



On the Computation of Geometric Features of Spectra of Linear Operators on Hilbert Spaces

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Abstract

Computing spectra is a central problem in computational mathematics with an abundance of applications throughout the sciences. However, in many applications gaining an approximation of the spectrum is not enough. Often it is vital to determine geometric features of spectra such as Lebesgue measure, capacity or fractal dimensions, different types of spectral radii and numerical ranges, or to detect gaps in essential spectra and the corresponding failure of the finite section method. Despite new results on computing spectra and the substantial interest in these geometric problems, there remain no general methods able to compute such geometric features of spectra of infinite-dimensional operators. We provide the first algorithms for the computation of many of these long-standing problems (including the above). As demonstrated with computational examples, the new algorithms yield a library of new methods. Recent progress in computational spectral problems in infinite dimensions has led to the solvability complexity index (SCI) hierarchy, which classifies the difficulty of computational problems. These results reveal that infinite-dimensional spectral problems yield an intricate infinite classification theory determining which spectral problems can be solved and with which type of algorithm. This is very much related to S. Smale's comprehensive program on the foundations of computational mathematics initiated in the 1980s. We classify the computation of geometric features of spectra in the SCI hierarchy, allowing us to precisely determine the boundaries of what computers can achieve (in any model of computation) and prove that our algorithms are optimal. We also provide a new universal technique for establishing lower bounds in the SCI hierarchy, which both greatly simplifies previous SCI arguments and allows new, formerly unattainable, classifications.

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1 Introduction

This paper resolves open computational spectral problems related to geometric features of spectra of operators. In other words, we consider the following problem:

Are there algorithms that given a bounded¹ operator $A \in \mathcal{B}(l^2(\mathbb{N}))$, approximate key geometric features (e.g. spectral gaps, notions of sizes and capacity, measures, topological features such as fractal dimensions, etc.) of the set $\text{Sp}(A)$ from a matrix representation of A ?

To answer this question, we use the newly established solvability complexity index (SCI) hierarchy [18, 51, 91], a classification tool that determines the boundaries of what is computationally possible. Classifying spectral problems and providing a library of optimal algorithms² remains largely uncharted territory in the foundations of computational mathematics. In exploring this territory, there will, necessarily, have to be many different types of algorithms, as different structures on the various classes of operators and different spectral properties require different techniques.

A famous example of the above question is the almost Mathieu operator on $l^2(\mathbb{Z})$ (see Sect. 4.4):

$$(H_\alpha x)_n = x_{n-1} + x_{n+1} + 2\lambda \cos(2\pi n\alpha)x_n,$$

which induces the Hofstadter butterfly [92]. The almost Mathieu operator plays an important role in physics [104], arising in the study of the quantum Hall effect [160], and has become a laboratory for exploring the spectral properties of ergodic Schrödinger operators [95]. When α is irrational, the Lebesgue measure of the spectrum is $4|\lambda| - |\lambda|$. This formula was conjectured based on the numerical work of Aubry and André [8] and became one of B. Simon’s problems for the twenty-first century [146]. It was later proven by Avila and Krikorian [11]. Similarly, M. Kac’s “Ten Martini Problem”, that the spectrum is a Cantor set for all irrational α and $\lambda > 0$, was conjectured by Azbel [13] and also became one of B. Simon’s problems. This problem attracted a host of numerical and analytical work (see the summary in [104]), before being proven by Avila and Jitomirskaya [9]. In both of these examples, we see a crucial interplay between computation, conjecture, and mathematical proof. The

¹ Many of our algorithms can also be extended to unbounded operators.

² For precise notions of algorithm, see Sect. 5.

above geometric features of spectra play an important role in the physics of the underlying quantum system [90, 99, 100, 147]. The almost Mathieu operator is by no means unique in this regard, and there is a growing literature on computational studies of geometric features of spectra in diverse areas of physics [14, 68, 83, 94, 103, 106, 110, 120, 125, 133, 138, 139, 156, 161].

However, there is a current lack of *rigorous computational theory* and *convergence analysis*, and no known algorithms can tackle general cases. Moreover, the foundations of computation (i.e. what is and what is not computationally possible) for computing geometric features of spectra are almost entirely unexplored. We solve these open problems and others by providing algorithms that compute geometric features of spectra and by classifying the computational problems in the SCI hierarchy.

1.1 The SCI Hierarchy

The SCI hierarchy has recently been used to resolve the problem of computing spectra of general bounded operators in infinite dimensions [18, 91] and is now being used to explore the foundations of computation in many diverse areas of mathematics [2, 15, 16, 19–23, 30, 52, 53, 55, 57, 59, 60, 64, 140, 141, 166].³ Whilst for some classes of operators one can compute spectra with error control [54, 60, 64], a potentially surprising consequence is that, for general operators, one needs several successive limits to compute the spectrum. Since traditional approaches are dominated by techniques based on one limit, this explains why many computational spectral problems remain unsolved and opens the door to an infinite classification theory. Moreover, this phenomenon is not just restricted to spectral problems but is shared by other areas of computational mathematics. An example is S. Smale’s problem of root-finding of polynomials with rational maps [149], which also requires several successive limits as established by McMullen [115, 116] and Doyle and McMullen [70]. These results can be expressed in terms of the SCI hierarchy [18], which generalises Smale’s seminal work [148, 150] with Blum et al. [28, 29, 66], and his program on the foundations of scientific computing and the existence of algorithms. Many other problems in the foundations of computations, such as the work by Weinberger [167], can also be viewed in the context of the SCI hierarchy.

The SCI hierarchy is further motivated by computer-assisted proofs. Computer-assisted proofs are rapidly becoming an essential part of modern mathematics [86] and, perhaps surprisingly, non-computable problems can be used in computer-assisted proofs. Examples include the recent proof of Kepler’s conjecture (Hilbert’s 18th problem) [87, 88] on optimal packings of 3-spheres, led by T. Hales, and the Dirac–Schwinger conjecture on the asymptotic behaviour of ground states of certain Schrödinger operators, proven in a series of papers by Fefferman and Seco [72–80]. Both of these proofs rely on computing non-computable problems. This apparent paradox can be explained by the SCI hierarchy (the Σ_1^A and Π_1^A classes described below become available for computer-assisted proofs); Hales, Fefferman and Seco implicitly

³ For related work on practical infinite-dimensional numerical linear algebra, see [63, 85, 93, 126–130, 166], and for rigorous data-driven algorithms for spectral properties of Koopman operators (operators on infinite-dimensional spaces that globally linearise nonlinear dynamical systems), see [56, 58, 65].

prove Σ_1^A classifications in the SCI hierarchy in their papers. Some of the problems we consider also lie in $\Sigma_1^A \cup \Pi_1^A$, meaning that they can be used for computer-assisted proofs.

1.2 The Problems Addressed in this Paper

The algorithms we provide are sharp in the SCI hierarchy, meaning that they realise the boundaries of what computers can achieve. Table 1 provides a summary of the main SCI classifications of this paper. The main theorems are contained in Sect. 3, including further motivations and classifications for different classes of operators. We provide resolutions to the following problems:

- (i) *Computing spectral radii, essential spectral radii, polynomial operator norms and capacity of spectra.* The spectral radius is perhaps the most basic geometric property of spectra and arises in stability analysis. We show that computing the spectral radius is high up in the SCI hierarchy for non-normal operators. In fact, it has the same classification in the SCI hierarchy for general bounded operators as that of computing the spectrum itself. Classifications are given for different types of operators (e.g. known column decay, control on resolvent norms) and also for the essential spectral radius. In many cases, the problem of computing polynomial operator norms is easier in the sense of SCI hierarchy. We also consider the problem of computing the logarithmic capacity of the spectrum, following the work of Halmos [89], which has applications in orthogonal polynomials, approximation theory and when studying the convergence of Krylov methods (see, for example, the work of Nevanlinna [121–123] and Miekkala and Nevanlinna [117]).
- (ii) *Computing essential numerical ranges, gaps in essential spectra, and determining whether spectral pollution occurs on sets.* We provide classification results for the essential numerical range, which also hold in the case of unbounded operators. In connection with computing spectra, there has been a substantial effort in studying the finite section method and locating gaps in essential spectra of operators (see the discussion in Sect. 3.4). When using the finite section method to approximate spectra of self-adjoint operators, spurious eigenvalues, known as spectral pollution, can occur anywhere within these gaps. Paradoxically, we show that determining if spectral pollution occurs on a given set is strictly harder in the sense of the SCI hierarchy than computing the spectrum itself. Hence, computing a failure flag for the finite section method is, in a certain sense, strictly harder than solving the original problem for which it was designed. Moreover, we establish the SCI of detecting gaps in essential spectra of self-adjoint operators, a problem that arises in areas such as perturbation theory and defect models.
- (iii) *Computing Lebesgue measure of spectra and pseudospectra, and determining if the spectrum is Lebesgue null.* An important property of the spectrum is its Lebesgue measure, with recent progress in the field of Schrödinger operators with random or almost periodic potentials [9, 11, 12, 17, 135]. If the spectrum of an operator is Lebesgue null; then, this implies the absence of absolutely continuous spectra,⁴

⁴ For algorithms that compute spectral measures and decompositions, see [53, 61, 63] and their recent physical applications in [62, 97].

which is related to transport properties if the operator represents a Hamiltonian. Whilst results are known for specific one-dimensional examples such as the almost Mathieu operator [11] or the Fibonacci Hamiltonian [154], very little is known in the general case or higher dimensions. This is reflected by the difficulty of performing rigorous numerical studies, despite many examples studied in the physics literature (see the references in [10, 24, 147]). We provide the first algorithms for computing the Lebesgue measure of spectra and pseudospectra, and determining whether the spectrum is Lebesgue null, for many different classes of operators.

- (iv) *Computing fractal dimensions of spectra.* Fractal dimensions of spectra are important in many applications. For example, in quantum mechanics, they lead to upper bounds on the spreading of wavepackets and are related to time-dependent quantities associated with wave functions [90, 99, 100]. Fractal spectra appear in a wide variety of contexts, such as exciting new results in multilayer materials (e.g. bilayer graphene) [68, 83, 94, 133], strained materials [120, 139] or quasicrystals [14, 103, 106, 156]. Another well-studied area where fractal spectral properties appear is optics [125, 138], following the analytical and numerical work of Berry and coauthors [25–27]. Despite the physical importance of fractal dimensions, analytical results are known only for a limited number of specific models. Moreover, there are currently no algorithms for computing fractal dimensions of spectra for general operators, or even tridiagonal self-adjoint operators. We provide the first algorithms for computing the box-counting and Hausdorff dimensions of spectra for many different classes of operators.

1.3 Contributions to the SCI Hierarchy Itself

Our final contribution is a new tool to prove lower bounds (impossibility results) in the SCI hierarchy. This is crucial for some of the classifications of the above problems and holds regardless of the model of computation. We show that for a certain special class of combinatorial problems, the SCI hierarchy is equivalent to the Baire hierarchy from descriptive set theory. (This equivalence does not hold in general.) By embedding these combinatorial problems into spectral problems,⁵ we provide the first technique for dealing with problems that have SCI greater than three and also greatly simplify the proofs of results lower down in the SCI hierarchy. However, it should be stressed that this is not a paper on descriptive set theory or mathematical logic. Our discussion is entirely self-contained and written for a wide audience from a primarily computational background.

1.4 Outline of Paper

In Sect. 2, we provide a brief summary of the SCI hierarchy and define the classes of operators for the interpretation of Table 1 and the main results. A detailed discussion of the SCI hierarchy is delayed until Sect. 5.1. In Sect. 3, we summarise our main results on the classification of computational spectral problems. Computational examples are

⁵ This technique is not restricted to spectral problems—it can be adapted to other scenarios.

then given in Sect. 4. For example, we provide numerical evidence that a portion of the spectrum of the graphical Laplacian on an infinite Penrose tile is Lebesgue null and fractal, with a fractal dimension of approximately 0.8, and that the whole spectrum has a logarithmic capacity of approximately 2.26. Mathematical preliminaries, including definitions of the SCI hierarchy and the new tool to provide lower bounds in the SCI hierarchy, are presented in Sect. 5. Proofs of our results are given in Sects. 6–9. To make the paper self-contained, we include a short appendix on the results/algorithms of [64], which are used in some of our proofs. Pseudocode for many of the new algorithms is provided in “Appendix B”.

2 Essentials of the SCI Hierarchy and Preliminary Definitions

2.1 A Brief Introduction to the SCI Hierarchy

2.1.1 Description of the SCI Hierarchy

First, we define a computational problem. The basic objects of a computational problem are:

- Ω , called the *domain*,
- Λ , a set of complex-valued functions on Ω , called the *evaluation set*,
- (\mathcal{M}, d) , a metric space,
- $\Xi : \Omega \rightarrow \mathcal{M}$ the *problem function*.

The set Ω is the set of objects that give rise to our computational problems, the goal being to compute the problem function $\Xi : \Omega \rightarrow \mathcal{M}$. The set Λ is the collection of functions that provide us with the information we are allowed to read as input to the algorithm. This leads to the following definition:

Definition 2.1 (*Computational problem*) Given a domain Ω ; an evaluation set Λ , such that for any $A_1, A_2 \in \Omega$, $A_1 = A_2$ if and only if $f(A_1) = f(A_2)$ for all $f \in \Lambda$; a metric space \mathcal{M} ; and a problem function $\Xi : \Omega \rightarrow \mathcal{M}$, we call the collection $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ a computational problem.

The definition of a computational problem is deliberately general. The SCI of a computational problem is the smallest number of successive limits needed to compute the solution to the problem. We call a corresponding suitably indexed family of algorithms a ‘tower of algorithms’. In addition, we will use finer notions of error control. For example, consider the case that (\mathcal{M}, d) is the space of non-empty compact subsets of \mathbb{C} , equipped with the Hausdorff metric. Then, the SCI hierarchy [18, 51] can be described as follows.

The SCI hierarchy Given a collection \mathcal{C} of computational problems,

- (i) $\Delta_0^\alpha = \Pi_0^\alpha = \Sigma_0^\alpha$ is the set of problems that can be computed in finite time (the SCI = 0). In other words, \exists an algorithm Γ such that $\Gamma(A) = \Xi(A), \forall A \in \Omega$.
- (ii) Δ_1^α is the set of problems that can be computed using one limit (the SCI = 1) with control of the error, i.e. \exists a sequence of algorithms $\{\Gamma_n\}$ such that $d(\Gamma_n(A), \Xi(A)) \leq 2^{-n}, \forall A \in \Omega$.

Table 1 Summary of the main results for the readable information Λ_1 consisting of matrix values

| Description of problem | SCI hierarchy classification | Theorem |
|--|--|---------|
| Computing the spectral radius. | Varies. e.g. Normal operators: $\in \Sigma_1^A, \notin \Delta_1^G$ Controlled resolvent: $\in \Sigma_2^A, \notin \Delta_2^G$ General bounded operators: $\in \Pi_3^A, \notin \Delta_3^G$ | 3.3 |
| Computing the essential spectral radius. | Varies. e.g. Most classes: $\in \Pi_2^A, \notin \Delta_2^G$ General bounded operators: $\in \Pi_3^A, \notin \Delta_3^G$ | 3.6 |
| Computing polynomial operator norms. | With bounded dispersion: $\in \Sigma_1^A, \notin \Delta_1^G$ Without bounded dispersion: $\in \Sigma_2^A, \notin \Delta_2^G$ | 3.7 |
| Computing the capacity of the spectrum. | With bounded dispersion: $\in \Pi_2^A, \notin \Delta_2^G$ Without bounded dispersion: $\in \Pi_3^A, \notin \Delta_3^G$ | 3.7 |
| Computing gaps in the essential spectrum. | $\in \Sigma_3^A, \notin \Delta_3^G$ | 3.10 |
| Computing the essential numerical range. | $\in \Pi_2^A, \notin \Delta_2^G$ | 3.10 |
| Determining whether spectral pollution can occur on a set (i.e. failure of finite section method). | $\in \Sigma_3^A, \notin \Delta_3^G$ | 3.10 |
| Computing the Lebesgue measure of the spectrum. | Varies. e.g. Bounded dispersion: $\in \Pi_2^A, \notin \Delta_2^G$ Self-adjoint and general bounded: $\in \Pi_3^A, \notin \Delta_3^G$ | 3.13 |
| Computing the Lebesgue measure of the pseudospectrum. | Varies. e.g. Bounded dispersion: $\in \Sigma_1^A, \notin \Delta_1^G$ Self-adjoint and general bounded: $\in \Sigma_2^A, \notin \Delta_2^G$ | 3.14 |
| Determining whether the Lebesgue measure of the spectrum is zero. | Varies. e.g. Bounded dispersion: $\in \Pi_3^A, \notin \Delta_3^G$ Self-adjoint and general bounded: $\in \Pi_4^A, \notin \Delta_4^G$ | 3.16 |
| Computing the box-counting dimension of the spectrum (when it exists). | Varies. e.g. Bounded dispersion: $\in \Pi_2^A, \notin \Delta_2^G$ Self-adjoint: $\in \Pi_3^A, \notin \Delta_3^G$ | 3.18 |
| Computing the Hausdorff dimension of the spectrum. | Varies. e.g. | 3.18 |

Table 1 continued

| Description of problem | SCI hierarchy classification | Theorem |
|------------------------|---|---------|
| | Bounded dispersion: $\in \Sigma_3^A, \notin \Delta_3^G$ | |
| | Self-adjoint: $\in \Sigma_4^A, \notin \Delta_4^G$ | |

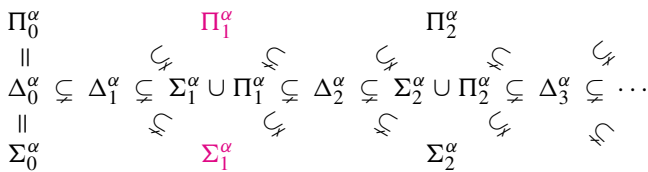
The main theorems contain classifications for different classes of operators (see Table 3 for the operator classes)

- (iii) Σ_1^α : We have $\Delta_1^\alpha \subset \Sigma_1^\alpha \subset \Delta_2^\alpha$ and Σ_1^α is the set of problems for which \exists a sequence of algorithms $\{\Gamma_n\}$ such that $\forall A \in \Omega$ we have $\Gamma_n(A) \rightarrow \Xi(A)$ as $n \rightarrow \infty$. Moreover, $\sup_{z \in \Gamma_n(A)} \text{dist}(z, \Xi(A)) \leq 2^{-n}$, where $\text{dist}(x, S)$ denotes the Euclidean distance of x to S .
- (iv) Π_1^α : We have $\Delta_1^\alpha \subset \Pi_1^\alpha \subset \Delta_2^\alpha$ and Π_1^α is the set of problems for which \exists a sequence of algorithms $\{\Gamma_n\}$ such that $\forall A \in \Omega$ we have $\Gamma_n(A) \rightarrow \Xi(A)$ as $n \rightarrow \infty$. Moreover, $\sup_{z \in \Xi(A)} \text{dist}(z, \Gamma_n(A)) \leq 2^{-n}$.
- (v) Δ_2^α is the set of problems that can be computed using one limit (SCI = 1) without error control, i.e. \exists a sequence of algorithms $\{\Gamma_n\}$ such that $\lim_{n \rightarrow \infty} \Gamma_n(A) = \Xi(A), \forall A \in \Omega$.
- (vi) Δ_{m+1}^α , for $m \in \mathbb{N}$, is the set of problems that can be computed by using m successive limits, (SCI $\leq m$), i.e. \exists a family of algorithms $\{\Gamma_{n_m, \dots, n_1}\}$ such that

$$\lim_{n_m \rightarrow \infty} \dots \lim_{n_1 \rightarrow \infty} \Gamma_{n_m, \dots, n_1}(A) = \Xi(A), \quad \forall A \in \Omega.$$

- (vii) Σ_m^α is the set of problems in Δ_{m+1}^α such that, letting $\Gamma_{n_m}(A) = \lim_{n_{m-1} \rightarrow \infty} \dots \lim_{n_1 \rightarrow \infty} \Gamma_{n_m, \dots, n_1}(A)$, $\sup_{z \in \Gamma_{n_m}(A)} \text{dist}(z, \Xi(A)) \leq 2^{-n_m}$. In other words, computing the m th limit is a Σ_1^α problem.
- (viii) Π_m^α is the set of problems in Δ_{m+1}^α such that $\sup_{z \in \Xi(A)} \text{dist}(z, \Gamma_{n_m}(A)) \leq 2^{-n_m}$. In other words, computing the m th limit is a Π_1^α problem.

