^{n} \left(X_i^2 u + Y_i^2 u - (X_i a_i + Y_i b_i)\partial_t u\right), \tag{8}$$

see also [5] for the proof of the same property in $\mathbb{R}^3$.

Moreover a_i and b_i satisfy the following crucial relation

$$X_i u = \left(\partial_{x_i} + \frac{u_{y_i} - u_{x_i} u_t}{1 + u_t^2} \partial_t\right) u = \frac{u_{x_i} + u_{y_i} u_t}{1 + u_t^2} = -b_i \tag{9}$$

and

$$Y_i u = \left(\partial_{y_i} - \frac{u_{x_i} + u_{y_i} u_t}{1 + u_t^2} \partial_t\right) u = \frac{u_{y_i} - u_{x_i} u_t}{1 + u_t^2} = a_i.$$

As a consequence we will represent $\mathcal{L}u$ as a sum of squares of vector fields, times a multiplying factor. Indeed

$$(X_i a_i + Y_i b_i)\partial_t u$$

(by (9))

$$= (X_i Y_i u - Y_i X_i u)\partial_t u = [X_i, Y_i] u \partial_t u = (X_i b_i - Y_i a_i)(\partial_t u)^2$$

(always by (9))

$$= -\left(X_i^2 u + Y_i^2 u\right)(\partial_t u)^2.$$

Hence, substituting in (8)

$$\mathcal{L}u = \sum_{i=1}^{n} \left(X_i^2 u + Y_i^2 u\right)\left(1 + (\partial_t u)^2\right). \tag{10}$$

Moreover, if u is a solution of equation (5), then the commutator can be computed as

$$\sum_{i=1}^{n}[X_i, Y_i] = \sum_{i=1}^{n}(X_i b_i - Y_i a_i)\partial_t \tag{11}$$

$$= -\sum_{i=1}^{n}\big(X_i^2 u + Y_i^2 u\big)\partial_t = -\frac{\mathcal{L}u}{1+(\partial_t u)^2}\partial_t = -\frac{q(\cdot, u, a(u), b(u), \partial_t u)}{1+(\partial_t u)^2}\partial_t.$$

Since k is always different from zero, then the last term in (11) does not vanish. Thus ∂_t is a derivative of "length" 2 in the direction of X_i and Y_i. Besides the vector fields

$$X_i,\ Y_i,\ \sum_{i=1}^{n}[X_i, Y_i]$$

are linearly independent at every point. Hence we can introduce a control distance d, associated to $\mathcal{L}$ exactly as in [9]. Since a_i and b_i are Lipschitz continuous with respect to the Euclidean metrics, for every $\xi_0 \in \Omega$ there exists an unique integral curve γ_i of X_i, starting from ξ_0. We will call γ_i the *exponential curve* and we will denote it by

$$\gamma_i(s) = \exp(sX_i)(\xi_0).$$

Definition 2.1. *Let K be a compact subset of Ω. For all $\xi, \bar{\xi} \in K$ there exists an unique $\Phi = (\Phi_1, \dots, \Phi_{2n+1}) \in \mathbb{R}^{2n+1}$ such that*

$$\xi = \exp\Big(\sum_{i=1}^{n}(\Phi_i X_i + \Phi_{n+i}Y_i) + \Phi_{2n+1}\partial_t\Big)(\bar{\xi}).$$

The numbers $\Phi_i = \Phi_i(\xi, \bar{\xi})$ are called local coordinates of ξ around $\bar{\xi}$ with respect to the vector fields X_i, Y_i, ∂_t.

Let us denote by $|\Phi|$, the induced norm,

$$|\Phi| = \Big(\Big(\sum_{i=1}^{2n}\Phi_i^2\Big)^2 + \Phi_{2n+1}^2\Big)^{1/4},$$

and d the associated quasidistance

$$d(\xi, \bar{\xi}) = |\Phi(\xi, \bar{\xi})|. \tag{12}$$

Definition 2.2. *If $\gamma_i = \exp(sX_i)(\xi_0)$ we define the Lie derivative of a function f in ξ_0 as*

$$X_i f(\xi_0) = \frac{d}{dh}(f \circ \gamma_i),$$

when the right-hand side exists and is finite. The set of functions f such that $X_i f$ and $Y_i f$ exist and are continuous for all $i = 1, \dots, n$ will be called $C^1_{\mathcal{L}}$. If a_i and b_i are of class $C^1_{\mathcal{L}}$, then we will say that a function f is of class $C^2_{\mathcal{L}}$ if $X_i f$ and $Y_i f$ are of class $C^1_{\mathcal{L}}$ and we will similarly introduce classes $C^m_{\mathcal{L}}$. Finally we say that f is of class $C^{m,\alpha}_{\mathcal{L}}$, if its derivatives X_i, Y_i of order m are Hölder continuous of exponent α, with respect to the quasidistance d.

3. The freezing method

We present here a generalization of the freezing method introduced in [5] for the Levi equation in R^3. Since the structure of $\mathcal{L}$ depends on X_i, Y_i and their commutators, it is necessary to introduce frozen vector fields X_{i,ξ_0} and Y_{i,ξ_0}, such that for every i and j, $[X_{i,\xi_0}, Y_{j,\xi_0}] = [X_i, Y_j]$ at the point ξ_0. This is made with the Taylor development at the first-order of the coefficients of X_i and Y_i. If the dimension of the space is 3, the resulting operator L_{ξ_0} is, up to a chance of variable, the Kohn Laplacian on the Heisenberg group, and its fundamental solution can be explicitly written. In a higher dimension the situation is completely different, since the structure of the Lie algebra generated by X_{i,ξ_0} and Y_{i,ξ_0} is different from one point to another and L_{ξ_0} is not invariant with respect to any family of dilations. Hence in this section we have to introduce a new definition of the singular integral, and study its properties in detail. As a consequence we will be able to deduce some properties of the fundamental solution of L_{ξ_0}, which will be crucial in the proof of the regular properties of the solution.

Every function $v \in C^{1,\alpha}_{\mathcal{L}}(\Omega)$ has the following Taylor development (see [5, Remark 2.3]):

$$v(\xi) = P^1_{\xi_0} v(\xi) + O\big(d^{1+\alpha}(\xi, \xi_0)\big), \tag{13}$$

where

$$P^1_{\xi_0} v(\xi) = v(\xi_0) + \sum_{i=1}^n X_i v(\xi_0)(x_i - x_{0,i}) + Y_i v(\xi_0)(y_i - y_{0,i})$$

and we have denoted $\xi_0 = (x_0, y_0, t_0) = (x_{0,1}, ..., x_{0,n}, y_{0,1}, ..., y_{0,n}, t_0)$.

Hence it is natural to call the frozen vector fields

$$X_{i,\xi_0} = \begin{pmatrix} e_i \\ 0 \\ P^1_{\xi_0} a_i \end{pmatrix} \quad \textit{and} \quad Y_{i,\xi_0} = \begin{pmatrix} 0 \\ e_i \\ P^1_{\xi_0} b_i \end{pmatrix}, \tag{14}$$

for every $i = 1, ..., n$. By simplicity we will also denote

$$D_{i,\xi_0} = X_{i,\xi_0}, \quad D_{n+i,\xi_0} = Y_{i,\xi_0}, \tag{15}$$

for all $i = 1, \ldots, n$.

The frozen operator of $\mathcal{L}$ will be formally defined as (8), but in terms of the linear vector fields X_{i,ξ_0} and Y_{i,ξ_0}:

$$L_{\xi_0} = \sum_{i=1}^n \big(X^2_{i,\xi_0} + Y^2_{i,\xi_0} - (X_i a_i + Y_i b_i)(\xi_0)\partial_t\big). \tag{16}$$

For a fixed ξ_0 the operator L_{ξ_0} is an Hörmander-type operator. Indeed X_{i,ξ_0} and Y_{i,ξ_0} have C^∞ coefficients and satisfy

$$[X_{i,\xi_0}, Y_{j,\xi_0}] = (X_i b_j - Y_j a_i)(\xi_0)\partial_t.$$

In particular if u is a solution of equation (5) then, arguing as in (11), we have

$$\sum_{i=1}^{n}[X_{i,\xi_0}, Y_{i,\xi_0}] = -\frac{q(\xi_0, u(\xi_0), a(\xi_0), b(\xi_0), \partial_t u(\xi_0))}{1+(\partial_t u(\xi_0))^2}\partial_t \tag{17}$$

and the Hörmander condition for hypoellipticity is satisfied. Also note that the Lie algebra generated by X_{i,ξ_0} and Y_{i,ξ_0} is nilpotent, since all the commutators of order 3 are zero. Hence there exists a control distance d_{ξ_0} and a fundamental solution Γ_{ξ_0} associated to L_{ξ_0} : d_{ξ_0} can be formally defined as in Definition 2.1 with X_i, Y_i replaced by X_{i,ξ_0}, Y_{i,ξ_0}. Moreover, because of condition (17), the results in [9] ensure that Γ_{ξ_0} is locally equivalent to $d_{\xi_0}^{-Q+2}$, where $Q = 2n + 2$. Let us note explicitly that, if $n > 1$ the Lie algebra generated by the frozen vector fields is not free, and this implies that there is no dilation group associated with the vector fields. Obviously this will be the main obstacle in the introduction of the singular integrals. However Γ_{ξ_0} and its derivatives satisfy the following estimate from above (see [11]): for every compact set $K \subset \Omega$ for every $\xi_0, \xi, \zeta \in K$,

$$|D_{j_1,\xi_0} \dots D_{j_s,\xi_0}\Gamma_{\xi_0}(\xi,\zeta)| \le C_s d_{\xi_0}^{2-Q-s}(\xi,\zeta), \tag{18}$$

where the constant C_s only depends on K, and the derivatives D_{i,ξ_0} defined in (15), act with respect to one of the two variables on which Γ_{ξ_0} depends. However this estimate of Γ_{ξ_0} and of its derivatives are not sufficient to obtain our regularity result, and we also have to study the dependence of $\Gamma_{\xi_0}(\xi,\zeta)$ on the variable ξ_0.

First of all we state a relation which holds between different frozen distances:

Remark 3.1. If ξ_0 and ξ_1 are two points fixed in Ω, the distances d_{ξ_0} and d_{ξ_1} are not equivalent, but they can be written almost explicitly as in [5, page 492]. Then for every compact set K, there exist M_1, M_2 and M_3 such that for every ξ_0, ξ_1, for every θ and ζ in K we have:

$$d_e(\theta,\zeta) \le d_{\xi_0}(\theta,\zeta),$$

$$d_{\xi_0}(\theta,\zeta) \le d_{\xi_1}(\theta,\zeta) + M_1 d_e^{1/2}(\theta,\zeta) d_e^{1/2}(\xi_1,\xi_0)$$

and

$$M_2 d(\xi_1,\zeta) \le d_{\xi_1}(\xi_1,\zeta) \le M_3 d(\xi_1,\zeta),$$

where d_e is the Euclidean distance (see [5, Remark 2.8], for the proof of the same assertion in $\mathbb{R}^3$).

Let us now begin the study of the fundamental solution with the following remark

Remark 3.2. For a fixed ξ_0 the function $\partial_t\Gamma_{\xi_0}$ satisfies the following conditions

$$|\partial_t\Gamma_{\xi_0}(\xi,\zeta)| \le C d_{\xi_0}^{-Q}(\xi,\zeta) \quad \textit{and} \quad \int_{a\le\Gamma_{\xi_0}^{-1}(\xi,\theta)\le b} \partial_t\Gamma_{\xi_0}(\xi,\theta)d\theta = 0.$$

Indeed the first one follows from (18), and the fact that the derivative ∂_t can be represented as (17), while

$$\int_{a\le\Gamma_{\xi_0}^{-1}(\xi,\theta)\le b}\partial_t\Gamma_{\xi_0}(\xi,\theta)d\theta=$$

integrating by parts

$$=\frac{1}{b}\int_{\Gamma_{\xi_0}^{-1}(\xi,\theta)=b}\frac{\partial_t\Gamma_{\xi_0}(\xi,\theta)}{|\nabla\Gamma_{\xi_0}(\xi,\theta)|}d\sigma(\theta)-\frac{1}{a}\int_{\Gamma_{\xi_0}^{-1}(\xi,\theta)=a}\frac{\partial_t\Gamma_{\xi_0}(\xi,\theta)}{|\nabla\Gamma_{\xi_0}(\xi,\theta)|}d\sigma(\theta).$$

Integrating again by parts we get the assertion.

