Foundations of Computational Mathematics
https://doi.org/10.1007/s10208-023-09621-y

FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
The Journal of the Society for the Foundations of Computational Mathematics

Error Analysis for 2D Stochastic Navier–Stokes Equations in Bounded Domains with Dirichlet Data

Dominic Breit[1,2] · Andreas Prohl[3]

Received: 5 October 2022 / Revised: 18 May 2023 / Accepted: 13 July 2023

Abstract

We study a finite-element based space-time discretisation for the 2D stochastic Navier–Stokes equations in a bounded domain supplemented with no-slip boundary conditions. We prove optimal convergence rates in the energy norm with respect to convergence in probability, that is convergence of order (almost) 1/2 in time and 1 in space. This was previously only known in the space-periodic case, where higher order energy estimates for any given (deterministic) time are available. In contrast to this, estimates in the Dirichlet-case are only known for a (possibly large) stopping time. We overcome this problem by introducing an approach based on discrete stopping times. This replaces the localised estimates (with respect to the sample space) from earlier contributions.

Keywords Stochastic Navier–Stokes equations · Error analysis · Space-time discretisation · Convergence rates

Mathematics Subject Classification 65M15 · 65C30 · 60H15 · 60H35

Communicated by Endre Süli.

✉ Dominic Breit
dominic.breit@tu-clausthal.de

Andreas Prohl
prohl@na.uni-tuebingen.de

1 Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, UK

2 Institute of Mathematics, TU Clausthal, Erzstraße 1, 38678 Clausthal-Zellerfeld, Germany

3 Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany

Published online: 26 October 2023

1 Introduction

We are concerned with the numerical approximation of the 2D stochastic Navier–Stokes equations in a smooth bounded domain $\mathcal{O} \subset \mathbb{R}^2$ supplemented with no-slip boundary conditions. They describe the flow of a homogeneous incompressible fluid in terms of the velocity field $\mathbf{u}$ and pressure function p defined on a filtered probability space $(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P})$ and read as

$$\begin{cases} \mathrm{d}\mathbf{u} = \mu\Delta\mathbf{u}\,\mathrm{d}t - (\mathbf{u}\cdot\nabla)\mathbf{u}\,\mathrm{d}t - \nabla p\,\mathrm{d}t + \Phi(\mathbf{u})\mathrm{d}W & \text{in } \mathcal{O}_T, \\ \mathrm{div}\mathbf{u} = 0 & \text{in } \mathcal{O}_T, \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \mathcal{O}, \end{cases} \tag{1.1}$$

$\mathbb{P}$-a.s. in $\mathcal{O}_T := (0, T) \times \mathcal{O}$, where $T > 0$, $\mu > 0$ is the viscosity and $\mathbf{u}_0$ is a given initial datum. The momentum equation is driven by a cylindrical Wiener process W and the diffusion coefficient Φ takes values in the space of Hilbert-Schmidt operators; see Sect. 2.1 for details.

Existence, regularity and long-time behaviour of solutions to (1.1) have been studied extensively over the last three decades, and we refer to [23] for a complete picture. Most of the available results consider (1.1) with respect to periodic boundary conditions. In some cases this is only for a simplification of the presentation. For instance, the existence of stochastically strong solutions to (1.1) is not affected by the boundary condition. Looking at the spatial regularity of solutions the situation is completely different:

- If $\mathcal{O} = \mathbb{T}^2$ — the two-dimensional torus — and (1.1) is supplemented with periodic boundary conditions one can obtain estimates in any Sobolev space provided the data (initial datum and diffusion coefficient) are sufficiently regular; cf. [23, Corollary 2.4.13].
- If, on the other hand, $\mathcal{O}$ is a bounded domain with smooth boundary and (1.1) is supplemented with the no-slip boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on } (0, T) \times \partial\mathcal{O}, \tag{1.2}$$

  it is still an open problem if the solution satisfies

$$\mathbb{E}\Big[\|\nabla\mathbf{u}(T)\|_{L^2_x}^2\Big] < \infty \tag{1.3}$$

  for any given $T < \infty$, cf. [18, 22]. Regularity estimates are only known until a (possibly large) stopping time and even with this restriction the spatial regularity seems limited; see Lemma 3.1 (c) and Remark 3.2.

Moment estimates such as (1.3) are crucial for the numerical analysis. If they are not at disposal it is unclear how to obtain convergence rates for a discretisation of (1.1). Consequently most, if not all available results are concerned with the space-periodic problem. In particular, it is shown in [5] and [12] for the space-periodic problem that for any $\xi > 0$

$$\mathbb{P}\left[\frac{\max_m \|\mathbf{u}(t_m) - \mathbf{u}_{h,m}\|^2_{L^2_x} + \sum_{m=1}^{M} \tau \|\nabla \mathbf{u}(t_m) - \nabla \mathbf{u}_{h,m}\|^2_{L^2_x}}{h^{2\beta} + \tau^{2\alpha}} > \xi\right] \to 0 \tag{1.4}$$

as $h, \tau \to 0$ (where $\alpha < \frac{1}{2}$ and $\beta < 1$ are arbitrary); see also [3, 4] for related results. Here $\mathbf{u}$ is the solution to (1.1) and $\mathbf{u}_{h,m}$ the approximation of $\mathbf{u}(t_m)$ with discretisation parameters $\tau = T/M$ (time) and h (space). The relation (1.4) tells us that the convergence in probability is of order (almost) 1/2 in time and 1 in space. It seems to be an intrinsic feature of SPDEs with general non-Lipschitz nonlinearities such as (1.1) that the more common concept of a pathwise error (an error measured in $L^2(\Omega)$) is too strong (see [26] for first contributions). Hence (1.4) is the best result we can hope for. The proof of (1.4) is based on estimates in $L^2(\Omega)$, which are localised with respect to the sample set. The size of the neglected sets shrinks asymptotically with respect to the discretisation parameters and is consequently not seen in (1.4). The localised $L^2(\Omega)$-estimates in question rely on an iterative argument in the m-th step of which one can only control the discrete solution up to the step $m-1$ (to avoid problems with $(\mathfrak{F}_t)$-adaptedness), while the continuous solution is estimated by means of the global regularity estimates being available in the periodic setting (recall the discussion above).

In this work, we consider for (1.1) the semi-implicit space-time discretisation scheme (4.1), with general stable mixed finite element pairings as detailed in (2.8)–(2.10). We remark that many pairings are available in the literature that satisfy criterion (2.10), see *e.g.* [6, 7, 17, 19]; the convective term is treated in a semi-implicit, symmetrized way, which is a well-known strategy in the deterministic setting that goes back to [29] to enhance stability of this discretisation of the nonlinearity in the context of only *discretely* divergence-free functions; see $(4.1)_2$. As a result, this amounts to solving linear (coupled) problems in the m-th iteration. It is due to the used Dirichlet data for (1.1) that a related error analysis of this scheme (4.1) is more difficult if compared to the periodic situation. In fact, in the Dirichlet-case estimates in stronger norms for the solution of (1.1) are only known for a (possibly large) stopping time since the equality $\int_{\mathcal{O}} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \Delta \mathbf{u} \,\mathrm{d}x = 0$ is no longer available. Incorporating the latter case into the framework of the localised estimates, the iterative argument just mentioned fails: controlling the continuous solution in the m-th step only until the time t_{m-1} is insufficient for the estimates, while "looking into" the interval $[t_{m-1}, t_m]$ in this set-up destroys the martingale character of certain stochastic integrals we have to estimate. We overcome this problem by using an approach based on discrete stopping times, which replaces the localised $L^2(\Omega)$-estimates from earlier contributions. This allows to control all quantities even in the interval $[t_{m-1}, t_m]$ and, at the same time, preserves the martingale property of the stochastic integrals (see also the discussion in Remark 4.3). As a result we obtain 'global-in-Ω' estimates up to the discrete stopping time; cf. Theorem 4.2. The discrete stopping times are constructed such that they converge to T, where T can be any given end-time. Consequently, the convergence in probability as in (1.4) follows for the Dirichlet-case, see our main result in Theorem 4.4. We believe that this strategy will be of use also for other SPDEs with non-Lipschitz nonlinearities.

We work under the structural assumption of a solenoidal diffusion coefficient which vanishes at the boundary. This is crucial in the regularity estimate from Lemma 3.1 (b) in order to control the correction term $V^N(t)$ in the proof. Due to the counterexamples concerning the regularity for stochastic PDEs in bounded domains, see [21], this seems to be unavoidable. In fact, the same assumptions are made in the analytical paper [18] on which we built on.

2 Mathematical Framework

2.1 Probability Setup

Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The process W is a cylindrical $\mathfrak{U}$-valued Wiener process, that is, $W(t) = \sum_{j\geq 1} \beta_j(t)e_j$ with $(\beta_j)_{j\geq 1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathfrak{F}_t)_{t\geq 0}$, and $(e_j)_{j\geq 1}$ a complete orthonormal system in a separable Hilbert space $\mathfrak{U}$. Let us now give the precise definition of the diffusion coefficient Φ taking values in the set of Hilbert-Schmidt operators $L_2(\mathfrak{U}; \mathbb{H})$, where $\mathbb{H}$ can take the role of various Hilbert spaces. We define $L^2_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$ and $W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$ to be the closure of $C^\infty_{c,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$ – the solenoidal $C^\infty_c(\mathcal{O}, \mathbb{R}^2)$-functions – in $L^2(\mathcal{O}, \mathbb{R}^2)$ and $W^{1,2}_0(\mathcal{O}, \mathbb{R}^2)$, respectively, see e.g. [15, Chapter III]. We also work with fractional Sobolev spaces $W^{\sigma,p}(0, T; X)$ for $p \in (1,\infty)$ and $\sigma \in (0, 1)$ and a Banach space $(X; \|\cdot\|_X)$ with norm given by

$$\|f\|^p_{W^{\sigma,p}(0,T;X)} := \|f\|^p_{L^p(0,T;X)} + \int_0^T \int_0^T \frac{\|f(t) - f(s)\|^p_X}{|t-s|^{1+\sigma p}} \,\mathrm{d}s\,\mathrm{d}t.$$

Similarly, $W^{\sigma,p}(\mathcal{O}, \mathbb{R}^2)$ is the fractional Sobolev space with norm given by

$$\|v\|^p_{W^{\sigma,p}_x} := \|v\|^p_{L^p_x} + \int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|v(x) - v(y)|^p}{|x-y|^{2+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y.$$

We assume that $\Phi(\mathbf{u}) \in L_2(\mathfrak{U}; L^2_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2))$ for $\mathbf{u} \in L^2_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$, and $\Phi(\mathbf{u}) \in L_2(\mathfrak{U}; W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2))$ for $\mathbf{u} \in W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$, together with

$$\|\Phi(\mathbf{u}) - \Phi(\mathbf{v})\|_{L_2(\mathfrak{U}; L^2_x)} \leq c\|\mathbf{u} - \mathbf{v}\|_{L^2_x} \qquad \forall \mathbf{u}, \mathbf{v} \in L^2_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2), \tag{2.1}$$

$$\|\Phi(\mathbf{u})\|_{L_2(\mathfrak{U}; W^{1,2}_x)} \leq c\big(1 + \|\mathbf{u}\|_{W^{1,2}_x}\big) \qquad \forall \mathbf{u} \in W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2), \tag{2.2}$$

$$\|D\Phi(\mathbf{u})\|_{L_2(\mathfrak{U}; \mathcal{L}(L^2_x; L^2_x))} \leq c \qquad \forall \mathbf{u} \in L^2_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2). \tag{2.3}$$

If we are interested in higher regularity, some further assumptions are in place and we require additionally that $\Phi(\mathbf{u}) \in L_2(\mathfrak{U}; W^{2,2}(\mathcal{O}, \mathbb{R}^2))$ for $\mathbf{u} \in W^{2,2} \cap W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$, together with

$$\|\Phi(\mathbf{u})\|_{L_2(\mathfrak{U};W^{2,2}_x)} \le c\big(1+\|\mathbf{u}\|^2_{W^{1,4}_x}+\|\mathbf{u}\|_{W^{2,2}_x}\big) \quad \forall \mathbf{u}\in W^{2,2}\cap W^{1,2}_{0,\mathrm{div}}(\mathcal{O},\mathbb{R}^2), \tag{2.4}$$

$$\|D^2\Phi(\mathbf{u})\|_{L_2(\mathfrak{U};\mathcal{L}(L^2_x\times L^2_x;L^2_x))} \le c \quad \forall \mathbf{u}\in L^2_{\mathrm{div}}(\mathcal{O},\mathbb{R}^2). \tag{2.5}$$

Assumption (2.1) allows us to define stochastic integrals. Given an $(\mathfrak{F}_t)$-adapted process $\mathbf{u}\in L^2(\Omega;C([0,T];L^2_{\mathrm{div}}(\mathcal{O})))$, the stochastic integral

$$t\mapsto \int_0^t \Phi(\mathbf{u})\,\mathrm{d}W$$

is a well-defined process taking values in $L^2_{\mathrm{div}}(\mathcal{O},\mathbb{R}^2)$; see [13] for a detailed construction. Moreover, we can multiply by test functions to obtain

$$\left(\int_0^t \Phi(\mathbf{u})\,\mathrm{d}W,\boldsymbol{\phi}\right)_{L^2_x} = \sum_{j\ge 1}\int_0^t (\Phi(\mathbf{u})e_j,\boldsymbol{\phi})_{L^2_x}\,\mathrm{d}\beta_j \qquad \forall \boldsymbol{\phi}\in L^2(\mathcal{O},\mathbb{R}^2).$$

Similarly, we can define stochastic integrals with values in $W^{1,2}_{0,\mathrm{div}}(\mathcal{O},\mathbb{R}^2)$ and $W^{2,2}(\mathcal{O},\mathbb{R}^2)$, respectively, if $\mathbf{u}$ belongs to the corresponding class.

2.2 The Concept of Solutions

In dimension two, pathwise uniqueness for analytically weak solutions is known under the assumption (2.1); we refer the reader for instance to Capiński–Cutland [11], Capiński [10]. Consequently, we may work with the definition of a weak pathwise solution.

Definition 2.1 Let $(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\ge0},\mathbb{P})$ be a given stochastic basis with a complete right-continuous filtration and an $(\mathfrak{F}_t)$-cylindrical Wiener process W. Let $\mathbf{u}_0$ be an $\mathfrak{F}_0$-measurable random variable with values in $L^2_{\mathrm{div}}(\mathcal{O},\mathbb{R}^2)$. Then $\mathbf{u}$ is called a *weak pathwise solution* to (1.1) with the initial condition $\mathbf{u}_0$ provided

(a) the velocity field $\mathbf{u}$ is $(\mathfrak{F}_t)$-adapted and

$$\mathbf{u}\in C_{loc}([0,\infty);L^2_{\mathrm{div}}(\mathcal{O},\mathbb{R}^2))\cap L^2_{\mathrm{loc}}(0,\infty;W^{1,2}_{0,\mathrm{div}}(\mathcal{O},\mathbb{R}^2)) \quad \mathbb{P}\text{-a.s.},$$

(b) the momentum equation

$$\begin{aligned}
&\int_{\mathcal{O}}\mathbf{u}(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x-\int_{\mathcal{O}}\mathbf{u}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}x\\
&\quad=-\int_0^t\int_{\mathcal{O}}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s+\mu\int_0^t\int_{\mathcal{O}}\nabla\mathbf{u}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s\\
&\qquad+\int_0^t\int_{\mathcal{O}}\Phi(\mathbf{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}W
\end{aligned}$$

holds $\mathbb{P}$-a.s. for all $\boldsymbol{\varphi}\in W^{1,2}_{0,\mathrm{div}}(\mathcal{O},\mathbb{R}^2)$ and all $t\ge 0$.

Theorem 2.2 *Suppose that Φ satisfies (2.1). Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration and an $(\mathfrak{F}_t)$-cylindrical Wiener process W. Let $\mathbf{u}_0$ be an $\mathfrak{F}_0$-measurable random variable such that $\mathbf{u}_0 \in L^r(\Omega; L^2_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2))$ for some $r > 2$. Then there exists a unique weak pathwise solution to (1.1) in the sense of Definition 2.1 with the initial condition $\mathbf{u}_0$.*

We give the definition of a strong pathwise solution to (1.1) which exists up to a stopping time $\mathfrak{t}$, cf. [18, 22]. The velocity field here belongs $\mathbb{P}$-a.s. to $C([0, \mathfrak{t}]; W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2))$.