Schematically, the SCI hierarchy can be viewed in the following way:



A visual demonstration of these classes is shown in Fig. 1. For the description for decision problems, see Sect. 5.1. The Σ_1^α and Π_1^α classes become crucial in computer-assisted proofs (see below).

Remark 2.2 (Computability, not complexity) It is important to note that (despite its name) the SCI hierarchy is a hierarchy for classifying computability, not complexity. Most computational spectral problems of interest are $\notin \Delta_1$ in the SCI hierarchy, and complexity theory only makes sense for problems in Δ_1 . Hence, it is impossible to build

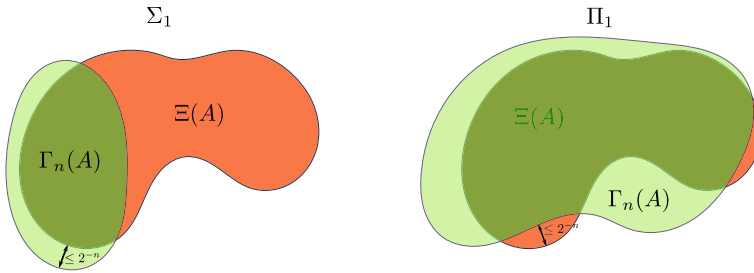


Fig. 1 Meaning of Σ_1 and Π_1 convergence for a problem function Ξ computed in the Hausdorff metric. The red areas represent $\Xi(A)$, whereas the green areas represent the output of the algorithm $\Gamma_n(A)$. Σ_1 convergence means convergence as $n \rightarrow \infty$ but each point in the output $\Gamma_n(A)$ is at most distance 2^{-n} from $\Xi(A)$. Similarly, in the case of Π_1 , we have convergence as $n \rightarrow \infty$ but any point in $\Xi(A)$ is at most distance 2^{-n} from $\Gamma_n(A)$ (Color figure online)

a complexity theory for most infinite-dimensional spectral problems. The scientific community computes with non-computable problems ($\notin \Delta_1$) on a daily basis (e.g. in quantum mechanics). This also happens in high-profile computer-assisted proofs (see below). □

2.1.2 The Model of Computation α

The α in the superscript indicates the model of computation, which is described in Sect. 5.1. For $\alpha = G$, the underlying algorithm is general (see Definition 5.1) and can use any tools at its disposal. The reader may think of a Blum–Shub–Smale (BSS) machine or a Turing machine with access to any oracle, although a general algorithm is even more powerful. However, for $\alpha = A$ this means that only arithmetic operations and comparisons are allowed. In particular, if rational inputs are considered, the algorithm is a Turing machine, and in the case of real inputs, a BSS machine. Hence, a result of the form

$$\notin \Delta_k^G \text{ is stronger than } \notin \Delta_k^A.$$

Indeed, a $\notin \Delta_k^G$ result is universal and holds for any model of computation. Moreover,

$$\in \Delta_k^A \text{ is stronger than } \in \Delta_k^G,$$

and similarly for the Π_k and Σ_k classes. In this paper, we prove lower bounds for $\alpha = G$ and upper bounds for $\alpha = A$, thus obtaining the strongest results. Remark 5.12 discusses further how the model of computation is of less importance in infinite dimensions.

2.1.3 Computer-Assisted Proofs

The class of problems Δ_1^A are precisely those that are computable according to Turing’s definition of computability (i.e. there exists an algorithm such that for any $\epsilon > 0$ the algorithm can produce an ϵ -accurate output). However, most infinite-dimensional

spectral problems are $\notin \Delta_1^A$. The simplest example is the problem of computing spectra of infinite diagonal matrices. Very few interesting infinite-dimensional spectral problems are actually in Δ_1^A , and most of the literature on spectral computations provides algorithms that yield Δ_2^A classification results. Such algorithms converge, but may not provide error control. In many cases, error control is impossible.

Problems not in Δ_1^A are a daily occurrence in the sciences due to suggestive numerical simulations or evidence based on experiments. However, in the field of computer-assisted proofs, this is not possible, since only 100% rigour is accepted. Nevertheless, there are many examples of famous conjectures that have been proven using computational problems that do not lie in Δ_1^A . For example, the proof of Kepler’s conjecture (Hilbert’s 18th problem) [87, 88] relies on decision problems that are not in Δ_1^A [15]. Another example is C. Fefferman and L. Seco’s proof of the Dirac–Schwinger conjecture on the asymptotics of ground states of certain Schrödinger operators [72–80]. The reason for this apparent paradox is that the Σ_1^A and Π_1^A classes are larger than Δ_1^A , but can still be used in computer-assisted proofs. Both of the above examples implicitly prove Σ_1^A classifications. For example, suppose we have a computational spectral problem that lies in Σ_1^A . This means that there is an algorithm that will converge and never provide incorrect output, up to a user-specified error bound. Thus, conjectures about operators never having spectra in a certain area (a common problem in stability analysis, for example) could be disproved by a computer-assisted proof. Recent results using computer-assisted proofs in spectral theory include [33, 111].

2.2 Evaluation Sets and Domains

Throughout this paper, unless otherwise specified, A will be a bounded operator acting on the canonical Hilbert space $l^2(\mathbb{N})$ (we define $\Omega_B := \mathcal{B}(l^2(\mathbb{N}))$), and realised as a matrix with respect to the canonical basis. However, the results of this paper extend to general separable Hilbert spaces \mathcal{H} through a choice of orthonormal basis e_1, e_2, \dots if one can compute the matrix values of the operators with respect to this basis (see the discussion of the evaluation sets below). For example, we can treat operators naturally defined on lattices such as \mathbb{Z}^d , or more generally on graphs. Such operators are abundant in mathematical physics. Below we give the evaluation sets and classes of operators treated in this paper. For convenience, this information is summarised in Tables 2 and 3.

Table 2 Summary of evaluation sets used in this paper

| Evaluation set | Information available to algorithm | Meaning |
|----------------|---|--|
| Δ_1 | $f_{i,j}^1 : A \mapsto \langle Ae_j, e_i \rangle$ for $i, j \in \mathbb{N}$ | Matrix entries of A |
| Δ_2 | $f_{i,j}^1 : A \mapsto \langle Ae_j, e_i \rangle$ for $i, j \in \mathbb{N}$ $f_{i,j}^2 : A \mapsto \langle Ae_j, Ae_i \rangle$ for $i, j \in \mathbb{N}$ $f_{i,j}^3 : A \mapsto \langle A^*e_j, A^*e_i \rangle$ for $i, j \in \mathbb{N}$ | Matrix entries of A, A^*A , and AA^* |

Table 3 Summary of classes of operators treated in this paper

| Domain (class of operators) | Meaning |
|-----------------------------|---|
| Ω_B | Bounded operators |
| Ω_N | Bounded normal operators |
| Ω_{SA} | Bounded self-adjoint operators |
| Ω_D | Bounded self-adjoint diagonal operators |
| Ω_f | Bounded operators with known f satisfying (2.1) |
| Ω_g | Bounded operators with known g satisfying (2.2) |

Note that all considered classes lie in Ω_B , and that $\Omega_D \subset \Omega_{SA} \subset \Omega_N$, $\Omega_D \subset \Omega_f$, and $\Omega_N \subset \Omega_{g:x \mapsto x}$

2.2.1 Evaluation Sets

We consider two natural sets of information that our algorithms can read. The first, Λ_1 , provides the entries of the matrix representation of A with respect to the canonical basis $\{e_i\}_{i \in \mathbb{N}}$:

$$\Lambda_1 = \{f_{i,j}^1 : A \mapsto \langle Ae_j, e_i \rangle | i, j \in \mathbb{N}\}.$$

The second, Λ_2 , appends Λ_1 with the entries of the matrix representations of A^*A and AA^* with respect to the canonical basis $\{e_i\}_{i \in \mathbb{N}}$:

$$\Lambda_2 = \Lambda_1 \cup \{f_{i,j}^2 : A \mapsto \langle Ae_j, Ae_i \rangle, f_{i,j}^3 : A \mapsto \langle A^*e_j, A^*e_i \rangle | i, j \in \mathbb{N}\}.$$

We include Λ_2 since it is natural for problems posed in variational form, and can often be evaluated through numerical integration. When considering classes with functions f (and $\{c_n\}$) and g as in (2.1) and (2.2) below, we will add these to the relevant evaluation set (evaluating g at rational points) and with an abuse of notation still use the notation Λ_i . A small selection of the problems also require additional information, such as when testing if a set intersects a spectral set, but any changes to Λ_i will be pointed out where appropriate.

2.2.2 Classes of Operators

Let Ω_N denote the class of normal operators in Ω_B , Ω_{SA} denote the class of self-adjoint operators in Ω_N , and Ω_D denote the class of self-adjoint diagonal operators in Ω_{SA} . For $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(n) \geq n + 1$ define

$$D_{f,n}(A) := \max \left\{ \|(I - P_{f(n)})AP_n\|, \|P_nA(I - P_{f(n)})\| \right\}, \tag{2.1}$$

where P_m is the orthogonal projection onto $\text{span}\{e_1, \dots, e_m\}$. Given such an f , we assume access to an estimate $D_{f,n}(A) \leq c_n(A) \in \mathbb{Q}_{\geq 0}$, where $c_n \rightarrow 0$ as $n \rightarrow \infty$. We let Ω_f denote the class of bounded operators with known function f and $\{c_n\}$.⁶ As

⁶ Sometimes the sequence $\{c_n\}$ is not needed and we will explicitly mention when this is the case.

a special case, if we know our matrix is sparse with finitely many nonzero entries in each column and row (and we know the positions of the nonzero entries) then we know an f with $c_n = 0$. Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a strictly increasing, continuous function that vanishes only at 0 with $\lim_{x \rightarrow \infty} g(x) = \infty$. Let Ω_g be the class of bounded operators with

$$g(\text{dist}(z, \text{Sp}(A))) \leq \|R(z, A)\|^{-1}, \tag{2.2}$$

for $z \in \mathbb{C}$, where $R(z, A) = (A - zI)^{-1}$. A simple compactness argument shows that such a g always exists for any given $A \in \Omega_B$. However, the classification of spectral problems in the SCI hierarchy generally depends on whether one knows an estimate for g or not. For example, in the self-adjoint and normal cases, $g(x) = x$ is the trivial choice of g . Operators with $g(x) = x$ are known as G_1 and include the well-studied class of hyponormal operators (operators with $A^*A - AA^* \geq 0$) [136]. A common assumption is that

$$\|R(z, A)\| \leq \frac{C}{\text{dist}(z, \text{Sp}(A))} \quad \forall z \notin \text{Sp}(A),$$

for some constant C , which is equivalent to $A \in \Omega_g$ with $g(x) = x/C$. For example, if A is similar to a normal operator with a similarity transformation S that has bounded condition number $\kappa(S)$, we can take $C = \kappa(S)$. Other examples with nonlinear g include perturbations of self-adjoint operators [84, e.g. Theorem 7.7.1]. More generally, one can view the function g as a measure of stability of the spectrum of A through the formula

$$\text{Sp}_\epsilon(A) := \text{Sp}(A) \cup \{z \notin \text{Sp}(A) : \|R(z, A)\| \geq 1/\epsilon\} = \bigcup_{B \in \Omega_B, \|B\| \leq \epsilon} \text{Sp}(A + B), \tag{2.3}$$

where $\text{Sp}_\epsilon(A)$ denotes the (ϵ -)pseudospectrum of A [162]. The function g is held fixed for a given class Ω_g and a smaller g leads to a larger class of operators Ω_g .

3 Main Results: The Foundations of Computing Geometric Features of Spectra

Our results classify computing geometric features of spectra in the SCI hierarchy. In other words, we are concerned with the foundations of computation for geometric features of spectra. There are two aspects of this classification: proving impossibility results (lower bounds), where we make use of the tools developed in Sect. 5 and Theorem 5.19, and proving upper bounds through the construction of algorithms. This ensures that our algorithms realise the boundary of what computers can achieve in spectral computations. We have included routines for some of the main algorithms in “Appendix B” and computational examples in Sect. 4.

Remark 3.1 (Bounding the operator norm) The proofs of lower bounds make clear that all classifications still hold if we replace the respective sub-class $\Omega \subset \Omega_B$ by the

restriction to operators in Ω having operator norm at most $M \in \mathbb{R}_{>0}$, adding such a value M (constant function) to the evaluation set Λ . □

Remark 3.2 (Computing the resolvent norm) Some of the algorithms are built on the local approximation of the functions (or similar functions) defined by

$$\gamma_n(z; A) = \min\{\sigma_{\text{inf}}((A - zI)|_{P_n\mathcal{H}}), \sigma_{\text{inf}}((A^* - \bar{z}I)|_{P_n\mathcal{H}})\},$$

where σ_{inf} denotes the smallest singular value or injection modulus:

$$\sigma_{\text{inf}}(T) = \inf\{\|Tv\| : \|v\| = 1\}.$$

The functions γ_n converge to the resolvent norm $\|R(z, A)\|^{-1}$ uniformly on compact subsets of \mathbb{C} from above as $n \rightarrow \infty$. This idea was crucial in [60, 64] to compute spectra with Σ_1^A error control for a large class of operators. A theme of some of our proofs, especially those concerning Lebesgue measure and fractal dimensions, is the extension of these ideas to compute geometric properties of the spectrum. □

3.1 Spectral Radii

We begin with a very simple geometric feature of the spectrum. The spectral radius, $r(A)$, of a bounded operator A is the supremum of the absolute values of members of the spectrum, which is attained. Spectral radii commonly appear in applications involving stability analysis. We set $\Xi_r(A) := r(A)$ and make the following initial observations:

- (i) One can easily show that the computational problem of the operator norm of any $A \in \Omega_B$ lies in Σ_1^A . Hence, since $r(A) \leq \|A\|$, we can easily get an upper bound for $\Xi_r(A)$ in one limit. Of course, if A is not normal, this upper bound may not agree with $\Xi_r(A)$.
- (ii) If an operator lies in Ω_g with $g(x) = x$, then the convex hull of the spectrum is equal to the closure of the numerical range (recall that the numerical range is $\{\langle Ax, x \rangle : \|x\| = 1\}$) [131]. Such operators are known as convexoid and the problem of computing $\Xi_r(A)$ for such operators lies in Σ_1^A .
- (iii) In light of Gelfand’s famous formula $\Xi_r(A) = \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$, one might expect that the computation of $\Xi_r(A)$ is strictly easier in the sense of the SCI hierarchy than that of the spectrum.

The following shows that the intuition in (iii) is misguided in general and only occurs if an operator is convexoid as in (ii). Computing $\Xi_r(A)$ is just as hard as computing the spectrum for the class Ω_B . Controlling the resolvent via a function g as in (2.2) makes the problem easier in the sense of SCI hierarchy than the general class Ω_B , but is not sufficient to reduce the SCI of the problem to 1.

Theorem 3.3 Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a strictly increasing, continuous function that vanishes only at 0 with $\lim_{x \rightarrow \infty} g(x) = \infty$. In addition, suppose that $g(x) \leq (1 - \delta)x$ for some $\delta \in (0, 1)$. Then:

$$\begin{aligned} \Delta_1^G \not\equiv \{\Xi_r, \Omega_D, \Lambda_1\} \in \Sigma_1^A, & \quad \Delta_1^G \not\equiv \{\Xi_r, \Omega_N, \Lambda_1\} \in \Sigma_1^A, & \quad \Delta_1^G \not\equiv \{\Xi_r, \Omega_f \cap \Omega_g, \Lambda_1\} \in \Sigma_1^A, \\ \Delta_2^G \not\equiv \{\Xi_r, \Omega_g, \Lambda_1\} \in \Sigma_2^A, & \quad \Delta_2^G \not\equiv \{\Xi_r, \Omega_f, \Lambda_1\} \in \Pi_2^A, & \quad \Delta_3^G \not\equiv \{\Xi_r, \Omega_B, \Lambda_1\} \in \Pi_3^A. \end{aligned}$$

When considering the evaluation set Λ_2 , the only changes are the following classifications:

$$\Delta_1^G \not\equiv \{\Xi_r, \Omega_g, \Lambda_2\} \in \Sigma_1^A, \quad \Delta_2^G \not\equiv \{\Xi_r, \Omega_B, \Lambda_2\} \in \Pi_2^A.$$

Remark 3.4 The Π_2^A algorithm for $\{\Xi_r, \Omega_f\}$ does not need a null sequence $\{c_n\}$ bounding the dispersion, $D_{f,n}(A) \leq c_n$, to be sharp in the SCI hierarchy since this is absorbed in the first limit. \square

Remark 3.5 The proofs of the lower bounds in Theorem 3.3 for Ω_g require g with the stated additional property and $\delta > 0$. In particular, the lower bound does not cover the smaller class of G_1 operators. \square

3.2 Essential Spectral Radii

Next, we consider the essential spectral radius. Define the essential spectrum of $A \in \Omega_B$ as

$$\text{Sp}_{\text{ess}}(A) = \bigcap_{B \in \Omega_K} \text{Sp}(A + B),$$

where Ω_K denotes the class of compact operators. The essential spectral radius, $\Xi_{er}(A)$, is simply the supremum of the absolute values over $\text{Sp}_{\text{ess}}(A)$.