The previous remark allows us to give the following definition

Definition 3.1. *If ξ_0 is fixed, f is a Hölder continuous function such that the application $\theta\to f(\theta)d_{\xi_0}^{-Q}(\theta,\xi_0)$ is integrable in a neighborhood of ∞, we define a principal value integral depending on ξ_0 as follows:*

$$PV_{\xi_0}\int\partial_t\Gamma_{\xi_0}(\xi,\theta)f(\theta)d\theta=$$

$$lim_{\epsilon\to 0}\int_{\epsilon\le\Gamma_{\xi_0}^{-1}(\xi,\theta)\le R}\partial_t\Gamma_{\xi_0}(\xi,\theta)(f(\theta)-f(\xi))d\theta+\int_{R\le\Gamma_{\xi_0}^{-1}(\xi,\theta)}\partial_t\Gamma_{\xi_0}(\xi,\theta)f(\theta)d\theta$$

for any fixed R, and the value does not depend on R.

Proposition 3.1. *Let ξ_0 and ξ_1 be fixed in Ω. If we denote Γ_{ξ_0} the fundamental solution of L_{ξ_0} and Γ_{ξ_1} the fundamental solution of L_{ξ_1} we have the following representation formula*

$$\Gamma_{\xi_0}(\xi,\zeta)=-\sum_{i=1}^{2n}\int D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)D_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$-\sum_{i=1}^{2n}PV_{\xi_0}\int D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)(D_{i,\xi_1}-D_{i,\xi_0})\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$-c(\xi_1)PV_{\xi_0}\int\Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

where $c(\xi_1)=\sum_{i=1}^n X_i a_i(\xi_1)+Y_i b_i(\xi_1)$, all the derivatives are taken with respect to θ, and D_{i,ξ_1} is defined in (15).

Proof. If $L_{\xi_1}^*$ denotes the formal adjoint of L_{ξ_1} and $\Gamma_{\xi_1}^*$ is its fundamental solution we have

$$0=\int_{\substack{\left\{\theta:\epsilon\le\Gamma_{\xi_0}^{-1}(\theta,\zeta)\le M\right\}\\ \left\{\theta:\epsilon\le\Gamma_{\xi_1}^{-1}(\xi,\theta)\right\}}}L_{\xi_1}^*\Gamma_{\xi_1}^*(\theta,\xi)\Gamma_{\xi_0}(\theta,\zeta)d\theta=$$

using the fact that $D^*_{i,\xi_1}\Gamma^*_{\xi_1}(\theta,\xi) = -D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)$

$$= \int_{\substack{\{\theta:\epsilon\le\Gamma^{-1}_{\xi_0}(\theta,\zeta)\le M\}\\ \{\theta:\epsilon\le\Gamma^{-1}_{\xi_1}(\xi,\theta)\}}} \Big(\sum_{i=1}^{2n} D^2_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta) - c(\xi_1)\partial_t\Gamma_{\xi_1}(\xi,\theta)\Big)\Gamma_{\xi_0}(\theta,\zeta)d\theta =$$

integrating by parts

$$= -\int_{\{\theta:\Gamma^{-1}_{\xi_0}(\theta,\zeta)=\epsilon\}} \frac{\sum_{i=1}^{2n} D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)D_{i,\xi_1}\Gamma_{\xi_0}(\theta,\zeta) - c(\xi_1)\Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)}{|\nabla\Gamma_{\xi_0}(\theta,\zeta)|}\Gamma_{\xi_0}(\theta,\zeta)d\sigma(\theta)$$

$$+ \int_{\{\theta:\Gamma^{-1}_{\xi_0}(\theta,\zeta)=M\}} \frac{\sum_{i=1}^{2n} D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)D_{i,\xi_1}\Gamma_{\xi_0}(\theta,\zeta) - c(\xi_1)\Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)}{|\nabla\Gamma_{\xi_0}(\theta,\zeta)|}\Gamma_{\xi_0}(\theta,\zeta)d\sigma(\theta)$$

$$- \int_{\{\theta:\Gamma^{-1}_{\xi_1}(\xi,\theta)=\epsilon\}} \frac{\sum_{i=1}^{2n}(D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta))^2 - c(\xi_1)\Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_1}(\xi,\theta)}{|\nabla\Gamma_{\xi_1}(\xi,\theta)|}\Gamma_{\xi_0}(\theta,\zeta)d\sigma(\theta)$$

$$- \int_{\substack{\{\theta:\epsilon\le\Gamma^{-1}_{\xi_0}(\theta,\zeta)\le M\}\\ \{\theta:\epsilon\le\Gamma^{-1}_{\xi_1}(\xi,\theta)\}}} \Big(\sum_{i=1}^{2n} D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)D_{i,\xi_1}\Gamma_{\xi_0}(\theta,\zeta)\Big) - c(\xi_1)\Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$= I_1 + I_2 + I_3 + I_4.$$

In I_1 we use the fact that Γ_{ξ_0} is constantly equal to ϵ^{-1} and we integrate by parts:

$$I_1 = \frac{1}{\epsilon}\sum_{i=1}^{n} \int_{\{\theta:\Gamma^{-1}_{\xi_0}(\theta,\zeta)\le\epsilon\}} L^*_{\xi_1}\Gamma^*_{\xi_1}(\theta,\xi)d\theta = 0.$$

In order to study I_2 we first note that $\{\theta : \Gamma^{-1}_{\xi_0}(\theta,\zeta) = M\} \subset \{\theta : c_1 M^{1/(Q-2)} \le d_{\xi_0}(\theta,\zeta) \le c_2 M^{1/(Q-2)}\}$, for suitable constants c_1 and c_2. We can choose a cut-off function ϕ such that $\phi = 1$ on the set $\{\theta : c_1 M^{1/(Q-2)}/2 \le d_{\xi_0}(\theta,\zeta) \le 2c_2 M^{1/(Q-2)}\}$, $\phi = 0$ on the set $\{\theta : c_1 M^{1/(Q-2)}/4 \ge d_{\xi_0}(\theta,\zeta),\ \ or\ d_{\xi_0}(\theta,\zeta) \ge 4c_2 M^{1/(Q-2)}\}$, and $|D_{i,\xi_0}\phi(\zeta)| \le M^{-1/(Q-2)}$. Then, using the fact that Γ_{ξ_0} is constant on the integration set, we have

$$I_2 = \frac{1}{M}\int_{\{\theta:\Gamma^{-1}_{\xi_0}(\theta,\zeta)=M\}} \phi(\theta)\frac{\sum_{i=1}^{2n} D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)D_{i,\xi_1}\Gamma_{\xi_0}(\theta,\zeta) - c(\xi_1)\Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)}{|\nabla\Gamma_{\xi_0}(\theta,\zeta)|}d\sigma(\theta)$$

integrating by parts, since $L_{\xi_1}\Gamma_{\xi_1} = 0$, we get

$$= \frac{1}{M}\int_{\{\theta:\Gamma^{-1}_{\xi_0}(\theta,\zeta)\le M\}} \sum_{i=1}^{2n} D_{i,\xi_1}\phi(\theta)D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta) - c(\xi_1)\partial_t\phi(\theta)\Gamma_{\xi_1}(\xi,\theta)d\theta.$$

By Remark 3.1 $d_{\xi_1}(\theta,\zeta) \geq d_{\xi_0}(\theta,\zeta) - M_1 d_{\xi_0}^{1/2}(\theta,\zeta)\, d_{\xi_0}^{1/2}(\xi_0,\xi_1) \geq C_2 M^{1/(Q-2)}$ $-M^{1/2(Q-2)} \geq CM^{1/(Q-2)}$. Consequently

$$|D_{i,\xi_1}\phi(\zeta) D_{i,\xi_1}\Gamma_{\xi_1}(\theta,\zeta)| \leq CM^{-1/(Q-2)} d_{\xi_1}^{-(Q-1)}(\theta,\zeta) \leq CM^{-Q/(Q-2)}$$

and,

$$I_2 \leq \frac{1}{M} \int\limits_{\substack{\{\theta: C_1 M^{1/(Q-2)} \leq d_{\xi_0}(\theta,\zeta) \\ d_{\xi_0}(\theta,\zeta) \leq C_2 M^{1/(Q-2)}\}}} CM^{-Q/(Q-2)} d\theta \leq C\frac{1}{M} \to 0 \text{ as } M \to \infty.$$

In order to estimate I_3 we first note that

$$\int\limits_{\{\theta: \Gamma_{\xi_1}^{-1}(\xi,\theta)=\epsilon\}} \frac{\sum_{i=1}^{2n}(D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta))^2 + \Gamma_{\xi_1}(\xi,\theta)\partial_t \Gamma_{\xi_1}(\xi,\theta)}{|\nabla \Gamma_{\xi_1}(\xi,\theta)|} d\sigma(\theta) = 1$$

for all $\epsilon > 0$. This assertion is standard. For example in [7] it is proved as a general representation formula for regular functions, which implies the stated assertion, if applied to the function constantly equal to 1. Hence

$$I_3 = -\Gamma_{\xi_0}(\xi,\zeta)$$

$$-\int\limits_{\{\theta: \Gamma_{\xi_1}^{-1}(\xi,\theta)=\epsilon\}} (\Gamma_{\xi_0}(\theta,\zeta) - \Gamma_{\xi_0}(\xi,\zeta)) \frac{\sum_{i=1}^{2n}(D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta))^2 + \Gamma_{\xi_1}(\xi,\theta)\partial_t \Gamma_{\xi_1}(\xi,\theta)}{|\nabla \Gamma_{\xi_1}(\xi,\theta)|} d\sigma(\theta).$$

Now arguing as in the estimate if I_2, we have

$$I_3 \to -\Gamma_{\xi_0}(\xi,\zeta) \quad as \;\; \epsilon \to 0.$$

Then

$$I_4 = -\sum_{i=1}^{2n} \int\limits_{\substack{\{\theta: \epsilon \leq \Gamma_{\xi_0}^{-1}(\theta,\zeta) \leq M\} \\ \{\theta: \epsilon \leq \Gamma_{\xi_1}^{-1}(\xi,\theta)\}}} D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta) D_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta) d\theta$$

$$-\sum_{i=1}^{2n} \int\limits_{\substack{\{\theta: \epsilon \leq \Gamma_{\xi_0}^{-1}(\theta,\zeta) \leq M\} \\ \{\theta: \epsilon \leq \Gamma_{\xi_1}^{-1}(\xi,\theta)\}}} D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)(D_{i,\xi_1} - D_{i,\xi_0})\Gamma_{\xi_0}(\theta,\zeta) d\theta$$

$$-c(\xi_1) \int\limits_{\substack{\{\theta: \epsilon \leq \Gamma_{\xi_0}^{-1}(\theta,\zeta) \leq M\} \\ \{\theta: \epsilon \leq \Gamma_{\xi_1}^{-1}(\xi,\theta)\}}} \Gamma_{\xi_1}(\xi,\theta)\partial_t \Gamma_{\xi_0}(\theta,\zeta) d\theta.$$

Since for example $X_{i,\xi_1} - X_{i,\xi_0} = (P^1_{\xi_1} a - P^1_{\xi_0} a)\partial_t$, the second and third integrals tend to singular integrals as $\epsilon \to 0$. Precisely

$$\lim_{\epsilon \to 0, M \to +\infty} I_4 = -\sum_{i=1}^{2n} \int D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta) D_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$-\sum_{i=1}^{2n} PV_{\xi_0} \int D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)(D_{i,\xi_1} - D_{i,\xi_0})\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$-c(\xi_1) PV_{\xi_0} \int \Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)d\theta.$$

Collecting all the terms we get the stated assertion.

Let us estimate the singular integrals in the preceding formula. The estimate is similar to the classical one, but we have to take into account that d_{ξ_0} and d_{ξ_1} are not equivalent.

Lemma 3.1. *Let K be a compact subset of Ω, let M_1, M_2, M_3 be the constants introduced in Remark 3.1, and let*

$$S = \left\{\zeta : d(\xi_1,\zeta) \geq 4M_1^2 M_3 M_2^{-1} d(\xi_1,\xi_0)\right\}.$$

Then for every ξ_0, and ξ_1 in K, for every $\zeta \in S$ the following estimate holds

$$\left|PV_{\xi_1} \int \partial_t\Gamma_{\xi_1}(\xi_1,\theta) X_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)\right| \leq C d^{-Q+1}(\xi_1,\zeta),$$

where C is a constant depending only on the compact K.