Definition 2.3 Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ be stochastic basis with a complete right-continuous filtration and an $(\mathfrak{F}_t)$-cylindrical Wiener process W. Let $\mathbf{u}_0$ be an $\mathfrak{F}_0$-measurable random variable with values in $W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$. The tuple $(\mathbf{u}, \mathfrak{t})$ is called a *strong pathwise solution* to (1.1) with the initial condition $\mathbf{u}_0$ provided

(a) $\mathfrak{t}$ is a $\mathbb{P}$-a.s. strictly positive $(\mathfrak{F}_t)$-stopping time;
(b) the velocity field $\mathbf{u}$ is $(\mathfrak{F}_t)$-adapted and

$$\mathbf{u}(\cdot \wedge \mathfrak{t}) \in C_{loc}([0, \infty); W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)) \cap L^2_{loc}(0, \infty; W^{2,2}(\mathcal{O}, \mathbb{R}^2)) \quad \mathbb{P}\text{-a.s.},$$

(c) the momentum equation

$$\begin{aligned} &\int_{\mathcal{O}} \mathbf{u}(t \wedge \mathfrak{t}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x - \int_{\mathcal{O}} \mathbf{u}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x \\ &\quad = - \int_0^{t\wedge \mathfrak{t}} \int_{\mathcal{O}} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}s + \mu \int_0^{t\wedge \mathfrak{t}} \int_{\mathcal{O}} \Delta \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}s \\ &\qquad + \int_{\mathcal{O}} \int_0^{t\wedge \mathfrak{t}} \Phi(\mathbf{u}) \cdot \boldsymbol{\varphi} \, \mathrm{d}W \, \mathrm{d}x \end{aligned} \tag{2.6}$$

holds $\mathbb{P}$-a.s. for all $\boldsymbol{\varphi} \in C^{\infty}_{c,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$ and all $t \geq 0$.

Note that (2.6) certainly implies the corresponding formulation in Definition 2.1. The reverse implication is only true for analytically strong solutions.

We finally define what a maximal strong pathwise solution is.

Definition 2.4 (Maximal strong pathwise solution) Fix a stochastic basis with a cylindrical Wiener process and an initial condition as in Definition 2.3. A triplet

$$(\mathbf{u}, (\mathfrak{t}_R)_{R\in\mathbb{N}}, \mathfrak{t})$$

is a maximal strong pathwise solution to system (1.1) provided

(a) $\mathfrak{t}$ is a $\mathbb{P}$-a.s. strictly positive $(\mathfrak{F}_t)$-stopping time;
(b) $(\mathfrak{t}_R)_{R\in\mathbb{N}}$ is an increasing sequence of $(\mathfrak{F}_t)$-stopping times such that $\mathfrak{t}_R < \mathfrak{t}$ on the set $[\mathfrak{t} < \infty]$, as well as $\lim_{R\to\infty} \mathfrak{t}_R = \mathfrak{t}$ $\mathbb{P}$-a.s., and

$$\mathfrak{t}_R := \inf \left\{ t \in [0, \infty) : \ \|\mathbf{u}(t)\|_{W^{1,2}_x} \geq R \right\} \quad \text{on} \quad [\mathfrak{t} < \infty], \tag{2.7}$$

with the convention that $\mathfrak{t}_R = \infty$ if the set above is empty;

(c) each tuple $(\mathbf{u}, \mathfrak{t}_R)$, for $R \in \mathbb{N}$, is a local strong pathwise solution in the sense of Definition 2.3.

We talk about a global solution if we have (in the framework of Definition 2.4) $\mathfrak{t} = \infty$ $\mathbb{P}$-a.s. Otherwise it is called a local solution. The following result concerning the existence of a global solution is shown in [24]; see also [18] for a similar statement. In the 3D case strong solutions are only known to exists locally, cf. [2, 8, 20].

Theorem 2.5 *Suppose that (2.1)–(2.3) hold and that $\mathbf{u}_0 \in L^2(\Omega, W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2))$. Then there is a unique global maximal strong pathwise solution to (1.1) in the sense of Definition 2.4.*

2.3 Finite Elements

We work with a standard finite element set-up for incompressible fluid mechanics; see e.g. [17]. We denote by $\mathscr{T}_h$ a quasi-uniform subdivision [6] of $\mathcal{O}$ into triangles of maximal diameter $h > 0$. For $K \subset \mathcal{O}$ and $\ell \in \mathbb{N}_0$ we denote by $\mathscr{P}_\ell(K)$ the polynomials on K of degree less than or equal to ℓ. Let us characterize the finite element spaces $V^h(\mathcal{O}, \mathbb{R}^2)$ and $P^h(\mathcal{O})$ as

$$V^{h,i}(\mathcal{O}, \mathbb{R}^2) := \{\mathbf{v}_h \in W^{1,2}_0(\mathcal{O}, \mathbb{R}^2) : \mathbf{v}_h|_K \in (\mathscr{P}_i(K))^2 \quad \forall K \in \mathscr{T}_h\}, \tag{2.8}$$

$$P^{h,j}(\mathcal{O}) := \{\pi_h \in L^2(\mathcal{O})/\mathbb{R} : \pi_h|_K \in \mathscr{P}_j(K) \quad \forall K \in \mathscr{T}_h\}, \tag{2.9}$$

where $i, j \geq 0$. In order to guarantee stability of our approximation we relate $V^h(\mathcal{O}, \mathbb{R}^2)$ and $P^h(\mathcal{O})$ by the discrete inf-sup condition, *i.e.*, there exists a positive constant C not depending on h such that

$$\sup_{\mathbf{v}_h \in V^{h,i}(\mathcal{O}, \mathbb{R}^2)} \frac{\int_{\mathcal{O}} \mathrm{div}\mathbf{v}_h \, \pi_h \, \mathrm{d}x}{\|\nabla \mathbf{v}_h\|_{L^2_x}} \geq C \, \|\pi_h\|_{L^2_x} \quad \forall \pi_h \in P^{h,j}(\mathcal{O}) \,. \tag{2.10}$$

A well-known class of inf-sup stable pairings are the 'conforming Stokes elements', with the simplest choice $i = 2$ in (2.8) and $j = 0$ in (2.9); see *e.g.* [7, Ch. 6] or [19, Rem. 3.4] for further admissible examples of pairings.

We define the space of discretely solenoidal finite element functions by

$$V^{h,i}_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2) := \left\{ \mathbf{v}_h \in V^{h,i}(\mathcal{O}, \mathbb{R}^2) : \int_{\mathcal{O}} \mathrm{div}\mathbf{v}_h \, \pi_h \, \mathrm{d}x = 0 \quad \forall \pi_h \in P^{h,j}(\mathcal{O}) \right\}.$$

Let $\Pi_h : L^2(\mathcal{O}, \mathbb{R}^2) \to V^{h,i}_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$ be the $L^2(\mathcal{O}, \mathbb{R}^2)$-orthogonal projection onto $V^{h,i}_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$. The following results concerning the approximability of Π_h are well-known (see, for instance [19, Lemma 4.3]): there is $c > 0$ independent of h such that we have

$$\|\mathbf{v} - \Pi_h \mathbf{v}\|_{L^2_x} + h \|\nabla \mathbf{v} - \nabla \Pi_h \mathbf{v}\|_{L^2_x} \leq c \, h \|\nabla \mathbf{v}\|_{L^2_x} \tag{2.11}$$

for all $\mathbf{v} \in W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$; moreover, the arguments in [19, Section 4] together with standard interpolations arguments (see e.g. [17, Lemma A.2]) also imply for $\beta \in (0, 1]$ that

$$\|\mathbf{v} - \Pi_h \mathbf{v}\|_{L^2_x} + h\|\nabla \mathbf{v} - \nabla \Pi_h \mathbf{v}\|_{L^2_x} \le c\, h^{1+\beta} \|\mathbf{v}\|_{W^{1+\beta,2}_x} \tag{2.12}$$

for all $\mathbf{v} \in W^{1+\beta,2} \cap W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$. Similarly, if $\Pi^\pi_h : L^2(\mathcal{O})/\mathbb{R} \to P^{h,j}(\mathcal{O})$ denotes the $L^2(\mathcal{O})$-orthogonal projection onto $P^{h,j}(\mathcal{O})$, we have

$$\|p - \Pi^\pi_h p\|^2_{L^2_x} \le ch\, \|\nabla p\|_{L^2_x} \tag{2.13}$$

for all $p \in W^{1,2}(\mathcal{O})/\mathbb{R}$.

3 Regularity of Solutions

In this section we analyse the regularity of the continuous solution as well as the associated pressure function. For various purposes we need the Helmholtz-projection $\mathcal{P} : L^p(\mathcal{O}, \mathbb{R}^2) \to L^2_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$, for $1 < p < \infty$, given by

$$\mathcal{P}\boldsymbol{\phi} := \boldsymbol{\phi} - \nabla \Delta^{-1}_{\mathcal{O}} \mathrm{div}\boldsymbol{\phi}. \tag{3.1}$$

Here $\Delta^{-1}_{\mathcal{O}}\mathrm{div}$ is the solution operator to the equation

$$\Delta \mathfrak{h} = \mathrm{div}\mathbf{g} \quad \text{in} \quad \mathcal{O}, \quad \nu_{\mathcal{O}} \cdot (\nabla \mathfrak{h} - \mathbf{g}) = 0 \quad \text{on} \quad \partial\mathcal{O},$$

where $\nu_{\mathcal{O}}$ denotes the unit normal of $\partial\mathcal{O}$. Note that $\nabla \Delta^{-1}_{\mathcal{O}}\mathrm{div}$ satisfies (since $\partial\mathcal{O}$ was assumed to be sufficiently smooth)

$$\nabla \Delta^{-1}_{\mathcal{O}}\mathrm{div} : W^{r,p}(\mathcal{O}, \mathbb{R}^2) \to W^{r,p}(\mathcal{O}, \mathbb{R}^2), \tag{3.2}$$

for all $p \in (1, \infty)$ and all $r \in \mathbb{N}$, where $W^{0,p}(\mathcal{O}, \mathbb{R}^2) = L^p(\mathcal{O}, \mathbb{R}^2)$; see [1] for the case $r \in \mathbb{N}$ and [15, Chapter IV] for the case $r = 0$. Clearly, (3.2) transfers to $\mathcal{P}$.

With the help of the Helmholtz projection we can define the Stokes operator as

$$\mathcal{A} := \mathcal{P}\Delta : W^{2,p} \cap W^{1,p}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2) \to L^p_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2). \tag{3.3}$$

Due to well-known estimates for the Stokes system there is $c > 0$ such that

$$\|\mathbf{u}\|_{W^{r+2,p}_x} \le c\, \|\mathcal{A}\mathbf{u}\|_{W^{r,p}_x}, \quad \mathbf{u} \in W^{r+2,p} \cap W^{1,p}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2), \tag{3.4}$$

for all $p \in (1, \infty)$ and all $r \in \mathbb{N}_0$, see, e.g., [15, Thm. IV. 6.1.], which uses sufficient smoothness of $\partial\mathcal{O}$. Moreover, there is a system of eigenfunctions to the Stokes operator $(\mathbf{u}_k) \subset W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$ with strictly positive eigenvalues (λ_k) such that $\lambda_k \to \infty$ as $k \to \infty$. It is possible to choose the $\mathbf{u}_k$'s such that the system $(\mathbf{u}_k)$ is orthonormal in

$L^2(\mathcal{O}, \mathbb{R}^2)$ and orthogonal in $W_0^{1,2}(\mathcal{O}, \mathbb{R}^2)$. Finally, we can assume that the $\mathbf{u}_k$'s are sufficiently smooth due to the assumed smoothness of $\partial\mathcal{O}$. Since $\mathcal{A}$ is positive, its root $\mathcal{A}^{1/2}$ is well-defined with domain $W_{0,\mathrm{div}}^{1,p}(\Omega, \mathbb{R}^2)$, and we have

$$\|\nabla \mathbf{u}\|_{L_x^p} \leq c \big\|\mathcal{A}^{1/2}\mathbf{u}\big\|_{L_x^p} \leq C\|\nabla \mathbf{u}\|_{L_x^p}, \quad \mathbf{u} \in W_{0,\mathrm{div}}^{1,p}(\mathcal{O}, \mathbb{R}^2), \tag{3.5}$$

$$\int_{\mathcal{O}} \mathcal{A}^{1/2}\mathbf{u} \cdot \mathbf{w}\,\mathrm{d}x = \int_{\mathcal{O}} \mathbf{u} \cdot \mathcal{A}^{1/2}\mathbf{w}\,\mathrm{d}x, \quad \mathbf{u} \in W_{0,\mathrm{div}}^{1,p}(\mathcal{O}, \mathbb{R}^2),\ \ \mathbf{w} \in W_{0,\mathrm{div}}^{1,p'}(\mathcal{O}, \mathbb{R}^2), \tag{3.6}$$

where $c, C > 0$; cf. [16].

3.1 Estimates for the Continuous Solution

In this section we derive crucial estimates for the maximal strong pathwise solution from Definition 2.4, which hold up to the stopping time $\mathfrak{t}_R$. Here $R > 0$ is a fixed truncation parameter and $T > 0$ an arbitrary but fixed time.

Lemma 3.1 *Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ be a given stochastic basis with a complete right-continuous filtration and an $(\mathfrak{F}_t)$-cylindrical Wiener process W.*

(a) Assume that $\mathbf{u}_0 \in L^r(\Omega, L^2_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2))$ for some $r \geq 2$ and that Φ satisfies (2.1). Then we have

$$\mathbb{E}\Bigg[\bigg(\sup_{0\leq t\leq T}\|\mathbf{u}(t)\|^2_{L^2_x} + \int_0^T \|\nabla \mathbf{u}\|^2_{L^2_x}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\Bigg] \leq c\,\mathbb{E}\Big[1 + \|\mathbf{u}_0\|^r_{L^2_x}\Big], \tag{3.7}$$

where $\mathbf{u}$ is the weak pathwise solution to (1.1); cf. Definition 2.1.

(b) Assume that $\mathbf{u}_0 \in L^r(\Omega, W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2))$ for some $r \geq 2$ and that Φ satisfies (2.1)–(2.3). Then we have

$$\begin{aligned}\mathbb{E}\Bigg[\bigg(\sup_{0\leq t\leq T}\|\mathbf{u}(t\wedge \mathfrak{t}_R)\|^2_{W^{1,2}_x} + \int_0^{T\wedge \mathfrak{t}_R} \|\mathbf{u}\|^2_{W^{2,2}_x}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\Bigg]& \\ \leq cR^{3r}\,\mathbb{E}\Big[1 + \|\mathbf{u}_0\|^r_{W^{1,2}_x}\Big]&,\end{aligned} \tag{3.8}$$

where $(\mathbf{u}, (\mathfrak{t}_R)_{R\in\mathbb{N}}, \mathfrak{t})$ is the maximal strong pathwise solution to (1.1); cf. Definition 2.4.

(c) Assume that $\mathbf{u}_0 \in L^r(\Omega, W^{2,2}(\mathcal{O}, \mathbb{R}^2)) \cap L^{5r}(\Omega, W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2))$ for some $r \geq 2$, we have $\mathcal{A}\mathbf{u}_0 - \mathcal{P}(\mathbf{u}_0 \cdot \nabla \mathbf{u}_0)|_{\partial\mathcal{O}} = 0$ $\mathbb{P}$-a.s. and that (2.1)–(2.5) holds. Then we have for all $\beta < 1$

$$\begin{aligned}\mathbb{E}\Bigg[\bigg(\sup_{0\leq t\leq T}\|\mathbf{u}(t\wedge \mathfrak{t}_R)\|^2_{W^{1+\beta}_x} + \int_0^{T\wedge \mathfrak{t}_R} \|\mathbf{u}\|^2_{W^{2+\beta,2}_x}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\Bigg]& \\ \leq cR^{5r}\,\mathbb{E}\Big[1 + \|\mathbf{u}_0\|^r_{W^{2,2}_x} + \|\mathbf{u}_0\|^{2r}_{W^{1,2}_x}\Big]&,\end{aligned} \tag{3.9}$$

where $(\mathbf{u}, (\mathfrak{t}_R)_{R\in\mathbb{N}}, \mathfrak{t})$ is the maximal strong pathwise solution to (1.1); cf. Definition 2.4.

Here $c = c(r, T, \beta) > 0$ is independent of R.

Proof **Part (a)** is the standard a priori estimate, which is a consequence of applying Itô's formula to $t \mapsto \|\mathbf{u}\|^2_{L^2_x}$.