Theorem 3.6 We have the following classifications for $i = 1, 2$:

$$\Delta_2^G \not\equiv \{\Xi_{er}, \Omega_D, \Lambda_i\} \in \Pi_2^A, \quad \Delta_2^G \not\equiv \{\Xi_{er}, \Omega_N, \Lambda_i\} \in \Pi_2^A, \quad \Delta_2^G \not\equiv \{\Xi_{er}, \Omega_f, \Lambda_i\} \in \Pi_2^A.$$

Whereas, for general operators,

$$\Delta_3^G \not\equiv \{\Xi_{er}, \Omega_B, \Lambda_1\} \in \Pi_3^A, \quad \Delta_2^G \not\equiv \{\Xi_{er}, \Omega_B, \Lambda_2\} \in \Pi_2^A.$$

3.3 Capacity and Polynomial Operator Norms

Given a polynomial p of degree at least two,⁷ we consider the problem of computing $\Xi_{r,p} = \|p(A)\|$ and the capacity of the spectrum defined by

$$\begin{aligned} \Xi_{cap}(A) &= \inf_{\text{monic polynomial } p} \|p(A)\|^{\frac{1}{\deg(p)}} \\ &= \lim_{d \rightarrow \infty} \inf \left\{ \|p(A)\|^{\frac{1}{d}} : \text{monic polynomial } p, \deg(p) = d \right\}. \end{aligned}$$

A theorem of Halmos shows that this definition of capacity agrees with the usual potential-theoretic definition of capacity of the set $\text{Sp}(A)$ [89]. Roughly speaking, the capacity measures the ability of $\text{Sp}(A)$ to hold electrical charge. We will also see some other measures of size in Sects. 3.5 and 3.6. The capacity of the spectrum is of particular interest in Krylov methods where, for instance, it is related to the speed of convergence⁸ [117, 119, 121–123]. The capacity is also an important object in local spectral theory [1, 105, 119], and related work [48, 124] includes methods for computing the polynomially convex hull of an operator. The following theorem provides the relevant SCI classifications.

Theorem 3.7 *We have the following classifications for $i = 1, 2$ and $\hat{\Omega} = \Omega_D$ or Ω_f :*

$$\Delta_1^G \not\equiv \{\Xi_{r,p}, \hat{\Omega}, \Lambda_i\} \in \Sigma_1^A, \quad \Delta_2^G \not\equiv \{\Xi_{cap}, \hat{\Omega}, \Lambda_i\} \in \Pi_2^A.$$

For $\tilde{\Omega} = \Omega_N, \Omega_g$ or Ω_B ,

$$\begin{aligned} \Delta_2^G \not\equiv \{\Xi_{r,p}, \tilde{\Omega}, \Lambda_1\} \in \Sigma_2^A, & \quad \Delta_3^G \not\equiv \{\Xi_{cap}, \tilde{\Omega}, \Lambda_1\} \in \Pi_3^A \\ \Delta_1^G \not\equiv \{\Xi_{r,p}, \tilde{\Omega}, \Lambda_2\} \in \Sigma_1^A, & \quad \Delta_2^G \not\equiv \{\Xi_{cap}, \tilde{\Omega}, \Lambda_2\} \in \Pi_2^A. \end{aligned}$$

The proof shows these problems have the same classifications for Ω_{SA} as Ω_N . Somewhat surprising is the result that the computation of $\|p(A)\|$ requires two successive limits for self-adjoint operators. The proof shows that one reason for this is spectral pollution associated with finite section methods.

⁷ We fix the polynomial p for the strongest possible negative results. However, the existence of the towers of algorithms also holds when considering the polynomial p itself as an input.

⁸ This is an idealisation since the capacity studies operator norms while true Krylov processes look at $p(A)x$ with one or several vectors x . However, from local spectral theory (e.g. [118]) it follows that, generically, the asymptotic speeds are the same.

3.4 Essential Numerical Range, Gaps in Essential Spectra and Detecting Failure of Finite Section

We now consider geometric features of spectra that are related to the finite section method, the most intensely studied computational method of approximating spectra [35, 36, 40, 41].⁹ The basic form of the finite section method approximates the spectrum of A by $\text{Sp}(P_n A|_{P_n \mathcal{H}})$, where $\{P_m\}$ is a sequence of finite-dimensional projections converging strongly to the identity as $m \rightarrow \infty$. The computation is often done with finite element, finite difference or spectral methods by discretising the operator on a suitable finite-dimensional space [31, 32, 47, 50, 102, 108, 137, 168]. Even when A is self-adjoint, spurious eigenvalues, that have nothing to do with $\text{Sp}(A)$, can accumulate anywhere within gaps of the essential spectrum as $n \rightarrow \infty$.¹⁰ This is known as *spectral pollution*. More precisely, the essential numerical range of $A \in \Omega_B$ is defined as

$$W_e(A) = \bigcap_{B \in \Omega_K} \overline{W(A + B)}, \tag{3.1}$$

where $W(A) = \{Ax, x : \|x\| = 1\}$ is the usual numerical range.¹¹ We recall the following two theorems.

Theorem 3.8 (Pokrzywa [132]) *Let $A \in \mathcal{B}(\mathcal{H})$ and let $\{P_n\}$ be a sequence of finite-dimensional projections converging strongly to the identity. Suppose that $S \subset W_e(A)$. Then there exists a sequence $\{Q_n\}$ of finite-dimensional projections such that $P_n < Q_n$ (so $Q_n \rightarrow I$ strongly) and*

$$d_H(\text{Sp}(P_n A|_{P_n \mathcal{H}}) \cup S, \text{Sp}(Q_n A|_{Q_n \mathcal{H}})) \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where d_H denotes the Hausdorff distance.

Theorem 3.9 (Pokrzywa [132]) *Let $A \in \mathcal{B}(\mathcal{H})$ and let $\{P_n\}$ be a sequence of finite-dimensional projections converging strongly to the identity. If $\lambda \notin W_e(A)$, then $\lambda \in \text{Sp}(A)$ if and only if*

$$\text{dist}(\lambda, \text{Sp}(P_n A|_{P_n \mathcal{H}})) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Theorems 3.8 and 3.9 show that spectral pollution is confined to the essential numerical range and can be arbitrarily bad in $W_e(A) \setminus \text{Sp}(A)$.¹² For self-adjoint operators, the

⁹ Arveson [3–7] and Brown [42–44] pioneered spectral computations from the point of view of C^* -algebras, both for the general spectral computation problem and for Schrödinger operators. This combination can be traced back to the work of Böttcher and Silberman [39]. Arveson also considered spectral computation in terms of densities, which is related to Szegő’s work [155] on finite section approximations.

¹⁰ Even when the finite section method converges, it typically only yields Δ_2^A classifications in the SCI hierarchy [37, 38, 45, 46].

¹¹ If A is hyponormal, then $W_e(A)$ is the convex hull of the essential spectrum [142].

¹² In the non-normal case, it is possible for finite sections to not capture all of the spectrum—parts of the spectrum may be unattainable. This is distinct from spectral pollution. Theorem 3.8 says that, up to a different choice of projections, this can be avoided on $W_e(A)$.

gaps in the essential spectrum correspond exactly to $W_e(A) \setminus \text{Sp}(A)$. As a result, there has been considerable attention towards methods that detect gaps in essential spectra and eigenvalues within these gaps [31, 49, 108, 144], as well as studying the precise nature of spectral pollution [107, 112, 113, 137].

A consequence of the main result of this section, Theorem 3.10, is that detecting these gaps is strictly harder in the sense of the SCI hierarchy than computing the spectrum for self-adjoint operators (which was classified in [18, 60, 64]). We define the problem function $\Xi_{we}(A) = W_e(A)$. For a given non-empty open set U in \mathbb{F} (with \mathbb{F} being \mathbb{C} or \mathbb{R}), let $\Xi_{poll}^{\mathbb{F}}$ be the decision problem

$$\Xi_{poll}^{\mathbb{F}}(A, U) = \begin{cases} 1, & \text{if } \overline{U} \cap (W_e(A) \setminus \text{Sp}(A)) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

$\Xi_{poll}^{\mathbb{F}}$ decides whether spectral pollution can occur on the closed set \overline{U} . For the self-adjoint case and $\mathbb{F} = \mathbb{R}$, this is equivalent to asking whether there exists a point in the open set U that also lies in a gap of the essential spectrum. To incorporate U into Λ_i , we allow access to a countable number of open balls $\{U_m\}_{m \in \mathbb{N}}$ whose union is U . If $\mathbb{F} = \mathbb{R}$, then each U_m is of the form (a_m, b_m) with $a_m, b_m \in \mathbb{Q} \cup \{\pm\infty\}$. If $\mathbb{F} = \mathbb{C}$, then each U_m is equal to $D_{r_m}(z_m)$ (the open ball of radius r_m centred at z_m) with $r_m \in \mathbb{Q}_+ \cup \{\infty\}$ and $z_m \in \mathbb{Q} + i\mathbb{Q}$. We add pointwise evaluations of the relevant sequences $\{(a_m, b_m)\}$ or $\{(r_m, z_m)\}$ to Λ_i .

Theorem 3.10 (Computation of essential numerical range and whether spectral pollution can occur on a set) *Let $\Omega = \Omega_N, \Omega_{SA}$ or Ω_B and let $i = 1, 2$. Then*

$$\Delta_2^G \not\preceq \{\Xi_{we}, \Omega, \Lambda_i\} \in \Pi_2^A.$$

Furthermore, for $i = 1, 2$ the following classifications hold, valid also if we restrict to the case $U = U_1$ or to $U = U_1 = \mathbb{F}$:

$$\Delta_3^G \not\preceq \{\Xi_{poll}^{\mathbb{R}}, \Omega_{SA}, \Lambda_i\} \in \Sigma_3^A, \quad \Delta_3^G \not\preceq \{\Xi_{poll}^{\mathbb{C}}, \Omega_B, \Lambda_i\} \in \Sigma_3^A.$$

Remark 3.11 (Computing spectra is easier than algorithmically determining whether spectral pollution can occur on a set) One can show that $\{\text{Sp}(\cdot), \Omega_{SA}, \Lambda_1\} \in \Sigma_2^A$ and $\{\text{Sp}(\cdot), \Omega_{SA}, \Lambda_2\} \in \Sigma_1^A$. Hence determining $\Xi_{poll}^{\mathbb{R}}$ is strictly harder than the spectral computational problem and requires two additional successive limits if $\Lambda = \Lambda_2$. Even in the general case, $\{\text{Sp}(\cdot), \Omega_B, \Lambda_2\} \in \Pi_2^A$ and hence the spectral problem is strictly easier in the sense of SCI hierarchy. The proofs also make clear that we get the same classification of $\Xi_{poll}^{\mathbb{F}}$ for other classes such as Ω_N, Ω_g etc. \square

Remark 3.12 (Unbounded operators) In Sect. 7.1, we show that computing the essential numerical range for closed unbounded operators T on $l^2(\mathbb{N})$ (under the condition that the linear span of the canonical basis forms a core of T) also lies in Π_2^A . The definition of the essential numerical range for such operators was recently given in [34]. This paper showed that $W_e(T)$ consists precisely of the essential spectrum of T together with all possible spectral pollution that may arise by applying projection

methods to approximate the spectrum of T , thus generalising Theorems 3.8 and 3.9. A computational example is given in Sect. 4.2. □

3.5 Lebesgue Measure of Spectra

A basic property of the set $\text{Sp}(A)$, also connected to physical applications, is its Lebesgue measure. Well-studied operators such as the almost Mathieu operator at critical coupling [11] or the Fibonacci Hamiltonian [154] have spectra with Lebesgue measure zero. Following [8], there have been many further numerical studies [157–159]. For further examples of operators with numerical approximations of the Lebesgue measure, see the references in [10, 24, 147]. Numerical studies typically look at periodic approximates [134], and computing the Lebesgue measure of periodic approximates of tridiagonal operators lies in Δ_1^A . The tools we develop are more general and do not assume such structure. Verification of our algorithms for the almost Mathieu operator is presented in Sect. 4.4.

The Lebesgue measure on \mathbb{C} will be denoted by Leb . When considering classes of self-adjoint operators, we use the Lebesgue measure on \mathbb{R} denoted by $\text{Leb}_{\mathbb{R}}$. We also define

$$\widehat{\text{Sp}}_{\epsilon}(A) = \{z \in \mathbb{C} : \|R(z, A)\|^{-1} < \epsilon\},$$

whose closure is $\text{Sp}_{\epsilon}(A)$. For a class $\Omega \subset \Omega_B$, there are three questions we answer in this section:

- (1) Given $A \in \Omega$, can we compute $\text{Leb}(\text{Sp}(A))$?
- (2) Given $A \in \Omega$ and $\epsilon > 0$, can we compute $\text{Leb}(\widehat{\text{Sp}}_{\epsilon}(A))$?¹³
- (3) Given $A \in \Omega$, can we determine whether $\text{Leb}(\text{Sp}(A)) = 0$?

For the first two questions, we consider the metric space $([0, \infty), d)$ with the Euclidean metric. For question three, we consider the discrete metric on $\{0, 1\}$, where 1 is interpreted as “Yes”, and 0 as “No”. We denote the above problem functions by Ξ_1^L , Ξ_2^L and Ξ_3^L , respectively. In analogy to computing spectra and pseudospectra, Ξ_2^L is the easiest to compute and can be done in one limit for a large class of operators. It also follows from the dominated convergence theorem that

¹³ We consider the computation of $\text{Leb}(\widehat{\text{Sp}}_{\epsilon}(A))$ instead of $\text{Leb}(\text{Sp}_{\epsilon}(A))$ since it is not clear that the level sets

$$S_{\epsilon}(A) := \{z \in \mathbb{C} : \|R(z, A)\|^{-1} = \epsilon\} \tag{3.2}$$

always have Lebesgue measure zero (this is currently an open problem for general bounded operators). This situation is analogous to the case of approximating the pseudospectra of bounded operators, where one uses the crucial property that pseudospectra cannot jump—the resolvent norm cannot be constant on open subsets of $\mathbb{C} \setminus \text{Sp}(A)$ for a bounded operator A acting on a separable Hilbert space [143]. The question of whether the sets in (3.2) are Lebesgue null is the measure theoretic equivalent. Note, however, that it is straightforward to show that $S_{\epsilon}(A)$ is null for $A \in \Omega_N$ through the formula $\|R(z, A)\|^{-1} = \text{dist}(z, \text{Sp}(A))$.

$$\lim_{\epsilon \downarrow 0} \text{Leb}(\widehat{\text{Sp}}_\epsilon(A)) = \text{Leb}(\text{Sp}(A)). \tag{3.3}$$

Theorem 3.13 (Lebesgue measure of spectra) *Given the above set-up, we have the following classifications*

$$\Delta_2^G \not\cong \{\Xi_1^L, \Omega_f, \Lambda_i\} \in \Pi_2^A, \quad \Delta_2^G \not\cong \{\Xi_1^L, \Omega_D, \Lambda_i\} \in \Pi_2^A \quad i = 1, 2,$$

and for $\Omega = \Omega_B, \Omega_{SA}, \Omega_N$ or Ω_g ,

$$\Delta_3^G \not\cong \{\Xi_1^L, \Omega, \Lambda_1\} \in \Pi_3^A, \quad \Delta_2^G \not\cong \{\Xi_1^L, \Omega, \Lambda_2\} \in \Pi_2^A.$$

The algorithm constructed in the proof of Theorem 3.13 is local and can be adapted to find the Lebesgue measure of $\text{Sp}(A)$ intersected with any compact interval or cube in one or two dimensions, respectively. Moreover, when considering Ω_f , we do not need the sequence $\{c_n\}$, and the algorithm can be restricted to \mathbb{R} , where it converges to $\text{Leb}_{\mathbb{R}}(\text{Sp}(A) \cap \mathbb{R})$. Our results also hold when considering bounded diagonal operators (dropping the restriction of self-adjointness) and using Leb instead of $\text{Leb}_{\mathbb{R}}$.

We now turn to the SCI classification of $\text{Leb}(\widehat{\text{Sp}}_\epsilon(A))$, which is useful since it provides a route to computing $\text{Leb}(\text{Sp}(A))$ for any $A \in \Omega_B$ via (3.3). This is a similar state of affairs to the computation of the spectrum itself—one can approximate the spectrum via pseudospectra.

Theorem 3.14 (Lebesgue measure of pseudospectra) *Given the above set-up, we have the following classifications*

$$\Delta_1^G \not\cong \{\Xi_2^L, \Omega_f, \Lambda_i\} \in \Sigma_1^A, \quad \Delta_1^G \not\cong \{\Xi_2^L, \Omega_D, \Lambda_i\} \in \Sigma_1^A \quad i = 1, 2,$$

and for $\Omega = \Omega_B, \Omega_{SA}, \Omega_N$ or Ω_g ,

$$\Delta_2^G \not\cong \{\Xi_2^L, \Omega, \Lambda_1\} \in \Sigma_2^A, \quad \Delta_1^G \not\cong \{\Xi_2^L, \Omega, \Lambda_2\} \in \Sigma_1^A.$$

Why is Ξ_2^L easier to compute than Ξ_1^L ? Heuristically, the pseudospectrum is less refined than the spectrum, making the measure easier to approximate. Another viewpoint is the continuity points of the maps Ξ_1^L and Ξ_2^L . For simplicity, consider these maps restricted to Ω_D and equip these diagonal operators with the operator norm topology. The following shows that Ξ_2^L is more stable than Ξ_1^L , explaining why it is easier to approximate. Again, this is the same state of affairs as comparing $\text{Sp}(A)$ and $\text{Sp}_\epsilon(A)$ as sets.

Proposition 3.15 *In the above set-up, the following hold:*

- (1) Ξ_1^L is continuous at $A \in \Omega_D$ if and only if $\text{Leb}_{\mathbb{R}}(\text{Sp}(A)) = 0$.
- (2) Ξ_2^L is continuous at all $A \in \Omega_D$.

Finally, when computing Ξ_3^L , we let (\mathcal{M}, d) be the set $\{0, 1\}$ endowed with the discrete topology and consider the problem function

$$\Xi_3^L(A) = \begin{cases} 0, & \text{if } \text{Leb}(\text{Sp}(A)) > 0 \\ 1, & \text{otherwise.} \end{cases}$$

It is straightforward to build a family of algorithms that converge in three successive limits for this problem using the algorithm constructed in Theorem 3.13 and its monotonicity. The next theorem shows that this is optimal, even for the set of diagonal self-adjoint bounded operators. This demonstrates how hard it is to solve decision problems about the spectrum with finite amounts of information, particularly when the problems involve an object that ignores countable sets, such as the Lebesgue measure.