Proof. Let us first note that, by Remark 3.1,

$$S \subset S_0 = \left\{\zeta : d_{\xi_1}(\xi_1,\zeta) \geq 4M_1^2 d_{\xi_1}(\xi_0,\xi_1)\right\},$$

hence we will prove the assertion for $\zeta \in S_0$. From the estimate of Γ_{ξ_0} and Γ_{ξ_1} we have

$$\left|PV_{\xi_1} \int \partial_t\Gamma_{\xi_1}(\xi_1,\theta) X_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)d\theta\right|$$

$$\leq \lim_{\epsilon \to 0} \int_{\epsilon \leq 4d_{\xi_1}(\xi_1,\theta) \leq d_{\xi_1}(\xi_1,\zeta)} \left|\partial_t\Gamma_{\xi_1}(\xi_1,\theta)(X_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta) - X_{i,\xi_0}\Gamma_{\xi_0}(\xi_1,\zeta))\right|d\theta$$

$$+ \int_{\substack{4d_{\xi_1}(\xi_1,\theta) \geq d_{\xi_1}(\xi_1,\zeta)\\ d_{\xi_1}(\xi_1,\theta) \leq 2d_{\xi_0}(\theta,\zeta)}} \left|\partial_t\Gamma_{\xi_1}(\xi_1,\theta) X_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)\right|d\theta$$

$$+ \int_{\substack{4d_{\xi_1}(\xi_1,\theta) \geq d_{\xi_1}(\xi_1,\zeta)\\ d_{\xi_1}(\xi_1,\theta) \geq 2d_{\xi_0}(\theta,\zeta)}} \left|\partial_t\Gamma_{\xi_1}(\xi_1,\theta) X_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)\right|d\theta$$

$$= I_1 + I_2 + I_3.$$

Let us first study I_1. We will call

$$S_1 = \{\theta : \epsilon \le 4d_{\xi_1}(\xi_1, \theta) \le d_{\xi_1}(\xi_1, \zeta)\}$$

Let $\gamma : [0, 1] \to R^{2n+1}$ be an integral curve of the vector fields X_{i,ξ_0}, Y_{i,ξ_0}, or ∂_t connecting ξ_1 and θ. By the mean value theorem there exists $\tilde{\theta} \in \gamma([0, 1])$ such that

$$\left|X_{i,\xi_0}\Gamma_{\xi_0}(\theta, \zeta) - X_{i,\xi_0}\Gamma_{\xi_0}(\xi_1, \zeta)\right|$$

$$\le C\Big(d_{\xi_0}(\theta, \xi_1)\sum_{i=1}^{2n}\left|D_{i,\xi_0}X_{i,\xi_0}\Gamma_{\xi_0}(\tilde{\theta}, \zeta)\right| + d^2_{\xi_0}(\theta, \xi_1)\left|\partial_t X_{i,\xi_0}\Gamma_{\xi_0}(\tilde{\theta}, \zeta)\right|\Big)$$

$$\le C\big(d_{\xi_0}(\theta, \xi_1)d^{-Q}_{\xi_0}(\tilde{\theta}, \zeta) + d^2_{\xi_0}(\theta, \xi_1)d^{-Q-1}_{\xi_0}(\tilde{\theta}, \zeta)\big).$$

In order to evaluate I_1 we have to estimate this expression in terms of the distance d_{ξ_1}. By Remark 3.1

$$d_{\xi_0}(\theta, \xi_1) \le d_{\xi_1}(\theta, \xi_1) + M_1 d^{1/2}_{\xi_1}(\theta, \xi_1)d^{1/2}_{\xi_1}(\xi_0, \xi_1)$$

(by the definition of S_0 and S_1)

$$\le C d^{1/2}_{\xi_1}(\theta, \xi_1)d^{1/2}_{\xi_1}(\zeta, \xi_1).$$

Analogously $d_{\xi_0}(\tilde{\theta}, \zeta)$ can also be expressed in terms of d_{ξ_1}. Indeed

$$d_{\xi_0}(\tilde{\theta}, \zeta) \ge d_{\xi_0}(\xi_1, \zeta) - d_{\xi_0}(\xi_1, \tilde{\theta})$$

(since $\tilde{\theta} \in \gamma([0, 1])$

$$\ge d_{\xi_0}(\xi_1, \zeta) - d_{\xi_0}(\xi_1, \theta)$$

(by Remark 3.1)

$$\ge d_{\xi_1}(\xi_1, \zeta) - M_1 d^{1/2}_{\xi_1}(\zeta, \xi_1)d^{1/2}_{\xi_1}(\xi_0, \xi_1) - d_{\xi_1}(\xi_1, \theta) - M_1 d^{1/2}_{\xi_1}(\theta, \xi_1)d^{1/2}_{\xi_1}(\xi_0, \xi_1)$$

(by the definition of S_0 and S_1)

$$\ge \frac{1}{8}d_{\xi_1}(\xi_1, \zeta).$$

Hence

$$d_{\xi_0}(\theta, \xi_1)d^{-Q}_{\xi_0}(\tilde{\theta}, \zeta) + d^2_{\xi_0}(\theta, \xi_1)d^{-Q-1}_{\xi_0}(\tilde{\theta}, \zeta)$$

$$\le C\Big(d^{1/2}_{\xi_1}(\theta, \xi_1)d^{\frac{1}{2}-Q}_{\xi_1}(\zeta, \xi_1) + d_{\xi_1}(\theta, \xi_1)d^{-Q}_{\xi_1}(\zeta, \xi_1)\Big)$$

(by the definition of the set S_1)

$$\le C d^{1/2}_{\xi_1}(\theta, \xi_1)d^{\frac{1}{2}-Q}_{\xi_1}(\zeta, \xi_1).$$

Collecting these estimates we get

$$I_1 \le d^{\frac{1}{2}-Q}_{\xi_1}(\zeta, \xi_1)\int_{S_1} d^{\frac{1}{2}-Q}_{\xi_1}(\theta, \xi_1)d\theta \le d^{1-Q}_{\xi_1}(\zeta, \xi_1) \le C d^{1-Q}(\zeta, \xi_1).$$

The estimate of I_2 can be done as follows:

$$I_2 \leq \int_{4d_{\xi_1}(\zeta,\xi_1)\leq d_{\xi_1}(\theta,\xi_1)} d_{\xi_1}^{-2Q+1}(\theta,\xi_1)d\theta$$
$$\leq d_{\xi_1}^{-Q+1}(\zeta,\xi_1) \leq d^{-Q+1}(\zeta,\xi_1).$$

Finally we can estimate I_3. On the set

$$S_3 = \left\{\theta : d_{\xi_1}(\zeta,\xi_1) \leq 4d_{\xi_1}(\theta,\xi_1),\, d_{\xi_0}(\theta,\zeta) \leq \frac{1}{2}d_{\xi_1}(\theta,\xi_1)\right\}$$

we have

$$d_{\xi_0}(\theta,\zeta) \leq \frac{1}{2}d_{\xi_1}(\xi_1,\theta) \leq \frac{1}{2}(d_{\xi_1}(\theta,\zeta) + d_{\xi_1}(\xi_1,\zeta)). \tag{19}$$

From Remark 3.1 it follows that

$$d_{\xi_1}(\theta,\zeta) \leq \frac{3}{2}d_{\xi_0}(\theta,\zeta) + \frac{M_1^2}{2}d_{\xi_1}(\xi_0,\xi_1),$$

hence we can deduce from (19) that

$$d_{\xi_0}(\theta,\zeta) \leq M_1^2 d_{\xi_1}(\xi_0,\xi_1) + 2d_{\xi_1}(\xi_1,\zeta) \leq Cd_{\xi_1}(\xi_1,\zeta),$$

by the definition of S_0. Then

$$I_3 \leq d_{\xi_1}^{-Q}(\xi_1,\zeta)\int_{d_{\xi_0}(\theta,\zeta)\leq d_{\xi_1}(\xi_1,\zeta)} d_{\xi_0}^{-Q+1}(\theta,\zeta)d\theta \leq Cd^{-Q+1}(\xi_1,\zeta).$$

This concludes the proof.

With a simple integration by parts we have

Remark 3.3.

$$PV_{\xi_1}\int \partial_t\Gamma_{\xi_1}(\xi,\theta)\Gamma_{\xi_0}(\theta,\zeta)d\theta = -PV_{\xi_0}\int \Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)d\theta.$$

Proof. Let us fix $M > 0$ such that

$$\left\{\theta : \Gamma_{\xi_1}^{-1}(\xi,\theta) \leq M\right\} \cap \left\{\theta : \Gamma_{\xi_0}^{-1}(\theta,\zeta) \leq M\right\} = \emptyset.$$

By definition we have

$$PV_{\xi_1}\int \partial_t\Gamma_{\xi_1}(\xi,\theta)\Gamma_{\xi_0}(\theta,\zeta)d\theta =$$
$$= \lim_{\epsilon\to 0}\int_{\left\{\theta:\epsilon\leq\Gamma_{\xi_1}^{-1}(\xi,\theta)\leq M\right\}} \partial_t\Gamma_{\xi_1}(\xi,\theta)(\Gamma_{\xi_0}(\theta,\zeta) - \Gamma_{\xi_0}(\xi,\zeta))d\theta$$
$$+ \int_{\left\{\theta:M\leq\Gamma_{\xi_1}^{-1}(\xi,\theta)\right\}} \partial_t\Gamma_{\xi_1}(\xi,\theta)\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

(since Γ_{ξ_0} is locally integrable)

$$= \lim_{\epsilon\to 0} \int\limits_{\left\{\theta:\epsilon\le\Gamma_{\xi_1}^{-1}(\xi,\theta)\le M\right\}} \partial_t\Gamma_{\xi_1}(\xi,\theta)\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_0}(\xi,\zeta)\big)d\theta$$

$$+ \int\limits_{\substack{\left\{\theta:M\le\Gamma_{\xi_1}^{-1}(\xi,\theta)\right\}\\ \left\{\theta:M\le\Gamma_{\xi_0}^{-1}(\theta,\zeta)\right\}}} \partial_t\Gamma_{\xi_1}(\xi,\theta)\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$+ \lim_{\delta\to 0} \int\limits_{\substack{\left\{\theta:\delta\le\Gamma_{\xi_0}^{-1}(\theta,\zeta)\le M\right\}\\ \left\{\theta:M\le\Gamma_{\xi_1}^{-1}(\xi,\theta)\right\}}} \partial_t\Gamma_{\xi_1}(\xi,\theta)\Gamma_{\xi_0}(\theta,\zeta)d\theta =$$

integrating by parts we have:

$$= -\lim_{\epsilon\to 0} \int\limits_{\left\{\theta:\epsilon=\Gamma_{\xi_1}^{-1}(\xi,\theta)\right\}} \Gamma_{\xi_1}(\xi,\theta)\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_0}(\xi,\zeta)\big)\frac{\partial_t\Gamma_{\xi_1}(\xi,\theta)}{|\nabla\Gamma_{\xi_1}(\xi,\theta)|}d\sigma(\theta)+$$

$$+ \int\limits_{\left\{\theta:\Gamma_{\xi_1}^{-1}(\xi,\theta)=M\right\}} \Gamma_{\xi_1}(\xi,\theta)\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_0}(\xi,\zeta)\big)\frac{\partial_t\Gamma_{\xi_1}(\xi,\theta)}{|\nabla\Gamma_{\xi_1}(\xi,\theta)|}d\sigma(\theta)+$$

$$- \lim_{\epsilon\to 0} \int\limits_{\left\{\theta:\epsilon\le\Gamma_{\xi_1}^{-1}(\xi,\theta)\le M\right\}} \Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)d\theta-$$

$$- \int\limits_{\left\{\theta:\Gamma_{\xi_1}^{-1}(\xi,\theta)=M\right\}} \Gamma_{\xi_1}(\xi,\theta)\Gamma_{\xi_0}(\theta,\zeta)\frac{\partial_t\Gamma_{\xi_1}(\xi,\theta)}{|\nabla\Gamma_{\xi_1}(\xi,\theta)|}d\sigma(\theta)+$$

$$- \int\limits_{\left\{\theta:\Gamma_{\xi_0}^{-1}(\theta,\zeta)=M\right\}} \Gamma_{\xi_1}(\xi,\theta)\Gamma_{\xi_0}(\theta,\zeta)\frac{\partial_t\Gamma_{\xi_0}(\theta,\zeta)}{|\nabla\Gamma_{\xi_0}(\theta,\zeta)|}d\sigma(\theta)+$$