For **part (b)** we follow [24], where the solution to a truncated problem is considered. For $R > 1$ and $\zeta \in C_c^\infty([0,1))$ with $0 \leq \zeta \leq 1$ and $\zeta = 1$ in $[0,1]$ we set $\zeta_R := \zeta(R^{-1}\cdot)$. Similar to Definition 2.1 we seek an $(\mathfrak{F}_t)$-adapted stochastic process $\mathbf{u}^R$ with

$$\mathbf{u}^R \in C([0,T]; L^2_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)) \cap L^2(0,T; W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)) \quad \mathbb{P}\text{-a.s.}$$

such that

$$\begin{aligned} \int_{\mathcal{O}} \mathbf{u}^R(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x &= \int_{\mathcal{O}} \mathbf{u}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_0^t \zeta_R(\|\nabla\mathbf{u}^R\|_{L^2_x})\int_{\mathcal{O}} \mathbf{u}^R\otimes\mathbf{u}^R : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \mu\int_0^t\int_{\mathcal{O}} \nabla\mathbf{u}^R : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathcal{O}} \Phi(\mathbf{u}^R)\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}W \end{aligned} \tag{3.10}$$

holds $\mathbb{P}$-a.s. for all $\boldsymbol{\varphi} \in W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$ and all $t \in [0,T]$. Arguing as in [24, Lemma 3.7] one can show that a unique global strong pathwise solution to (3.10) exists in the class $C([0,T]; W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2))$,[1] and that it satisfies

$$\mathbb{E}\bigg[\sup_{0\leq t\leq T}\|\nabla\mathbf{u}^R(t)\|^2_{L^2_x}\,\mathrm{d}x + \int_0^T \|\nabla^2\mathbf{u}^R\|^2_{L^2_x}\,\mathrm{d}t\bigg] \leq c(r, R, T). \tag{3.11}$$

The proof of (3.11) in [24] is based on a Galerkin approximation which we mimick now in order to prove (3.8) and (3.9).

1) Galerkin approximation. Let $(\mathbf{u}_k) \subset W^{1,2}_{0,\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$ be a system of eigenfunctions to the Stokes operator, cf. (3.3). For $N \in \mathbb{N}$ let $\mathbb{H}^N := \mathrm{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_N\}$, and consider the unique solution $\mathbf{u}^{R,N}$ to

$$\begin{aligned} \int_{\mathcal{O}} \mathbf{u}^{R,N}(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x &= \int_{\mathcal{O}} \mathbf{u}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}x - \mu\int_0^t\int_{\mathcal{O}} \nabla\mathbf{u}^{R,N} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t \zeta_R(\|\nabla\mathbf{u}^{R,N}\|_{L^2_x})\int_{\mathcal{O}} \mathbf{u}^{R,N}\otimes\mathbf{u}^{R,N} : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathcal{O}} \Phi(\mathbf{u}^{R,N})\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}W \end{aligned} \tag{3.12}$$

for all $\boldsymbol{\phi} \in \mathbb{H}^N$. By $\mathcal{P}_N$ we denote the $L^2(\mathcal{O}, \mathbb{R}^2)$-projection onto $\mathbb{H}^N$. Problem (3.12) can be written as a system of SDEs with Lipschitz-continuous coefficients. Hence it is clear that there is a unique strong solution, i.e., an $(\mathfrak{F}_t)$-adapted process defined on

[1] Different from the solution obtained in Theorem 2.5 it can be constructed for any given deterministic $T > 0$.

$(\Omega, \mathfrak{F}, \mathbb{P})$ with values in $C([0,T];\mathbb{H}^N)$ and moments of order r. Arguing as in [24, Prop. 3.2] one can prove that as $N \to \infty$

$$\sup_{0\leq t\leq T} \|\mathbf{u}^R(t) - \mathbf{u}^{R,N}(t)\|_{L_x^2}^2 + \int_0^T \|\nabla(\mathbf{u}^R - \mathbf{u}^{R,N})\|_{L_x^2}^2 \,\mathrm{d}x\,\mathrm{d}t \to 0 \tag{3.13}$$

in probability. Applying Itô's formula to $t \mapsto \|\mathbf{u}^{R,N}\|_{L_x^2}^2$ and using the cancellation of the convective term one can prove for $r \geq 2$

$$\mathbb{E}\bigg[\bigg(\sup_{0\leq t\leq T} \|\mathbf{u}^{R,N}(t)\|_{L_x^2}^2 + \int_0^T \|\nabla\mathbf{u}^{R,N}\|_{L_x^2}^2 \,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg] \leq c\,\mathbb{E}\Big[1 + \|\mathbf{u}_0\|_{L_x^2}^r\Big], \tag{3.14}$$

where $c = c(r,T)$ is independent of N and R.

2) Proof of (3.8). By construction we have $\mathcal{A}\mathbf{u}^{R,N} \in C([0,T];\mathbb{H}^N)$ $\mathbb{P}$-a.s. such that we can apply Itô's formula to $t \mapsto (\mathbf{u}^{R,N}(t), \mathcal{A}\mathbf{u}^{R,N}(t))_{L_x^2}$ and use (3.12). This yields using $\mathbf{u}^{R,N}|_{\partial\mathcal{O}} = 0$

$$\begin{aligned}
\|\nabla\mathbf{u}^{R,N}(t)\|_{L_x^2}^2 &= -\big(\mathbf{u}^{R,N}(t), \Delta\mathbf{u}^{R,N}(t)\big)_{L_x^2} = -\big(\mathbf{u}^{R,N}(t), \mathcal{A}\mathbf{u}^{R,N}(t)\big)_{L_x^2}\\
&= \|\mathcal{P}_N\nabla\mathbf{u}_0\|_{\frac{2}{x}}^2 + 2\int_0^t \zeta_R(\|\nabla\mathbf{u}^{R,N}\|_{L_x^2})\big((\mathbf{u}^{R,N}\cdot\nabla)\mathbf{u}^{R,N}, \mathcal{A}\mathbf{u}^{R,N}\big)_{L_x^2}\,\mathrm{d}s\\
&\quad - 2\mu\int_0^t \|\mathcal{A}\mathbf{u}^{R,N}\|_{L_x^2}^2\,\mathrm{d}s + 2\sum_{k=1}^N \int_0^t \big(\Phi(\mathbf{u}^{R,N})e_k, \mathcal{A}\mathbf{u}^{R,N}\big)_{L_x^2}\,\mathrm{d}\beta_k\\
&\quad + \sum_{k=1}^N \lambda_k \int_0^t \big(\Phi(\mathbf{u}^{R,N})e_k, \mathbf{u}_k\big)_{L_x^2}^2\,\mathrm{d}s\\
&=: \mathrm{I}^N(t) + \cdots + \mathrm{V}^N(t)
\end{aligned} \tag{3.15}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. We estimate now the terms II^N, IV^N and V^N. First of all, we have by definition of ζ_R

$$\begin{aligned}
\mathrm{II}^N(t) &\leq 2\int_0^t \zeta_R(\|\nabla\mathbf{u}^{R,N}\|_{L_x^2})\|\mathbf{u}^{R,N}\|_{L_x^4}\|\nabla\mathbf{u}^{R,N}\|_{L_x^4}\|\mathcal{A}\mathbf{u}^{R,N}\|_{L_x^2}\mathrm{d}s\\
&\leq 2\int_0^t \zeta_R(\|\nabla\mathbf{u}^{R,N}\|_{L_x^2})\|\mathbf{u}^{R,N}\|_{L_x^2}^{\frac{1}{2}}\|\nabla\mathbf{u}^{R,N}\|_{L_x^2}\|\mathcal{A}\mathbf{u}^{R,N}\|_{L_x^2}^{\frac{3}{2}}\mathrm{d}s\\
&\leq cR^{3/2}\int_0^t \|\mathcal{A}\mathbf{u}^{R,N}\|_{L_x^2}^{\frac{3}{2}}\mathrm{d}s \leq \delta\int_0^t \|\mathcal{A}\mathbf{u}^{R,N}\|_{L_x^2}^2\mathrm{d}s + c(\delta)R^6,
\end{aligned}$$

where $\delta > 0$ is arbitrary. Moreover, we obtain by definition of $\mathbf{u}_k$ and using (3.6) (and recalling that $\Phi(\mathbf{u}^{R,N})e_k \in W_{0,\mathrm{div}}^{1,2}(\mathcal{O},\mathbb{R}^2)$ for all $k \in \mathbb{N}$ by assumption)

$$\begin{aligned}\mathrm{V}^N(t) &= \sum_{k=1}^N \int_0^t \big(\Phi(\mathbf{u}^{R,N})e_k, \sqrt{\lambda_k}\mathbf{u}_k\big)_{L_x^2}^2 \,\mathrm{d}s = \sum_{k=1}^N \int_0^t \big(\Phi(\mathbf{u}^{R,N})e_k, \mathcal{A}^{1/2}\mathbf{u}_k\big)^2 \,\mathrm{d}s \\ &= \sum_{k=1}^N \int_0^t \big(\mathcal{A}^{1/2}\Phi(\mathbf{u}^{R,N})e_k, \mathbf{u}_k\big)_{L_x^2}^2 \,\mathrm{d}s.\end{aligned}$$

Furthermore, since $\|\mathbf{u}_k\|_{L_x^2} = 1$,

$$\begin{aligned}\mathrm{V}^N(t) &\le \sum_{k\ge 1} \int_0^t \|\mathcal{A}^{1/2}\Phi(\mathbf{u}^{R,N})e_k\|_{L_x^2}^2 \|\mathbf{u}_k\|_{L_x^2}^2 \,\mathrm{d}s \le c \sum_{k\ge 1} \int_0^t \|\nabla\Phi(\mathbf{u}^{R,N})e_k\|_{L_x^2}^2 \,\mathrm{d}s \\ &= c \int_0^t \|\Phi(\mathbf{u}^{R,N})\|_{L_2(\mathfrak{U};W_x^{1,2})}^2 \,\mathrm{d}s \le c \int_0^t \big(1 + \|\mathbf{u}^{R,N}\|_{W_x^{1,2}}^2\big) \,\mathrm{d}s,\end{aligned}$$

using (2.2) in the last step. The expectation of the right-hand side is bounded by (3.14). Finally, by Burkholder-Davis-Gundy inequality and (2.1),

$$\begin{aligned}&\mathbb{E}\Bigg[\bigg(\sup_{0\le t\le T} |\mathrm{IV}(t)|\bigg)^{\frac{r}{2}}\Bigg] \\ &\le \mathbb{E}\Bigg[\bigg(\sup_{0\le t\le T}\bigg|\int_0^t \sum_{k=1}^N \big(\Phi(\cdot,\mathbf{u}^{R,N})e_k, \mathcal{A}\mathbf{u}^{R,N}\big)_{L_x^2} \,\mathrm{d}\beta_k\bigg|\bigg)^{\frac{r}{2}}\Bigg] \\ &\le c\,\mathbb{E}\Bigg[\bigg(\sum_{k\ge 1}\int_0^T \big(\Phi(\cdot,\mathbf{u}^{R,N})e_k \cdot \mathcal{A}\mathbf{u}^{R,N}\big)_{L_x^2}^2 \,\mathrm{d}t\bigg)^{\frac{r}{4}}\Bigg] \\ &\le c\,\mathbb{E}\Bigg[\bigg(\sum_{k\ge 1}\int_0^T \|\Phi_k(\mathbf{u}^{R,N})e_k\|_{L_x^2}^2 \|\mathcal{A}\mathbf{u}^{R,N}\|_{L_x^2}^2 \,\mathrm{d}t\bigg)^{\frac{r}{4}}\Bigg] \\ &\le c\,\mathbb{E}\Bigg[\bigg(\int_0^T \big(1 + \|\mathbf{u}^{R,N}\|_{L_x^2}^2\big)\|\mathcal{A}\mathbf{u}^{R,N}\|_{L_x^2}^2 \,\mathrm{d}t\bigg)^{\frac{r}{4}}\Bigg] \\ &\le c(\delta)\,\mathbb{E}\Bigg[\bigg(1 + \sup_{0\le t\le T}\|\mathbf{u}^{R,N}\|_{L_x^2}^2\bigg)^{\frac{r}{2}}\Bigg] + \delta\,\mathbb{E}\Bigg[\bigg(\int_0^T \|\mathcal{A}\mathbf{u}^{R,N}\|_{L_x^2}^2 \,\mathrm{d}t\bigg)^{\frac{r}{2}}\Bigg] \\ &\le c(\delta) + \delta\,\mathbb{E}\Bigg[\bigg(\int_0^T \|\mathcal{A}\mathbf{u}^{R,N}\|_{L_x^2}^2 \,\mathrm{d}t\bigg)^{\frac{r}{2}}\Bigg]\end{aligned}$$

using (3.14), where again $\delta > 0$ is arbitrary. Choosing δ small enough and using (3.4) we conclude that

$$\begin{aligned}\mathbb{E}\Bigg[\bigg(\sup_{0\le t\le T}\int_{\mathcal{O}} \|\nabla\mathbf{u}^{R,N}(t)\|_{L_x^2}^2 + \int_0^T \|\nabla^2\mathbf{u}^{R,N}\|_{L_x^2}^2 \,\mathrm{d}t\bigg)^{\frac{r}{2}}\Bigg] \\ \le cR^{3r}\,\mathbb{E}\Big[1 + \|\mathbf{u}_0\|_{W_x^{1,2}}^r\Big],\end{aligned} \tag{3.16}$$

uniformly in N. This implies that $(\mathbf{u}^{R,N})_{N\in\mathbb{N}}$ is a bounded sequence in the function space generated by the left-hand side of (3.16). After taking a subsequence we obtain a limit object $\mathbf{u}^R$ which is the unique global strong solution to (3.10) recalling (3.13). Furthermore, we can pass to the limit $N\to\infty$ and obtain a corresponding estimate for $\mathbf{u}^R$ due to lower semi-continuity of the involved functionals. Since $\mathbf{u}^R(\cdot\wedge\mathfrak{t}_R)=\mathbf{u}(\cdot\wedge\mathfrak{t}_R)$ we obtain (3.8).

3) Proof of (3.9). The verification of **part (c)** proceeds in two steps. In the first step we show an improved version of (3.16). Applying Itô's formula to the mapping

$$t\mapsto\|\nabla\mathbf{u}^{R,N}(t)\|_{L^2_x}^2\big(\mathbf{u}^{R,N}(t),\mathcal{A}\mathbf{u}^{R,N}(t)\big)_{L^2_x},$$

equation (3.15) yields

$$\begin{aligned}\|\nabla\mathbf{u}^{R,N}(t)\|_{L^2_x}^4&=\|\mathcal{P}_N\nabla\mathbf{u}_0\|_{L^2_x}^4-4\mu\int_0^t\|\nabla\mathbf{u}^{R,N}\|_{L^2_x}^2\|\mathcal{A}\mathbf{u}^{R,N}\|_{L^2_x}^2\,\mathrm{d}s\\&\quad+4\int_0^t\zeta_R(\|\nabla\mathbf{u}^{R,N}\|_{L^2_x})\|\nabla\mathbf{u}^{R,N}\|_{L^2_x}^2\big((\mathbf{u}^{R,N}\cdot\nabla)\mathbf{u}^{R,N},\mathcal{A}\mathbf{u}^{R,N}\big)_{L^2_x}\,\mathrm{d}s\\&\quad+4\sum_{k=1}^N\int_0^t\|\nabla\mathbf{u}^{R,N}\|_{L^2_x}^2\big(\Phi(\mathbf{u}^{R,N})e_k,\mathcal{A}\mathbf{u}^{R,N}\big)_{L^2_x}\,\mathrm{d}\beta_k\\&\quad+2\sum_{k=1}^N\lambda_k\int_0^t\|\nabla\mathbf{u}^{R,N}\|_{L^2_x}^2\big(\Phi(\mathbf{u}^{R,N})e_k,\mathbf{u}_k\big)_{L^2_x}^2\,\mathrm{d}s\\&\quad+2\sum_{k=1}^N\int_0^t\big(\Phi(\cdot,\mathbf{u}^{R,N})e_k,\mathcal{A}\mathbf{u}^{R,N}\big)_{L^2_x}^2\,\mathrm{d}t.\end{aligned}$$

Following now step by step the arguments from the proof of (3.16) above we arrive at

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\sup_{0\le t\le T}\|\nabla\mathbf{u}^{R,N}(t)\|_{L^2_x}^4+\int_0^T\|\nabla\mathbf{u}^{R,N}\|_{L^2_x}^2\|\nabla^2\mathbf{u}^{R,N}\|_{L^2_x}^2\,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg]\\\le cR^{3r}\,\mathbb{E}\Big[1+\|\mathbf{u}_0\|_{W^{1,2}_x}^{2r}\Big].\end{aligned}$$

Again we can pass to the limit in N obtaining

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\sup_{0\le t\le T}\|\nabla\mathbf{u}^{R}(t)\|_{L^2_x}^4+\int_0^T\|\nabla\mathbf{u}^{R}\|_{L^2_x}^2\|\nabla^2\mathbf{u}^{R}\|_{L^2_x}^2\,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg]\\\le cR^{3r}\,\mathbb{E}\Big[1+\|\mathbf{u}_0\|_{W^{1,2}_x}^{2r}\Big].\end{aligned}\tag{3.17}$$

Now we turn to the proof of (3.9) stated in **part (c)** for which we use the mild formulation of (3.10).