Theorem 3.16 (Is the spectrum Lebesgue null?) *Given the above set-up, we have the following classifications*

$$\Delta_3^G \not\cong \{\Xi_3^L, \Omega_f, \Lambda_i\} \in \Pi_3^A, \quad \Delta_3^G \not\cong \{\Xi_3^L, \Omega_D, \Lambda_i\} \in \Pi_3^A, \quad i = 1, 2,$$

and for $\Omega = \Omega_B, \Omega_{SA}, \Omega_N$ or Ω_g ,

$$\Delta_4^G \not\cong \{\Xi_3^L, \Omega, \Lambda_1\} \in \Pi_4^A, \quad \Delta_3^G \not\cong \{\Xi_3^L, \Omega, \Lambda_2\} \in \Pi_3^A.$$

Remark 3.17 These are the first examples of computational spectral problems that require four successive limits to compute in the SCI hierarchy. To prove this, we need some tools from descriptive set theory in Sect. 5. Note that we prove the lower bounds for *general* algorithms, so regardless of the model of computation. \square

3.6 Fractal Dimensions of Spectra

When considering operators from physical models, such as Schrödinger operators in quantum mechanics, fractal dimensions of spectra are related to important phenomena, such as the spreading of an initially localised wavepacket [101]. Further applications and numerical studies are already discussed in Sect. 1. However, estimating the fractal dimension is extremely difficult. This can be explained by the SCI hierarchy—the $\text{SCI} > 1$, even for computing the box-counting dimension, the most basic definition of fractal dimension. The Hausdorff dimension is even worse and has $\text{SCI} \geq 3$. In this section, we exclusively treat self-adjoint operators and hence seek fractal dimensions of $\text{Sp}(A) \subset \mathbb{R}$.¹⁴

¹⁴ The proofs for general self-adjoint operators can be adapted with an additional successive limit and the use of two-dimensional covering boxes to treat the class of general bounded operators. Some care is needed to deal with the boundaries of covering boxes for the Hausdorff dimension, but we omit the details.

Box-Counting Dimension: Let F be a bounded set in \mathbb{R} and let $N_\delta(F)$ be the number of closed intervals of length $\delta > 0$ required to cover F . We define the upper and lower box-counting dimensions as

$$\overline{\dim}_B(F) = \limsup_{\delta \downarrow 0} \frac{\log(N_\delta(F))}{\log(1/\delta)}, \quad \underline{\dim}_B(F) = \liminf_{\delta \downarrow 0} \frac{\log(N_\delta(F))}{\log(1/\delta)}.$$

When $\overline{\dim}_B(F) = \underline{\dim}_B(F)$, we can replace the \liminf and \limsup by \lim , and the common value is the box-counting dimension $\dim_B(F)$, an example of a fractal dimension. A possible drawback of the box-counting dimension is its lack of countable stability. For example, $\dim_B(\{0, 1, 1/2, 1/3, \dots\}) = 1/2$. Let Ω_f^{BD} be the class of self-adjoint operators in Ω_f (see (2.1)) whose upper and lower box-counting dimensions of the spectrum agree. Let Ω_{SA}^{BD} be the class of self-adjoint operators whose upper and lower box-counting dimensions of the spectrum agree, and denote by Ω_D^{BD} the class of diagonal operators in Ω_{SA}^{BD} .

Hausdorff Dimension: A more complicated, yet robust notion of fractal dimension is related to the Hausdorff measure [71, 114]. Let $F \subset \mathbb{R}^n$ be a bounded Borel set and let $\mathcal{C}_\delta(F)$ denote the class of (countable) δ -covers¹⁵ of F . One first defines the quantities (for $d \geq 0$)

$$\mathcal{H}_\delta^d(F) = \inf \left\{ \sum_i \text{diam}(U_i)^d : \{U_i\} \in \mathcal{C}_\delta(F) \right\}, \quad \mathcal{H}^d(F) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^d(F).$$

There is a unique $d' = \dim_H(F) \geq 0$, the Hausdorff dimension of F , such that $\mathcal{H}^d(F) = 0$ for $d > d'$ and $\mathcal{H}^d(F) = \infty$ for $d < d'$. One can prove that

$$\dim_H(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F).$$

With these definitions in hand, we can now present the main theorem of this section.

Theorem 3.18 (Fractal dimensions of spectra) *Let $\Xi_B(A) = \dim_B(\text{Sp}(A))$ and $\Xi_H = \dim_H(\text{Sp}(A))$. Then for $i = 1, 2$,*

$$\begin{aligned} \Delta_2^G \not\cong \{\Xi_B, \Omega_f^{BD}, \Lambda_i\} \in \Pi_2^A, & \quad \Delta_2^G \not\cong \{\Xi_B, \Omega_D^{BD}, \Lambda_i\} \in \Pi_2^A \\ \Delta_3^G \not\cong \{\Xi_H, \Omega_f \cap \Omega_{SA}, \Lambda_i\} \in \Sigma_3^A, & \quad \Delta_3^G \not\cong \{\Xi_H, \Omega_D, \Lambda_i\} \in \Sigma_3^A, \end{aligned}$$

whereas

$$\begin{aligned} \Delta_3^G \not\cong \{\Xi_B, \Omega_{SA}^{BD}, \Lambda_1\} \in \Pi_3^A, & \quad \Delta_2^G \not\cong \{\Xi_B, \Omega_{SA}^{BD}, \Lambda_2\} \in \Pi_2^A \\ \Delta_4^G \not\cong \{\Xi_H, \Omega_{SA}, \Lambda_1\} \in \Sigma_4^A, & \quad \Delta_3^G \not\cong \{\Xi_H, \Omega_{SA}, \Lambda_2\} \in \Sigma_3^A. \end{aligned}$$

¹⁵ That is, the set of covers $\{U_i\}_{i \in I}$ with I at most countable and with $\text{diam}(U_i) \leq \delta$.

Remark 3.19 (When $\underline{\dim}_B(\text{Sp}(A)) \neq \overline{\dim}_B(\text{Sp}(A))$) The algorithms for Ξ_B also converge without the assumption that the upper and lower box-counting dimensions of $\text{Sp}(A)$ agree, to a quantity $\Gamma(A)$ with

$$\underline{\dim}_B(\text{Sp}(A)) \leq \Gamma(A) \leq \overline{\dim}_B(\text{Sp}(A)).$$

One of the properties that makes the Hausdorff dimension harder to compute than the box-counting dimension is its countable stability, meaning that if F is countable then $\dim_H(F) = 0$. □

Remark 3.20 Some of our results have interpretations for real bounded sequences. Given such a sequence $\{a_i\}_{i \in \mathbb{N}} \subset \mathbb{R}$, we can ask the same questions about $\overline{\{a_1, a_2, \dots\}}$ as we have asked about the spectrum. We can embed these problems as spectral problems for the class Ω_D of bounded self-adjoint diagonal operators by simply considering diagonal operators with entries $\{a_1, a_2, \dots\}$. Theorems 3.13, 3.16 and 3.18 immediately then give the classifications. With regard to fractal dimensions, the key problem is to try and relate the amount of data that has been seen to the resolution obtained from the data (as highlighted in the computational example below). Once we have the framework of the SCI, we can immediately see why the problem is so difficult—the computational problem requires three successive limits for the Hausdorff dimension. □

Finally, the following lemma is used in the construction of the tower of algorithms for computing the Hausdorff dimension but is interesting in its own right so is listed here.

Lemma 3.21 *Let $(a, b) \subset \mathbb{R}$ be a finite open interval and let $A \in \Omega_f \cap \Omega_{SA}$. Then determining whether $\text{Sp}(A) \cap (a, b) \neq \emptyset$ using Λ_i is a problem with $\text{SCI}_A = 1$. Furthermore, we can design an algorithm that halts if and only if the answer is “Yes”, that is, the problem lies in Σ_1^A . Similarly, the problem lies in Σ_2^A when considering Ω_{SA} with Λ_1 (or Σ_1^A when we allow access to Λ_2).*

4 Computational Examples

In this section, we demonstrate that the SCI-sharp algorithms constructed in this paper can be efficiently implemented for large-scale computations. Moreover, the algorithms have desirable convergence properties, converging monotonically or being eventually constant, as captured by the Σ/Π classification. Generically, this monotonicity holds in all of the successive limits, and not just the final limit; many of the towers of algorithms undergo *oscillation phenomena* where each subsequent limit is monotone but in the opposite sense/direction than the limit beforehand. We can take advantage of this when analysing the algorithms numerically. The algorithms also highlight suitable information that lowers the SCI classification to Σ_1/Π_1 . Other advantages of the algorithms based on approximating the resolvent norm include locality, numerical stability and speed/parallelisation. In the examples that follow, we remind the reader what each parameter n_k intuitively does in the relevant algorithm and simplified routines for many of the algorithms can be found in “Appendix B”. Finally, we point the

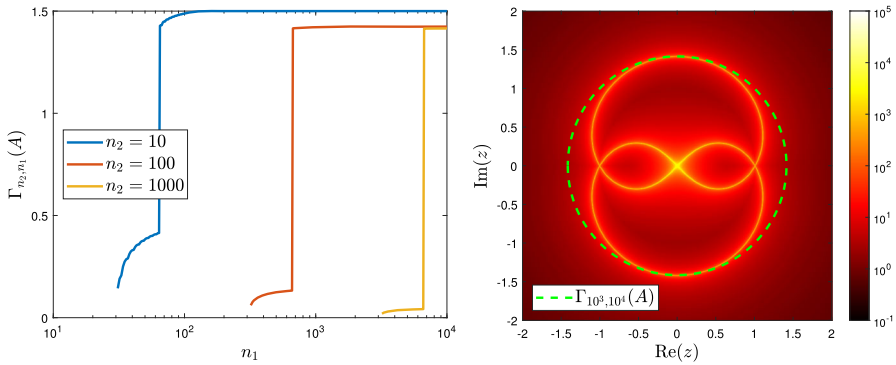


Fig. 2 Left: Output of the algorithm for computing the spectral radius. Right: Pseudospectrum computed using the method of [64] (the colour scale corresponds to the resolvent norm $\|(A - zI)^{-1}\|$ which provides error control. We have shown the output of $\Gamma_{10^3, 10^4}(A)$ via the green dashed circle (Color figure online)

reader to Remark 5.12—all of the algorithms can be implemented rigorously using arithmetic operations over the rationals or with methods such as interval arithmetic.

4.1 Spectral Radius

We begin with the spectral radius and consider the upper-triangular non-normal operator on $l^2(\mathbb{Z})$ defined by its action on the canonical basis via

$$Ae_j = e_{j-2} + i^j e_{j-1}.$$

In this case, the operator norm of A is 2 and the approximation of the spectrum by finite section is $\{0\}$. Hence, to compute the spectral radius, one must resort to the techniques used in our algorithms based on rectangular truncations. Recall that the SCI classification for computing the spectral radius of such operators (where the dispersion is known) is Π_2^A (see Theorem 3.3 for further classifications). The first parameter, n_1 , controls the size of the rectangular truncation¹⁶ (as well as the grid resolution), whereas the second, n_2 , controls the resolvent norm cut-off ($\epsilon = 1/n_2$).

Figure 2 (left) shows the output of $\Gamma_{n_2, n_1}(A)$ for computing the spectral radius. We see the expected monotonicity; $\Gamma_{n_2, n_1}(A)$ is increasing in n_1 but decreasing in n_2 . It appears that $\lim_{n_1 \rightarrow \infty} \Gamma_{10^2, n_1}(A) \approx \lim_{n_1 \rightarrow \infty} \Gamma_{10^3, n_1}(A) \approx 1.4149$. The fact that these two values for different n_2 are similar suggests that we have reached convergence. Though, of course, the proof that the problem does not lie in Δ_2^G shows that we can never apply a choice of subsequences to gain convergence in one limit over the whole class Ω_f . Nevertheless, the approximate value of 1.4149 is confirmed in Fig. 2 (right) where we have shown pseudospectra, computed using the algorithm in [64].

¹⁶ For this example and other operators on $l^2(\mathbb{Z})$ below, we reorder the basis so that the operator A acts on $l^2(\mathbb{N})$.

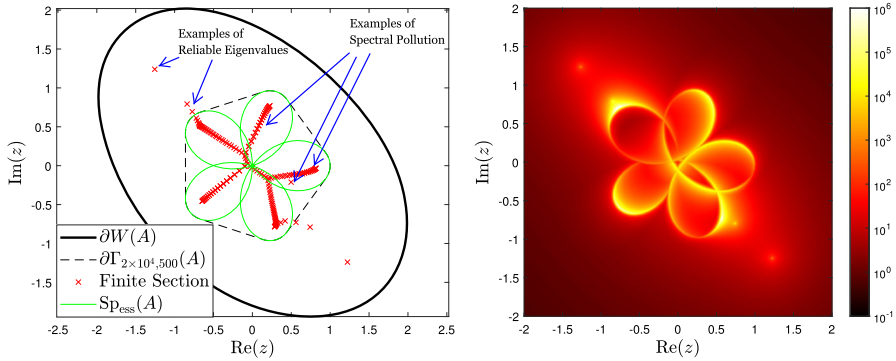


Fig. 3 Left: The boundaries $\partial W(A)$ and $\partial \Gamma_{2 \times 10^4, 500}(A)$. We have also shown the essential spectrum of A (whose convex hull, in this example, corresponds to $W_e(A)$) and the output of finite section for a 200×200 truncation. Right: Pseudospectrum computed using the method of [64] (the colour scale corresponds to the resolvent norm $\|(A - zI)^{-1}\|$) which provides error control. This confirms that eigenvalues, computed using finite section, outside $\partial \Gamma_{2 \times 10^4, 500}(A)$ are accurate and, in this example, indicates that the other eigenvalues correspond to spectral pollution (Color figure online)

4.2 Essential Numerical Range

To demonstrate the algorithm for computing the essential numerical range, we first consider the Laurent operator A_0 acting on $l^2(\mathbb{Z})$ with the symbol

$$a(t) = \frac{t^4 + t^{-1}}{2}.$$

In this case, $\text{Sp}(A_0) = \text{Sp}_{\text{ess}}(A_0) = \{a(z) : |z| = 1\}$. We consider the operator $A = A_0 + E$ where the compact perturbation E is given by

$$Ee_j = -\frac{3i}{1 + |j|}e_{j-1}.$$

Recall that the SCI classification for computing the essential numerical range is Π_2^A (see Theorem 3.10). The first parameter, n_1 , controls the size of the truncation, whereas the second, n_2 , controls how far along the matrix the truncations $(I - P_{n_2})P_{n_1+n_2}A|_{P_{n_1+n_2}(I - P_{n_2})\mathcal{H}}$ are taken with respect to the canonical basis.

Figure 3 (left) shows the output of the algorithm $\Gamma_{n_2, n_1}(A)$ to compute the essential numerical range for $n_2 = 20,000$ and $n_1 = 500$. We show the boundary $\partial \Gamma_{n_2, n_1}(A)$ since the essential numerical range is convex. In this example, $W_e(A)$ is the convex hull of $\text{Sp}_{\text{ess}}(A_0)$, which allows us to verify the output of the algorithm. We also show 200 eigenvalues of finite section (computed using extended precision to avoid numerical instabilities associated with non-normal truncations), the majority of which are due to truncation and provide an example of spectral pollution. This is confirmed when we compare to the pseudospectrum, also shown in Fig. 3 (right), computed using the algorithm in [64]. However, eigenvalues outside $W_e(A)$ correspond to true eigenvalues of A (see Theorem 3.9).

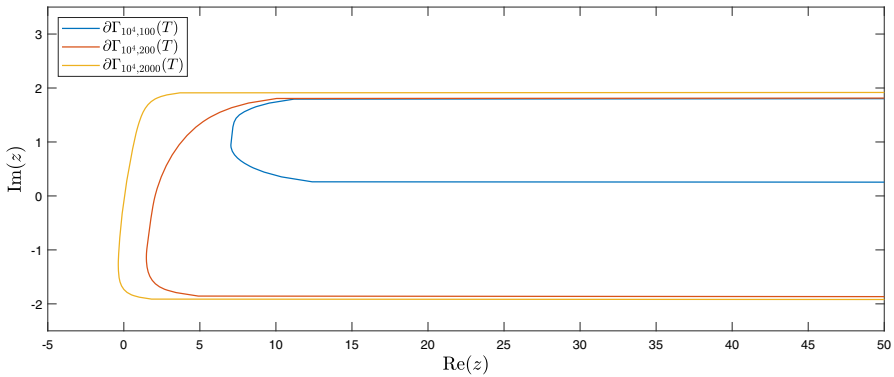


Fig. 4 The output of the algorithm for computing the essential numerical range of closed operators, applied to the complex Schrödinger operator T in (4.1)

The algorithm can also be extended to unbounded operators, as outlined in Sect. 7.1. For example, we consider the complex Schrödinger operator

$$T = -\frac{d^2}{dx^2} + (2i + 1) \cos(x). \tag{4.1}$$

By using a Gabor basis, we can represent T as a closed operator on $l^2(\mathbb{N})$ such that the linear span of the canonical basis (corresponding to the Gabor basis) forms a core. This allows us to use Corollary 7.5, where we can compute the matrix elements (corresponding to inner products with the basis functions) with error control using quadrature. Figure 4 shows the output for $n_2 = 10^4$ and various n_1 . We see the expected monotonicity as n_1 increases and the output for $n_1 = 2000$ has converged to visible accuracy in the plot.

4.3 Capacity

We now consider a transport Hamiltonian on a Penrose tile for which few analytical results are known. Quasicrystals were discovered in 1982 by Shechtman [145] who was awarded the Nobel prize in 2011 for his discovery. Over the past 30 years, there has been considerable interest in their often exotic properties [67, 151]. The Penrose tile is the standard two-dimensional model [69, 165], and a finite portion of the tiling is shown in Fig. 5 (left). However, unlike one-dimensional models, very little is known about the spectral properties of two-dimensional quasicrystals. Let G be the graph consisting of the vertices, $V(G)$, of the Penrose tiling and $E(G)$ the set of edges. If there is an edge connecting two vertices x and y , we write $x \sim y$. The (negative) Laplacian, H , acts on $\psi \in l^2(V(G)) \cong l^2(\mathbb{N})$ by

$$(H\psi)(x) = \sum_{y \sim x} (\psi(y) - \psi(x)). \tag{4.2}$$

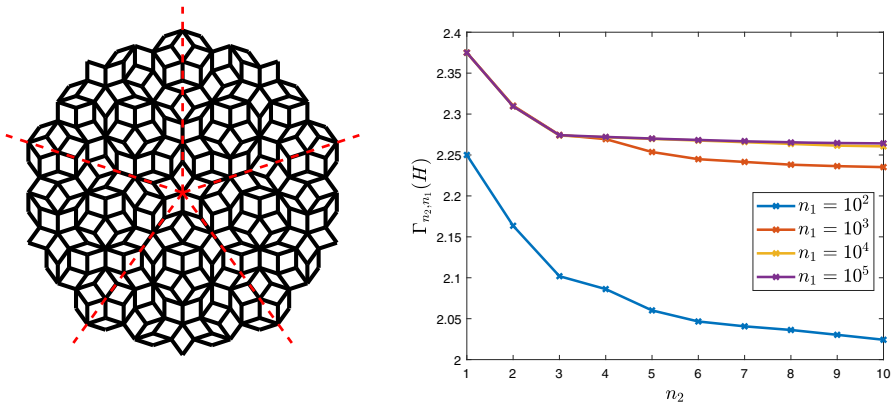


Fig. 5 Left: Finite portion of the Penrose tiling showing a fivefold rotational symmetry. Right: Output of the algorithm for computing the capacity of $\text{Sp}(H)$, where H is the operator in (4.2)

By choosing a suitable ordering of the vertices, we can represent H as an operator acting on $l^2(\mathbb{N})$ of bounded dispersion with $f(n) - n \sim \mathcal{O}(\sqrt{n})$. Recall that the SCI classification for computing the capacity of the spectrum of such operators is Π_2^A (see Theorem 3.7 for further classifications). The first parameter, n_1 , controls the size of the truncation used to test if intervals intersect the spectrum via Lemma 3.21, whereas the second, n_2 , controls the spacings of the interval coverings (which have width 2^{-n_2}). In this example, we used the conformal mapping method of [109] to accurately and rapidly compute the capacity of finite unions of intervals in \mathbb{R} (see also Remark 6.4).