$$- \int\limits_{\substack{\left\{\theta:\Gamma_{\xi_0}^{-1}(\theta,\zeta)\ge M\right.\\ \left.\Gamma_{\xi_1}^{-1}(\xi,\theta)\ge M\right\}}} \Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)d\theta+$$

$$- \lim_{\delta\to 0} \int\limits_{\left\{\theta:\delta\le\Gamma_{\xi_0}^{-1}(\theta,\zeta)\le M\right\}} \Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)d\theta-$$

$$- \lim_{\delta\to 0} \int\limits_{\left\{\theta:\Gamma_{\xi_0}^{-1}(\theta,\zeta)=\delta\right\}} \Gamma_{\xi_1}(\xi,\theta)\Gamma_{\xi_0}(\theta,\zeta)\frac{\partial_t\Gamma_{\xi_0}(\theta,\zeta)}{|\nabla\Gamma_{\xi_0}(\theta,\zeta)|}d\sigma(\theta)+$$

$$+ \int_{\left\{\theta:\Gamma_{\xi_0}^{-1}(\theta,\zeta)=M\right\}} \Gamma_{\xi_1}(\xi,\theta)\Gamma_{\xi_0}(\theta,\zeta)\frac{\partial_t\Gamma_{\xi_0}(\theta,\zeta)}{|\nabla\Gamma_{\xi_0}(\theta,\zeta)|}d\sigma(\theta) =$$

$$= lim_{\epsilon\to 0,\delta\to 0} I_1 + \ldots + I_9.$$

Note that

$$I_1 \to 0 \ \ as \ \ \epsilon \to 0,$$

$$I_2 + I_4 = \frac{\Gamma_{\xi_0}(\xi,\zeta)}{M} \int_{\left\{\theta:\Gamma_{\xi_1}^{-1}(\xi,\theta)=M\right\}} \frac{\partial_t\Gamma_{\xi_1}(\xi,\theta)}{|\nabla\Gamma_{\xi_1}(\xi,\theta)|}d\sigma(\theta) = 0,$$

and, in the same way

$$I_5 + I_9 = 0$$

while, by Remark 3.2

$$I_7 = -\int_{\left\{\theta:\delta\le\Gamma_{\xi_0}^{-1}(\theta,\zeta)\le M\right\}} \big(\Gamma_{\xi_1}(\xi,\theta) - \Gamma_{\xi_1}(\xi,\zeta)\big)\partial_t\Gamma_{\xi_0}(\theta,\zeta)d\theta.$$

Hence

$$lim_{\epsilon\to 0,\delta\to 0} I_1 + \ldots + I_9 = lim_{\epsilon\to 0,\delta\to 0} I_3 + I_6 + I_7 =$$

$$= -PV_{\xi_0}\int \Gamma_{\xi_1}(\xi,\theta)\partial_t\Gamma_{\xi_0}(\theta,\zeta)d\theta.$$

Proposition 3.2. *Let K be a compact subset of Ω, and let S be defined as in Lemma 3.1. Then for every ξ_0 and ξ_1 in K, for every $\zeta \in S$, Γ_{ξ_0} and Γ_{ξ_1} satisfy the following estimate: there exists $c > 0$ such that*

$$|\Gamma_{\xi_0}(\xi_0,\zeta) - \Gamma_{\xi_1}(\xi_1,\zeta)| \le c\left(\frac{d(\xi_0,\xi_1)}{d^{Q-1}(\xi_1,\zeta)} + \frac{d^{\alpha}(\xi_0,\xi_1)}{d^{Q-2}(\xi_1,\zeta)}\right), \tag{20}$$

where c depends only on K.

Note that, on the set S, $\Gamma_{\xi_0}(\xi_0,\zeta)$ and $\Gamma_{\xi_1}(\xi_1,\zeta)$ are of class C^∞, since

$$d(\xi_1,\zeta) \ge 2d(\xi_1,\xi_0)$$

and

$$d(\xi_0,\zeta) \ge d(\xi_1,\zeta) - d(\xi_0,\xi_1) \ge d(\xi_1,\xi_0).$$

Proof. We apply the representation formula to the function $\Gamma^*_{\xi_1}$ using the fact that $\Gamma^*_{\xi_0}$ is the fundamental solution of $L^*_{\xi_0}$. Then we get

$$\Gamma^*_{\xi_1}(\zeta,\xi) = -\int\sum_{i=1}^{2n} D^*_{i,\xi_0}\Gamma^*_{\xi_0}(\zeta,\theta)D^*_{i,\xi_1}\Gamma^*_{\xi_1}(\theta,\xi)d\theta$$

$$-\sum_{i=1}^{2n} PV_{\xi_1}\int D^*_{i,\xi_0}\Gamma^*_{\xi_0}(\zeta,\theta)\big(D^*_{i,\xi_0} - D^*_{i,\xi_1}\big)\Gamma^*_{\xi_1}(\theta,\xi)d\theta$$

$$-c(\xi_0)PV_{\xi_1}\int \Gamma^*_{\xi_0}(\zeta,\theta)\partial_t\Gamma^*_{\xi_1}(\theta,\xi)d\theta =$$

using the fact that $D^*_{i,\xi_0}\Gamma^*_{\xi_0}(\zeta,\theta) = -D_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)$,

$$= -\sum_{i=1}^{2n}\int D_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)d\theta$$

$$-\sum_{i=1}^{2n} PV_{\xi_1}\int D_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)(D_{i,\xi_0}-D_{i,\xi_1})\Gamma_{\xi_1}(\xi,\theta)d\theta$$

$$+c(\xi_0)PV_{\xi_1}\int \Gamma_{\xi_0}(\theta,\zeta)\partial_t\Gamma_{\xi_1}(\xi,\theta)d\theta.$$

Subtracting from this relation the expression of Γ_{ξ_0} provided in Proposition 3.1, and using Remark 3.3 we get

$$\Gamma_{\xi_1}(\xi,\zeta)-\Gamma_{\xi_0}(\xi,\zeta) =$$

$$-\sum_{i=1}^{2n} PV_{\xi_1}\int D_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)(D_{i,\xi_0}-D_{i,\xi_1})\Gamma_{\xi_1}(\xi,\theta)d\theta$$

$$+\sum_{i=1}^{2n} PV_{\xi_0}\int D_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)(D_{i,\xi_1}-D_{i,\xi_0})\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$+(c(\xi_0)-c(\xi_1))PV_{\xi_1}\int \partial_t\Gamma_{\xi_1}(\xi,\theta)\Gamma_{\xi_0}(\theta,\zeta)d\theta.$$

Since all these terms have the same behavior, we will consider only the first one:

$$PV_{\xi_1}\int \big(X^\theta_{i,\xi_1}-X^\theta_{i,\xi_0}\big)\Gamma_{\xi_1}(\xi,\theta)X^\theta_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$= PV_{\xi_1}\int \big(P^1_{\xi_1}a_i(\theta)-P^1_{\xi_0}a_i(\theta)\big)\partial_t\Gamma_{\xi_1}(\xi,\theta)X^\theta_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$= \big(P^1_{\xi_1}a_i(\xi_0)-a_i(\xi_0)\big)PV_{\xi_1}\int \partial_t\Gamma_{\xi_1}(\xi,\theta)X^\theta_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$+\sum_{j=1}^{n}(X_ja_i(\xi_1)-X_ja_i(\xi_0))PV_{\xi_1}\int \partial_t\Gamma_{\xi_1}(\xi,\theta)(x_{\theta,j}-x_{0,j})X^\theta_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

$$+\sum_{j=1}^{n}(Y_ja_i(\xi_1)-Y_ja_i(\xi_0))PV_{\xi_1}\int \partial_t\Gamma_{\xi_1}(\xi,\theta)(y_{\theta,j}-y_{0,j})X^\theta_{i,\xi_0}\Gamma_{\xi_0}(\theta,\zeta)d\theta$$

where we have denoted $\theta = (x_{\theta,1},\dots,x_{\theta,n},y_{\theta,1},\dots,y_{\theta,n},t_\theta)$ and X^θ_{i,ξ_0} the derivative with respect to the variable θ.

If we choose $\xi = \xi_1$, all the integrals appearing here can be estimated as in Lemma 3.1, and, since $a\in C^{1,\alpha}$, by (13) we obtain

$$|\Gamma_{\xi_0}(\xi_1,\zeta)-\Gamma_{\xi_1}(\xi_1,\zeta)| \le c\frac{d^{1+\alpha}(\xi_0,\xi_1)}{d^{Q-1}(\xi_1,\zeta)}. \tag{21}$$

Finally,

$$|\Gamma_{\xi_0}(\xi_0,\zeta)-\Gamma_{\xi_1}(\xi_1,\zeta)|$$

$$\le |\Gamma_{\xi_0}(\xi_0,\zeta)-\Gamma_{\xi_0}(\xi_1,\zeta)|+|\Gamma_{\xi_0}(\xi_1,\zeta)-\Gamma_{\xi_1}(\xi_1,\zeta)|$$

(by the mean value theorem applied to $\Gamma_{\xi_0}(\cdot,\zeta)$, and the estimate of its derivatives, and by (21))

$$\le c\left(\frac{d(\xi_0,\xi_1)}{d^{Q-1}(\widetilde{\xi},\zeta)}+\frac{d^2(\xi_0,\xi_1)}{d^{Q}(\widetilde{\xi},\zeta)}+\frac{d^{1+\alpha}(\xi_0,\xi_1)}{d^{Q-1}(\xi_1,\zeta)}\right),$$

for a suitable $\widetilde{\xi}$, which belongs to an integral curve of X_{i,ξ_0}, Y_{i,ξ_0} or ∂_t connecting ξ_0 and ξ_1. Hence also applying the definition of S,

$$\frac{1}{4}d(\xi_1,\zeta)\le d(\widetilde{\xi},\zeta)\le 4d(\xi_1,\zeta).$$

Then (20) follows immediately.

In order to estimate the derivatives of Γ_{ξ_0}, we also need the following remark:

Remark 3.4. If we denote $\xi=(\xi_1,\dots,\xi_{2n+1})$, and $\theta=(\theta_1,\dots,\theta_{2n+1})$, a direct computation proves that

$$D^{\xi}_{k,\xi_1}\Gamma_{\xi_1}(\xi,\theta)=-D^{\theta}_{k,\xi_1}\Gamma_{\xi_1}(\xi,\theta)-\sum_{i=1}^{2n}(\xi_i-\theta_i)[D_{i,\xi_1},D_{k,\xi_1}]^{\theta}\Gamma_{\xi_1}(\xi,\theta) \tag{22}$$

$$=-D^{\theta}_{k,\xi_1}\Gamma_{\xi_1}(\xi,\theta)-\sum_{i=1}^{2n}(\xi_i-\theta_i)c_{i,k}(\xi_1)\partial^{\theta}_t\Gamma_{\xi_1}(\xi,\theta),$$

for functions $c_{i,k}$ of class $C^{\alpha}_{\mathcal{L}}$ (see [10, page 295, line 5] from below, for the proof of the same assertion in a more general situation).