($\mathbf{c}_1$) Due to the regularity proved in (3.16) and (3.17), [25, Proposition F.0.5, (i)] applies and we can write

$$\mathbf{u}^R(t) = e^{-t\mathcal{A}}\mathbf{u}_0 + \int_0^t e^{-(t-s)\mathcal{A}}\mathbf{g}_R \,\mathrm{d}s + \int_0^t e^{-(t-s)\mathcal{A}}\Phi(\mathbf{u}^R)\,\mathrm{d}W,$$
$$\text{where} \quad \mathbf{g}_R := \zeta_R(\|\nabla \mathbf{u}^R\|_{L^2_x})\mathcal{P}[(\mathbf{u}^R \cdot \nabla)\mathbf{u}^R].$$

Here $(e^{-t\mathcal{A}})_{t\geq 0}$ denotes the analytic semigroup on $L^2_{\mathrm{div}}(\mathcal{O}, \mathbb{R}^2)$ generated by the Stokes operator $\mathcal{A}$. Setting

$$\mathbf{Y}^R(t) := e^{-t\mathcal{A}}\mathbf{u}_0 + \int_0^t e^{-(t-s)\mathcal{A}}\mathbf{g}_R \,\mathrm{d}s,$$
$$\mathbf{Z}^R(t) := \int_0^t e^{-(t-s)\mathcal{A}}\Phi(\mathbf{u}^R)\,\mathrm{d}W,$$

we consider now the deterministic and stochastic contribution separately. We note that $\mathbf{Y}^R$ is the unique solution to a deterministic Stokes problem with initial datum $\mathbf{u}_0$ and forcing $\mathbf{g}_R$, whereas $\mathbf{Z}^R$ solves a stochastic Stokes problem with homogeneous initial datum and diffusion coefficient $\Phi(\mathbf{u}^R)$ – both equipped with homogeneous Dirichlet boundary conditions.

By Ladyshenskaya's inequality we have

$$\begin{aligned}
\mathbb{E}\bigg[\bigg(\int_0^T \|\mathbf{g}_R\|^2_{L^2_x}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg] &\leq \mathbb{E}\bigg[\bigg(\int_0^T \|\mathbf{u}^R\|^2_{L^4_x}\|\nabla\mathbf{u}^R\|^2_{L^4_x}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg]\\
&\leq c\,\mathbb{E}\bigg[\bigg(\int_0^T \|\mathbf{u}^R\|_{L^2_x}\|\nabla\mathbf{u}^R\|^2_{L^2_x}\|\nabla^2\mathbf{u}^R\|_{L^2_x}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg]\\
&\leq cR^{3r}\,\mathbb{E}\Big[1 + \|\mathbf{u}_0\|^{2r}_{W^{1,2}_x}\Big],
\end{aligned}$$

where we used (3.17) in the last step.

($\mathbf{c}_2$) Interpolating $W^{1/2,2}(0, T; W^{1,2}(\mathcal{O}, \mathbb{R}^2))$ between $W^{1,2}(0, T; L^2(\mathcal{O}, \mathbb{R}^2))$ and $L^2(0, T; W^{2,2}(\mathcal{O}, \mathbb{R}^2))$ and applying $\mathbb{P}$-a.s. classical estimates for the Stokes system yields

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\|\mathbf{Y}^R\|^2_{W^{1/2}(0,T;W^{1,2}_x)}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg]\\
&\quad\leq \mathbb{E}\bigg[\bigg(\|\mathbf{Y}^R\|^2_{W^{1,2}(0,T;L^2_x)} + \|\mathbf{Y}^R\|^2_{L^2(0,T;W^{2,2}_x)}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg]\\
&\quad\leq c\,\mathbb{E}\bigg[\bigg(\int_0^T \|\mathbf{g}_R\|^2_{L^2_x}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg] \leq cR^{3r}\,\mathbb{E}\Big[1 + \|\mathbf{u}_0\|^{2r}_{W^{1,2}_x}\Big].
\end{aligned} \tag{3.18}$$

($\mathbf{c}_3$) For $\mathbf{Z}^R$ we apply the recent results from [30, Theorems 25 and 28] proving for any $\sigma < 1$

$$
\begin{aligned}
&\mathbb{E}\bigg[\|\mathbf{Z}^R\|^2_{C^{\sigma/2}([0,T];L^2_x)} + \|\mathbf{Z}^R\|^2_{W^{\sigma/2,2}(0,T;W^{1,2}_x)}\bigg] \\
&\quad \le c\,\mathbb{E}\bigg[1 + \sup_{0\le t\le T}\|\mathbf{u}^R\|^4_{W^{1,2}_x}\bigg] \\
&\quad \le c\,\mathbb{E}\bigg[1 + \|\mathbf{u}_0\|^4_{W^{1,2}_x}\bigg]
\end{aligned}
\tag{3.19}
$$

using also (2.2) and (3.17). Combining (3.18) and (3.19) and recalling that $\mathbf{u}^R$ is the sum of $\mathbf{Y}^R$ and $\mathbf{Z}^R$ gives

$$
\mathbb{E}\bigg[\|\mathbf{u}^R\|^2_{C^{\sigma/2}([0,T];L^2_x)} + \|\mathbf{u}^R\|^2_{W^{\sigma/2,2}(0,T;W^{1,2}_x)}\bigg] \le c\,\mathbb{E}\bigg[1 + \|\mathbf{u}_0\|^4_{W^{1,2}_x}\bigg]. \tag{3.20}
$$

($\mathbf{c}_4$) Due to our assumption on the noise from (2.4) we know that $\Phi(\mathbf{u}^R)e_k$, with $k \in \mathbb{N}$, belongs to the domain of the Stokes operator such that we can write

$$
\mathcal{A}\mathbf{Z}^R(t) = \int_0^t e^{-(t-s)\mathcal{A}}\mathcal{A}\Phi(\mathbf{u}^R)\,\mathrm{d}W.
$$

We conclude that $\mathcal{A}\mathbf{Z}^R$ is the unique weak pathwise solution to the stochastic Stokes problem with zero initial datum, homogeneous boundary conditions and diffusion coefficient $\mathcal{A}\Phi(\mathbf{u}^R)$. It is standard to derive for $r \ge 2$ the estimate

$$
\begin{aligned}
&\mathbb{E}\bigg[\bigg(\sup_{0\le t\le T}\|\mathcal{A}\mathbf{Z}^R\|^2_{L^2_x} + \int_0^T \|\nabla\mathcal{A}\mathbf{Z}^R\|^2_{L^2_x}\,\mathrm{d}s\bigg)^{\frac{r}{2}}\bigg] \\
&\quad \le c\,\mathbb{E}\bigg[\bigg(\int_0^T \|\mathcal{A}\Phi(\mathbf{u}^R)\|^2_{L_2(\mathfrak{U};L^2_x)}\,\mathrm{d}s\bigg)^{\frac{r}{2}}\bigg] \\
&\quad \le c\,\mathbb{E}\bigg[\bigg(\int_0^T \|\Phi(\mathbf{u}^R)\|^2_{L_2(\mathfrak{U};W^{2,2}_x)}\,\mathrm{d}s\bigg)^{\frac{r}{2}}\bigg] \\
&\quad \le c\bigg[\bigg(\int_0^t \big(1 + \|\mathbf{u}^R\|^2_{W^{1,2}_x}\|\mathbf{u}^R\|^2_{W^{2,2}_x} + \|\mathbf{u}^R\|^2_{W^{2,2}_x}\big)\,\mathrm{d}s\bigg)^{\frac{r}{2}}\bigg],
\end{aligned}
$$

applying Itô's formula to $t \mapsto \|\mathcal{A}\mathbf{u}^R\|^2_{L^2_x}$ and using Burkholder-Davis-Gundy inequality (and (2.4) in the last step). The properties of the Stokes operator from (3.4) yield

$$
\begin{aligned}
&\mathbb{E}\bigg[\bigg(\sup_{0\le t\le T}\|\mathbf{Z}^R\|^2_{W^{2,2}_x} + \int_0^T \|\mathbf{Z}^R\|^2_{W^{3,2}_x}\,\mathrm{d}s\bigg)^{\frac{r}{2}}\bigg] \\
&\quad \le c\bigg[\bigg(\int_0^t \big(1 + \|\mathbf{u}^R\|^2_{W^{1,2}_x}\|\mathbf{u}^R\|^2_{W^{2,2}_x} + \|\mathbf{u}^R\|^2_{W^{2,2}_x}\big)\,\mathrm{d}s\bigg)^{\frac{r}{2}}\bigg].
\end{aligned}
$$

($\mathbf{c}_5$) To sharpen the estimates for $\mathbf{Y}^R$ is slightly more involved as the convective term $\mathbf{g}_R$ does not lie in the domain of the Stokes operator since it does not necessarily have a zero trace. We can choose $p < 2$ such that the embedding $W^{1,p}(\mathcal{O}) \hookrightarrow W^{\sigma,2}(\mathcal{O})$ holds. We obtain by continuity of $\mathcal{P}$, cf. (3.2),

$$\begin{aligned}\|\mathbf{g}_R\|_{W^{\sigma,2}_x} &\le c\,\|\mathbf{g}_R\|_{W^{1,p}_x} \le c\|(\mathbf{u}^R\cdot\nabla)\mathbf{u}^R\|_{W^{1,p}_x}\\ &\le c\|\nabla\mathbf{u}^R\|^2_{L^{2p}_x} + c\|\mathbf{u}^R\|_{L^q_x}\|\nabla^2\mathbf{u}_R\|_{L^2_x} \le c\|\nabla\mathbf{u}^R\|_{L^2_x}\|\nabla^2\mathbf{u}_R\|_{L^2_x},\end{aligned}$$

where we used Hölder's inequality with exponents $2/p$ and $q := 2/(2-p)$ as well as Sobolev's embedding $W^{1,2}(\mathcal{O},\mathbb{R}^2) \hookrightarrow L^q(\mathcal{O},\mathbb{R}^2)$ and Ladyshenskaya's inequality. By (3.17) we conclude that

$$\mathbf{g}_R \in L^2(0,T;W^{\sigma,2}(\mathcal{O},\mathbb{R}^2)) \quad \mathbb{P}\text{-a.s.} \tag{3.21}$$

We argue now similarly for the temporal regularity of order $\sigma/2$ obtaining for any $\sigma' \in (\sigma,1)$

$$\begin{aligned}&\|\mathbf{g}_R\|^p_{W^{\sigma/2,p}(0,T;L^p_x)}\\ &\le c\int_0^T\int_0^T \frac{\|\mathbf{u}^R(t)\nabla\mathbf{u}^R(t)-\mathbf{u}^R(s)\nabla\mathbf{u}^R(s)\|^p_{L^p_x}}{|t-s|^{1+p\sigma/2}}\,\mathrm{d}t\,\mathrm{d}s\\ &\le c\int_0^T\int_0^T\left(\frac{\|\mathbf{u}^R(t)-\mathbf{u}^R(s)\|_{L^2_x}}{|t-s|^{\sigma'/2}}\|\nabla\mathbf{u}^R(t)\|_{L^q_x}\right)^p\frac{\mathrm{d}t\,\mathrm{d}s}{|t-s|^{1+\frac{(\sigma-\sigma')p}{2}}}\\ &\quad + c\int_0^T\int_0^T\left(\frac{\|\mathbf{u}^R(s)\|_{L^q_x}\|\nabla\mathbf{u}^R(t)-\nabla\mathbf{u}^R(s)\|_{L^2_x}}{|t-s|^{\sigma/2}}\right)^p\frac{\mathrm{d}t\,\mathrm{d}s}{|t-s|}\\ &\le c\|\mathbf{u}^R\|^p_{C^{\sigma'/2}([0,T];L^2_x)}\int_0^T\|\nabla\mathbf{u}^R(t)\|^p_{L^q_x}\,\mathrm{d}t\\ &\quad + c\sup_{0\le s\le t}\|\mathbf{u}^R(s)\|^p_{L^q_x}\int_0^T\int_0^T\frac{\|\nabla\mathbf{u}^R(t)-\nabla\mathbf{u}^R(s)\|^p_{L^2_x}}{|t-s|^{1+p\sigma/2}}\,\mathrm{d}t\,\mathrm{d}s\\ &\le c\|\mathbf{u}^R\|^p_{C^{\sigma'/2}([0,T];L^2_x)}\int_0^T\left(1+\|\mathbf{u}^R(t)\|^2_{W^{2,2}_x}\right)\mathrm{d}t\\ &\quad + c\sup_{0\le s\le t}\|\mathbf{u}^R(s)\|^p_{W^{1,2}_x}\|\mathbf{u}^R\|^p_{W^{\sigma/2,p}(0,T;W^{1,2}_x)}\\ &\le c\left(\|\mathbf{u}^R\|^2_{C^{\sigma'/2}([0,T];L^2_x)} + \|\mathbf{u}^R\|^2_{W^{\sigma'/2,2}(0,T;W^{1,2}_x)}+1\right)\\ &\quad + c\left(\sup_{0\le s\le t}\|\mathbf{u}^R(s)\|^2_{W^{1,2}_x}+\int_0^T\|\mathbf{u}^R\|^2_{W^{2,2}_x}\,\mathrm{d}t\right)^q.\end{aligned}$$

The expectation of the right-hand side is bounded using (3.9) and (3.20); in particular, for any $\sigma < 1$

$$\mathbf{g}_R \in W^{\sigma/2,2}(0,T;L^2(\mathcal{O},\mathbb{R}^2)) \quad \mathbb{P}\text{-a.s.} \tag{3.22}$$

using the embedding decreasing the value of σ and using $W^{\sigma/2,p}(0,T) \hookrightarrow W^{\sigma'/2,2}(0,T)$ for an appropriate choice of $\sigma > \sigma'$ and $p < 2$. By (3.21) and (3.22) classical results on the Stokes system (see [28, Thm. 15] and note the compability assumption $\mathcal{A}\mathbf{u}_0 - \mathcal{P}(\mathbf{u}_0 \cdot \nabla \mathbf{u}_0)|_{\partial\mathcal{O}} = 0$ $\mathbb{P}$-a.s.) and interpolation yield

$$\mathbf{Y}^R \in W^{1+\sigma/2}(0,T;L^2(\mathcal{O},\mathbb{R}^2)) \cap L^2(0,T;W^{2+\sigma,2}(\mathcal{O},\mathbb{R}^2)) \quad \mathbb{P}\text{-a.s.}$$

and thus, again by interpolation and appropriate choice of $\sigma \in (\beta, 1)$ and the embedding $W^{\alpha,2}(0,T) \hookrightarrow L^\infty(0,T)$ for $\alpha > 1/2$,

$$\mathbf{Y}^R \in L^\infty(0,T;W^{1+\beta,2}(\mathcal{O},\mathbb{R}^2)) \cap L^2(0,T;W^{2+\beta,2}(\mathcal{O},\mathbb{R}^2)) \quad \mathbb{P}\text{-a.s.} \tag{3.23}$$

together with

$$\begin{aligned}
&\sup_{0\le t\le T} \|\mathbf{Y}^R\|^2_{W^{1+\beta,2}_x} + \int_0^T \|\mathbf{Y}^R\|^2_{W^{2+\beta,2}_x}\,\mathrm{d}s \\
&\quad \le c\left[\|\mathbf{u}_0\|^2_{W^{1+\sigma,2}_x} + \|\mathbf{g}_R\|^2_{W^{\sigma/2,2}_t(L^2_x)} + \int_0^T \|\mathbf{g}_R\|^2_{W^{\sigma,2}_x}\,\mathrm{d}s\right] \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

Combining the estimates for $\mathbf{Y}^R$ and $\mathbf{Z}^R$, choosing κ sufficiently small and using (3.16) and (3.17) we arrive at

$$\begin{aligned}
&\mathbb{E}\left[\left(\sup_{0\le t\le T} \|\mathbf{u}^R(t)\|^2_{W^{1+\sigma,2}_x}\,\mathrm{d}x + \int_0^T \|\mathbf{u}^R\|^2_{W^{2+\sigma,2}_x}\,\mathrm{d}t\right)^{\frac{r}{2}}\right] \\
&\qquad\qquad \le cR^{5r}\,\mathbb{E}\left[1 + \|\mathbf{u}_0\|^r_{W^{2,2}_x} + \|\mathbf{u}_0\|^{2r}_{W^{1,2}_x}\right].
\end{aligned}$$

uniformly in R. □

Remark 3.2 **1.** It seems not possible to prove Lemma 3.1 (c) for $\beta \ge 1$, see (3.23). In fact, even for the deterministic Stokes system high regularity is only possible if the forcing is regular in space *and* time or belongs to the domain of the Stokes operator. Since neither is true for the convective term $\mathcal{P}(\mathbf{u}\cdot\nabla)\mathbf{u}$ (its temporal regularity is restricted by that of the driving Wiener process) we conjecture that the spatial regularity from 3.1 (c) is optimal. Interestingly, this is just enough to prove an optimal convergence rate for the discretisation of (1.1) in Theorem 4.4.