Figure 5 (right) shows the output of $\Gamma_{n_2, n_1}(H)$, and we see the expected monotonicity; the output is increasing in n_1 but decreasing in n_2 . By comparing the outputs for $n_1 = 10^4$ and $n_1 = 10^5$, it appears we have convergence up to around $n_2 = 8$. This suggests an upper bound (since the output is non-increasing in n_2) of approximately 2.26 for the capacity of $\text{Sp}(H)$ ($\text{Sp}(H)$ is shown in Fig. 6).

4.4 Lebesgue Measure

First, we consider the almost Mathieu operator, which is related to a wealth of mathematical and physical problems such as the Ten Martini Problem [9]. The operator acts on $l^2(\mathbb{Z})$ via

$$(H_\alpha x)_n = x_{n-1} + x_{n+1} + 2\lambda \cos(2\pi n\alpha)x_n. \tag{4.3}$$

The choice of $\lambda = 1$ was studied in Hofstadter’s classic paper [92], giving rise to the famous Hofstadter butterfly. In this case, the Hamiltonian represents a crystal electron in a uniform magnetic field and the spectrum can be interpreted as the allowed energies of the system. For irrational α , we have [11]

$$\text{Leb}_{\mathbb{R}}(\text{Sp}(H_\alpha)) = 4|1 - |\lambda|| \tag{4.4}$$

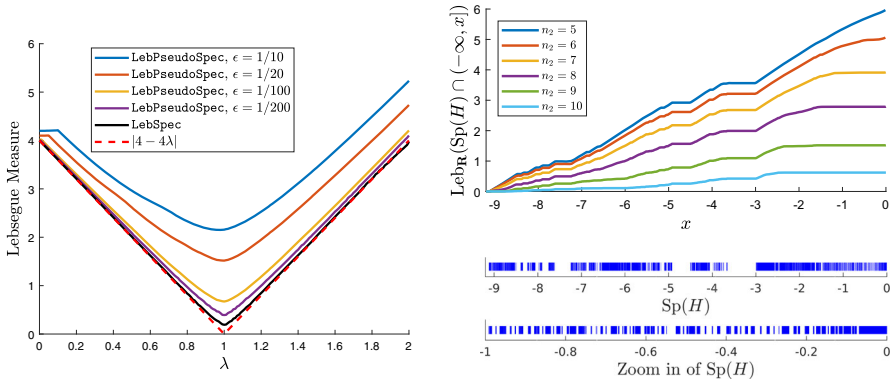


Fig. 6 Left: Output of the algorithm LebSpec to compute $\text{Leb}_{\mathbb{R}}(\text{Sp}(H_{\alpha}))$ as well as the algorithm LebPseudoSpec for $\text{Leb}_{\mathbb{R}}(\text{Sp}_{\epsilon}(H_{\alpha}))$ (which converges to $\text{Leb}_{\mathbb{R}}(\text{Sp}(H_{\alpha}))$ as $\epsilon \downarrow 0$). These were computed using $n_1 = 10^4$ and $n_2 = 7$. Right: Estimates for $\text{Leb}_{\mathbb{R}}(\text{Sp}(H) \cap (-\infty, x])$, where H is the Laplacian on a Penrose tiling in (4.2), obtained by letting $n_1 = 10^5$ and selecting different n_2 . The estimate above -3 appears to be well resolved, suggesting a region of Lebesgue measure 0

and we consider the case $\alpha = (\sqrt{5} - 1)/2$. Recall that the SCI classification for computing the Lebesgue measure of the spectrum of such operators (where the dispersion is known) is Π_2^A , whereas the SCI classification of computing the Lebesgue measure of the pseudospectrum is Σ_1^A (see Theorems 3.13, 3.14 and 3.16 for the further classifications). For computing the Lebesgue measure of the spectrum, the first parameter, n_1 , controls the size of the truncation used to compute the approximation of the resolvent norm, whereas the second, n_2 , controls the grid refinement (the spacings are 2^{-n_2}). For the pseudospectrum, n_1 controls the size of the truncations and the grid spacings.

Figure 6 (left) shows the output of the algorithms computing $\text{Leb}_{\mathbb{R}}(\text{Sp}(H_{\alpha}))$ (LebSpec) and also $\text{Leb}_{\mathbb{R}}(\text{Sp}_{\epsilon}(H_{\alpha}))$ (LebPseudoSpec) for a range of values of ϵ . We chose values of $n_1 = 10^4$ and a grid spacing of $1/128$ ($n_2 = 7$). One can clearly see that the estimates for $\text{Leb}_{\mathbb{R}}(\text{Sp}_{\epsilon}(H_{\alpha}))$ are decreasing to the true value of $\text{Leb}_{\mathbb{R}}(\text{Sp}(H_{\alpha}))$, which is well approximated by LebSpec .

Next, we consider the operator H in (4.2), for which the Lebesgue measure of $\text{Sp}(H)$ is unknown. We set $n_1 = 10^5$ and look at the average estimated error of the output via DistSpec (see ‘‘Appendix A’’). This was of the order 10^{-3} , so we consider grid refinements of spacing $1/32, 1/64, \dots, 1/1024$ corresponding to $n_2 = 5, 6, \dots, 10$. Figure 6 (right) shows the output as a cumulative Lebesgue measure, that is, an estimate of $\text{Leb}_{\mathbb{R}}(\text{Sp}(A) \cap (-\infty, x])$ for a given x , along with the computed spectrum (for a grid spacing of 10^{-5}). The figure provides strong evidence that the part of the spectrum closest to 0 is resolved by the algorithm and has Lebesgue measure zero. We shall see more evidence for this in Sect. 4.5.

4.5 Fractal Dimension

For this example, we again consider the operator H in (4.2), for which the fractal dimension of $\text{Sp}(H)$ is unknown. In Fig. 7, we plot $N_{1/n_2}(\tilde{\Gamma}_{10^5}(H) \cap [-3, \infty))$ against

n_2 (recall that $N_\delta(F)$ is the number of closed intervals of length $\delta > 0$ required to cover F). This corresponds to a rectangular truncation with $n_1 = 10^5$ columns. Recall that $\tilde{\Gamma}_n$ denotes the algorithm that converges to the spectrum with error control, in particular avoiding spectral pollution (see “Appendix A”). We also show a linear fit of slope 0.8. The error control provided by the algorithm $\tilde{\Gamma}_n$ allows us to deduce the region where the fit holds, corresponding to a reliable resolution of the spectrum (this is at least as large as the region shown in the plot). In other words, we can ensure that n_2 is not too large so that the spacings of the coverings are not smaller than the numerically resolved spectrum. As expected, when n_2 is too large we see the effect of the grid spacing and the unresolved spectrum (by choosing larger n_1 , we can take n_2 larger). The figure suggests that the spectrum above -3 is fractal with box-counting dimension ≈ 0.8 and hence has Lebesgue measure zero, in agreement with the findings in Fig. 6.

Figure 7 also shows what happens when one performs the same experiment but with a finite section replacing $\tilde{\Gamma}_n$ (now using a square $10^5 \times 10^5$ truncation). There are two noticeable features. First, for small n_2 , using a finite section produces an overestimate of the size of the covering and the corresponding slope of the graph due to spectral pollution. In other words, finite section prevents us from detecting the fractal spectrum. Second, the covering estimate via finite section breaks down at smaller n_2 and it is impossible to predict suitable values of n_2 so that the spacings of the coverings do not go beyond the resolution of the computed spectrum. Together, these issues highlight why the finite section method is unsuitable in general¹⁷ for approximating fractal dimensions and why the new algorithms in this paper (which are proven to converge) are needed.

¹⁷ There do exist examples of operators, typically with a lot of structure, where one can use periodic versions of finite section.

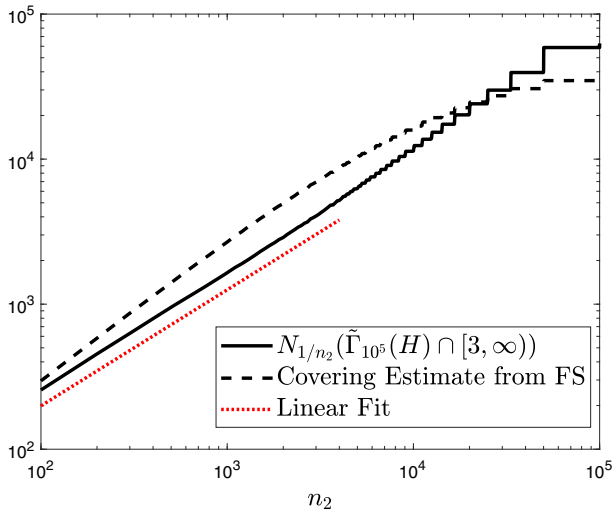


Fig. 7 A plot of $N_{1/n_2}(\tilde{\Gamma}_{10^5}(H) \cap [-3, \infty))$ against n_2 . We found a scaling region with estimated box-counting dimension ≈ 0.80 . Note that for large $n_2 \gtrsim 5000$, scalings are not resolved by $\tilde{\Gamma}_{10^5}$ (we can predict when this happens using the Σ_1^A property of $\tilde{\Gamma}_n$). We have also shown the approximation using finite sections (square $10^5 \times 10^5$ matrix truncations), as a dashed line, which overestimate the size of coverings, cannot detect the fractal structure, and break down for smaller n_2

5 Mathematical Preliminaries and Combinatorial Problems in the SCI Hierarchy

In this section, we begin by providing formal definitions of the SCI hierarchy. We then link the SCI hierarchy, in a certain specific case, to the Baire hierarchy on a suitable topological space. As well as being interesting in its own right, this provides a useful method of providing canonical problems high up in the SCI hierarchy. In particular, the results we prove hold for towers of *general* algorithms (see Definition 5.1) without the restrictions of arithmetic operations or notions of recursivity etc. This will be used extensively in the proofs of lower bounds for spectral problems that have $SCI > 2$, where we typically reduce the problems discussed here to the given spectral problem. It should be stressed that such links to existing hierarchies only exist in special cases when Ω and \mathcal{M} are particularly well-behaved. Even when such a link does exist, the induced topology on Ω is often too complicated, unnatural or strong to be useful from a computational viewpoint. We also take the view that, for problems of scientific interest, the mappings Λ and metric space \mathcal{M} are often given to us a priori from the corresponding applications and are typically not compatible with topological viewpoints of computation.

5.1 The SCI Hierarchy

We begin by defining the solvability complexity index (SCI) hierarchy, allowing us to show that our algorithms realise the boundary of what computers can achieve. We have

already presented the definition of a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ in §2.1. Recall that the goal is to find algorithms that approximate the function Ξ . More generally, the main pillar of our framework is the concept of a tower of algorithms, which is needed to describe problems that need several successive limits in the computation. However, first one needs the definition of a general algorithm.

Definition 5.1 (*General Algorithm*) Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, a general algorithm is a mapping $\Gamma : \Omega \rightarrow \mathcal{M}$ such that for each $A \in \Omega$

- (i) there exists a (non-empty) finite subset of evaluations $\Lambda_\Gamma(A) \subset \Lambda$,
- (ii) the action of Γ on A only depends on $\{A_f\}_{f \in \Lambda_\Gamma(A)}$ where $A_f := f(A)$,
- (iii) for every $B \in \Omega$ such that $B_f = A_f$ for every $f \in \Lambda_\Gamma(A)$, it holds that $\Lambda_\Gamma(B) = \Lambda_\Gamma(A)$.

The definition of a general algorithm is more general than the definition of a Turing machine [164] or a BSS machine [28]. A general algorithm has no restrictions on the operations allowed. The only restriction is that it can only take a finite amount of information, though it is allowed to *adaptively* choose the finite amount of information it reads depending on the input. Condition (iii) ensures that the algorithm consistently reads the information. With a definition of a general algorithm, we can define the concept of towers of algorithms.

Definition 5.2 (*Tower of Algorithms*) Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, a tower of algorithms of height k for $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ is a family of sequences of functions

$$\Gamma_{n_k} : \Omega \rightarrow \mathcal{M}, \Gamma_{n_k, n_{k-1}} : \Omega \rightarrow \mathcal{M}, \dots, \Gamma_{n_k, \dots, n_1} : \Omega \rightarrow \mathcal{M},$$

where $n_k, \dots, n_1 \in \mathbb{N}$ and the functions Γ_{n_k, \dots, n_1} at the lowest level of the tower are general algorithms in the sense of Definition 5.1. Moreover, for every $A \in \Omega$,

$$\Xi(A) = \lim_{n_k \rightarrow \infty} \Gamma_{n_k}(A), \Gamma_{n_k, \dots, n_{j+1}}(A) = \lim_{n_j \rightarrow \infty} \Gamma_{n_k, \dots, n_j}(A) \quad j = k - 1, \dots, 1.$$

In addition to a general tower of algorithms, we focus on arithmetic towers.

Definition 5.3 (*Arithmetic Tower*) Given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, where Λ is countable, we define the following: An arithmetic tower of algorithms of height k for $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ is a tower of algorithms where the lowest functions $\Gamma = \Gamma_{n_k, \dots, n_1} : \Omega \rightarrow \mathcal{M}$ satisfy the following: For each $A \in \Omega$ the mapping $(n_k, \dots, n_1) \mapsto \Gamma_{n_k, \dots, n_1}(A) = \Gamma_{n_k, \dots, n_1}(\{A_f\}_{f \in \Lambda})$ is recursive, and $\Gamma_{n_k, \dots, n_1}(A)$ is a finite string of complex numbers that can be identified with an element in \mathcal{M} . For arithmetic towers, we let $\alpha = A$.

Remark 5.4 By recursive we mean the following. If $f(A) \in \mathbb{Q}$ (or $\mathbb{Q} + i\mathbb{Q}$) for all $f \in \Lambda, A \in \Omega$, and Λ is countable, then $\Gamma_{n_k, \dots, n_1}(\{A_f\}_{f \in \Lambda})$ can be executed by a Turing machine [164], that takes (n_k, \dots, n_1) as input, and that has an oracle tape consisting of $\{A_f\}_{f \in \Lambda}$. If $f(A) \in \mathbb{R}$ (or \mathbb{C}) for all $f \in \Lambda$, then $\Gamma_{n_k, \dots, n_1}(\{A_f\}_{f \in \Lambda})$ can be executed by a BSS machine [28] that takes (n_k, \dots, n_1) , as input, and that has an oracle that can access any A_f for $f \in \Lambda$. □

Given the definitions above we can now define the key concept, namely the solvability complexity index:

Definition 5.5 (*Solvability Complexity Index*) A computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ is said to have solvability complexity index $\text{SCI}(\Xi, \Omega, \mathcal{M}, \Lambda)_\alpha = k$, with respect to a tower of algorithms of type α , if k is the smallest integer for which there exists a tower of algorithms of type α of height k . If no such tower exists, then $\text{SCI}(\Xi, \Omega, \mathcal{M}, \Lambda)_\alpha = \infty$. If there exists a tower $\{\Gamma_n\}_{n \in \mathbb{N}}$ of type α and height one such that $\Xi = \Gamma_{n_1}$ for some $n_1 < \infty$, then we define $\text{SCI}(\Xi, \Omega, \mathcal{M}, \Lambda)_\alpha = 0$. The type α may be General, or Arithmetic, denoted, respectively, G and A. We may sometimes write $\text{SCI}(\Xi, \Omega)_\alpha$ to simplify notation when \mathcal{M} and Λ are obvious.

We will let $\text{SCI}(\Xi, \Omega)_A$ and $\text{SCI}(\Xi, \Omega)_G$ denote the SCI with respect to an arithmetic tower and a general tower, respectively. Note that a general tower means just a tower of algorithms as in Definition 5.2, where there are no restrictions on the mathematical operations. Thus, clearly $\text{SCI}(\Xi, \Omega)_A \geq \text{SCI}(\Xi, \Omega)_G$. The definition of the SCI immediately induces the SCI hierarchy:

Definition 5.6 (*The Solvability Complexity Index Hierarchy*) Consider a collection \mathcal{C} of computational problems and let \mathcal{T} be the collection of all towers of algorithms of type α for the computational problems in \mathcal{C} . Define

$$\begin{aligned} \Delta_0^\alpha &:= \{\{\Xi, \Omega\} \in \mathcal{C} \mid \text{SCI}(\Xi, \Omega)_\alpha = 0\} \\ \Delta_{m+1}^\alpha &:= \{\{\Xi, \Omega\} \in \mathcal{C} \mid \text{SCI}(\Xi, \Omega)_\alpha \leq m\}, \quad m \in \mathbb{N}, \end{aligned}$$

as well as

$$\Delta_1^\alpha := \{\{\Xi, \Omega\} \in \mathcal{C} \mid \exists \{\Gamma_n\}_{n \in \mathbb{N}} \in \mathcal{T} \text{ s.t. } \forall A \ d(\Gamma_n(A), \Xi(A)) \leq 2^{-n}\}.$$

When there is additional structure on the metric space, such as in the spectral case when one considers the Attouch–Wets or the Hausdorff metric, one can extend the SCI hierarchy. For non-empty closed sets, we consider the *Attouch–Wets metric* defined by

$$d_{AW}(C_1, C_2) = \sum_{n=1}^\infty 2^{-n} \min \left\{ 1, \sup_{|x| \leq n} |\text{dist}(x, C_1) - \text{dist}(x, C_2)| \right\}, \quad (5.1)$$

for $C_1, C_2 \in \text{Cl}(\mathbb{C})$, where $\text{Cl}(\mathbb{C})$ denotes the set of closed non-empty subsets of \mathbb{C} . This generalises the familiar Hausdorff metric to unbounded closed sets and corresponds to local uniform converge on compact subsets of \mathbb{C} .