Proposition 3.3. *Let K be a compact subset of Ω, and let S be defined as in Lemma 3.1. Then there exists $C>0$ such that for every ξ_0 and ξ_1 in K, $\zeta\in S$, the following estimate holds*

$$|D_{j_1,\xi_0}\dots D_{j_s,\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta)-D_{j_1,\xi_1}\dots D_{j_s,\xi_1}\Gamma_{\xi_1}(\xi_1,\zeta)|$$

$$\le C\left(\frac{d(\xi_0,\xi_1)}{d(\zeta,\xi_1)^{Q+s-1}}+\frac{d^{\alpha}(\xi_0,\xi_1)}{d(\zeta,\xi_1)^{Q+s-2}}\right).$$

Proof. We denote by ψ a function of class $C^{\infty}(\mathbb{R},\mathbb{R})$ such that $\psi(t)=0$ if $|t|<\frac{1}{4}$, $\psi(t)=1$ if $|t|>\frac{3}{4}$. Then we call

$$\phi(\theta)=\psi\left(\frac{d(\theta,\zeta)}{d(\xi,\zeta)}\right),$$

so that $\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta) \in C^\infty(\mathbb{R}^{2n+1},\mathbb{R})$. Since Γ_{ξ_1} is the fundamental solution of L_{ξ_1}, then

$$\Gamma_{\xi_0}(\xi,\zeta)-\Gamma_{\xi_1}(\xi,\zeta)=\int \Gamma_{\xi_1}(\xi,\theta)L_{\xi_1}\big(\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_1}(\theta,\zeta)\big)\phi(\theta)\big)d\theta$$

$$=\int \Gamma_{\xi_1}(\xi,\theta)L_{\xi_1}\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_1}(\theta,\zeta)\big)\phi(\theta)d\theta$$

$$+2\sum_{i=1}^{2n}\int \Gamma_{\xi_1}(\xi,\theta)D_{i,\xi_1}\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_1}(\theta,\zeta)\big)D_{i,\xi_1}\phi(\theta)d\theta$$

$$+\int \Gamma_{\xi_1}(\xi,\theta)\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_1}(\theta,\zeta)\big)L_{\xi_1}\phi(\theta)d\theta$$

(since $L_{\xi_1}\Gamma_{\xi_1}(\theta,\zeta)=0$ and $L_{\xi_0}\Gamma_{\xi_0}(\theta,\zeta)=0$ on the support of ϕ).

$$=\int \Gamma_{\xi_1}(\xi,\theta)(L_{\xi_1}-L_{\xi_0})\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta)d\theta$$

$$+2\sum_{i=1}^{2n}\int \Gamma_{\xi_1}(\xi,\theta)D_{i,\xi_1}\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_1}(\theta,\zeta)\big)D_{i,\xi_1}\phi(\theta)d\theta$$

$$+\int \Gamma_{\xi_1}(\xi,\theta)\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_1}(\theta,\zeta)\big)L_{\xi_1}\phi(\theta)d\theta$$

(integrating by parts)

$$=\int \Gamma_{\xi_1}(\xi,\theta)(L_{\xi_1}-L_{\xi_0})\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta)d\theta$$

$$-2\sum_{i=1}^{2n}\int D^{\theta}_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_1}(\theta,\zeta)\big)D_{i,\xi_1}\phi(\theta)d\theta$$

$$-2\sum_{i=1}^{2n}\int \Gamma_{\xi_1}(\xi,\theta)\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_1}(\theta,\zeta)\big)D^2_{i,\xi_1}\phi(\theta)d\theta$$

$$-\int \Gamma_{\xi_1}(\xi,\theta)\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_1}(\theta,\zeta)\big)L_{\xi_1}\phi(\theta)d\theta.$$

We will now differentiate this formula. We will study only the following two terms

$$A(\xi,\zeta)=\int \Gamma_{\xi_1}(\xi,\theta)(L_{\xi_1}-L_{\xi_0})\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta)d\theta,$$

and

$$B(\xi,\zeta)=\int D^{\theta}_{i,\xi_1}\Gamma_{\xi_1}(\xi,\theta)\big(\Gamma_{\xi_0}(\theta,\zeta)-\Gamma_{\xi_1}(\theta,\zeta)\big)D_{i,\xi_1}\phi(\theta)d\theta,$$

since all the others have the same behavior as B.

The domain of integration of B is $\{\theta : D_{i,\xi_1}\phi(\theta) \neq 0\} \subset \{\theta : \frac{1}{4}d(\xi,\zeta) \le d(\theta,\zeta) \le \frac{3}{4}d(\xi,\zeta)\}$. Hence on this set we have

$$d(\xi,\theta) \ge d(\xi,\zeta) - d(\theta,\zeta) \ge d(\xi,\zeta) - \frac{3}{4}d(\xi,\zeta) = \frac{1}{4}d(\xi,\zeta)$$

and the integrand has no pole. Hence we can take the derivatives under the integral sign, and, also using (21), we get

$$|D_{j_1,\xi_1} \dots D_{j_s,\xi_1} B(\xi,\zeta)| \le C\left(\frac{d^{1+\alpha}(\xi,\xi_0)}{d(\xi,\zeta)^{Q+s-1}} + \frac{d^{\alpha}(\xi_0,\xi_1)d(\xi,\xi_0)}{d(\zeta,\xi_1)^{Q+s-1}}\right). \tag{23}$$

Let us now take the derivative of $A(\xi,\xi_0)$:

$$D^{\xi}_{j_1,\xi_1} \int \Gamma_{\xi_1}(\xi,\theta)(L_{\xi_1} - L_{\xi_0})\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta)d\theta$$

$$= \int D^{\xi}_{j_1,\xi_1}\Gamma_{\xi_1}(\xi,\theta)(L_{\xi_1} - L_{\xi_0})\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta)d\theta$$

(by formula (22))

$$= -\int D^{\theta}_{j_1,\xi_1}\Gamma_{\xi_1}(\xi,\theta)(L_{\xi_1} - L_{\xi_0})\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta)d\theta$$

$$-\sum_{i=1}^{2n} c_{i,j}(\xi_1)\int (\xi_i - \theta_i)\partial^{\theta}_t\Gamma_{\xi_1}(\xi,\theta)(L_{\xi_1} - L_{\xi_0})\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta)d\theta$$

integrating by parts

$$= -\int \Gamma_{\xi_1}(\xi,\theta)D^{\theta}_{j_1,\xi_1}\big((L_{\xi_1} - L_{\xi_0})\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta)\big)d\theta$$

$$-\sum_{i=1}^{2n} c_{i,j}(\xi_1)\int (\xi_i - \theta_i)\Gamma_{\xi_1}(\xi,\theta)\partial^{\theta}_t\big((L_{\xi_1} - L_{\xi_0})\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta)\big)d\theta.$$

Since this term has the same structure as A, it can be further differentiated and we get

$$|D_{j_1,\xi_1} \dots D_{j_s,\xi_1} A(\xi,\zeta)|$$

$$= \Big|\int \Gamma_{\xi_1}(\xi,\theta)D^{\zeta}_{j_1,\xi_1} \dots D^{\zeta}_{j_s,\xi_1}\big((L_{\xi_1} - L_{\xi_0})\Gamma_{\xi_0}(\theta,\zeta)\phi(\theta)\big)d\theta$$

$$+ \mathit{similar\ terms}\Big|$$

(arguing as in the proof of Proposition 3.2)

$$\le c\left(\frac{d^{1+\alpha}(\xi,\xi_0)}{d(\xi,\zeta)^{Q+s-1}} + \frac{d^{\alpha}(\xi_0,\xi_1)d(\xi,\xi_0)}{d(\zeta,\xi_1)^{Q+s-1}}\right).$$

Hence, also using (23), the same estimate holds for

$$|D_{j_1,\xi_0} \dots D_{j_s,\xi_0}\Gamma_{\xi_0}(\xi_0,\zeta) - D_{j_1,\xi_1} \dots D_{j_s,\xi_1}\Gamma_{\xi_1}(\xi_1,\zeta)|$$

and the thesis is proved.

4. Regularity results

In this section we conclude the proof of Theorem 1.1. First we introduce a definition of higher-order difference quotients in the direction of the vector fields and we give a sufficient condition of differentiability in terms of them. Then we apply this result to the representation formulas stated in the Appendix, and establish regularity results for the solutions of the linearized operator L_u. This result can be applied, in particular, to the solution u of $\mathcal{L}$, and ensures that, if u is of class $C^{2,\alpha}$ then it is of class $C^{3,\alpha}_{\mathcal{L}}$. Iterating this theorem we deduce Theorem 1.1.

Let us first introduce a simpler notation for the derivatives in the direction of the vector fields: we will call

$$D_i = X_i, \quad D_{n+i} = Y_i, \tag{24}$$

for all $i = 1, \dots, n$,. Then we give the following definition:

Definition 4.1. *If $g : \Omega \to R$, we define difference quotients in the direction of the vector fields D_i:*

$$\Delta_i g(h)(\xi) = \frac{g(e^{hD_i}(\xi)) - g(\xi)}{h},$$

and in the Euclidean directions e_i

$$\Delta_{e_i} g(h)(\xi) = \frac{g(\xi + h^2 e_i) - g(\xi)}{h^2},$$

where e_i is the i-th element of the canonical basis in $\mathbb{R}^{2n+1}$. For every $m \geq 2$ we set

$$\Delta^m_{i_1\dots i_m} g(h)(\xi) = \Delta_{i_1}\big(\Delta^{m-1}_{i_2\dots i_m} g(h)\big)(h)(\xi).$$[1]

Remark 4.1. It immediately follows from the definition that, if there exists $D^m_{i_1\dots i_m} g(\xi)$, then these also exists

$$\lim_{h\to 0} \Delta^m_{i_1\dots i_m} g(h)(\xi),$$

and they have the same value.

Vice versa we have the following result:

Remark 4.2. If $g \in C^1_{\mathcal{L}}(\Omega)$ and

$$\Delta^2_{ij} g(h)(\xi) \overset{h\to 0}{\to} f(\xi), \tag{25}$$

uniformly in ξ, then there exists $D^2_{ij} g$ and

$$D^2_{ij} g = f.$$

[1] Let us note explicitly that this definition is weaker than the one introduced in [5], and it seems to work better, because it requires the same increment in any directions.

Proof. By definition we have

$$\Delta_{ij}^2 g(h)(\xi) = \frac{1}{h^2}\big(g(e^{hD_j}e^{hD_i}\xi) - g(e^{hD_i}\xi) - g(e^{hD_j}\xi) + g(\xi)\big) \tag{26}$$
$$= \frac{1}{h^2}\int_0^h D_j g(e^{\lambda D_j}e^{hD_i}\xi)d\lambda - \frac{1}{h^2}\int_0^h D_j g(e^{\lambda D_j}\xi)d\lambda$$

If we choose $\xi = e^{sD_i}\xi_0$, and integrate (25) on the interval $[0, \tau]$ we get

$$\int_0^\tau f(e^{sD_i}\xi_0)ds = \lim_{h\to 0}\int_0^\tau \Delta_{ij}^2(h)g(e^{sD_i}\xi_0)ds = \tag{27}$$

(by (26))

$$= \lim_{h\to 0}\frac{1}{h^2}\int_0^\tau\Big(\int_0^h D_j g(e^{\lambda D_j}e^{(h+s)D_i}\xi_0)d\lambda\Big)ds - \frac{1}{h^2}\int_0^\tau\Big(\int_0^h D_j g(e^{\lambda D_j}e^{sD_i}\xi_0)d\lambda\Big)ds$$

(using the change of variable $h + s = s'$ in the first integral)

$$= \lim_{h\to 0}\frac{1}{h^2}\int_h^{\tau+h}\Big(\int_0^h D_j g(e^{\lambda D_j}e^{sD_i}\xi_0)d\lambda\Big)ds - \frac{1}{h^2}\int_0^\tau\Big(\int_0^h D_j g(e^{\lambda D_j}e^{sD_i}\xi_0)d\lambda\Big)ds$$

$$= \lim_{h\to 0}\frac{1}{h^2}\int_\tau^{\tau+h}\Big(\int_0^h D_j g(e^{\lambda D_j}e^{sD_i}\xi_0)d\lambda\Big)ds - \frac{1}{h^2}\int_0^h\Big(\int_0^h D_j g(e^{\lambda D_j}e^{sD_i}\xi_0)d\lambda\Big)ds$$

since $g \in C^1_{\mathcal{L}}$

$$= D_j g(e^{\tau D_i}\xi_0) - D_j g(\xi_0).$$

Because of (27) we obtain

$$D_j g(e^{\tau D_i}\xi_0) - D_j g(\xi_0) = \int_0^\tau f(e^{sD_i}\xi_0)ds,$$

and, since f is continuous because of (25), then there exists $D_{ij}^2 g(\xi_0)$, and

$$D_{ij}^2 g(\xi_0) = f(\xi_0)$$

for all $\xi_0 \in \Omega$. Higher-order difference quotients can be treated in an analogous way and we have

Remark 4.3. If $g \in C^m_{\mathcal{L}}(\Omega)$ and

$$\Delta^{m+1}_{i_1\ldots i_{m+1}} g(h)(\xi) \overset{h\to 0}{\to} f(\xi), \tag{28}$$

uniformly in ξ, then there exists $D^{m+1}_{i_1\ldots i_{m+1}} g$ and

$$D^{m+1}_{i_1\ldots i_{m+1}} g = f.$$

The same result also holds for mixed difference quotients:

Remark 4.4. If $g \in C^2_{\mathcal{L}}(\Omega)$ and $\partial_t g \in C^1_{\mathcal{L}}(\Omega)$

$$\Delta^2_{ij}(\Delta_{e_{2n+1}} g(h))(h)(\xi) \overset{h\to 0}{\to} f(\xi), \tag{29}$$

uniformly in ξ, then there exists

$$D^2_{ij}\partial_t g = f.$$

We can now apply these results to the representation formulas stated in the Appendix. Since $u \in C^{2,\alpha}$ is a fixed solution of equation (5), the coefficients $a_i = a_i(u)$ and $b_i = b_i(u)$ are fixed, and the squares of the vector fields $X_i = X_i(u)$ and $Y_i = Y_i(u)$ can act on any function $v \in C^{2,\alpha}_{\mathcal{L}}$. According to (8) the linear operator L_u may be written as

$$L_u v = \sum_{i=1}^{n} \big(X_i^2 v + Y_i^2 v - (X_i a_i + Y_i b_i)\partial_t v\big),$$

and here we study the solutions of the equation

$$L_u v = g. \tag{30}$$

Proposition 4.1. *If a and b are in $C^{1,\alpha}$, $v \in C^{2,\alpha}_{\mathcal{L}}(\Omega)$ is a solution of (30) with $\partial_t v, g \in C^{1,\alpha}_{\mathcal{L}}(\Omega)$ and $\partial_t g \in C^{\alpha}_{\mathcal{L}}(\Omega)$ then $\nabla v \in C^{2,\beta}_{\mathcal{L}}(\Omega)$, for all $\beta < \alpha$.*

Proof. Let us prove that $\partial_t v \in C^{2,\beta}_{\mathcal{L}}(\Omega)$, to begin with.