2. Using a recent result from [30] we can show that the gradient of the velocity field and hence the convective has a fractional time derivative of order $\beta/2 < 1/2$. This is optimal in view of the limited regularity of the driving Wiener process in the momentum equation. It is classical for deterministic parabolic equations (see [28] for the Stokes equations and [27] for the heat equation) that the solution gains two spatial and one temporal derivatives compared to the right-hand side. Hence the regularity of the latter has to be measured in space and time with respect to the parabolic scaling; pure space regularity does not transfer unless additional assumptions are in place such that we can only hope for $2+\beta$ spatial derivatives.

3.2 Regularity of the Pressure

Since we will be working with discretely divergence-free function spaces in the finite-element analysis for (4.1) in Sect. 4, it is inevitable to introduce the pressure function. Note that the strong formulation of the momentum equation in (2.6) even allows test functions from the class $L^2_{\mathrm{div}}(\mathcal{O},\mathbb{R}^2)$ (using a standard smooth approximation argument), i.e., functions which do not have zero traces on $\partial\mathcal{O}$. Hence for $\boldsymbol{\phi}\in C^\infty_c(\mathcal{O},\mathbb{R}^2)$ we can insert

$$\mathcal{P}\boldsymbol{\phi}=\boldsymbol{\phi}-\nabla\Delta_{\mathcal{O}}^{-1}\mathrm{div}\boldsymbol{\phi}$$

with the Helmholz projection $\mathcal{P}$; cf. (3.1). We obtain

$$\begin{aligned}\int_{\mathcal{O}}\mathbf{u}(t\wedge\mathfrak{t}_R)\cdot\boldsymbol{\varphi}\,\mathrm{d}x-\int_0^{t\wedge\mathfrak{t}_R}\int_{\mathcal{O}}\mu\Delta\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}\sigma+\int_0^{t\wedge\mathfrak{t}_R}\int_{\mathcal{O}}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}\sigma\\=\int_{\mathcal{O}}\mathbf{u}(0)\cdot\boldsymbol{\varphi}\,\mathrm{d}x+\int_0^{t\wedge\mathfrak{t}_R}\int_{\mathcal{O}}\pi\,\mathrm{div}\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}\sigma\\+\int_{\mathcal{O}}\int_0^{t\wedge\mathfrak{t}_R}\Phi(\mathbf{u})\,\mathrm{d}W\cdot\boldsymbol{\varphi}\,\mathrm{d}x,\end{aligned}\tag{3.24}$$

where

$$\pi=-\Delta_{\mathcal{O}}^{-1}\mathrm{div}\big((\mathbf{u}\cdot\nabla)\mathbf{u}-\mu\Delta\mathbf{u}\big).$$

In the following we will analyse how the regularity of $\mathbf{u}$ transfers to π, where again $R>0$ is a fixed truncation parameter and $T>0$ an arbitrary but fixed time.

Lemma 3.3 *(a) Under the assumptions of Lemma 3.1 (b) we have*

$$\mathbb{E}\bigg[\bigg(\int_0^{T\wedge\mathfrak{t}_R}\|\pi\|^2_{W^{1,2}_x}\,\mathrm{d}t\bigg)^{\frac{r}{4}}\bigg]\leq cR^{3r}\,\mathbb{E}\Big[1+\|\mathbf{u}_0\|^r_{W^{1,2}_x}\Big].$$

(b) Under the assumptions of Lemma 3.1 (c) we have

$$\mathbb{E}\bigg[\bigg(\int_0^{T\wedge\mathfrak{t}_R}\|\pi\|^2_{W^{2,2}_x}\,\mathrm{d}t\bigg)^{\frac{r}{4}}\bigg]\leq cR^{5r}\,\mathbb{E}\Big[1+\|\mathbf{u}_0\|^r_{W^{2,2}_x}\Big].$$

Here $c=c(r,T)>0$ is independent of R.

Proof Ad **(a)**. Arguing as in [5, Corollary 2.5] and using (3.2) we obtain

$$\mathbb{E}\bigg[\bigg(\int_0^{T\wedge \mathfrak{t}_R} \|\pi\|^2_{W^{1,2}_x}\,\mathrm{d}t\bigg)^{\frac{r}{4}}\bigg] \leq c\,\mathbb{E}\bigg[\bigg(1+\sup_{t\in[0,T\wedge\mathfrak{t}_R]}\|\mathbf{u}\|^2_{W^{1,2}_x}+\int_0^{T\wedge\mathfrak{t}_R}\|\mathbf{u}\|^2_{W^{2,2}_x}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg].$$

Consequently, Lemma 3.1 (b) implies (a).

Ad **(b)**. Using (3.2) we have for $p>2$ close to 2 and $q:=\frac{2p}{p-2}$

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\int_0^{T\wedge \mathfrak{t}_R} \|\pi\|^2_{W^{2,2}_x}\,\mathrm{d}t\bigg)^{\frac{r}{4}}\bigg] &\leq c\,\mathbb{E}\bigg[\bigg(\int_0^{T\wedge \mathfrak{t}_R} \|\mathbf{u}\cdot\nabla\mathbf{u}\|^2_{W^{1,2}_x}\,\mathrm{d}t\bigg)^{\frac{r}{4}}\bigg]\\
&\leq c\,\mathbb{E}\bigg[\bigg(\int_0^{T\wedge \mathfrak{t}_R} \|\nabla\mathbf{u}\|^4_{L^4_x}\,\mathrm{d}t+\int_0^{T\wedge \mathfrak{t}_R}\|\mathbf{u}\|^2_{L^q_x}\|\nabla^2\mathbf{u}\|^2_{L^p_x}\,\mathrm{d}t\bigg)^{\frac{r}{4}}\bigg]\\
&\leq c\,\mathbb{E}\bigg[\bigg(\int_0^{T\wedge \mathfrak{t}_R} \|\mathbf{u}\|^4_{W^{1+\beta,2}_x}\,\mathrm{d}t+\sup_{0\leq t\leq T\wedge\mathfrak{t}_R}\|\mathbf{u}\|^2_{W^{1,2}_x}\int_0^{T\wedge \mathfrak{t}_R}\|\mathbf{u}\|^2_{W^{2+\beta,2}_x}\,\mathrm{d}t\bigg)^{\frac{r}{4}}\bigg]\\
&\leq c\,\mathbb{E}\bigg[\bigg(\sup_{0\leq t\leq T\wedge\mathfrak{t}_R}\|\mathbf{u}\|^2_{W^{1+\beta,2}_x}+\int_0^{T\wedge \mathfrak{t}_R}\|\mathbf{u}\|^2_{W^{2+\beta,2}_x}\,\mathrm{d}t\bigg)^{\frac{r}{2}}\bigg]\end{aligned}$$

using the embeddings $W^{1+\beta,2}(\mathcal{O},\mathbb{R}^2)\hookrightarrow W^{1,4}(\mathcal{O},\mathbb{R}^2)$ and $W^{2+\beta,2}(\mathcal{O},\mathbb{R}^2)\hookrightarrow W^{2,p}(\mathcal{O},\mathbb{R}^2)$, which hold for an appropriate choice of $\beta\in(0,1)$. Hence using Lemma 3.1 (c) completes the proof. □

Corollary 3.4 *(a) Let the assumptions of Lemma 3.1 (b) be satisfied for some $r>2$. For all $\alpha<\frac{1}{2}$ we have*

$$\mathbb{E}\bigg[\Big(\|\mathbf{u}(\cdot\wedge\mathfrak{t}_R)\|_{C^\alpha([0,T];L^2_x)}\Big)^{\frac{r}{2}}\bigg]\leq cR^{3r}\,\mathbb{E}\Big[1+\|\mathbf{u}_0\|^r_{W^{1,2}_x}\Big]. \tag{3.25}$$

(b) Let the assumptions of Lemma 3.1 (c) be satisfied for some $r>2$. For all $\alpha<\frac{1}{2}$ we have

$$\mathbb{E}\bigg[\Big(\|\mathbf{u}(\cdot\wedge\mathfrak{t}_R)\|_{C^\alpha([0,T];W^{1,2}_x)}\Big)^{\frac{r}{2}}\bigg]\leq cR^{5r}\,\mathbb{E}\Big[1+\|\mathbf{u}_0\|^r_{W^{2,2}_x}+\|\mathbf{u}_0\|^{2r}_{W^{1,2}_x}\Big]. \tag{3.26}$$

Here $c=c(r,T,\alpha)>0$ is independent of R.

Proof As in [5, Corollary 2.6] we can combine Lemmas 3.1 and 3.3 to conclude the result concerning the time regularity of $\mathbf{u}$ form **(a)**. As far is **(b)** is concerned we analyse each term in equation (3.24) separately. Lemma 3.1 **(b)** implies

$$\int_0^{\cdot\wedge\mathfrak{t}_R}\Delta\mathbf{u}\,\mathrm{d}\sigma\in L^r(\Omega;L^2(0,T;W^{\beta,2}(\mathcal{O},\mathbb{R}^2))),$$

whereas Lemmas 3.1 **(c)** and 3.3 **(b)** yield

$$\int_0^{\cdot\wedge\mathfrak{t}_R}\big(\mathrm{div}(\mathbf{u}\otimes\mathbf{u})+\nabla\pi\big)\,\mathrm{d}\sigma\in L^{\frac{r}{2}}(\Omega;L^2(0,T;W^{\beta,2}(\mathcal{O},\mathbb{R}^2))).$$

Finally, we have

$$\int_0^{\cdot\wedge\mathfrak{t}_R}\Phi(\mathbf{u})\,\mathrm{d}W\in L^r(\Omega;C^\alpha([0,T];L^2(\mathcal{O},\mathbb{R}^2))).$$

by combing Lemma 3.1 **(a)** with (2.2). We conclude that

$$\mathbb{E}\Big[\Big(\|\mathbf{u}(\cdot\wedge\mathfrak{t}_R)\|_{C^\alpha([0,T];W^{\beta,2}_x)}\Big)^{\frac{r}{2}}\Big]\leq cR^{5r}\,\mathbb{E}\Big[1+\|\mathbf{u}_0\|^r_{W^{2,2}_x}+\|\mathbf{u}_0\|^{2r}_{W^{1,2}_x}\Big].$$

for all $\beta<1$. Interpolating this with the estimate from Lemma 3.1 **(c)** gives the claim. □

4 Error Analysis: Direct Comparison

Now we consider a fully practical scheme combining a semi-implicit Euler scheme in time with a finite element approximation in space. It is defined on the given filtered probability space $(\Omega,\mathfrak{F},(\mathfrak{F}_t),\mathbb{P})$ on which W as well as the maximal strong pathwise solution to (1.1) are defined. For a given $h>0$ let $\mathbf{u}_{h,0}$ be an $\mathfrak{F}_0$-measurable random variable with values in $V^{h,i}_{\mathrm{div}}(\mathcal{O},\mathbb{R}^2)$ (for instance $\Pi_h\mathbf{u}_0$; see (2.11)). We aim at constructing iteratively a sequence of random variables $(\mathbf{u}_{h,m},p_{h,m})$ such that for every $(\boldsymbol{\phi},\chi)\in V^{h,i}(\mathcal{O},\mathbb{R}^2)\times P^{h,j}(\mathcal{O})$ it holds true $\mathbb{P}$-a.s.

$$\begin{aligned}&\int_{\mathcal{O}}\mathbf{u}_{h,m}\cdot\boldsymbol{\varphi}\,\mathrm{d}x+\tau\int_{\mathcal{O}}\big((\mathbf{u}_{h,m-1}\cdot\nabla)\mathbf{u}_{h,m}+(\mathrm{div}\mathbf{u}_{h,m-1})\mathbf{u}_{h,m}\big)\cdot\boldsymbol{\phi}\,\mathrm{d}x\\&\quad+\mu\,\tau\int_{\mathcal{O}}\nabla\mathbf{u}_{h,m}:\nabla\boldsymbol{\phi}\,\mathrm{d}x-\tau\int_{\mathcal{O}}p_{h,m}\,\mathrm{div}\boldsymbol{\varphi}\,\mathrm{d}x\\&=\int_{\mathcal{O}}\mathbf{u}_{h,m-1}\cdot\boldsymbol{\varphi}\,\mathrm{d}x+\int_{\mathcal{O}}\Phi(\mathbf{u}_{h,m-1})\,\Delta_mW\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,,\\&\int_{\mathcal{O}}\mathrm{div}\mathbf{u}_{h,m}\cdot\chi\,\mathrm{d}x=0\,,\end{aligned}\tag{4.1}$$

where $\Delta_mW=W(t_m)-W(t_{m-1})$. Here the interval $[0,T]$ is decomposed into an equidistant grid of time points $t_m=m\tau=m\frac{T}{M}$ with $M\in\mathbb{N}$. For our theoretical analysis it is convenient to work with the pressure-free formulation of (4.1): For every $\boldsymbol{\phi}\in V^{h,i}_{\mathrm{div}}(\mathcal{O},\mathbb{R}^2)$ it holds true $\mathbb{P}$-a.s.

$$\int_{\mathcal{O}} \mathbf{u}_{h,m} \cdot \boldsymbol{\varphi}\, \mathrm{d}x + \tau \int_{\mathcal{O}} \big((\mathbf{u}_{h,m-1} \cdot \nabla)\mathbf{u}_{h,m} + (\mathrm{div}\mathbf{u}_{h,m-1})\mathbf{u}_{h,m}\big) \cdot \boldsymbol{\phi}\, \mathrm{d}x$$
$$= \int_{\mathcal{O}} \mathbf{u}_{h,m-1} \cdot \boldsymbol{\varphi}\, \mathrm{d}x - \mu\, \tau \int_{\mathcal{O}} \nabla \mathbf{u}_{h,m} : \nabla \boldsymbol{\phi}\, \mathrm{d}x \tag{4.2}$$
$$+ \int_{\mathcal{O}} \Phi(\mathbf{u}_{h,m-1})\, \Delta_m W \cdot \boldsymbol{\varphi}\, \mathrm{d}x.$$

We quote the following result concerning the solution $(\mathbf{u}_{h,m})_{m=1}^{M}$ to (4.2) from [9, Lemma 3.1].

Lemma 4.1 *Fix $T > 0$. Assume that $\mathbf{u}_{h,0} \in L^{2^q}(\Omega, V_{\mathrm{div}}^{h,i}(\mathcal{O}, \mathbb{R}^2))$ with $q \in \mathbb{N}$ is an $\mathfrak{F}_0$-measurable random variable. Suppose that Φ satisfies (2.1). Then the iterates $(\mathbf{u}_{h,m})_{m=1}^{M}$ given by (4.2) are $(\mathfrak{F}_{t_m})$-measurable. Moreover, the following estimate holds uniformly in M and h:*

$$\mathbb{E}\bigg[\max_{1\le m\le M} \|\mathbf{u}_{h,m}\|_{L_x^2}^{2^q} + \tau \sum_{k=1}^{M} \|\mathbf{u}_{h,m}\|_{L_x^2}^{2^q-2} \|\nabla \mathbf{u}_{h,m}\|_{L_x^2}^2\bigg]$$
$$\le c(q,T)\mathbb{E}\Big[1 + \|\mathbf{u}_{h,0}\|_{L_x^2}^{2^q}\Big]. \tag{4.3}$$

Our error analysis for (4.2) is based on an auxiliary problem which coincides with (4.2) until a discrete stopping time. As we shall see below both problems coincide with high probability. For every $m \ge 1$ we introduce the discrete stopping time

$$\mathfrak{t}_m^R := \max_{1\le n\le m} \big\{t_n : t_n \le \mathfrak{t}_R\big\}, \tag{4.4}$$

which is obviously $(\mathfrak{F}_{t_m})$-stopping time (but not an $(\mathfrak{F}_t)$-stopping time). Setting $\tau_m^R := \mathfrak{t}_m^R - \mathfrak{t}_{m-1}^R$ we introduce $\mathbf{u}_{h,m}^R$ as the $V_{\mathrm{div}}^{h,i}(\mathcal{O}, \mathbb{R}^2)$-valued solution to

$$\int_{\mathcal{O}} \mathbf{u}_{h,m}^R \cdot \boldsymbol{\varphi}\, \mathrm{d}x + \tau_m^R \int_{\mathcal{O}} \big((\mathbf{u}_{h,m-1}^R \cdot \nabla)\mathbf{u}_{h,m}^R + (\mathrm{div}\mathbf{u}_{h,m-1}^R)\mathbf{u}_{h,m}^R\big) \cdot \boldsymbol{\phi}\, \mathrm{d}x$$
$$= \int_{\mathcal{O}} \mathbf{u}_{h,m-1}^R \cdot \boldsymbol{\varphi}\, \mathrm{d}x - \mu\, \tau_m^R \int_{\mathcal{O}} \nabla \mathbf{u}_m^R : \nabla \boldsymbol{\phi}\, \mathrm{d}x \tag{4.5}$$
$$+ \frac{\tau_m^R}{\tau} \int_{\mathcal{O}} \Phi(\mathbf{u}_{h,m-1}^R)\, \Delta_m W \cdot \boldsymbol{\varphi}\, \mathrm{d}x$$

for every $\boldsymbol{\phi} \in V_{\mathrm{div}}^{h,i}(\mathcal{O}, \mathbb{R}^2)$. Obviously $\mathbf{u}_{h,m}^R = \mathbf{u}_{h,m}$ in $[t_m = \mathfrak{t}_m^R]$. Our main effort is dedicated to the proof of an error estimate for (4.5) in the following theorem, for which $R > 0$ is a fixed truncation parameter and $T > 0$ an arbitrary but fixed time.