Definition 5.7 (*The SCI Hierarchy (Attouch–Wets/Hausdorff metric)*) Given the set-up in Definition 5.6, and suppose in addition that (\mathcal{M}, d) has the Attouch–Wets or the Hausdorff metric induced by another metric space (\mathcal{M}', d') , define, for $m \in \mathbb{N}$,

$$\begin{aligned} \Sigma_0^\alpha &= \Pi_0^\alpha = \Delta_0^\alpha, \\ \Sigma_1^\alpha &= \{\{\Xi, \Omega\} \in \Delta_2^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T}, \{X_n(A)\} \subset \mathcal{M} \text{ s.t. } \Gamma_n(A) \underset{\mathcal{M}'}{\subset} X_n(A)\}, \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \Gamma_n(A) &= \Xi(A), \quad d(X_n(A), \Xi(A)) \leq 2^{-n} \quad \forall A \in \Omega, \\ \Pi_1^\alpha &= \{ \{\Xi, \Omega\} \in \Delta_2^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T}, \{X_n(A)\} \subset \mathcal{M} \text{ s.t. } \Xi(A) \subset_{\mathcal{M}'} X_n(A), \\ &\quad \lim_{n \rightarrow \infty} \Gamma_n(A) = \Xi(A), \quad d(X_n(A), \Gamma_n(A)) \leq 2^{-n} \quad \forall A \in \Omega, \end{aligned}$$

where $\subset_{\mathcal{M}'}$ means inclusion in the metric space \mathcal{M}' , and $\{X_n(A)\}$ is a sequence where $X_n(A) \in \mathcal{M}$ depends on A . Moreover,

$$\begin{aligned} \Sigma_{m+1}^\alpha &= \{ \{\Xi, \Omega\} \in \Delta_{m+2}^\alpha \mid \exists \{\Gamma_{n_{m+1}, \dots, n_1}\} \in \mathcal{T}, \\ &\quad \{X_{n_{m+1}}(A)\} \subset \mathcal{M} \text{ s.t. } \Gamma_{n_{m+1}}(A) \subset_{\mathcal{M}'} X_{n_{m+1}}(A), \\ &\quad \lim_{n_{m+1} \rightarrow \infty} \Gamma_{n_{m+1}}(A) = \Xi(A), \quad d(X_{n_{m+1}}(A), \Xi(A)) \leq 2^{-n_{m+1}} \quad \forall A \in \Omega, \\ \Pi_{m+1}^\alpha &= \{ \{\Xi, \Omega\} \in \Delta_{m+2}^\alpha \mid \exists \{\Gamma_{n_{m+1}, \dots, n_1}\} \in \mathcal{T}, \\ &\quad \{X_{n_{m+1}}(A)\} \subset \mathcal{M} \text{ s.t. } \Xi(A) \subset_{\mathcal{M}'} X_{n_{m+1}}(A), \\ &\quad \lim_{n_{m+1} \rightarrow \infty} \Gamma_{n_{m+1}}(A) = \Xi(A), \quad d(X_{n_{m+1}}(A), \Gamma_{n_{m+1}}(A)) \leq 2^{-n_{m+1}} \quad \forall A \in \Omega, \end{aligned}$$

where d can be either d_H or d_{AW} .

Note that to build a Σ_1 algorithm, it is enough (by taking subsequences of n) to construct $\Gamma_n(A)$ such that $\Gamma_n(A) \subset \mathcal{N}_{E_n(A)}(\Xi(A))$ with some computable $E_n(A)$ that converges to zero. The same idea can be applied to the real line with the usual metric, or $\{0, 1\}$ with the discrete metric (we interpret 1 as “Yes”).

Definition 5.8 (*The SCI Hierarchy (totally ordered set)*) Given the set-up in Definition 5.6 and suppose in addition that \mathcal{M} is a totally ordered set. Define

$$\begin{aligned} \Sigma_0^\alpha &= \Pi_0^\alpha = \Delta_0^\alpha, \\ \Sigma_1^\alpha &= \{ \{\Xi, \Omega\} \in \Delta_2^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T} \text{ s.t. } \Gamma_n(A) \nearrow \Xi(A) \quad \forall A \in \Omega, \\ \Pi_1^\alpha &= \{ \{\Xi, \Omega\} \in \Delta_2^\alpha \mid \exists \{\Gamma_n\} \in \mathcal{T} \text{ s.t. } \Gamma_n(A) \searrow \Xi(A) \quad \forall A \in \Omega, \end{aligned}$$

where \nearrow and \searrow denotes convergence from below and above, respectively, as well as, for $m \in \mathbb{N}$,

$$\begin{aligned} \Sigma_{m+1}^\alpha &= \{ \{\Xi, \Omega\} \in \Delta_{m+2}^\alpha \mid \exists \{\Gamma_{n_{m+1}, \dots, n_1}\} \in \mathcal{T} \text{ s.t. } \Gamma_{n_{m+1}}(A) \nearrow \Xi(A) \quad \forall A \in \Omega, \\ \Pi_{m+1}^\alpha &= \{ \{\Xi, \Omega\} \in \Delta_{m+2}^\alpha \mid \exists \{\Gamma_{n_{m+1}, \dots, n_1}\} \in \mathcal{T} \text{ s.t. } \Gamma_{n_{m+1}}(A) \searrow \Xi(A) \quad \forall A \in \Omega. \end{aligned}$$

Remark 5.9 ($\Delta_1^\alpha \subsetneq \Sigma_1^\alpha \subsetneq \Delta_2^\alpha$) Note that the inclusions are strict. For example, if Ω_K consists of the set of compact infinite matrices acting on $l^2(\mathbb{N})$ and $\Xi(A) = \text{Sp}(A)$ (the spectrum of A) then $\{\Xi, \Omega_K\} \in \Delta_2^\alpha$ but not in $\Sigma_1^\alpha \cup \Pi_1^\alpha$ for α representing either towers of arithmetical or general type (see [18] for a proof). Moreover, as was demonstrated in [64], if $\tilde{\Omega}$ is the set of discrete Schrödinger operators on $l^2(\mathbb{Z})$, then $\{\Xi, \tilde{\Omega}\} \in \Sigma_1^\alpha$ but not in Δ_1^α . \square

Suppose we are given a computational problem $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$, and that $\Lambda = \{f_j\}_{j \in \beta}$, where β is some index set that can be finite or infinite. Obtaining f_j may be a computational task on its own, which is exactly the problem in most areas of computational mathematics. In particular, for $A \in \Omega$, $f_j(A)$ could be the number $e^{\frac{\pi}{j}i}$ for example. Hence, we cannot access $f_j(A)$, but rather $f_{j,n}(A)$ where $f_{j,n}(A) \rightarrow f_j(A)$ as $n \rightarrow \infty$. Or, just as for problems that are high up in the SCI hierarchy, it could be that we need several successive limits, in particular one may need mappings $f_{j,n_m,\dots,n_1} : \Omega \rightarrow \mathbb{D} + i\mathbb{D}$, where \mathbb{D} denotes the dyadic rational numbers, such that

$$\lim_{n_m \rightarrow \infty} \dots \lim_{n_1 \rightarrow \infty} \|f_{j,n_m,\dots,n_1}(A) - f_j(A)\|_\infty = 0 \quad \forall j \in \beta, \forall A \in \Omega. \tag{5.2}$$

In particular, we may view the problem of obtaining $f_j(A)$ as a problem in the SCI hierarchy, where Δ_1 classification would correspond to the existence of mappings $f_{j,n} : \Omega \rightarrow \mathbb{D} + i\mathbb{D}$ such that

$$\|f_{j,n}(A) - f_j(A)\|_\infty \leq 2^{-n} \quad \forall j \in \beta, \forall A \in \Omega. \tag{5.3}$$

This idea is formalised in the following definition.

Definition 5.10 (*Δ_m -information*) Let $\{\Xi, \Omega, \mathcal{M}, \Lambda\}$ be a computational problem. For $m \in \mathbb{N}$, we say that Λ has Δ_{m+1} -information if each $f_j \in \Lambda$ is not available, however, there are mappings $f_{j,n_m,\dots,n_1} : \Omega \rightarrow \mathbb{D} + i\mathbb{D}$ such that (5.2) holds. Similarly, for $m = 0$ there are mappings $f_{j,n} : \Omega \rightarrow \mathbb{D} + i\mathbb{D}$ such that (5.3) holds. Finally, if $k \in \mathbb{N}$ and $\hat{\Lambda}$ is a collection of such functions described above such that Λ has Δ_k -information, we say that $\hat{\Lambda}$ provides Δ_k -information for Λ . Moreover, we denote the family of all such $\hat{\Lambda}$ by $\mathcal{L}^k(\Lambda)$.

We want algorithms that can handle all computational problems $\{\Xi, \Omega, \mathcal{M}, \hat{\Lambda}\}$ when $\hat{\Lambda} \in \mathcal{L}^m(\Lambda)$. To formalise this, we define a computational problem with Δ_m -information.

Definition 5.11 (*Computational problem with Δ_m -information*) Given $m \in \mathbb{N}$ with $m > 1$, a computational problem where Λ has Δ_m -information is denoted by $\{\Xi, \Omega, \mathcal{M}, \Lambda\}^{\Delta_m} := \{\tilde{\Xi}, \tilde{\Omega}, \mathcal{M}, \tilde{\Lambda}\}$, where

$$\tilde{\Omega} = \left\{ \tilde{A} = \{f_{j,n_{m-1},\dots,n_1}(A)\}_{j,n_{m-1},\dots,n_1 \in \beta \times \mathbb{N}^{m-1}} \mid A \in \Omega, \{f_j\}_{j \in \beta} = \Lambda, \right. \\ \left. f_{j,n_{m-1},\dots,n_1} \text{ satisfy (5.2)} \right\}.$$

Moreover, $\tilde{\Xi}(\tilde{A}) = \Xi(A)$, and we have $\tilde{\Lambda} = \{\tilde{f}_{j,n_{m-1},\dots,n_1}\}_{j,n_{m-1},\dots,n_1 \in \beta \times \mathbb{N}^{m-1}}$ where $\tilde{f}_{j,n_{m-1},\dots,n_1}(\tilde{A}) = f_{j,n_{m-1},\dots,n_1}(A)$. Note that $\tilde{\Xi}$ is well-defined by Definition 2.1 of a computational problem. Similarly, we define Δ_1 -information using (5.3).

The SCI and the SCI hierarchy, given Δ_m -information, are then defined in the standard obvious way.

Remark 5.12 (Classifications in this paper) For the problems considered in this paper, the SCI classifications do not change if we consider arithmetic towers with Δ_1 -information. This is easy to see through Church’s thesis and an analysis of the stability of our algorithms. For example, when the input is rational we have been careful to restrict all relevant operations to \mathbb{Q} rather than \mathbb{R} , and errors incurred from Δ_1 -information can be removed in the first limit. Explicitly, for the algorithms based on `DistSpec` (see “Appendix A”) it is possible to carry out an error analysis. We can also bound numerical errors (e.g. using interval arithmetic [163]) and incorporate this uncertainty for the estimation of $\|R(z, A)\|^{-1}$ to gain the same classification of our problems. Similarly, for other algorithms based on similar functions. In other words, for the results of this paper, it does not matter which model of computation one uses for a definition of ‘algorithm’. From a classification point of view, they are equivalent for these spectral problems. This leads to rigorous Σ_k^α or Π_k^α type error control suitable for verifiable numerics. In particular, for Σ_1^α or Π_1^α towers of algorithms, this could be useful for computer-assisted proofs. \square

5.2 Recalling Some Results from Descriptive Set Theory

We briefly recall the definition of the Borel hierarchy as well as some well-known theorems from descriptive set theory. It is beyond the scope of this paper to provide an extensive discussion of descriptive set theory, but we refer the reader to [98, Chapter 2] for an excellent introduction that covers the main ideas.

Let X be a metric space and define

$$\Sigma_1^0(X) = \{U \subset X : U \text{ is open}\}, \quad \Pi_1^0(X) = \sim \Sigma_1^0(X) = \{F \subset X : F \text{ is closed}\},$$

where for a class \mathcal{U} , $\sim\mathcal{U}$ denotes the class of complements (in X) of elements of \mathcal{U} . Inductively define

$$\begin{aligned} \Sigma_\xi^0(X) &= \{\cup_{n \in \mathbb{N}} A_n : A_n \in \Pi_{\xi_n}^0, \xi_n < \xi\}, \text{ if } \xi > 1, \\ \Pi_\xi^0(X) &= \sim \Sigma_\xi^0(X), \quad \Delta_\xi^0(X) = \Sigma_\xi^0(X) \cap \Pi_\xi^0(X). \end{aligned}$$

The full Borel hierarchy extends to all $\xi < \omega_1$ (ω_1 being the first uncountable ordinal) by transfinite induction but we do not need this here.

Definition 5.13 Given a class of subsets, \mathcal{U} , of a metric space X and given another metric space Y , we say that the function $f : X \rightarrow Y$ is \mathcal{U} -measurable if $f^{-1}(U) \in \mathcal{U}$ for every open set $U \subset Y$.

Given metric spaces X and Y , the Baire hierarchy is defined as follows. A function $f : X \rightarrow Y$ is of Baire class 1, written $f \in \mathcal{B}_1$, if it is $\Sigma_2^0(X)$ -measurable. For $1 < \xi < \omega_1$, a function $f : X \rightarrow Y$ is of Baire class ξ , written $f \in \mathcal{B}_\xi$, if it is the pointwise limit of a sequence of functions f_n in \mathcal{B}_{ξ_n} with $\xi_n < \xi$. The following Theorem is well-known [98, Section 24] and provides a useful link between the Borel and Baire hierarchies.

Theorem 5.14 (Lebesgue, Hausdorff, Banach) *Let X, Y be metric spaces with Y separable and $1 \leq \xi < \omega_1$. Then, $f \in \mathcal{B}_\xi$ if and only if it is $\Sigma_{\xi+1}^0(X)$ measurable. Furthermore, if X is zero-dimensional (Hausdorff with a basis of clopen sets) and $f \in \mathcal{B}_1$, then f is the pointwise limit of a sequence of continuous functions.*

The assumption that X is zero-dimensional in the last statement is important. Without any assumptions, the final statement of the theorem is false, as is easily seen by considering $X = \mathbb{R}$. Examples of zero-dimensional spaces include products of the discrete space $\{0, 1\}$ or the Cantor space. Any such space is necessarily totally disconnected, meaning that the connected components in the space are the one-point sets (the converse is true for locally compact Hausdorff spaces). Our primary interest will be when Y is equal to $\{0, 1\}$ or $[0, 1]$, both with their natural topologies.

5.3 Linking the SCI Hierarchy to the Baire Hierarchy in a Special Case

Definition 5.15 Given the triple $\{\Omega, \mathcal{M}, \Lambda\}$, a class of algorithms \mathcal{A} is closed under search with respect to $\{\Omega, \mathcal{M}, \Lambda\}$ if whenever

- (1) \mathcal{I} is an index set,
- (2) $\{n_i\}_{i \in \mathcal{I}}$ a family of natural numbers,
- (3) $\{\Gamma_{i,l} : \Omega \rightarrow \mathcal{M}\}_{i \in \mathcal{I}, l \leq n_i} \subset \mathcal{A}$,
- (4) $\{U_{i,l}\}_{i \in \mathcal{I}, l \leq n_i}$ family of basic open sets in \mathcal{M} with $\cup_{i \in \mathcal{I}} \cap_{l \leq n_i} \Gamma_{i,l}^{-1}(U_{i,l}) = \Omega$,
- (5) $\{c_i\}_{i \in \mathcal{I}}$ a family of points in some arbitrary dense subset of \mathcal{M} ,

then there is some $\Gamma \in \mathcal{A}$ such that for every $x \in \Omega$ there exists some $i \in \mathcal{I}$ with $\Gamma(x) = c_i$ and for all $l \leq n_i$ we have $\Gamma_{i,l}(x) \in U_{i,l}$.

Proposition 5.16 *Suppose that \mathcal{A} is closed under search with respect to $\{\Omega, \mathcal{M}, \Lambda\}$, then there exists a topology \mathcal{T} on Ω such that $\Delta_1^{\mathcal{A}}$ is precisely the set of continuous functions from (Ω, \mathcal{T}) to \mathcal{M} .*

Proof Let \mathcal{T} be the topology generated by $\{\Gamma^{-1}(B) : \Gamma \in \mathcal{A}, B \subset \mathcal{M} \text{ basic open}\}$. Any $\Gamma \in \mathcal{A}$ is continuous with respect to this topology. Uniform limits of continuous functions into metric spaces are also continuous, and hence any function in $\Delta_1^{\mathcal{A}}$ is continuous with respect to \mathcal{T} .

For the other direction, suppose that $f : (\Omega, \mathcal{T}) \rightarrow \mathcal{M}$ is continuous. Choose $\{c_i\}_{i \in \mathcal{I}} \subset \mathcal{M}$ such that $\mathcal{M} \subset \cup_{i \in \mathcal{I}} D(c_i, 2^{-n})$. Continuity of f implies that $f^{-1}(D(c_i, 2^{-n}))$ are open. This implies that there is an index set \mathcal{J} , natural numbers $\{n_{i,j}\}_{j \in \mathcal{J}}$, a family $\{\Gamma_{i,j,l}\}_{i \in \mathcal{I}, j \in \mathcal{J}, l \leq n_{i,j}}$ (in \mathcal{A}) and a family of basic open sets $\{U_{i,j,l}\}_{i \in \mathcal{I}, j \in \mathcal{J}, l \leq n_{i,j}}$ with the property that

$$f^{-1}(D(c_i, 2^{-n})) = \bigcup_{j \in \mathcal{J}} \bigcap_{l \leq n_{i,j}} \Gamma_{i,j,l}^{-1}(U_{i,j,l}).$$

It follows that

$$\bigcup_{i \in \mathcal{I}, j \in \mathcal{J}} \bigcap_{l \leq n_{i,j}} \Gamma_{i,j,l}^{-1}(U_{i,j,l}) = \Omega.$$

Since \mathcal{A} is closed under search, there exists $f_n \in \mathcal{A}$ such that for every $x \in \Omega$ there exists some $i \in \mathcal{I}$ and $j \in \mathcal{J}$ with $f_n(x) = c_i$ and for all $l \leq n_{i,j}$, $x \in \Gamma_{i,j,l}^{-1}(U_{i,j,l})$. But this implies that $d(f_n(x), f(x)) < 2^{-n}$. Since n was arbitrary, we have $f \in \Delta_1^{\mathcal{A}}$. \square

The generated topology can be very perverse and not every class of algorithms is closed under search. However, we do have the following useful theorem when Ω (and Λ) is a particularly simple discrete space.

Theorem 5.17 *Suppose that $\Omega = \{0, 1\}^{\mathbb{N}} = \{ \{a_i\}_{i \in \mathbb{N}} : a_i \in \{0, 1\} \}$ with the set of evaluation functions Λ equal to the set of pointwise evaluations $\{ \lambda_j(a) := a_j : j \in \mathbb{N} \}$ and let \mathcal{M} be an arbitrary separable metric space with at least two separated points. Endow Ω with the product topology, $\tilde{\mathcal{T}}$, induced by the discrete topology on $\{0, 1\}$ and consider the Baire hierarchy, $\{ \mathcal{B}_\xi((\Omega, \tilde{\mathcal{T}}), \mathcal{M}) = \mathcal{B}_\xi \}_{\xi < \omega_1}$, of functions $f : \Omega \rightarrow \mathcal{M}$. Then for any problem function $\Xi : \Omega \rightarrow \mathcal{M}$ and $m \in \mathbb{N}$,*

$$\{ \Xi, \Omega, \Lambda \} \in \Delta_{m+1}^G \Leftrightarrow \Xi \in \mathcal{B}_m. \tag{5.4}$$

In other words, the SCI corresponds to the Baire hierarchy index.