We can obviously represent v as in Theorem A.2, and we will differentiate this formula. The proof is divided in three steps: in the first one we show that the terms denoted as $B_{i,1}$, which are the most singular, can be uniformly approximated by a family of smooth functions. In the second step we prove the existence of the third derivative of v, and in the third we show that the derivative is of class C^β. We only give a short outline of the first and the third step, since they are a generalization of [5, Theorem 4.2]. We will prove in detail only the second step, where we use the properties of the new difference quotients introduced here.

Step 1. Let us assume that v is represented as in Theorem A.2, and let us study the last term in $B_{i,1}$:

$$w(\xi) = \int \partial_t \Gamma_{\xi_0}(\xi,\zeta)\big(b_i(\zeta) - P^1_{\xi_0} b_i(\zeta)\big)^2\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)\phi(\zeta)d\zeta.$$

Note that this is a principal value integral (in the sense of Definition 3.1). However we can define

$$w_1(\xi_0) = \int D^2_{lj}\partial_t^2 \Gamma_{\xi_0}(\xi_0,\zeta)\big(b_i(\zeta) - P^1_{\xi_0} b_i(\zeta)\big)^2\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)\phi(\zeta)d\zeta$$

and the integral is well defined, since

$$\big|D^2_{lj}\partial_t^2 \Gamma_{\xi_0}(\xi_0,\zeta)\big(b_i(\zeta) - P^1_{\xi_0} b_i(\zeta)\big)^2\big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)\big| \le d^{-Q-1+3\alpha}(\xi_0,\zeta).$$

Let us fix a function $\theta \in C^\infty(\mathbb{R})$ such that $0 \le \theta \le 1$, $\theta(\tau) = 0$ for all $\tau \le 1$ and $\theta(\tau) = 1$ for all $\tau \ge 2$. For every $\varepsilon > 0$ let us define

$$w_\varepsilon(\xi) = \int \partial_t \Gamma_{\xi_0}(\xi,\zeta)\big(b_i(\zeta) - P^1_{\xi_0} b_i(\zeta)\big)^2 \big(\partial_t v(\zeta) - P^1_{\xi_0}\partial_t v(\zeta)\big)\phi(\zeta)\theta\left(\frac{d_{\xi_0}(\xi,\zeta)}{\varepsilon}\right) d\zeta.$$

Arguing as in [5], by estimates (28) and (29) it is possible to prove that

$$\sup_{d_{\xi_0}(\xi,\xi_0)<\varepsilon/2} |w_\varepsilon(\xi) - w(\xi)| \le c_1 \varepsilon^{3+3\alpha}, \tag{31}$$

and

$$\sup_{d_{\xi_0}(\xi,\xi_0)<\varepsilon/2} \left|D^2_{lj}\partial_t w_\varepsilon(\xi) - w_1(\xi)\right| \le c_2 \varepsilon^{3\alpha-1}, \tag{32}$$

where the positive constants c_1, c_2 do not depend on ξ_0.

Step 2. In order to show that v is three times differentiable, here we study its third-order difference quotients. Let us start again with the last term in $B_{i,1}$, always called w. By (31), choosing $\varepsilon = |h|$, we get

$$\Delta^2_{lj}\Delta_{e_{2n+1}} w(h)(\xi_0) = \Delta^2_{lj}\Delta_{e_{2n+1}} w_{|h|}(h)(\xi_0) + O(|h|^{3\alpha-1})$$

(for a suitable $\tilde{\xi}$)

$$= D^2_{lj}\partial_t w_{|h|}(\tilde{\xi}) + O(|h|^{3\alpha-1}) = w_1(\xi_0) + o(1) \ \ as \ \ h \to 0$$

Hence there exists

$$\lim_{h\to 0} \Delta^2_{lj}\Delta_{e_{2n+1}} w(h)(\xi_0) = w_1(\xi_0)$$

uniformly for $\xi_0 \in \Omega$. Since all the terms in $B_{i,1}$ can be treated in the same way, there exists

$$\lim_{h\to 0} \Delta^2_{lj}\Delta_{e_{2n+1}} B_{i,1}(h)(\xi_0). \tag{33}$$

Let us now consider $A_1(\xi,\xi_0)$. It is standard to see that

$$\partial_t A_1(\xi,\xi_0) = -\int \Gamma_{\xi_0}(\xi,\zeta)\partial_t(L_u v\phi)(\zeta)d\zeta \in C^{1,\alpha}_{\mathcal{L}},$$

and, since $D_j = D_{j,\xi_0} + (a_j - P^1_{\xi_0} a_j)(\xi)\partial_t$, then

$$D_j\partial_t A_1(\xi,\xi_0) = -\int D_{j,\xi_0}\Gamma_{\xi_0}(\xi,\zeta)\partial_t(L_u v\phi)(\zeta)d\zeta -$$

$$-\big(a_j - P^1_{\xi_0} a_j\big)(\xi)\int \partial_t\Gamma_{\xi_0}(\xi,\zeta)\partial_t(L_u v\phi)(\zeta)d\zeta.$$

The structure of this term is similar to $B_{i,1}$, and arguing as in the proof of (33), but using first-order difference quotient, it is possible to show that there exists

$$\lim_{h\to 0} \Delta_l D_j \partial_t A_1(h)(\xi_0),$$

which, by definition, means that there exists $D^2_{lj}\partial_t A_1(\xi_0, \xi_0)$. Then, by Remark 4.1 it follows that there exists

$$\lim_{h\to 0} \Delta^2_{lj}\Delta_{e_{2n+1}} A_1(h)(\xi_0) \tag{34}$$

uniformly for $\xi_0 \in \Omega$. Finally note that the terms $C_i(\xi, \xi_0)$ and $F_i(\xi, \xi_0)$ are of class C^∞, since they are the convolution of Γ_{ξ_0} with a function of class C^α, identically zero in a neighborhood of the pole of Γ_{ξ_0}. Hence, by Remark 4.1 there exist

$$\lim_{h\to 0} \Delta^2_{lj}\Delta_{e_{2n+1}} C_i(h)(\xi_0) \quad \text{and} \quad \lim_{h\to 0} \Delta^2_{lj}\Delta_{e_{2n+1}} F_i(h)(\xi_0). \tag{35}$$

Summing up all the terms we deduce that there exists f such that

$$\Delta^2_{lj}\Delta_{e_{2n+1}} v(h)(\xi_0) \to f(\xi_0)$$

uniformly in Ω. Then, by Remark 4.4, there exists $D^2_{lj}\partial_t v = f$.

Step 3. Let us prove the regularity of the third derivatives of v. The expression of the function f is explicit, and depends on Γ_{ξ_0} and its derivatives. Using the estimates of Γ_{ξ_0} proved in the previous section, and arguing exactly as in [5, Theorem 5.1], we obtain a Hölder estimate of $D^2_{lj}\partial_t v$, which implies that $\partial_t v \in C^{2,\beta}_{\mathcal{L}}$ for all $\beta < \alpha$.

The proof that $\partial_{x_i} v$ and $\partial_{y_i} v$ are of class $C^{2,\beta}_{\mathcal{L}}$ is similar. Indeed, since $v \in C^{2,\alpha}_{\mathcal{L}}$ and $\partial_t v \in C^{1,\alpha}_{\mathcal{L}}$ then $\partial_{x_i} v = X_i v - a_i \partial_t v \in C^{1,\alpha}_{\mathcal{L}}$ and $\partial_{y_i} v = Y_i v - b_i \partial_t v \in C^{1,\alpha}_{\mathcal{L}}$. Besides, since $g \in C^{1,\alpha}_{\mathcal{L}}$ and $\partial_t g \in C^\alpha_{\mathcal{L}}$ then $\partial_{x_i} g \in C^\alpha_{\mathcal{L}}$ and $\partial_{y_i} g \in C^\alpha_{\mathcal{L}}$. Hence arguing as before we also get the existence of $D^2_{lj}\partial_{x_k} v$ and $D^2_{lj}\partial_{y_k} v$, and their Hölder estimates.

If we assume that the coefficients of the operator are more regular, we can prove a better regularity result.

Proposition 4.2. *If $a, b \in C^{2,\alpha}_{\mathcal{L}}(\Omega)$, $v \in C^{2,\alpha}_{\mathcal{L}}(\Omega)$ is a solution of (30) with $g \in C^{1,\alpha}_{\mathcal{L}}(\Omega)$ then $v \in C^{3,\beta}_{\mathcal{L}}(\Omega)$ for all $\beta < \alpha$.*

Sketch of the proof. The proof is similar to Proposition 4.1 but makes use of Theorem A.1, instead of Theorem A.2. In this way it is possible to show that there exists $f \in C^\beta_{\mathcal{L}}$, $\beta < \alpha$, such that

$$\Delta^3_{ijk} v(h)(\xi_0) \to f(\xi_0)$$

uniformly for $\xi_0 \in \Omega$. Then, by Remark 4.3, there exists $D^3_{ijk} v = f$.

Iterating this result we get the following:

Theorem 4.1. *If L_u is a linear operator with coefficients a_i and b_i of class $C^{m,\alpha}_{\mathcal{L},loc}$ with $m \geq 2$, and $v \in C^{2,\alpha}_{\mathcal{L}}$ is a solution of*

$$L_u v = g \in C^{m-1,\alpha}_{\mathcal{L},loc}, \tag{36}$$

then $v \in C^{m+1,\beta}_{\mathcal{L},loc}$ for all $\beta < \alpha$.

Proof. The proof can be carried out by induction arguing as in [5, Theorem 6.1] (where the same assertion is proved in R^3), but making use of Propositions 4.1 and 4.2, just proved.

Now we can apply these results to function u, since it is a solution of

$$L_u u = q(\cdot, u, a(u), b(u), \partial_t u).$$

Proposition 4.3. *If $u \in C^{2,\alpha}(\Omega)$ is a solution of (5), then*

$$a(u), \quad b(u) \quad \text{and} \quad \partial_t u \in C^{2,\beta}_{\mathcal{L}}(\Omega)$$

for all $\beta < \alpha$. In particular $u \in C^{3,\beta}_{\mathcal{L}}(\Omega)$.

Proof. By hypothesis $u \in C^{2,\alpha}(\Omega)$, so that $q(\cdot, u, a(u), b(u), \partial_t u) \in C^{1,\alpha}(\Omega)$. By Proposition 4.1 we deduce that $\nabla u \in C^{2,\beta}_{\mathcal{L}}$ for all $\beta < \alpha$, hence by the definition of a_i and b_i given in (2) it follows that $a, b \in C^{2,\beta}_{\mathcal{L}}$. Because of (9) this implies that $u \in C^{3,\beta}_{\mathcal{L}}$.

The derivatives in the directions ∂_t and X_i do not commute, but the following relation holds:

Remark 4.5. If $\partial_t u \in C^{m,\alpha}_{\mathcal{L}}$, then $\partial_t X_i u$ and $\partial_t Y_i u \in C^{m-1,\alpha}_{\mathcal{L}}$.