Theorem 4.2 *Let $\mathbf{u}_0 \in L^8(\Omega, W^{2,2}(\mathcal{O}, \mathbb{R}^2)) \cap L^{20}(\Omega; W_{0,\mathrm{div}}^{1,2}(\mathcal{O}, \mathbb{R}^2))$ be $\mathfrak{F}_0$-measurable, we have $\mathcal{A}\mathbf{u}_0 - \mathcal{P}(\mathbf{u}_0 \cdot \nabla \mathbf{u}_0)|_{\partial\mathcal{O}} = 0$ $\mathbb{P}$-a.s. and assume that Φ satisfies (2.1)–(2.5). Let*

$$(\mathbf{u}, (\mathfrak{t}_R)_{R\in\mathbb{N}}, \mathfrak{t})$$

be the unique maximal global strong pathwise solution to (1.1) in the sense of Definition 2.4. Let $(\mathfrak{t}_m^R)_{m=1}^M$ be defined by (4.4). Then we have for all $R \in \mathbb{N}$ and all $\alpha < \frac{1}{2}$, $\beta < 1$

$$\mathbb{E}\bigg[\max_{1\le m\le M}\|\mathbf{u}(\mathfrak{t}_m^R)-\mathbf{u}_{h,m}^R\|_{L_x^2}^2+\sum_{m=1}^M\tau_m^R\|\nabla\mathbf{u}(\mathfrak{t}_m^R)-\nabla\mathbf{u}_{h,m}^R\|_{L_x^2}^2\bigg)\bigg] \le ce^{cR^4}\left(h^{2\beta}+\tau^{2\alpha}\right), \tag{4.6}$$

where $(\mathbf{u}_{h,m}^R)_{m=1}^M$ is the solution to (4.5) with $\mathbf{u}_{h,0}^R=\Pi_h\mathbf{u}_0$. The constant c in (4.6) is independent of τ, h and R.

Remark 4.3 **1.** In previous papers concerning the periodic problem, in particular [12], the idea is to consider the equation for the error in the m-th step and multiply by the indicator function of a set $\Omega_{m-1}^{h,\tau}\subset\Omega$. Hereby $\Omega_{m-1}^{h,\tau}\subset\Omega$ is $\mathfrak{F}_{t_{m-1}}$-measurable and certain quantities up to time t_{m-1} are bounded in $\Omega_{m-1}^{h,\tau}$. It is, however, not necessary to control the continuous solution in this way since global estimates are available, see, e.g., [12, Lemma 2.1] or [5, Lemma 2].

In our situation, having only stopped estimates as in Lemma 3.1, it is necessary to also control the continuous solution. For certain quantities, having control until time t_{m-1} is not sufficient (see, for instance, the estimates for $I_2(m)$ and $I_3(m)$ below, where norms of $\mathbf{u}$ over $[t_{m-1},t_m]$ appear). Using $\mathfrak{F}_{t_m}$-measurable sets $\Omega_m^{h,\tau}\subset\Omega$ instead is not possible either as it destroys the martingale property of $\mathscr{M}^1$ given below in (4.7).

Both problems are overcome by the use of the discrete stopping time $\mathfrak{t}_m^R$: we can control norms of $\mathbf{u}$ over $[t_{m-1},t_m]$, and $\mathscr{M}^1$ is estimated at time $\mathfrak{t}_R\ge\mathfrak{t}_m^R$ such that the martingale property can be used.

2. The (discrete) gradient of the noise term in (4.1) need not be subtracted here, as is in [14], since a *simultaneous space-time* error analysis is used to prove Theorem 4.4 below.

Our main result is now a direct consequence of Theorem 4.2: Setting for $\varepsilon>0$ arbitrary $R=c^{-1/4}\sqrt[4]{-2\varepsilon\log\min\{h,\tau\}}$, we have for any $\xi>0$

$$\begin{aligned}&\mathbb{P}\left[\frac{\max_{1\le m\le M}\|\mathbf{u}(t_m)-\mathbf{u}_{h,m}\|_{L_x^2}^2+\sum_{m=1}^M\tau\|\nabla\mathbf{u}(t_m)-\nabla\mathbf{u}_{h,m}\|_{L_x^2}^2}{h^{2\beta-2\varepsilon}+\tau^{2\alpha-2\varepsilon}}>\xi\right]\\&\le\mathbb{P}\left[\frac{\max_{1\le m\le M}\|\mathbf{u}(\mathfrak{t}_m^R)-\mathbf{u}_{h,m}^R\|_{L_x^2}^2+\sum_{m=1}^M\tau_m^R\|\nabla\mathbf{u}(\mathfrak{t}_m^R)-\nabla\mathbf{u}_{h,m}^R\|_{L_x^2}^2}{h^{2\beta-2\varepsilon}+\tau^{2\alpha-2\varepsilon}}>\xi\right]\\&\quad+\mathbb{P}\big[\{\mathfrak{t}_R<T\}\big]\to0\end{aligned}$$

as $h,\tau\to0$ (recall that $\mathfrak{t}_R\to\infty$ $\mathbb{P}$-a.s. by Theorem 2.5 and that $\mathfrak{t}_M^R<t_M$ implies $\mathfrak{t}_R<T$). Relabeling α and β we have proved the following result.

Theorem 4.4 *Let $(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\ge0},\mathbb{P})$ be a given stochastic basis with a complete right-continuous filtration and an $(\mathfrak{F}_t)$-cylindrical Wiener process W. Let $\mathbf{u}_0\in$*

$L^8(\Omega, W^{2,2}(\mathcal{O},\mathbb{R}^2)) \cap L^{20}(\Omega; W^{1,2}_{0,\mathrm{div}}(\mathcal{O},\mathbb{R}^2))$ be $\mathfrak{F}_0$-measurable, we have $\mathcal{A}\mathbf{u}_0 - \mathcal{P}(\mathbf{u}_0\cdot\nabla\mathbf{u}_0)|_{\partial\mathcal{O}} = 0$ $\mathbb{P}$-a.s. and assume that Φ satisfies (2.1)–(2.5). Let

$$(\mathbf{u}, (\mathfrak{t}_R)_{R\in\mathbb{N}}, \mathfrak{t})$$

be the unique maximal global strong pathwise solution to (1.1) from Theorem 2.5. Then we have for any $\xi > 0$, $\alpha < \frac{1}{2}$, $\beta < 1$

$$\mathbb{P}\left[\frac{\max_{1\le m\le M}\|\mathbf{u}(t_m)-\mathbf{u}_{h,m}\|^2_{L^2_x} + \sum_{m=1}^M \tau\|\nabla\mathbf{u}(t_m)-\nabla\mathbf{u}_{h,m}\|^2_{L^2_x}}{h^{2\beta}+\tau^{2\alpha}} > \xi\right] \to 0$$

as $h,\tau\to 0$, where $(\mathbf{u}_{h,m})_{m=1}^M$ is the solution to (4.2) with $\mathbf{u}_{h,0} = \Pi_h\mathbf{u}_0$.

In order to finish the proof of our main result stated in Theorem 4.4 above, we focus now on proving the error estimate from Theorem 4.2 concerning the auxiliary problem (4.5).

Proof of Theorem 4.2 Define the error $\mathbf{e}_{h,m} = \mathbf{u}(\mathfrak{t}_m^R) - \mathbf{u}_{h,m}^R$. Subtracting (3.24) and (4.5) and recalling that functions from $W_0^{1,2}(\mathcal{O},\mathbb{R}^2)$ are admissible in (3.24) we obtain

$$\begin{aligned}
&\int_{\mathcal{O}} \mathbf{e}_{h,m}\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_{\mathfrak{t}_{m-1}^R}^{\mathfrak{t}_m^R}\int_{\mathcal{O}} \mu\nabla\mathbf{u}(\sigma) : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}\sigma - \tau_m^R\int_{\mathcal{O}}\mu\nabla\mathbf{u}_{h,m}^R : \nabla\boldsymbol{\phi}\,\mathrm{d}x\\
&= \int_{\mathcal{O}} \mathbf{e}_{h,m-1}\cdot\boldsymbol{\varphi}\,\mathrm{d}x - \int_{\mathfrak{t}_{m-1}^R}^{\mathfrak{t}_m^R}\int_{\mathcal{O}} (\mathbf{u}(\sigma)\cdot\nabla)\mathbf{u}(\sigma)\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&\quad + \tau_m^R\int_{\mathcal{O}}\Big((\mathbf{u}_{h,m-1}^R\cdot\nabla)\mathbf{u}_{h,m}^R + (\mathrm{div}\mathbf{u}_{h,m-1}^R)\mathbf{u}_{h,m}^R\Big)\cdot\boldsymbol{\phi}\,\mathrm{d}x\\
&\quad + \int_{\mathcal{O}}\int_{\mathfrak{t}_{m-1}^R}^{\mathfrak{t}_m^R}\Phi(\mathbf{u}(\sigma))\,\mathrm{d}W\cdot\boldsymbol{\varphi}\,\mathrm{d}x - \int_{\mathcal{O}}\int_{\mathfrak{t}_{m-1}^R}^{\mathfrak{t}_m^R}\Phi(\mathbf{u}_{h,m-1}^R)\,\mathrm{d}W\cdot\boldsymbol{\varphi}\,\mathrm{d}x\\
&\quad + \int_{\mathfrak{t}_{m-1}^R}^{\mathfrak{t}_m^R}\int_{\mathcal{O}}\pi(\sigma)\,\mathrm{div}\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}\sigma
\end{aligned}$$

for every $\boldsymbol{\phi}\in V^{h,i}(\mathcal{O},\mathbb{R}^2)$, which is equivalent to

$$\begin{aligned}
&\int_{\mathcal{O}} \mathbf{e}_{h,m}\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \tau_m^R\int_{\mathcal{O}}\mu\Big(\nabla\mathbf{u}(\mathfrak{t}_m^R) - \nabla\mathbf{u}_{h,m}^R\Big) : \nabla\boldsymbol{\phi}\,\mathrm{d}x\\
&= \int_{\mathcal{O}} \mathbf{e}_{h,m-1}\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_{\mathfrak{t}_{m-1}^R}^{\mathfrak{t}_m^R}\int_{\mathcal{O}}\mu\big(\nabla\mathbf{u}(\mathfrak{t}_m^R) - \nabla\mathbf{u}(\sigma)\big) : \nabla\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&\quad + \int_{\mathfrak{t}_{m-1}^R}^{\mathfrak{t}_m^R}\int_{\mathcal{O}}\Big((\mathbf{u}(\mathfrak{t}_{m-1}^R)\cdot\nabla)\mathbf{u}(\mathfrak{t}_m^R) - (\mathbf{u}(\sigma)\cdot\nabla)\mathbf{u}(\sigma)\Big)\cdot\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&\quad - \tau_m^R\int_{\mathcal{O}}\Big((\mathbf{u}(\mathfrak{t}_{m-1}^R)\cdot\nabla)\mathbf{u}(\mathfrak{t}_m^R) - \big((\mathbf{u}_{h,m-1}^R\cdot\nabla)\mathbf{u}_{h,m}^R + (\mathrm{div}\mathbf{u}_{h,m-1}^R)\mathbf{u}_{h,m}^R\big)\Big)\cdot\boldsymbol{\phi}\,\mathrm{d}x
\end{aligned}$$

$$+\int_{\mathcal{O}}\int_{\mathrm{t}_{m-1}^R}^{\mathrm{t}_m^R}\big(\Phi(\mathbf{u}(\sigma))-\Phi(\mathbf{u}_{h,m-1}^R)\big)\,\mathrm{d}W\cdot\boldsymbol{\varphi}\,\mathrm{d}x+\int_{\mathrm{t}_{m-1}^R}^{\mathrm{t}_m^R}\int_{\mathcal{O}}\pi(\sigma)\,\mathrm{div}\boldsymbol{\phi}\,\mathrm{d}x\,\mathrm{d}\sigma.$$

Setting $\boldsymbol{\phi}=\Pi_h\mathbf{e}_{h,m}$ and applying the identity $\mathbf{a}\cdot(\mathbf{a}-\mathbf{b})=\frac{1}{2}\big(|\mathbf{a}|^2-|\mathbf{b}|^2+|\mathbf{a}-\mathbf{b}|^2\big)$ (which holds for any $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$) we gain

$$\begin{aligned}
&\frac{1}{2}\big(\|\Pi_h\mathbf{e}_{h,m}\|_{L_x^2}^2-\|\Pi_h\mathbf{e}_{h,m-1}\|_{L_x^2}^2+\|\Pi_h\mathbf{e}_{h,m}-\Pi_h\mathbf{e}_{h,m-1}\|_{L_x^2}^2\big)+\tau_m^R\mu\|\nabla\mathbf{e}_{h,m}\|_{L_x^2}^2\\
&\quad=\tau_m^R\int_{\mathcal{O}}\mu\nabla\mathbf{e}_{h,m}:\nabla\big(\mathbf{u}(\mathrm{t}_m^R)-\Pi_h\mathbf{u}(\mathrm{t}_m^R)\big)\,\mathrm{d}x\\
&\qquad+\int_{\mathrm{t}_{m-1}^R}^{\mathrm{t}_m^R}\int_{\mathcal{O}}\mu\big(\nabla\mathbf{u}(\mathrm{t}_m^R)-\nabla\mathbf{u}(\sigma)\big):\nabla\Pi_h\mathbf{e}_{h,m}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&\qquad+\int_{\mathrm{t}_{m-1}^R}^{\mathrm{t}_m^R}\int_{\mathcal{O}}\Big((\mathbf{u}(\mathrm{t}_{m-1}^R\cdot\nabla)\mathbf{u}(\mathrm{t}_m^R)-(\mathbf{u}(\sigma)\cdot\nabla)\mathbf{u}(\sigma)\Big)\cdot\Pi_h\mathbf{e}_{h,m}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&\qquad-\tau_m^R\int_{\mathcal{O}}\Big((\mathbf{u}(\mathrm{t}_{m-1}^R)\cdot\nabla)\mathbf{u}(\mathrm{t}_m^R)-\big((\mathbf{u}_{h,m-1}^R\cdot\nabla+\mathrm{div}\mathbf{u}_{h,m-1}^R)\mathbf{u}_{h,m}^R\big)\Big)\cdot\Pi_h\mathbf{e}_{h,m}\,\mathrm{d}x\\
&\qquad+\int_{\mathcal{O}}\int_{\mathrm{t}_{m-1}^R}^{\mathrm{t}_m^R}\big(\Phi(\mathbf{u}(\sigma))-\Phi(\mathbf{u}_{h,m-1}^R)\big)\,\mathrm{d}W\cdot\Pi_h\mathbf{e}_{h,m}\,\mathrm{d}x\\
&\qquad+\int_{\mathrm{t}_{m-1}^R}^{\mathrm{t}_m^R}\int_{\mathcal{O}}\pi(\sigma)\,\mathrm{div}\Pi_h\mathbf{e}_{h,m}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&\quad=:I_1(m)+\cdots+I_6(m).
\end{aligned}$$

Eventually, we will take the maximum with respect to $m\in\{1,\ldots,M\}$ and apply expectations. Let us explain how to deal with $\mathbb{E}\big[\max_m I_1(m)],\ldots,\mathbb{E}[\max_m I_6(m)]$ independently.