Remark 5.18 The proof shows that we can replace Ω by $\{0, 1\}^{\mathbb{N} \times \mathbb{N}}$ or any other such product space (induced by a discrete topology) of the form A^B with A, B countable, with Λ the corresponding component-wise evaluations, as long as \mathcal{M} has at least $|A|$ jointly separated points and is separable. \square

Proof First we show that general algorithms are closed under search and that the topology \mathcal{T} in Proposition 5.16 is equal to the product topology $\tilde{\mathcal{T}}$. Without loss of generality, we can assume that \mathcal{I} is well-ordered by $<$. Given $x \in \Omega$, let $k \in \mathbb{N}$ be minimal such that there exists $i \in \mathcal{I}$ with $x \in \bigcap_{l \leq n_i} \Gamma_{i,l}^{-1}(U_{i,l})$ and $\Lambda_{\Gamma_{i,l}}(x) \subset \{ \lambda_j : j \leq k \}$ for $l \leq n_i$. Let i_0 be the $<$ -least witness for k and then define $\Gamma(x) = c_{i_0}$. The well-ordering of \mathcal{I} implies that Γ is a general algorithm, and it clearly satisfies the requirements in the definition of closed under search. Note that this part of the proof only uses countability of Λ .

To equate the topologies, suppose that $\Gamma \in \Delta_0^G$ is a general algorithm. For each $a \in \Omega$, $\Lambda_\Gamma(a)$ is finite and we can assume without loss of generality that it is equal to $\{ \lambda_j : j \leq I(a) \}$ for some finite $I(a)$. In particular, there exists an open set U_a such that any $b \in U_a$ has $\lambda_j(b) = \lambda_j(a)$ for $j \leq I(a)$ and hence $\Gamma(b) = \Gamma(a)$. Then for any open set $B \subset \mathcal{M}$

$$\Gamma^{-1}(B) = \bigcup_{a \in \Gamma^{-1}(B)} U_a$$

is open. Hence each Γ is continuous with respect to the product topology on Ω . It follows that $\mathcal{T} \subset \tilde{\mathcal{T}}$. To prove the converse, we must show that each projection map λ_j is continuous with respect to \mathcal{T} . Let x_1, x_2 be separated points in \mathcal{M} and consider $f : \{0, 1\} \rightarrow \mathcal{M}$ with $f(0) = x_1$ and $f(1) = x_2$. Then the composition $f \circ \lambda_j$ is a general algorithm and hence continuous with respect to \mathcal{T} . But this implies that λ_j is

continuous. It follows from Proposition 5.16 that $\{\Xi, \Omega, \Lambda\} \in \Delta_1^G$ if and only if Ξ is continuous.

Now the space (Ω, \mathcal{T}) is zero-dimensional and \mathcal{M} is separable, hence by Theorem 5.14, any element of \mathcal{B}_1 is a limit of continuous functions. The converse holds in greater generality. It follows that $\Xi \in \mathcal{B}_m$ if and only if there are $f_{n_m, \dots, n_1} \in \Delta_1^G$ with

$$\Xi(a) = \lim_{n_m \rightarrow \infty} \cdots \lim_{n_1 \rightarrow \infty} f_{n_m, \dots, n_1}(a). \tag{5.5}$$

If this holds, then there exist general algorithms Γ_{n_m, \dots, n_1} such that for all $a \in \Omega$,

$$d(\Gamma_{n_m, \dots, n_1}(a), f_{n_m, \dots, n_1}(a)) \leq 2^{-n_1}$$

and hence

$$\lim_{n_m \rightarrow \infty} \cdots \lim_{n_1 \rightarrow \infty} \Gamma_{n_m, \dots, n_1}(a) = \Xi(a)$$

so that $\{\Xi, \Omega, \Lambda\} \in \Delta_{m+1}^G$. Conversely if $\{\Xi, \Omega, \Lambda\} \in \Delta_{m+1}^G$ with tower of algorithms $\{\Gamma_{n_m, \dots, n_1}\}$, then since each general algorithm is continuous, (5.5) holds with $f_{n_m, \dots, n_1}(a) = \Gamma_{n_m, \dots, n_1}$. \square

5.4 Combinatorial Problems High up in the SCI Hierarchy

We can now combine the results of the previous two subsections and obtain combinatorial problems high up in the SCI hierarchy. Let $k \in \mathbb{N}_{\geq 2}$ and let Ω_k denote the collection of all infinite arrays $\{a_{m_1, \dots, m_k}\}_{m_1, \dots, m_k \in \mathbb{N}}$ with entries $a_{m_1, \dots, m_k} \in \{0, 1\}$. As usual, Λ_k is the set of component-wise evaluations/projections. Consider the formulas

$$P(a, m_1, \dots, m_{k-2}) = \begin{cases} 1, & \text{if } \exists i \forall j \exists n > j \text{ s.t. } a_{m_1, \dots, m_{k-2}, n, i} = 1 \\ 0, & \text{otherwise} \end{cases},$$

$$Q(a, m_1, \dots, m_{k-2}) = \begin{cases} 1, & \text{if } \forall^\infty i \forall j \exists n > j \text{ s.t. } a_{m_1, \dots, m_{k-2}, n, i} = 1 \\ 0, & \text{otherwise} \end{cases},$$

where \forall^∞ means ‘‘for all but a finite number of’’. In words, P decides whether the corresponding matrix has a column with infinitely many 1’s, whereas Q decides whether the matrix has only finitely many columns with only finitely many 1’s. For $R = P$ or Q , consider the problem function for $a \in \Omega_k$

$$\Xi_{k,R}(a) = \begin{cases} \exists m_1 \forall m_2 \dots \forall m_{k-2} R(a, m_1, \dots, m_{k-2}), & \text{if } k \text{ is even} \\ \forall m_1 \exists m_2 \dots \forall m_{k-2} R(a, m_1, \dots, m_{k-2}), & \text{otherwise} \end{cases},$$

that is, so that all quantifier types alternate.

Theorem 5.19 *Let \mathcal{M} be either $\{0, 1\}$ with the discrete metric or $[0, 1]$ with the usual metric and consider the above problems $\{\Xi_k, \Omega_k, \mathcal{M}, \Lambda_k\}$. For $k \in \mathbb{N}_{\geq 2}$ and $R = P$ or Q ,*

$$\Delta_{k+1}^G \not\equiv \{\Xi_k, R, \Omega_k, \mathcal{M}, \Lambda_k\} \in \Delta_{k+2}^A.$$

In other words, we can solve the problem via a height $k + 1$ arithmetic tower, but it is impossible to do so with a height k general tower.

Remark 5.20 Note that we allow both *discrete* and *continuous* spaces \mathcal{M} , which will be important for our reduction arguments when proving lower bounds for classifications of spectral problems for non-discrete \mathcal{M} . The lower bound is a strong result in the sense that it holds regardless of the model of computation. In other words, it is the intrinsic combinatorial complexity of the problems that make the problems hard. \square

Proof We deal with the case of $R = P$ since the case of $R = Q$ is completely analogous. It is easy to see that $\{\Xi_k, P, \Omega_k, \mathcal{M}, \Lambda_k\} \in \Delta_{k+2}^A$. First consider the case $k = 2$ and set

$$\Gamma_{n_3, n_2, n_1}(a) = \max_{j \leq n_3} \chi_{(n_2, \infty)} \left(\sum_{i=1}^{n_1} a_{i,j} \right),$$

where χ_C denotes the indicator function of a set C . This is the decision problem that decides whether there exists a column with index at most n_3 such that there are at least n_2 1's in the first n_1 rows. This is clearly an arithmetic tower and it is straightforward to show that this converges to $\Xi_{2,P}$ in \mathcal{M} (in either of the $\{0, 1\}$ and $[0, 1]$ cases). For $k > 2$, we simply alternate taking products (which corresponds to minima in this case) and maxima. Explicitly, we set

$$\begin{aligned} & \Gamma_{n_{k+1}, \dots, n_1}(a) \\ &= \begin{cases} \max_{m_1 \leq n_{k+1}} \prod_{m_2=1}^{n_k} \cdots \prod_{m_{k-2}=1}^{n_4} \left\{ \max_{j \leq n_3} \chi_{(n_2, \infty)} \left(\sum_{i=1}^{n_1} a_{m_1, \dots, m_{k-2}, i, j} \right) \right\}, & \text{if } k \text{ is even} \\ \prod_{m_1=1}^{n_{k+1}} \max_{m_2 \leq n_k} \cdots \prod_{m_{k-2}=1}^{n_4} \left\{ \max_{j \leq n_3} \chi_{(n_2, \infty)} \left(\sum_{i=1}^{n_1} a_{m_1, \dots, m_{k-2}, i, j} \right) \right\}, & \text{otherwise.} \end{cases} \end{aligned}$$

Again, this is an arithmetic tower and it is straightforward to show that this converges to $\Xi_{k,P}$ in \mathcal{M} . It also holds that $\{\Xi_k, P, \Omega_k, \mathcal{M}, \Lambda_k\} \in \Sigma_{k+1}^A$ if k is even and $\{\Xi_k, P, \Omega_k, \mathcal{M}, \Lambda_k\} \in \Pi_{k+1}^A$ if k is odd (not to be confused with the notation for the Borel hierarchy).

Recall the topology \mathcal{T} on Ω_k from Theorem 5.17. For the lower bound, we note that P is Σ_3^0 complete (in the literature it is known as the problem “ S_3 ”, see for example [98, Section 23]). Since (Ω_k, \mathcal{T}) is zero-dimensional, a theorem of Wadge implies that this means that P is the indicator function of a set, also denoted by P , which lies in $\Sigma_3^0(\Omega_k)$ but not $\Pi_3^0(\Omega_k)$. It also follows that $\Xi_{k,P}$ is $\Sigma_{k+1}^0(\Omega_k)$ complete if

k is even and $\Pi_{k+1}^0(\Omega_k)$ complete otherwise. Now suppose for a contradiction that $\{\Xi_{k,P}, \Omega_k, \mathcal{M}, \Lambda_k\} \in \Delta_{k+1}^G$. But then Theorem 5.17 implies that $\Xi_{k,P} \in \mathcal{B}_k(\Omega_k, \mathcal{M})$ and hence by Theorem 5.14, $\Xi_{k,P}$ is $\Sigma_{k+1}^0(\Omega_k)$ measurable. $\Xi_{k,P}$ is the indicator function of a set, which we denote by $\Xi_{k,P}$ with an abuse of notation, which is either $\Sigma_{k+1}^0(\Omega_k)$ or $\Pi_{k+1}^0(\Omega_k)$ complete depending on the parity of k . But 0 and 1 are separated in \mathcal{M} and hence since $\Xi_{k,P}$ is $\Sigma_{k+1}^0(\Omega_k)$ measurable, $\Xi_{k,P}$ and its complement both lie in $\Sigma_{k+1}^0(\Omega_k)$. It follows that $\Xi_{k,P} \in \Sigma_{k+1}^0(\Omega_k) \cap \Pi_{k+1}^0(\Omega_k)$, contradicting the stated completeness. \square

For our applications to spectral problems, we will use $\tilde{\Omega}$ to denote Ω_k and consider

$$\begin{aligned} \tilde{\Xi}_1 &= \Xi_{2,P}, & \tilde{\Xi}_2 &= \Xi_{2,Q}, \\ \tilde{\Xi}_3 &= \Xi_{3,P}, & \tilde{\Xi}_4 &= \Xi_{3,Q}. \end{aligned} \tag{5.6}$$

Theorem 5.19 holds for a much wider class of decision problems, but these four are the only ones we shall use in the sequel. The decision problems $\tilde{\Xi}_1$ and $\tilde{\Xi}_2$ were shown to have $\text{SCI}_G = 3$ in [18], but only with regard to the discrete space $\mathcal{M} = \{0, 1\}$ and the proof used a somewhat complicated Baire category argument. Theorem 5.19 is much more general, can be extended to arbitrarily large SCI, and has a much slicker proof, making clear a beautiful connection with the Baire hierarchy for well-behaved Ω .

6 Proofs Concerning Spectral Radii, Essential Spectral Radii, Capacity and Operator Norms

Here we prove the theorems found in Sects. 3.1–3.3. First, we briefly recall Σ_1^A algorithms for spectral problems presented in [64], that are sharp in the SCI hierarchy. The algorithms constructed in [64] are shown as pseudocode in ‘‘Appendix A’’, where we also refer the reader to a more detailed account. The following was proven in [64] and was generalised in [60] to unbounded operators:

Theorem 6.1 *For each Ω_f and $\Omega_f \cap \Omega_g$, consider the family Λ consisting of Λ_1 , together with pointwise evaluation of f , $\{c_n\}$ (and evaluation of g at rational points if considering $\Omega_f \cap \Omega_g$). The algorithms presented in ‘‘Appendix A’’ achieve Σ_1^A error control. In particular the following classification holds:*

$$\Delta_1^G \not\preceq \{\text{Sp}(\cdot), \Omega_f \cap \Omega_g, \Lambda_1\} \in \Sigma_1^A, \quad \Delta_1^G \not\preceq \{\text{Sp}_\epsilon(\cdot), \Omega_f, \Lambda_1\} \in \Sigma_1^A.$$

We now turn to the proof of Theorem 3.3, dealing first with the evaluation set Λ_1 . Suppose that $\{\tilde{\Gamma}_{n_k, \dots, n_1}\}$ is a Π_k^A tower of algorithms to compute the spectrum of a class of operators, where the output is a finite set for each n_1, \dots, n_k . It is then clear that

$$\Gamma_{n_k, \dots, n_1}(A) = \sup_{z \in \tilde{\Gamma}_{n_k, \dots, n_1}(A)} |z| + \frac{1}{2^{n_k}}$$

provides a Π_k^A tower of algorithms for the spectral radius. Strictly speaking, the above may not be an arithmetic tower owing to the absolute value. But it can be approximated to arbitrary precision (from above say), the error of which can be absorbed in the first limit. In what follows, we always assume this is done without further comment. Similarly if $\{\tilde{\Gamma}_{n_k, \dots, n_1}\}$ provides a Σ_k^A tower of algorithms for the spectrum (and outputs a finite set for each n_1, \dots, n_k),

$$\Gamma_{n_k, \dots, n_1}(A) = \sup_{z \in \tilde{\Gamma}_{n_k, \dots, n_1}(A)} |z| - \frac{1}{2^{n_k}}$$

provides a Σ_k^A tower of algorithms for the spectral radius. If we only have a height k tower with no Σ_k or Π_k type error control for the spectrum, then taking the supremum of absolute values shows that we get a height k tower for the spectral radius.

The fact that $\{\Xi_r, \Omega_D\} \in \Sigma_1^A$, $\{\Xi_r, \Omega_f \cap \Omega_g\} \in \Sigma_1^A$, $\{\Xi_r, \Omega_g\} \in \Sigma_2^A$, $\{\Xi_r, \Omega_f\} \in \Pi_2^A$ and $\{\Xi_r, \Omega_B\} \in \Pi_3^A$ hence follow from Theorems 6.1 and the results of [18]. It is clear that $\{\Xi_r, \Omega_D\} \notin \Delta_1^G$ and this also shows that $\{\Xi_r, \Omega_N\} \notin \Delta_1^G$ and $\{\Xi_r, \Omega_f \cap \Omega_g\} \notin \Delta_1^G$. Hence, we must show the positive result that $\{\Xi_r, \Omega_N\} \in \Sigma_1^A$ and prove the lower bounds $\{\Xi_r, \Omega_g\} \notin \Delta_2^G$, $\{\Xi_r, \Omega_f\} \notin \Delta_2^G$ and $\{\Xi_r, \Omega_B\} \notin \Delta_3^G$.

Proof of Theorem 3.3 for Λ_1 Throughout this proof, we use the evaluation set Λ_1 , which we drop from the notation for convenience.

Step 1 $\{\Xi_r, \Omega_N\} \in \Sigma_1^A$. Recall that the spectral radius of a normal operator $A \in \Omega_B$ is equal to its operator norm. Consider the finite section matrices $P_n A P_n \in \mathbb{C}^{n \times n}$. It is straightforward to show that

$$\|P_n A P_n\| \uparrow \|A\| \quad \text{as } n \rightarrow \infty.$$

The norm $\|P_n A P_n\|$ is the square root of the largest eigenvalue of the semi-positive definite self-adjoint matrix $(P_n A P_n)^*(P_n A P_n)$. This can be estimated from below to an accuracy of $1/n$ using Corollary 6.9 of [60], which then yields a Σ_1^A algorithm for $\{\Xi_r, \Omega_N\}$.

Step 2 $\{\Xi_r, \Omega_g\} \notin \Delta_2^G$. Recall that we assumed the existence of a $\delta \in (0, 1)$ such that $g(x) \leq (1 - \delta)x$. Let $\epsilon > 0$, then it is easy to see that the matrices

$$S_{\pm}(\epsilon) = \begin{pmatrix} 1 & 0 \\ \pm\epsilon & 1 \end{pmatrix}$$

have norm bounded by $1 + \epsilon + \epsilon^2$ and are clearly inverse of each other. Choose ϵ small such that $(1 + \epsilon + \epsilon^2)^2 \leq 1/(1 - \delta)$. If $B \in \mathbb{C}^{2 \times 2}$ is normal, it follows that $\hat{B} := S_+(\epsilon) B S_-(\epsilon)$ lies in Ω_g and has the same spectrum as B . We choose

$$\hat{B} = S_+(\epsilon) \begin{pmatrix} 1 & -\epsilon \\ -\epsilon & 0 \end{pmatrix} S_-(\epsilon) = \begin{pmatrix} 1 + \epsilon^2 & -\epsilon \\ \epsilon^3 & -\epsilon^2 \end{pmatrix}.$$

The crucial property of \hat{B} is that the first entry $1 + \epsilon^2$ is strictly greater in magnitude than the two eigenvalues $(1 \pm \sqrt{1 + 4\epsilon^2})/2$.