The derivatives $\partial_t X_i u$ and $\partial_t Y_i u$, obviously exist, since $u \in C^2$, and we only have to prove that they are of class $C^{m-1,\alpha}_{\mathcal{L}}$. Indeed

$$\partial_t X_i u = X_i \partial_t u + \partial_t a_i \partial_t u$$

(by (9))

$$= X_i \partial_t u + \partial_t Y_i u \partial_t u = X_i \partial_t u + (Y_i \partial_t u + \partial_t b_i \partial_t u)\partial_t u$$

(by (9))

$$= X_i \partial_t u + (Y_i \partial_t u)\partial_t u - \partial_t X_i u (\partial_t u)^2,$$

so that

$$\partial_t X_i u \big(1 + u_t^2\big) = X_i \partial_t u + u_t Y_i \partial_t u. \tag{37}$$

By dividing up equality (37) by $1 + u_t^2$ we get

$$\partial_t X_i u = \frac{X_i \partial_t u + u_t Y_i \partial_t u}{1 + u_t^2}. \tag{38}$$

Analogously we get

$$\partial_t Y_i u = \frac{Y_i \partial_t u - u_t X_i \partial_t u}{1 + u_t^2}. \tag{39}$$

The claim is proved, since $X_i \partial_t u$ and $Y_i \partial_t u$ are of class $C^{m-1,\alpha}_{\mathcal{L}}$ by hypothesis.

Let us now prove Theorem 1.1, by using Theorem 4.1 and an iteration procedure. We can not apply the iteration procedure directly to the functions $X_i u$ and $Y_i u$, since these functions satisfy equations analogous to (36), but with the second member not regular enough to apply Theorem 4.1. On the contrary we apply the iteration to the derivatives in the directions ∂_t, $\partial_{x_i} = X_i - a_i \partial_t$, and $\partial_{y_i} = Y_i - b_i \partial_t$.

Proof of Theorem 1.1. As we have already noted in Proposition 4.3, we can assume that

$$u \in C^{3,\beta}_{\mathcal{L},loc} \quad and \quad \partial_t u \in C^{2,\beta}_{\mathcal{L},loc}(\Omega). \tag{40}$$

Hence, by Remark 4.5 we also have

$$\partial_t Y_i u \ \ \partial_t X_i u \in C^{1,\beta}_{\mathcal{L},loc}(\Omega). \tag{41}$$

Differentiating equation (5) with respect to t, we get

$$L_u \partial_t u = -\sum_{i=1}^{n} \big(2\partial_t Y_i u X_i \partial_t u - 2\partial_t X_i u Y_i \partial_t u\big) + \partial_t\big(q(\xi, u, a, b, \partial_t u)\big)$$

(see [5, page 522]). In order to take the derivative of q, we will call $(\xi, s, p, \tau) \in \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}$ a generic element in the domain of q. Hence $v = \partial_t u$ is a solution of

$$\left(L_u - \frac{\partial q}{\partial \tau}(\xi, u, a, b, \partial_t u)\partial_t\right) v = -\sum_{i=1}^{n} \big(2\partial_t Y_i u X_i \partial_t u - 2\partial_t X_i u Y_i \partial_t u\big)$$

$$+\partial_t q(\xi, u, a, b, \partial_t u) + \partial_s q(\xi, u, a, b, \partial_t u)\partial_t u$$

$$+\sum_{i=1}^{n} \frac{\partial q}{\partial p_i}(\xi, u, a, b, \partial_t u)\partial_t Y_i u - \sum_{i=1}^{n} \frac{\partial q}{\partial p_{n+i}}(\xi, u, a, b, \partial_t u)\partial_t X_i u.$$

Obviously $L_u - \frac{\partial q}{\partial \tau}(\xi, u, a, b, \partial_t u)\partial_t$ is a linear operator with $C^{2,\beta}_{\mathcal{L},loc}$ coefficients, hence we can apply Theorem 4.1, and deduce that

$$\partial_t u \in C^{3,\beta}_{\mathcal{L},loc}.$$

Now we can differentiate equation (5), with respect to the vector field $X_i - a_i\partial_t$, writing $\mathcal{L}u$ as in (10). Let us begin with the derivative with respect to X_i.

$$X_i\left(X_j^2 + Y_j^2\right)u =$$

$$\left(X_j^2 + Y_j^2\right)X_i u + [X_i, X_j]X_j u + X_j[X_i, X_j]u + [X_i, Y_j]Y_j u + Y_j[X_i, Y_j]u =$$

$$= \left(X_j^2 + Y_j^2\right)X_i u + X_j[X_i, X_j]u + Y_j[X_i, Y_j]u + T^1_{i,j}, \tag{42}$$

where

$$T^1_{i,j} = [X_i, X_j]X_j u + [X_i, Y_j]Y_j u$$

$$= (X_i a_j - X_j a_i)\partial_t X_j u + (X_i b_j - Y_j a_i)\partial_t Y_j u \in C^{1,\beta}_{\mathcal{L},loc}$$

by (40) and (41). The second and third term in (42) can be evaluated as follows

$X_j\left([X_i, X_j]u\right) + Y_j\left([X_i, Y_j]u\right) = X_j\left((X_i a_j - X_j a_i)\partial_t u\right) + Y_j\left((X_i b_j - Y_j a_i)\partial_t u\right) =$

(by (9))

$$= X_j(X_i Y_j u - X_j Y_i u)\partial_t u + Y_j(-X_i X_j u - Y_j Y_i u)\partial_t u$$
$$+(X_i a_j - X_j a_i)X_j\partial_t u + (X_i b_j - Y_j a_i)Y_j\partial_t u =$$

$\left(\text{if we set } T^2_{i,j} = (X_i a_j - X_j a_i)X_j\partial_t u + (X_i b_j - Y_j a_i)Y_j\partial_t u \in C^{1,\beta}_{\mathcal{L},loc}\right)$

$$= (X_j X_i Y_j u - X_j X_j Y_i u - Y_j X_i X_j u - Y_j Y_j Y_i u)\partial_t u + T^2_{i,j} =$$

$$= \left([X_j, X_i]Y_j u + X_i([X_j, Y_j]u) + [X_i, Y_j]X_j u - X_j^2 Y_i u - Y_j^2 Y_i u\right)\partial_t u + T^2_{i,j} =$$
$$= X_i([X_j, Y_j]u)\partial_t u - \left(X_j^2 + Y_j^2\right)Y_i u\partial_t u + T^3_{i,j},$$

where we have called $T^3_{i,j} = T^2_{i,j} + \left([X_j, X_i]Y_j u + [X_i, Y_j]X_j u\right)\partial_t u$.

Inserting this identity in (42) we get

$$X_i\left(X_j^2 + Y_j^2\right)u = \left(X_j^2 + Y_j^2\right)X_i u + X_i([X_j, Y_j]u)\partial_t u - \left(X_j^2 + Y_j^2\right)Y_i u\partial_t u + T^1_{i,j} + T^3_{i,j} =$$

$$= \left(X_j^2 + Y_j^2\right)X_i u + X_i((X_j b_j - Y_j a_j)\partial_t u)\partial_t u - \left(X_j^2 + Y_j^2\right)Y_i u\partial_t u + T^1_{i,j} + T^3_{i,j} =$$

(using (9) in the second term)

$$= \left(X_j^2 + Y_j^2\right)X_i u - X_i\left(X_j^2 u + Y_j^2 u\right)(\partial_t u)^2 + (X_j b_j - Y_j a_j)X_i\partial_t u\partial_t u$$
$$-\left(X_j^2 + Y_j^2\right)Y_i u\partial_t u + T^1_{i,j} + T^3_{i,j} =$$
$$= \left(X_j^2 + Y_j^2\right)X_i u - X_i\left(X_j^2 u + Y_j^2 u\right)(\partial_t u)^2 - \left(X_j^2 + Y_j^2\right)Y_i u\partial_t u + T_{i,j},$$

where

$$T_{ij}u = T^1_{i,j} + T^3_{i,j} - (X_j b_j - Y_j a_j)X_i\partial_t u.$$

Bringing the second term to the left hand side we deduce that

$$\left(1 + (\partial_t u)^2\right)X_i\left(X_j^2 + Y_j^2\right)u \tag{43}$$
$$= \left(X_j^2 + Y_j^2\right)X_i u - \left(X_j^2 + Y_j^2\right)Y_i u\partial_t u + T_{ij}u.$$

Now we will compute the derivative with respect to $a_i\partial_t$. As in [5, page 522], we have

$$-a_i\partial_t\left(X_j^2 + Y_j^2\right)u = -a_i\left(\left(X_j^2 + Y_j^2\right)\partial_t u + \partial_t X_j a_j\partial_t u + \partial_t Y_j b_j\partial_t u\right)$$
$$-a_i\left(2\partial_t a_j X_j\partial_t u + 2\partial_t b_j Y_j\partial_t u\right)$$
$$= -a_i\left(\left(X_j^2 + Y_j^2\right)\partial_t u + \partial_t\left([X_j, Y_j]u\right)\partial_t u\right)$$
$$-a_i\left(2\partial_t a_j X_j\partial_t u + 2\partial_t b_j Y_j\partial_t u\right).$$

Let us remark that

$$\partial_t\left([X_j, Y_j]u\right) = \partial_t\left((X_j b_j - Y_j a_j)\partial_t u\right)$$

(by (9))

$$= \partial_t\left(\left(-X_j^2 - Y_j^2\right)u\right)\partial_t u + (X_j b_j - Y_j a_j)\partial^2_{tt}u.$$

Hence

$$-\left(1+(\partial_t u)^2\right) a_i \partial_t \left(X_j^2+Y_j^2\right) u = -a_i \left(\left(X_j^2+Y_j^2\right)\partial_t u\right) + K_{ij}u, \tag{44}$$

where

$$K_{ij}u = -a_i\left((X_jb_j - Y_ja_j)\partial_{tt}^2 u\partial_t u + 2\partial_t a_j X_j\partial_t u + 2\partial_t b_j Y_j \partial_t u\right)$$

is a function of class $C^{1,\beta}_{\mathcal{L},loc}$.

Finally, summing up (43) and (44) we get

$$\left(1+(\partial_t u)^2\right)(X_i - a_i\partial_t)\left(X_j^2+Y_j^2\right) u$$

$$= \left(X_j^2+Y_j^2\right) X_i u - \left(X_j^2+Y_j^2\right) Y_i u \partial_t u - Y_i u \left(X_j^2+Y_j^2\right)\partial_t u + T_{ij}u + K_{ij}u$$

$$= \left(X_j^2+Y_j^2\right)(X_i - a_i\partial_t)u + 2X_ja_iX_j\partial_t u + 2Y_ja_iY_j\partial_t u + T_{ij}u + K_{ij}u,$$

where

$$2X_ja_iX_j\partial_t u + 2Y_ja_iY_j\partial_t u + T_{ij}u + K_{ij}u \in C^{1,\beta}_{\mathcal{L},loc}.$$

Then

$$\mathcal{L}((X_i - a_i\partial_t)u) = \left(1+(\partial_t u)^2\right)\sum_{j=1}^n \left(X_j^2+Y_j^2\right)(X_i - a_i\partial_t)u$$

$$= \left(1+(\partial_t u)^2\right)^2 (X_i - a_i\partial_t)\left(\frac{q(\cdot, u, a, b, \partial_t u)}{1+(\partial_t u)^2}\right)$$

$$-\left(1+(\partial_t u)^2\right)\sum_{j=1}^n \left(2X_ja_iX_j\partial_t u + 2Y_ja_iY_j\partial_t u + T_{ij}u + K_{ij}u\right) \in C^{1,\beta}_{\mathcal{L}}$$

and, by Theorem 4.1,

$$(X_i - a_i\partial_t)u \in C^{3,\beta}_{\mathcal{L},loc},$$

for every $i = 1, ..., n$.

In the same way one can prove that

$$(Y_i - b_i\partial_t)u \in C^{3,\beta}_{\mathcal{L},loc},$$

for every $i = 1, ..., n$.

Hence

$$X_i u = \frac{(X_i - a_i\partial_t)u + (Y_i - b_i\partial_t)u\partial_t u}{1+(\partial_t u)^2} \in C^{3,\beta}_{\mathcal{L},loc},$$

$$Y_i u = -\frac{(Y_i - b_i\partial_t)u - (X_i - a_i\partial_t)u\partial_t u}{1+(\partial_t u)^2} \in C^{3,\beta}_{\mathcal{L},loc}$$

and $u \in C^{4,\beta}_{\mathcal{L},loc}$. We have proved that

$$u \in C^{4,\beta}_{\mathcal{L},loc} \quad \textit{and} \quad \partial_t u \in C^{3,\beta}_{\mathcal{L},loc}(\Omega).$$

Finally, iterating the procedure that we have applied to deduce this from (40), we get the result.