We clearly have for any $\kappa>0$

$$\begin{aligned}
I_1(m)&\le\kappa\tau_m^R\|\nabla\mathbf{e}_{h,m}\|_{L_x^2}^2+c(\kappa)\tau_m^R\|\nabla(\mathbf{u}(\mathrm{t}_m^R)-\Pi_h\mathbf{u}(\mathrm{t}_m^R)\|_{L_x^2}^2\\
&\le\kappa\tau_m^R\|\nabla\mathbf{e}_{h,m}\|_{L_x^2}^2+c(\kappa)\tau h^{2\beta}\|\mathbf{u}(\mathrm{t}_m^R)\|_{W_x^{1+\beta,2}}^2
\end{aligned}$$

due to the $W_x^{1,2}$-stability of Π_h, cf. (2.12). Note that the expectation of the last term may be bounded with the help of Lemma 3.1 (c) using $\mathrm{t}_m^R\le\mathrm{t}_R$. We continue with $I_2(m)$, for which we obtain

$$\begin{aligned}
I_2(m)&\le\kappa\tau_m^R\|\nabla\Pi_h\mathbf{e}_{h,m}\|_{L_x^2}^2+c(\kappa)\int_{\mathrm{t}_{m-1}^R}^{\mathrm{t}_m^R}\|\nabla(\mathbf{u}(\mathrm{t}_m^R)-\mathbf{u}(\sigma))\|_{L_x^2}^2\,\mathrm{d}\sigma\\
&\le\kappa\tau_m^R\|\nabla\mathbf{e}_{h,m}\|_{L_x^2}^2+\kappa\tau_m^R\|\nabla(\mathbf{u}(\mathrm{t}_m^R)-\Pi_h\mathbf{u}(\mathrm{t}_m^R))\|_{L_x^2}^2\\
&\qquad+c(\kappa)\tau^{1+2\alpha}\|\nabla\mathbf{u}\|_{C^\alpha([\mathrm{t}_{m-1}^R,\mathrm{t}_m^R];L_x^2)}^2,
\end{aligned}$$

where the last term can be controlled by Corollary 3.4 and the second last one by (2.12) and 3.1 (c) as for $I_2(m)$. We proceed by

$$\begin{aligned}
I_3(m) &= -\int_{t_{m-1}^R}^{t_m^R}\int_{\mathcal{O}}\Big(\mathbf{u}(t_m^R)\otimes\mathbf{u}(t_{m-1}^R)-\mathbf{u}(\sigma)\otimes\mathbf{u}(\sigma)\Big):\nabla\Pi_h\mathbf{e}_{h,m}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&\leq \kappa\tau_m^R\|\nabla\Pi_h\mathbf{e}_{h,m}\|_{L_x^2}^2 + c(\kappa)\int_{t_{m-1}^R}^{t_m^R}\|\mathbf{u}(t_m^R)\otimes\mathbf{u}(t_{m-1}^R)-\mathbf{u}(\sigma)\otimes\mathbf{u}(\sigma)\|_{L_x^2}^2\,\mathrm{d}\sigma\\
&\leq \kappa\tau_m^R\|\nabla\mathbf{e}_{h,m}\|_{L_x^2}^2 + \kappa\tau_m^R\|\nabla(\mathbf{u}(t_m^R)-\Pi_h\mathbf{u}(t_m^R))\|_{L_x^2}\\
&\quad + c(\kappa)\tau^{1+2\alpha}\|\mathbf{u}\|^2_{L^\infty((t_{m-1}^R,t_m^R)\times\mathcal{O})}\|\mathbf{u}\|^2_{C^\alpha([t_{m-1}^R,t_m^R];L_x^2)}\\
&\leq \kappa\tau_m^R\|\nabla\mathbf{e}_{h,m}\|_{L_x^2}^2 + c(\kappa)\tau h^{2\beta}\|\mathbf{u}(t_m^R)\|^2_{W_x^{1+\beta,2}}\\
&\quad + c(\kappa)\tau^{1+2\alpha}\|\mathbf{u}\|^2_{L^\infty(t_{m-1}^R,t_m^R;W_x^{2,2})}\|\mathbf{u}\|^2_{C^\alpha([t_{m-1}^R,t_m^R];L_x^2)},
\end{aligned}$$

where we used Sobolev's embedding $W^{2,2}(\mathcal{O},\mathbb{R}^2)\hookrightarrow L^\infty(\mathcal{O},\mathbb{R}^2)$. We can apply again Lemma 3.1 (c) and Corollary 3.4 to the last term. The term $I_4(m)$ can be decomposed as

$$\begin{aligned}
I_4(m) &= I_4^1(m)+I_4^2(m)+I_4^3(m),\\
I_4^1(m) &= -\tau_m^R\int_{\mathcal{O}}(\mathbf{u}(t_{m-1}^R\cdot\nabla)\mathbf{e}_{h,m}\cdot\big(\mathbf{u}(t_m^R)-\Pi_h\mathbf{u}(t_m^R)\big)\,\mathrm{d}x,\\
I_4^2(m) &= \tau_m^R\int_{\mathcal{O}}(\mathbf{e}_{h,m-1}\cdot\nabla)\mathbf{e}_{h,m}\cdot\big(\mathbf{u}(t_m^R)-\Pi_h\mathbf{u}(t_m^R)\big)\,\mathrm{d}x\\
&\qquad + \tau_m^R\int_{\mathcal{O}}(\mathrm{div}\mathbf{e}_{h,m-1})\mathbf{e}_{h,m}\cdot\big(\mathbf{u}(t_m^R)-\Pi_h\mathbf{u}(t_m^R)\big)\,\mathrm{d}x,\\
I_4^3(m) &= -\tau_m^R\int_{\mathcal{O}}(\mathbf{e}_{h,m-1}\cdot\nabla)\Pi_h\mathbf{e}_{h,m}\cdot\mathbf{u}(t_m^R)\,\mathrm{d}x\\
&\qquad - \tau_m^R\int_{\mathcal{O}}(\mathrm{div}\mathbf{e}_{h,m-1})\Pi_h\mathbf{e}_{h,m}\cdot\mathbf{u}(t_m^R)\,\mathrm{d}x.
\end{aligned}$$

We obtain for any $\kappa>0$

$$\begin{aligned}
I_4^1(m) &\leq \tau_m^R\|\nabla\mathbf{e}_{h,m}\|_{L_x^2}^2\|\mathbf{u}(t_{m-1}^R)\|_{L_x^4}\|\mathbf{u}(t_m^R)-\Pi_h\mathbf{u}(t_m^R)\|_{L_x^4}\\
&\leq c\tau_m^R\|\nabla\mathbf{e}_{h,m}\|_{L_x^2}^2\|\mathbf{u}(t_{m-1}^R)\|_{W_x^{1,2}}\|\mathbf{u}(t_m^R)-\Pi_h\mathbf{u}(t_m^R)\|_{L_x^2}^{\frac12}\|\mathbf{u}(t_m^R)-\Pi_h\mathbf{u}(t_m^R)\|_{W_x^{1,2}}^{\frac12}\\
&\leq c\tau_m^R\|\nabla\mathbf{e}_{h,m}\|_{L_x^2}h^{1+\beta/2}R\|\mathbf{u}(t_m^R)\|_{W_x^{1+\beta,2}}\\
&\leq \kappa\tau_m^R\|\nabla\mathbf{e}_{h,m}\|_{L_x^2}^2 + c(\kappa)h^{2+\beta}R^2\tau_m^R\|\mathbf{u}(t_m^R)\|^2_{W_x^{1+\beta,2}}
\end{aligned}$$

by the embedding $W^{1,2}(\mathcal{O},\mathbb{R}^2)\hookrightarrow L^4(\mathcal{O},\mathbb{R}^2)$, Ladyshenskaya's inequality, the definition of t_m^R, and (2.12). The first term can be absorbed for κ small enough, whereas the second one (in summed form and expectation) is bounded by $h^{2+\beta}R^{12}$ due Lemma 3.1 (c). Similarly, we have

$$
\begin{aligned}
I_4^2(m) &\le \tau_m^R \|\nabla \mathbf{e}_{h,m}\|_{L_x^2} \|\mathbf{e}_{h,m-1}\|_{L_x^4} \|\mathbf{u}(\mathrm{t}_m^R) - \Pi_h \mathbf{u}(\mathrm{t}_m^R)\|_{L_x^4} \\
&\quad + \tau_m^R \|\nabla \mathbf{e}_{h,m-1}\|_{L_x^2} \|\mathbf{e}_{h,m}\|_{L_x^4} \|\mathbf{u}(\mathrm{t}_m^R) - \Pi_h \mathbf{u}(\mathrm{t}_m^R)\|_{L_x^4} \\
&\le c\tau_m^R h^{1+\beta/2} \|\nabla \mathbf{e}_{h,m}\|_{L_x^2} \|\mathbf{e}_{h,m-1}\|_{L_x^2}^{\frac12} \|\nabla \mathbf{e}_{h,m-1}\|_{L_x^2}^{\frac12} \|\mathbf{u}(\mathrm{t}_m^R)\|_{W_x^{1+\beta,2}} \\
&\quad + c\tau_m^R h^{1+\beta/2} \|\nabla \mathbf{e}_{h,m-1}\|_{L_x^2} \|\mathbf{e}_{h,m}\|_{L_x^2}^{\frac12} \|\nabla \mathbf{e}_{h,m}\|_{L_x^2}^{\frac12} \|\mathbf{u}(\mathrm{t}_m^R)\|_{W_x^{1+\beta,2}} \\
&\le \kappa \tau_m^R \Big(\|\nabla \mathbf{e}_{h,m-1}\|_{L_x^2}^2 + \|\nabla \mathbf{e}_{h,m}\|_{L_x^2}^2 \Big) \\
&\quad + c(\kappa)\, \tau_m^R\, h^{4+2\beta} \Big(\max_{1\le n\le m} \|\mathbf{e}_{h,n}\|_{L_x^2}^2 \Big) \|\mathbf{u}(\mathrm{t}_m^R)\|_{W_x^{1+\beta}}^4 .
\end{aligned}
$$

The last term (in summed form and expectation) can be controlled by Lemma 3.1 (c) (with $r = 8$) and Lemma 4.1 (with $q = 2$). Note that we have either have $\mathbf{u}_{h,m} = \mathbf{u}_{h,m}^R$ or $\tau_m^R = 0$. Finally, by definition of t_m^R,

$$
\begin{aligned}
I_4^3(m) &\le \tau_m^R \|\nabla \Pi_h \mathbf{e}_{h,m}\|_{L_x^2} \|\mathbf{e}_{h,m-1}\|_{L_x^4} \|\mathbf{u}(\mathrm{t}_m^R)\|_{L_x^4} \\
&\quad + \tau_m^R \|\nabla \mathbf{e}_{h,m-1}\|_{L_x^2} \|\Pi_h \mathbf{e}_{h,m}\|_{L_x^4} \|\mathbf{u}(\mathrm{t}_m^R)\|_{L_x^4} \\
&\le \tau_m^R \|\nabla \mathbf{e}_{h,m}\|_{L_x^2} \|\mathbf{e}_{h,m-1}\|_{L_x^2}^{\frac12} \|\nabla \mathbf{e}_{h,m-1}\|_{L_x^2}^{\frac12} \|\mathbf{u}(\mathrm{t}_m^R)\|_{W_x^{1,2}} \\
&\quad + \tau_m^R \|\nabla \mathbf{e}_{h,m-1}\|_{L_x^2} \|\Pi_h \mathbf{e}_{h,m}\|_{L_x^2}^{\frac12} \|\nabla \Pi_h \mathbf{e}_{h,m}\|_{L_x^2}^{\frac12} \|\mathbf{u}(\mathrm{t}_m^R)\|_{W_x^{1,2}} \\
&\le \kappa \tau_m^R \Big(\|\nabla \mathbf{e}_{h,m-1}\|_{L_x^2}^2 + \|\nabla \Pi_h \mathbf{e}_{h,m}\|_{L_x^2}^2 \Big) + c_\kappa\, \tau_m^R R^4 \big(\|\Pi_h \mathbf{e}_{h,m}\|_{L_x^2}^2 + \|\mathbf{e}_{h,m-1}\|_{L_x^2}^2 \big) \\
&\le \kappa \tau_m^R \Big(\|\nabla \mathbf{e}_{h,m-1}\|_{L_x^2}^2 + \|\nabla \mathbf{e}_{h,m}\|_{L_x^2}^2 \Big) + c(\kappa)\, \tau_m^R R^4 \big(\|\Pi_h \mathbf{e}_{h,m}\|_{L_x^2}^2 + \|\Pi_h \mathbf{e}_{h,m-1}\|_{L_x^2}^2 \big) \\
&\quad + c(\kappa)\, \tau_m^R \|\nabla (\mathbf{u}(\mathrm{t}_m^R) - \Pi_h \mathbf{u}(\mathrm{t}_m^R))\|_{L_x^2}^2 + c(\kappa)\, \tau_m^R R^4 \|\mathbf{u}(\mathrm{t}_{m-1}^R) - \Pi_h \mathbf{u}(\mathrm{t}_{m-1}^R)\|_{L_x^2}^2 .
\end{aligned}
$$

The second last term will be dealt with by Gronwall's lemma leading to a constant of the form ce^{cR^4}. The final line is bounded by $c(\kappa)\, \tau_m^R R^4\, h^{2\beta} \|\mathbf{u}(\mathrm{t}_{m-1}^R)\|_{W_x^{1+\beta,2}}^2$ using (2.12) and hence can be controlled by Lemma 3.1 (c).

In order to estimate the stochastic term we write

$$
\begin{aligned}
\mathscr{M}_m &= \sum_{n=1}^m I_5(n) = \sum_{n=1}^m \int_{\mathcal{O}} \int_{\mathrm{t}_{n-1}^R}^{\mathrm{t}_n^R} \big(\Phi(\mathbf{u}) - \Phi(\mathbf{u}_{h,n-1}^R) \big) \,\mathrm{d}W \cdot \Pi_h \mathbf{e}_{h,n} \,\mathrm{d}x \\
&= \sum_{n=1}^m \int_{\mathcal{O}} \int_{\mathrm{t}_{n-1}^R}^{\mathrm{t}_n^R} \big(\Phi(\mathbf{u}) - \Phi(\mathbf{u}_{h,n-1}^R) \big) \,\mathrm{d}W \cdot \Pi_h \mathbf{e}_{h,n-1} \,\mathrm{d}x \\
&\quad + \sum_{n=1}^m \int_{\mathcal{O}} \int_{\mathrm{t}_{n-1}^R}^{\mathrm{t}_n^R} \big(\Phi(\mathbf{u}) - \Phi(\mathbf{u}_{h,n-1}^R) \big) \,\mathrm{d}W \cdot \Pi_h (\mathbf{e}_{h,n} - \mathbf{e}_{h,n-1}) \,\mathrm{d}x \\
&= \int_0^{\mathrm{t}_m^R} \sum_{n=1}^M \mathbf{1}_{[t_{n-1},t_n)} \int_{\mathcal{O}} \big(\Phi(\mathbf{u}) - \Phi(\mathbf{u}_{h,n-1}^R) \big) \,\mathrm{d}W \cdot \Pi_h \mathbf{e}_{h,n-1} \,\mathrm{d}x
\end{aligned}
$$

$$+\sum_{n=1}^{m}\int_{\mathcal{O}}\int_{\mathrm{t}_{n-1}^R}^{\mathrm{t}_n^R}\big(\Phi(\mathbf{u})-\Phi(\mathbf{u}_{h,n-1}^R)\big)\,\mathrm{d}W\cdot\Pi_h(\mathbf{e}_{h,n}-\mathbf{e}_{h,n-1})\,\mathrm{d}x$$
$$=: \mathscr{M}^1(\mathrm{t}_m^R)+\mathscr{M}_m^2. \tag{4.7}$$