A. Appendix

We assume that $u \in C^{2,\alpha}$ is a fixed solution of equation (5) and for every function $v \in C^{2,\alpha}_{\mathcal{L}}$ we state a representation formula in terms of $L_u v$. This formula is the principal tool in the proof of Theorem 1.1. Since this is a local result, we can always fix three open sets Ω, Ω_1 and Ω_2 such that $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$, and a function $\phi \in C^\infty_0(\Omega)$ such that $\phi \equiv 1$ in Ω_1. Then we study only $v_{|\Omega_2} = v\phi_{|\Omega_2}$.

Theorem A.1. *If $v \in C^{2,\alpha}_{\mathcal{L}}(\Omega)$, then for every ξ and $\xi_0 \in \Omega_2$ we can represent $v(\xi) = v\phi(\xi)$ in the following way:*

$$\phi v(\xi) = A(\xi, \xi_0) + \sum_{i=1}^n B_{i0}(\xi, \xi_0) + \partial_t v(\xi_0) \sum_{i=1}^n C_i(\xi, \xi_0) + \sum_{i=0}^n E_i(\xi, \xi_0),$$

where

$$A(\xi, \xi_0) = \int \Gamma_{\xi_0}(\xi, \zeta) L_u v(\zeta)\phi(\zeta) d\zeta,$$

if $P^0_{\xi_0}\partial_t v(\zeta) = \partial_t v(\xi_0)$ then for $k = 0, 1$ B_{ik} is the following function

$$\begin{aligned}
B_{ik}(\xi, \xi_0) = 2 &\int \Gamma_{\xi_0}(\xi, \zeta)\big(X_i a_i(\zeta) - X_i a_i(\xi_0)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^k_{\xi_0}\partial_t v(\zeta)\big) d\zeta \\
&-2 \int X_{i,\xi_0}\Gamma_{\xi_0}(\xi, \zeta)\big(a_i(\zeta) - P^1_{\xi_0} a_i(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^k_{\xi_0}\partial_t v(\zeta)\big) d\zeta \\
&-2 \sum_{k=1}^n (X_k a_i - X_i a_k)(\xi_0) \int (x_{\xi,k} - x_{\zeta,k})\partial_t \Gamma_{\xi_0}(\xi, \zeta)\cdot \\
&\qquad \cdot\big(a_i(\zeta) - P^1_{\xi_0} a_i(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^k_{\xi_0}\partial_t v(\zeta)\big) d\zeta \\
&-2 \sum_{k=1}^n (Y_k a_i - X_i b_k)(\xi_0) \int (y_{\xi,k} - y_{\zeta,k})\partial_t \Gamma_{\xi_0}\cdot \\
&\qquad \cdot\big(a_i(\zeta) - P^1_{\xi_0} a_i(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^k_{\xi_0}\partial_t v(\zeta)\big) d\zeta \\
&- \int \partial_t \Gamma_{\xi_0}(\xi, \zeta)\big(a_i(\zeta) - P^1_{\xi_0} a_i(\zeta)\big)^2\phi(\zeta)\big(\partial_t v(\zeta) - P^k_{\xi_0}\partial_t v(\zeta)\big) d\zeta \\
&+2 \int \Gamma_{\xi_0}(\xi, \zeta)\big(Y_i b_i(\zeta) - Y_i b_i(\xi_0)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^k_{\xi_0}\partial_t v(\zeta)\big) d\zeta \\
&-2 \int Y_{i,\xi_0}\Gamma_{\xi_0}(\xi, \zeta)\big(b_i(\zeta) - P^1_{\xi_0} b_i(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^k_{\xi_0}\partial_t v(\zeta)\big) d\zeta \\
&-2 \sum_{k=1}^n (X_k b_i - Y_i a_k)(\xi_0) \int (x_{\xi,k} - x_{\zeta,k})\partial_t \Gamma_{\xi_0}\cdot \\
&\qquad \cdot\big(b_i(\zeta) - P^1_{\xi_0} b_i(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta) - P^k_{\xi_0}\partial_t v(\zeta)\big) d\zeta
\end{aligned}$$

$$-2\sum_{k=1}^{n}(Y_k b_i - Y_i b_k)(\xi_0)\int(y_{\xi,k}-y_{\zeta,k})\partial_t\Gamma_{\xi_0}\cdot$$

$$\cdot\big(b_i(\zeta)-P^1_{\xi_0}b_i(\zeta)\big)\phi(\zeta)\big(\partial_t v(\zeta)-P^k_{\xi_0}\partial_t v(\zeta)\big)d\zeta$$

$$-\int\partial_t\Gamma_{\xi_0}(\xi,\zeta)\big(b_i(\zeta)-P^1_{\xi_0}b_i(\zeta)\big)^2\phi(\zeta)\big(\partial_t v(\zeta)-P^k_{\xi_0}\partial_t v(\zeta)\big)d\zeta,$$

$$C_i(\xi,\xi_0) = -2\int\Gamma_{\xi_0}(\xi,\zeta)\big(a_i(\zeta)-P^1_{\xi_0}a_i(\zeta)\big)X_{i,\xi_0}\phi(\zeta)d\zeta$$

$$-2\int\Gamma_{\xi_0}(\xi,\zeta)\big(b_i(\zeta)-P^1_{\xi_0}b_i(\zeta)\big)Y_{i,\xi_0}\phi(\zeta)d\zeta$$

$$-\int\Gamma_{\xi_0}(\xi,\zeta)\big(a_i(\zeta)-P^1_{\xi_0}a_i(\zeta)\big)^2\partial_t\phi(\zeta)d\zeta-\int\Gamma_{\xi_0}(\xi,\zeta)\big(b_i(\zeta)-P^1_{\xi_0}b_i(\zeta)\big)^2\partial_t\phi(\zeta)d\zeta,$$

$$E_0(\xi,\xi_0) = \int\Gamma_{\xi_0}(\xi,\zeta)v(\zeta)L_{\xi_0}\phi(\zeta)d\zeta,$$

$$E_i(\xi,\xi_0) = 2\int\Gamma_{\xi_0}(\xi,\zeta)X_i v(\zeta)X_{i,\xi_0}\phi(\zeta)d\zeta$$

$$+2\int\Gamma_{\xi_0}(\xi,\zeta)Y_i v(\zeta)Y_{i,\xi_0}\phi(\zeta)d\zeta$$

$$+\int\Gamma_{\xi_0}(\xi,\zeta)\big(a_i(\zeta)-P^1_{\xi_0}a_i(\zeta)\big)^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta$$

$$+\int\Gamma_{\xi_0}(\xi,\zeta)\big(b_i(\zeta)-P^1_{\xi_0}b_i(\zeta)\big)^2\partial_t v(\zeta)\partial_t\phi(\zeta)d\zeta,$$

We omit the proof, since it is similar to the analogous result in [5]: here we have used (22) in the expression of B_{ik}.

If v is more regular we also have the following formula:

Theorem A.2. *Assume that $v \in C^{2,\alpha}_{\mathcal{L}}(\Omega)$ and $\partial_t v \in C^{1,\alpha}_{\mathcal{L}}(\Omega)$. Then for every $\phi \in C^\infty_0(\Omega)$ we have*

$$\phi v(\xi) = A_1(\xi,\xi_0)+\sum_{i=1}^{n}B_{i1}(\xi,\xi_0)+\partial_t v(\xi_0)\sum_{i=1}^{n}C_i(\xi,\xi_0)+D_1(\xi,\xi_0)+\sum_{i=0}^{n}F_i(\xi,\xi_0)$$

where B_{i1} and C_i have been defined in the previous theorem, and

$$D_1(\xi,\xi_0) = t\big(P^1_{\xi_0}\partial_t v(\xi)-\partial_t v(\xi_0)\big),$$

$$A_1(\xi,\xi_0) = \int\Gamma_{\xi_0}(\xi,\zeta)\Big(L_u v(\zeta)-2\sum_{i=1}^{n}(a_i X_i\partial_t v(\xi_0)+b_i Y_i\partial_t v(\xi_0))\Big)\phi(\zeta)d\zeta,$$

$$F_0(\xi,\xi_0) = \int\Gamma_{\xi_0}(\xi,\zeta)(v(\zeta)-t(P^1_{\xi_0}\partial_t v(\xi)-\partial_t v(\xi_0)))L_{\xi_0}\phi(\zeta)d\zeta$$

$$F_i(\xi,\xi_0)=2\int \Gamma_{\xi_0}(\xi,\zeta)\big(X_i v(\zeta)-tX_i\partial_t v(\xi_0)-a_i\big(P^1_{\xi_0}\partial_t v(\xi)-\partial_t v(\xi_0)\big)\big)X_{i,\xi_0}\phi(\zeta)d\zeta$$

$$+2\int \Gamma_{\xi_0}(\xi,\zeta)\big(Y_i v(\zeta)-tY_i\partial_t v(\xi_0)-b_i\big(P^1_{\xi_0}\partial_t v(\xi)-\partial_t v(\xi_0)\big)\big)Y_{i,\xi_0}\phi(\zeta)d\zeta$$

$$+\int \Gamma_{\xi_0}(\xi,\zeta)\big(a_i(\zeta)-P^1_{\xi_0}a_i(\zeta)\big)^2\big(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta)\big)\partial_t\phi(\zeta)d\zeta$$

$$+\int \Gamma_{\xi_0}(\xi,\zeta)\big(b_i(\zeta)-P^1_{\xi_0}b_i(\zeta)\big)^2\big(\partial_t v(\zeta)-P^1_{\xi_0}\partial_t v(\zeta)\big)\partial_t\phi(\zeta)d\zeta.$$

Proof. The assertion can be proved by applying Theorem A.1 to the function

$$w(\xi)=v(\xi)-t\big(P^1_{\xi_0}\partial_t v(\xi)-\partial_t v(\xi_0)\big).$$

References

1. Bony, J.: Principe du Maximum, Inégalitéde Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés. Ann. Inst. Fourier (Grenoble) **19**, 277–304 (1969)
2. Bedford, E., Gaveau, B.: Hypersurfaces with Bounded Levi Form. Indiana Univ. Math. J. **27**, 867–873 (1978)
3. Bedford, E., Gaveau, B.: Envelopes of holomorphy of certain 2-spheres in C^2. Amer. J. Math. **105**, 975–1009 (1983)
4. Citti, G.: A comparison theorem for the Levi equation. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei, serie IX, vol. IV, 3, 207–212 (1993)
5. Citti, G.: C^∞ regularity of solutions of a quasilinear equation related to the Levi operator. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **23**, 483–529 (1996)
6. Citti, G.: C^∞ regularity of solutions of the Levi equation. Ann. Inst. H. Poincaré, Anal. non Linéaire **15**, 4, 517–534 (1998)
7. Citti, G., Garofalo, N., Lanconelli, E.: Harnack's inequality for sum of squares of vector fields plus a potential. Amer. J. Math. **115**, 699–734 (1993)
8. Debiard, A., Gaveau, B.: Problème de Dirichlet pour l'équation de Lévi. Bull. Sci. Math., 2^e série **102**, 369–386 (1978)
9. Nagel, A., Stein, E.M., Wainger, S.: Balls and metrics defined by vector fields I: Basic properties. Acta Math. **155**, 103–147 (1985)
10. Rothschild, L., Stein, E.M.: Hypoelliptic differential operators and nilpotent Lie groups. Acta Math. **137**, 247–320 (1977)
11. Sánchez-Calle, A.: Fundamental solutions and geometry of the sum of squares of vector fields. Invent. math. **78**, 143–160 (1984)
12. Slodkowski, Z., Tomassini, G.: Weak solutions for the Levi equation and Envelope of Holomorphy. J. Funct. Anal. **101**, no. 4, 392–407 (1991)
13. Slodkowski, Z., Tomassini, G.: The Levi equation in higher dimension and relationship to the envelope of holomorphy. Amer. J. Math. **116**, 479–499 (1994)
14. Tomassini, G.: Geometric Properties of Solutions of the Levi Equation. Ann. Mat. Pura Appl. (4) **152**, 331–344 (1988)