Since the process $(\mathscr{M}^1(t\wedge \mathrm{t}_R))_{t\geq 0}$ is an $(\mathfrak{F}_t)$-martingale we gain by the Burgholder-Davis-Gundy inequality (using that $\mathrm{t}_M^R \leq \mathrm{t}_R$ by definition)

$$\mathbb{E}\bigg[\max_{1\leq m\leq M}\big|\mathscr{M}^1(\mathrm{t}_m^R)\big|\bigg]\leq \mathbb{E}\bigg[\sup_{s\in[0,\mathrm{t}_M^R]}\big|\mathscr{M}^1(s)\big|\bigg]\leq \mathbb{E}\bigg[\sup_{s\in[0,T]}\big|\mathscr{M}^1(s\wedge \mathrm{t}_R)\big|\bigg]$$
$$\leq c\,\mathbb{E}\bigg[\bigg(\int_0^{T\wedge \mathrm{t}_R}\sum_{n=1}^{M}\mathbf{1}_{[t_{n-1},t_n)}\|\Phi(\mathbf{u})-\Phi(\mathbf{u}_{h,n-1}^R)\|_{L_2(\mathfrak{U},L_x^2)}^2\|\Pi_h\mathbf{e}_{h,n-1}\|_{L_x^2}^2\,\mathrm{d}t\bigg)^{\frac{1}{2}}\bigg]$$
$$\leq c\,\mathbb{E}\bigg[\max_{1\leq n\leq M}\|\Pi_h\mathbf{e}_{h,n}\|_{L_x^2}\bigg(\int_0^{T\wedge \mathrm{t}_R}\sum_{n=1}^{M}\mathbf{1}_{[t_{n-1},t_n)}\|\Phi(\mathbf{u})-\Phi(\mathbf{u}_{h,n-1}^R)\|_{L_2(\mathfrak{U},L_x^2)}^2\,\mathrm{d}t\bigg)^{\frac{1}{2}}\bigg]$$
$$\leq \kappa\,\mathbb{E}\bigg[\max_{1\leq n\leq M}\|\Pi_h\mathbf{e}_{h,n}\|_{L_x^2}^2\bigg]+c_\kappa\,\mathbb{E}\bigg[\int_0^{T\wedge \mathrm{t}_R}\sum_{n=1}^{M}\mathbf{1}_{[t_{n-1},t_n)}\|\mathbf{u}-\mathbf{u}_{h,n-1}^R\|_{L_x^2}^2\,\mathrm{d}t\bigg].$$

Here, we also used (2.1) as well as Young's inequality for $\kappa > 0$ arbitrary. Since $\mathbf{u}_{h,n-1}^R=\mathbf{e}_{h,n-1}+\mathbf{u}(\mathrm{t}_{n-1}^R)$ is $V_{\mathrm{div}}^{h,i}(\mathcal{O},\mathbb{R}^2)$-valued, we further estimate

$$\mathbb{E}\bigg[\max_{1\leq m\leq M}\big|\mathscr{M}^1(\mathrm{t}_m^R)\big|\bigg]$$
$$\leq \kappa\,\mathbb{E}\bigg[\max_{1\leq n\leq M}\|\Pi_h\mathbf{e}_{h,n}\|_{L_x^2}^2\bigg]+c(\kappa)\,\mathbb{E}\bigg[\int_0^{T\wedge \mathrm{t}_R}\sum_{n=1}^{M}\mathbf{1}_{[t_{n-1},t_n)}\|\mathbf{u}-\mathbf{u}(\mathrm{t}_{n-1}^R)\|_{L_x^2}^2\,\mathrm{d}t\bigg]$$
$$+\,c(\kappa)\,\mathbb{E}\bigg[\int_0^{T\wedge \mathrm{t}_R}\sum_{n=1}^{M}\mathbf{1}_{[t_{n-1},t_n)}\|\mathbf{u}(\mathrm{t}_{n-1}^R)-\Pi_h\mathbf{u}(\mathrm{t}_{n-1}^R)\|_{L_x^2}^2\,\mathrm{d}t\bigg]$$
$$+\,c(\kappa)\,\mathbb{E}\bigg[\int_0^{T\wedge \mathrm{t}_R}\sum_{n=1}^{M}\mathbf{1}_{[t_{n-1},t_n)}\|\Pi_h\mathbf{e}_{h,n-1}\|_{L_x^2}^2\,\mathrm{d}t\bigg]$$

We bound the last term by

$$\mathbb{E}\bigg[\int_0^{T\wedge \mathrm{t}_R}\sum_{n=1}^{M}\mathbf{1}_{[t_{n-1},t_n)}\|\Pi_h\mathbf{e}_{h,n-1}\|_{L_x^2}^2\,\mathrm{d}t\bigg]\leq \mathbb{E}\bigg[\sum_{n=1}^{M+1}\tau_n^R\|\Pi_h\mathbf{e}_{h,n-1}\|_{L_x^2}^2\,\mathrm{d}t\bigg]$$
$$\leq \mathbb{E}\bigg[\sum_{n=0}^{M}\tau_n^R\|\Pi_h\mathbf{e}_{h,n}\|_{L_x^2}^2\,\mathrm{d}t\bigg]$$

using that $\mathfrak{t}_R \wedge t_M \le \mathfrak{t}_{M+1}^R$ and $\tau_n^R \le \tau_{n-1}^R$ with $\tau_0^R := \tau$. Applying (2.12) as well as Lemma 3.1 (b) and Corollary 3.4 (b) we gain

$$\begin{aligned}
&\mathbb{E}\bigg[\max_{1\le m\le M}\big|\mathscr{M}^1(\mathfrak{t}_m^R)\big|\bigg]\\
&\quad\le \kappa\,\mathbb{E}\bigg[\max_{1\le n\le M}\|\Pi_h\mathbf{e}_{h,n}\|_{L_x^2}^2\bigg]+c(\kappa)\,\mathbb{E}\bigg[\sum_{n=0}^{M}\tau_n^R\|\Pi_h\mathbf{e}_{h,n}\|_{L_x^2}^2\bigg]\\
&\qquad+c(\kappa)\tau^{2\alpha}\mathbb{E}\Big[\|\mathbf{u}\|^2_{C^\alpha([0,T\wedge\mathfrak{t}_R],L_x^2)}\Big]+c(\kappa)h^{1+\beta}\mathbb{E}\bigg[\sup_{t\in[0,T]}\int_{\mathcal{O}}|\nabla\mathbf{u}(t\wedge\mathfrak{t}_R)|^2\,\mathrm{d}x\bigg]\\
&\quad\le \kappa\,\mathbb{E}\bigg[\max_{1\le n\le M}\|\Pi_h\mathbf{e}_{h,n}\|_{L_x^2}^2\bigg]+c(\kappa)\,\mathbb{E}\bigg[\sum_{n=0}^{M}\tau\|\Pi_h\mathbf{e}_{h,n}\|_{L_x^2}^2\bigg]\\
&\qquad+c(\kappa)\tau^{2\alpha}R^{20}+c(\kappa)h^{1+\beta}R^6.
\end{aligned}$$

Similarly: on using Cauchy-Schwartz inequality, Young's inequality, Itô-isometry and (2.1) we have for $\kappa > 0$

$$\begin{aligned}
&\mathbb{E}\bigg[\max_{1\le m\le M}|\mathscr{M}_m^2|\bigg]\\
&\quad\le \mathbb{E}\bigg[\sum_{n=1}^{M}\bigg(\kappa\|\Pi_h(\mathbf{e}_{h,n}-\mathbf{e}_{h,n-1})\|_{L_x^2}^2+c(\kappa)\bigg\|\int_{\mathfrak{t}_{n-1}^R}^{\mathfrak{t}_n^R}\big(\Phi(\mathbf{u})-\Phi(\mathbf{u}_{h,n-1}^R)\big)\,\mathrm{d}W\bigg\|_{L_x^2}^2\bigg)\bigg]\\
&\quad\le \kappa\mathbb{E}\bigg[\sum_{n=1}^{M}\|\Pi_h(\mathbf{e}_{h,n}-\mathbf{e}_{h,n-1})\|_{L_x^2}^2\bigg]+c(\kappa)\mathbb{E}\bigg[\sum_{n=1}^{M}\int_{\mathfrak{t}_{n-1}^R}^{\mathfrak{t}_n^R}\|\mathbf{u}-\mathbf{u}_{h,n-1}^R\|_{L_x^2}^2\,\mathrm{d}t\bigg]\\
&\quad\le \kappa\mathbb{E}\bigg[\sum_{n=1}^{M}\|\Pi_h(\mathbf{e}_{h,n}-\mathbf{e}_{h,n-1})\|_{L_x^2}^2\bigg]+c(\kappa)\,\mathbb{E}\bigg[\sum_{n=1}^{M}\int_{\mathfrak{t}_{n-1}^R}^{\mathfrak{t}_n^R}\|\mathbf{u}-\mathbf{u}(t_{n-1}^R)\|_{L_x^2}^2\,\mathrm{d}t\bigg]\\
&\qquad+c_\kappa\,\mathbb{E}\bigg[\sum_{n=1}^{M}\tau_{n,h}^R\|\mathbf{u}(\mathfrak{t}_{n-1}^R)-\Pi_h\mathbf{u}(\mathfrak{t}_{n-1}^R)\|_{L_x^2}^2\bigg]+c(\kappa)\,\mathbb{E}\bigg[\sum_{n=1}^{M}\tau_n^R\|\Pi_h\mathbf{e}_{h,n-1}\|_{L_x^2}^2\bigg]\\
&\quad\le \kappa\mathbb{E}\bigg[\sum_{n=1}^{M}\|\Pi_h(\mathbf{e}_{h,n}-\mathbf{e}_{h,n-1})\|_{L_x^2}^2\bigg]+c(\kappa)\,\mathbb{E}\bigg[\sum_{n=1}^{M}\tau_n^R\|\Pi_h\mathbf{e}_{h,n-1}\|_{L_x^2}^2\bigg]\\
&\qquad+c(\kappa)\tau^{2\alpha}R^{20}+c(\kappa)h^2R^6
\end{aligned}$$

as a consequence Lemma 3.1 (b) (using also (2.11)) and Corollary 3.4 (b).

Finally, we have by (2.13)

$$\begin{aligned}
I_6(m)&=\int_{\mathfrak{t}_{m-1}^R}^{\mathfrak{t}_m^R}\int_{\mathcal{O}}\big(\pi-\Pi_h^\pi\pi\big)\mathrm{div}\Pi_h\mathbf{e}_{h,m}\,\mathrm{d}x\,\mathrm{d}\sigma\\
&\le c(\kappa)\int_{\mathfrak{t}_{m-1}^R}^{\mathfrak{t}_m^R}\|\pi-\Pi_h^\pi\pi\|_{L_x^2}^2\,\mathrm{d}\sigma+\kappa\tau\,\|\nabla\Pi_h\mathbf{e}_{h,m}\|_{L_x^2}^2
\end{aligned}$$

$$\leq c(\kappa)h^2 \int_{t_{m-1}^R}^{t_m^R} \|\nabla\pi\|_{L_x^2}^2 \,\mathrm{d}\sigma + \kappa\tau\, \|\nabla \mathbf{e}_{h,m}\|_{L_x^2}^2,$$

where $\kappa > 0$ is arbitrary. The first term is summable in expectation with bound $c(\kappa)h^2R^{12}$ due to Lemma 3.3 (b) and the last one can be absorbed. Collecting all estimates, choosing κ small enough implies

$$\mathbb{E}\bigg[\max_{1\leq m\leq M} \|\Pi_h \mathbf{e}_{h,m}\|_{L_x^2}^2 + \sum_{m=1}^M \tau_m^R \|\Pi_h(\mathbf{e}_{h,m} - \mathbf{e}_{h,m-1})\|_{L_x^2}^2 + \sum_{m=1}^M \tau_m^R \|\nabla \mathbf{e}_{h,m}\|_{L_x^2}^2\bigg)\bigg]$$

$$\leq cR^4\,\mathbb{E}\left[\sum_{m=1}^M \tau_m^R \max_{1\leq n\leq m} \|\mathbf{e}_{h,n}\|_{L_x^2}^2\right] + ce^{cR^4}\left(h^{2\beta} + \tau^{2\alpha}\right).$$

Controlling the error between $\mathbf{e}_{h,m}$ and $\Pi_h\mathbf{e}_{h,m}$ by (2.11) as well as Lemma 3.1 (b) and applying Gronwall's lemma yields the claim. □

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. S. Agmon, A. Douglis, and L. Nirenberg: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Comm. Pure Appl. Math. 12 (1959), 623–727.
2. A. Bensoussan and J. Frehse: Local solutions for stochastic Navier–Stokes equations. M2AN Math. Model. Numer. Anal. 34 (2000), 241–273. (Special issue for R. Temam's 60th birthday)
3. H. Bessaih, A. Millet: Strong L^2 convergence of time numerical schemes for the stochastic 2D Navier-Stokes equation, IMA Journal of Numerical Analysis, 39-4 (2019), pp. 2135–2167.
4. H. Bessaih, A. Millet: Strong rates of convergence of space-time discretization schemes for the 2D Navier-Stokes equations with additive noise, Stoch. Dyn. 22, 2240005. (2022)
5. D. Breit & A. Dodgson: Convergence rates for the numerical approximation of the 2D stochastic Navier–Stokes equations. Numer. Math. 147, 553–578. (2021)
6. S.C. Brenner, L.R. Scott: The mathematical theory of finite element methods Springer. (2002)
7. F. Brezzi, M. Fortin: Mixed and hybrid finite element methods, Springer. (1991)
8. Z. Brzeźniak, S. Peszat: Strong local and global solutions for stochastic Navier–Stokes equations. Infinite Dimensional Stochastic Analysis (Amsterdam), Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad. Wet., vol. 52, R. Neth. Acad. Arts Sci., Amsterdam, 85–98. (1999/2000)
9. Z. Brzeźniak, E. Carelli, A. Prohl (2013): Finite-element-based discretizations of the incompressible Navier–Stokes equations with multiplicative random forcing, IMA J. Num. Anal. 33, pp. 771–824.
10. M. Capiński: A note on uniqueness of stochastic Navier-Stokes equations, Univ. Iagell. Acta Math. 30 (1993), pp. 219–228.
11. M. Capiński, N. J. Cutland: Stochastic Navier-Stokes equations, Acta Appl. Math. 25 (1991), pp. 59–85.

12. E. Carelli, J. A. Prohl (2012): Rates of convergence for discretizations of the stochastic incom- pressible Navier-Stokes equations. SIAM J. Numer. Anal. 50(5), pp. 2467–2496.
13. G. Da Prato, J. Zabczyk (1992): Stochastic Equations in Infinite Dimensions, Encyclopedia Math. Appl., vol. 44, Cambridge University Press, Cambridge.
14. X. Feng, A. Prohl, L. Vo: Optimally convergent mixed finite element methods for the stochastic Stokes equations. IMA J. Num. Anal. (2021).
15. G. P. Galdi: An Introduction to the Mathemaical Theory of the Navier–Stokes equations. Steady-Sate Problems. 2nd Edition. Springer Monographs in Mathematics. Springer, New York Dordrecht Heidelberg London. (2011)
16. G. P. Galdi, C. G. Simader, H. Sohr: A class of solutions to stationary Stokes and Navier-Stokes equations with boundary data in $W^{-1/q,q}$, Math. Ann. 331, 41–74. (2005)
17. V. Girault, P.A. Raviart: Finite element methods for Navier-Stokes equations, Springer, New York (1986).
18. N. Glatt-Holz, M. Ziane: Strong Pathwise Solutions of the Stochastic Navier-Stokes System. Adv. Diff. Equ. 14, 567–600. (2009)
19. J.G. Heywood & R. Rannacher: Finite element approximation of the nonstationary Navier–Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. SIAM J. Numer. Anal. 19, 275–311. (1982)
20. J. U. Kim: Strong Solutions of the Stochastic Navier–Stokes Equations in $\mathbb{R}^3$. Indiana Univ. Math. J. 59, 1417–1450. (2010)
21. N. V. Krylov: A $W^{n,2}$-theory of the Dirichlet problem for SPDEs in general smooth domains. Probab. Theory Rel. Fields 98, 389–421. (1994)
22. I. Kukavica, V. Vicol. On moments for strong solutions of the 2D stochastic Navier–Stokes equations in a bounded domain. Asympt. Anal. 90 (2014), no. 3-4, 189–206.
23. S. Kuksin and A. Shirikyan. Mathematics of two-dimensional turbulence, volume 194 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2012.
24. R. Mikulevicius: On strong H^1_2-solutions of stochastic Navier–Stokes equations in a bounded domain. SIAM J. Math. Anal. Vol. 41, No. 3, pp. 1206–1230. (2009)
25. C. Prévôt, M. Röckner (2007): A concise course on stochastic partial differential equations. Lecture Notes in Mathematics, 1905. Springer, Berlin.
26. J. Printems: On the discretization in time of parabolic stochastic partial differential equations. Math. Mod. Numer. Anal. 35, pp. 1055–1078. (2001)
27. V. A. Solonnikov: A priori estimates for solutions of second-order equations of parabolic type. (Russian) Trudy Mat. Inst. Steklov. 70, 133–212. (1964)
28. V. A. Solonnikov: Estimates for solutions of a non-stationary linearized system of Navier–Stokes equations, (Russian) Trudy Mat. Inst. Steklov. 70, 213–317. (1964)
29. R. Temam: Navier-Stokes equations — Theory and numerical analysis, AMS Chelsea Publishing, AMS (1984).
30. J. Wichmann: Temporal regularity of symmetric stochastic p-Stokes systems. Preprint at arXiv:2209.02796.