Now suppose for a contradiction that a height one tower, $\{\Gamma_n\}$, solves the problem. We will gain a contradiction by showing that $\Gamma_n(A)$ does not converge for an operator of the form,

$$A = \bigoplus_{r=1}^{\infty} A_{l_r}, \quad A_m := \begin{pmatrix} 1 + \epsilon^2 & & -\epsilon \\ & 0 & \\ & \ddots & \\ \epsilon^3 & & 0 & \\ & & & -\epsilon^2 \end{pmatrix} \in \mathbb{C}^{m \times m},$$

where we only consider $l_k \geq 3$. Each A_m is unitarily equivalent to the matrix $\hat{B} \oplus 0 \in \mathbb{C}^{m \times m}$ and has spectrum equal to $\{0, (1 \pm \sqrt{1 + 4\epsilon^2})/2\}$. Any A of the above form is unitarily equivalent to a direct sum of an infinite number of \hat{B} 's and the zero operator and hence lies in Ω_g . Now suppose that l_1, \dots, l_k have been chosen and consider the operator

$$B_k = A_{l_1} \oplus \dots \oplus A_{l_k} \oplus C, \quad C = \text{diag}\{1 + \epsilon^2, 0, \dots\}.$$

The spectrum of B_k is $\{0, (1 \pm \sqrt{1 + 4\epsilon^2})/2, 1 + \epsilon^2\}$, and hence, there exists $\eta > 0$ and $n(k) \geq k$ such that $\Gamma_{n(k)}(B_k) > (1 + \sqrt{1 + 4\epsilon^2})/2 + \eta$. But $\Gamma_{n(k)}(B_k)$ can only depend on the evaluations of the matrix entries $\{B_k\}_{ij} = \langle B_k e_j, e_i \rangle$ with $i, j \leq N(B_k, n(k))$ (as well as evaluations of the function g) into account. If we choose $l_{k+1} > N(B_k, n(k))$ then by the assumptions in Definition 5.1, $\Gamma_{n(k)}(A) = \Gamma_{n(k)}(B_k) > (1 + \sqrt{1 + 4\epsilon^2})/2 + \eta$. But $\Gamma_n(A)$ must converge to $(1 + \sqrt{1 + 4\epsilon^2})/2$, a contradiction.

Step 3 $\{\Xi_r, \Omega_f\} \notin \Delta_2^G$. Suppose for a contradiction that a height one tower, $\{\Gamma_n\}$, solves the problem. We will gain a contradiction by showing that $\Gamma_n(A)$ does not converge for an operator of the form

$$A = \bigoplus_{r=1}^{\infty} C_{l_r} \oplus A_{l_r}, \quad A_m := \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ & & & & 0 \end{pmatrix} \in \mathbb{C}^{m \times m},$$

$$C_m = \text{diag}\{0, 0, \dots, 0\} \in \mathbb{C}^{m \times m},$$

where we assume that $l_r \geq r$ to ensure that the spectrum of A is equal to the unit disc $B_1(0)$. Note that the function $f(n) = n + 1$ will do for the bounded dispersion with $c_n = 0$. Now suppose that l_1, \dots, l_k have been chosen and consider the operator

$$B_k = (C_{l_1} \oplus A_{l_1}) \oplus \dots \oplus (C_{l_k} \oplus A_{l_k}) \oplus C, \quad C = \text{diag}\{0, 0, \dots\}.$$

The spectrum of B_k is $\{0\}$ and hence there exist $n(k) \geq k$ such that $\Gamma_{n(k)}(B_k) < 1/4$. But $\Gamma_{n(k)}(B_k)$ can only depend on the evaluations of the matrix entries $\{B_k\}_{ij} = \langle B_k e_j, e_i \rangle$ with $i, j \leq N(B_k, n(k))$ (as well as evaluations of the function f) into

account. If we choose $l_{k+1} > N(B_k, n(k))$, then by the assumptions in Definition 5.1, $\Gamma_{n(k)}(A) = \Gamma_{n(k)}(B_k) < 1/4$. But $\Gamma_n(A)$ must converge to 1, a contradiction.

Step 4 $\{\Xi_r, \Omega_B\} \notin \Delta_3^G$. Suppose for a contradiction that $\{\Gamma_{n_2, n_1}\}$ is a height two (general) tower and without loss of generality, assume it to be nonnegative. We use the results of Sect. 5. Let (\mathcal{M}, d) be the space $[0, 1]$ with the usual metric (note in particular this is not discrete so we use Remark 5.20), let $\tilde{\Omega}$ denote the collection of all infinite matrices $\{a_{i,j}\}_{i,j \in \mathbb{N}}$ with entries $a_{i,j} \in \{0, 1\}$ and recall the problem function

$\tilde{\Xi}_1(\{a_{i,j}\})$: Does $\{a_{i,j}\}$ have a column containing infinitely many nonzero entries?

Theorem 5.19 in Sect. 5 shows that $\text{SCI}(\tilde{\Xi}_1, \tilde{\Omega})_G = 3$. We will gain a contradiction by using the supposed height two tower to solve $\{\tilde{\Xi}_1, \tilde{\Omega}\}$.

Without loss of generality, identify Ω_B with $\mathcal{B}(X)$ where $X = \bigoplus_{j=1}^\infty X_j$ in the l^2 -sense with $X_j = l^2(\mathbb{N})$. Now let $\{a_{i,j}\} \in \tilde{\Omega}$ and define $B_j \in \mathcal{B}(X_j)$ with the matrix representation

$$(B_j)_{k,i} = \begin{cases} 1, & \text{if } k = i \text{ and } a_{k,j} = 0 \\ 1, & \text{if } k < i \text{ and } a_{l,j} = 0 \text{ for } k < l < i \\ 0, & \text{otherwise } 0 \leq n \leq 1. \end{cases}$$

Let \mathcal{I}_j be the index set of all i where $a_{i,j} = 1$. B_j acts as a unilateral shift on $\overline{\text{span}}\{e_k : k \in \mathcal{I}_j\}$ and the identity on its orthogonal complement. It follows that

$$\text{Sp}(B_j) = \begin{cases} 1, & \text{if } \mathcal{I}_j = \emptyset \\ \{0, 1\}, & \text{if } \mathcal{I}_j \text{ is finite and non-empty} \\ \mathbb{D} \text{ (the unit disc)}, & \text{if } \mathcal{I}_j \text{ is infinite.} \end{cases}$$

For the matrix $\{a_{i,j}\}$ define $A \in \Omega_B$ by

$$A = \bigoplus_{j=1}^\infty (B_j - \frac{1}{2}I_j),$$

where I_j denotes the identity operator on $\mathbb{C}^{j \times j}$, then $\text{Sp}(A) = \overline{\bigcup_{j=1}^\infty \text{Sp}(B_j)} - \frac{1}{2}$.

Hence, we see that

$$\Xi_r(A) = \begin{cases} \frac{1}{2}, & \text{if } \tilde{\Xi}_1(\{a_{i,j}\}) = 0 \\ \frac{3}{2}, & \text{if } \tilde{\Xi}_1(\{a_{i,j}\}) = 1. \end{cases}$$

We then set $\tilde{\Gamma}_{n_2, n_1}(\{a_{i,j}\}) = \min\{\max\{\Gamma_{n_2, n_1}(A) - 1/2, 0\}, 1\}$. It is clear that this defines a generalised algorithm mapping into $[0, 1]$. In particular, given N we can evaluate $\{A_{k,l} : k, l \leq N\}$ using only finitely many evaluations of $\{a_{i,j}\}$, where we can use a bijection between canonical bases of $l^2(\mathbb{N})$ and $\bigoplus_{j=1}^\infty X_j$ to view A as acting

on $l^2(\mathbb{N})$. But then $\{\tilde{\Gamma}_{n_2, n_1}\}$ provides a height two tower for $\{\tilde{\Xi}_1, \tilde{\Omega}\}$, a contradiction. □

Remark 6.2 The algorithm in step 1 of the above proof works for any operator whose operator norm is equal to its spectral radius. If, instead, the operator is spectraloid, meaning the spectral radius is equal to the numerical radius

$$w(A) := \sup\{|\langle Ax, x \rangle| : \|x\| = 1\},$$

then a similar argument will hold by estimating $w(P_n A P_n)$. To do this, we need a way of computing $w(A)$ to a given accuracy using finitely many arithmetic operations and comparisons (e.g. Lemma 7.1 below). □

Proof of Theorem 3.3 for Λ_2 Here we prove the changes for Ξ_r when we consider the evaluation set Λ_2 . It is clear that the classifications in Σ_1^A do not change. It is also easy to use the algorithm in Theorem 6.1 (now using Λ_2 to collapse the first limit and approximate γ_n —see “Appendix A”) to prove $\{\Xi_r, \Omega_g, \Lambda_2\} \in \Sigma_1^A$. Similarly we can use the algorithm for the spectrum of operators in Ω_f for Ω_B using Λ_2 to collapse the first limit and hence $\{\Xi_r, \Omega_B, \Lambda_2\} \in \Pi_2^A$. Since $\Omega_f \subset \Omega_B$, it follows that we only need to prove $\{\Xi_r, \Omega_f, \Lambda_2\} \notin \Delta_2^G$. This can be proven using exactly the same example and a similar argument to step 3 of the proof of Theorem 3.3 (hence omitted). □

Proof of Theorem 3.6 We begin by proving the results for Λ_1 . For the lower bounds, it is enough to show that $\{\Xi_{er}, \Omega_D, \Lambda_1\} \notin \Delta_2^G$ and $\{\Xi_{er}, \Omega_B, \Lambda_1\} \notin \Delta_3^G$. For the upper bounds, we must show that $\{\Xi_{er}, \Omega_f, \Lambda_1\} \in \Pi_2^A$, $\{\Xi_{er}, \Omega_B, \Lambda_1\} \in \Pi_3^A$ and $\{\Xi_{er}, \Omega_N, \Lambda_1\} \in \Pi_2^A$. The lower bounds for Λ_2 follow from $\{\Xi_{er}, \Omega_D, \Lambda_1\} \notin \Delta_2^G$ and for the upper bounds it is enough to prove $\{\Xi_{er}, \Omega_B, \Lambda_2\} \in \Pi_2^A$.

Step 1 $\{\Xi_{er}, \Omega_D, \Lambda_1\} \notin \Delta_2^G$. This is the same argument as in step 3 of the proof of Theorem 3.3; however, now we replace A_m by $A_m = \text{diag}\{1, 1, \dots, 1\} \in \mathbb{C}^{m \times m}$ and use the fact that $\Xi_{er}(B_k) = 0$. It follows that given the proposed height one tower $\{\Gamma_n\}$ and the constructed A , $\Xi_{er}(A) = 1$ but $\Gamma_{n(k)}(A) < 1/4$, the required contradiction.

Step 2 $\{\Xi_{er}, \Omega_B, \Lambda_1\} \notin \Delta_3^G$. This is the same argument as step 4 of the proof of Theorem 3.3.

Step 3 $\{\Xi_{er}, \Omega_f, \Lambda_1\} \in \Pi_2^A$, $\{\Xi_{er}, \Omega_B, \Lambda_1\} \in \Pi_3^A$ and $\{\Xi_{er}, \Omega_B, \Lambda_2\} \in \Pi_2^A$. $\{\Xi_{er}, \Omega_f, \Lambda_1\} \in \Pi_2^A$ follows immediately from the existence of a Π_2^A tower of algorithms for the essential spectrum of operators in Ω_f proven in [18]. The output of this tower is a finite collection of rectangles with complex rational vertices; hence, we can gain an approximation of the maximum absolute value over this output to any given precision. This can be used to construct a Π_2^A tower for $\{\Xi_{er}, \Omega_f, \Lambda_1\}$. Similarly, $\{\Xi_{er}, \Omega_B, \Lambda_1\} \in \Pi_3^A$ follows from the Π_3^A tower of algorithms for $\{\text{Sp}_{\text{ess}}, \Omega_B, \Lambda_1\}$ constructed in [18]. Finally, we can use Λ_2 to collapse the first limit of the algorithm for the essential spectrum in [18], giving a Π_2^A algorithm, and this can be used to show $\{\Xi_{er}, \Omega_B, \Lambda_2\} \in \Pi_2^A$.

Step 4 $\{\Xi_{er}, \Omega_N, \Lambda_1\} \in \Pi_2^A$. A Π_2^A tower is constructed in the proof of Theorem 3.10 for the essential numerical range, $W_e(A)$, of normal operators (using Λ_1), and this outputs a finite collection of points. For normal operators A , $W_e(A)$ is the

convex hull of the essential spectrum and hence $\sup_{z \in W_e(A)} |z|$ is equal to $\Xi_{er}(A)$. Hence, a Π_2^A tower for $\{\Xi_{er}, \Omega_N, \Lambda_1\}$ follows by taking the maximum absolute value over the tower for $W_e(A)$. \square

Proof of Theorem 3.7 Note that given a height k arithmetical tower $\{\widehat{\Gamma}_{n_k, \dots, n_1}(\cdot, p)\}$ for $\Xi_{r,p}$ and a class Ω' , we can build a Π_{k+1}^A tower for $\{\Xi_{cap}, \Omega'\}$ as follows. Let p_1, p_2, \dots be an enumeration of the monic polynomials with rational coefficients and $\Gamma_{n_k, \dots, n_1}(\cdot, p)$ be an approximation to $|\widehat{\Gamma}_{n_k, \dots, n_1}(\cdot, p)|^{1/\deg(p)}$ to accuracy $1/n_1$ using finitely many arithmetic operations and comparisons. Define

$$\Gamma_{n_{k+1}, \dots, n_1}(A) = \min_{1 \leq m \leq n_{k+1}} \tilde{\Gamma}_{n_k, \dots, n_1}(A, p_m).$$

The fact that this is a convergent Π_{k+1}^A tower is clear. This, together with inclusions of the considered classes of operators, means that to prove the positive results we only need to prove $\{\Xi_{r,p}, \Omega_f, \Lambda_1\} \in \Sigma_1^A$, $\{\Xi_{r,p}, \Omega_B, \Lambda_1\} \in \Sigma_2^A$ and $\{\Xi_{r,p}, \Omega_B, \Lambda_2\} \in \Sigma_1^A$. Likewise, for the negative results we only need to prove $\{\Xi_{cap}, \Omega_D, \Lambda_2\} \notin \Delta_2^G$ (the fact that $\{\Xi_{r,p}, \Omega_D, \Lambda_2\} \notin \Delta_1^G$ is obvious), $\{\Xi_{cap}, \Omega_N, \Lambda_1\} \notin \Delta_3^G$ and $\{\Xi_{r,p}, \Omega_N, \Lambda_2\} \notin \Delta_2^G$. We shall prove these results with Ω_N replaced by the class of self-adjoint bounded operators denoted by Ω_{SA} .

Step 1 $\{\Xi_{r,p}, \Omega_f, \Lambda_1\} \in \Sigma_1^A$. The function f and sequence $\{c_n\}$ allow us to compute the matrix elements of $p(A)$ for any $A \in \Omega_f$ and polynomial p to arbitrary accuracy. We can then use the same argument as step 1 of the proof of Theorem 3.3, approximating $\|P_n p(A) P_n\|$ instead of $\|P_n A P_n\|$.

Step 2 $\{\Xi_{r,p}, \Omega_B, \Lambda_1\} \in \Sigma_2^A$ and $\{\Xi_{r,p}, \Omega_B, \Lambda_2\} \in \Sigma_1^A$. For the first result, we note that

$$\lim_{m \rightarrow \infty} \|P_n p(P_m A P_m) P_n\| = \|P_n p(A) P_n\|$$

and let $\Gamma_{n,m}(A, p)$ be an approximation of $\|P_n p(P_m A P_m) P_n\|$ to accuracy $1/m$, which can be computed in finitely many arithmetic operations and comparisons. To prove $\{\Xi_{r,p}, \Omega_B, \Lambda_2\} \in \Sigma_1^A$, for any given $A \in \Omega_B$ we can use Λ_2 to compute a function f_A and sequence $\{c_n(A)\}$ bounding the dispersion such that $A \in \Omega^{f_A}$ and use step 1.

Step 3 $\{\Xi_{cap}, \Omega_{SA}, \Lambda_1\} \notin \Delta_3^G$. Suppose for a contradiction that $\{\Gamma_{n_2, n_1}\}$ is a height two (general) tower for the problem and without loss of generality, assume it to be nonnegative. Our strategy will be as in the proof of Theorem 3.3 (recall also the results of Sect. 5). Let (\mathcal{M}, d) be the space $[0, 1]$ with the usual metric (note in particular this is not discrete so we use remark 5.20), let $\tilde{\Omega}$ denote the collection of all infinite matrices $\{a_{i,j}\}_{i,j \in \mathbb{N}}$ with entries $a_{i,j} \in \{0, 1\}$ and consider the problem function

$$\tilde{\Xi}_2(\{a_{i,j}\}) : \text{Does } \{a_{i,j}\} \text{ have (only) finitely many columns with (only) finitely many 1's?}$$

Recall that it is shown in Sect. 5 that $\text{SCI}(\tilde{\Xi}_2, \tilde{\Omega})_G = 3$. We will gain a contradiction by using the supposed height two tower to solve $\{\tilde{\Xi}_2, \tilde{\Omega}\}$. Without loss of generality, identify Ω_{SA} with self-adjoint operators in $\mathcal{B}(X)$ where $X = \bigoplus_{j=1}^\infty X_j$ in the l^2 -sense

with $X_j = l^2(\mathbb{N})$. To proceed, we need the following elementary lemma, which will be useful in constructing examples of spectral pollution.

Lemma 6.3 *Let $z_1, z_2, \dots, z_k \in [-1, 1]$ and let $a_j = \sqrt{1 - z_j^2}$ (say positive square root). Then, the symmetric matrix*

$$B(z_1, \dots, z_k) = \left(\begin{array}{ccc|ccc} z_1 & 0 & \cdots & a_1 & 0 & \cdots \\ 0 & z_2 & 0 & \cdots & 0 & a_2 & 0 & \cdots \\ \vdots & 0 & \ddots & & \vdots & 0 & \ddots & \\ & & & & \vdots & & & \\ & & & z_k & & & & a_k \\ \hline a_1 & 0 & \cdots & -z_1 & 0 & \cdots \\ 0 & a_2 & 0 & \cdots & 0 & -z_2 & 0 & \cdots \\ \vdots & 0 & \ddots & & \vdots & 0 & \ddots & \\ & & & & \vdots & & & \\ & & & a_k & & & & -z_k \end{array} \right) \in \mathbb{C}^{2k \times 2k}$$

has eigenvalues ± 1 (repeated k times).

Proof By a change of basis, the above matrix is equivalent to a block diagonal matrix with blocks

$$\begin{pmatrix} z_j & a_j \\ a_j & -z_j \end{pmatrix}.$$

These blocks have eigenvalues $\{-1, 1\}$. □

Now choose a sequence of rational numbers $\{z_j\}_{j \in \mathbb{N}} \in [-1, 1]$ that is also dense in $[-1, 1]$ and let $B_j = B(z_1, \dots, z_j)$. For each column of a given $\{a_{i,j}\} \in \tilde{\Omega}$, let the infinite matrix $C^{(j)}$ be defined as follows. If $k, l < j + 1$ then $C_{kl}^{(j)} = z_k \delta_{k,l}$. Let $r(i)$ denote the row of the i th one of the column $\{a_{i,j}\}_{i \in \mathbb{N}}$ (with $r(i) = \infty$ if $\sum_m a_{m,j} < i$ and $r(0) = 0$). If $r(i) < \infty$ then for $k \leq l$ define

$$C_{kl}^{(j)} = \begin{cases} a_p \delta_{k,l - (r(i) - r(i-1) - 1)}, & p = 1, \dots, j, l = r(i) + j \cdot (2i - 1) + p - 1 \\ -z_p \delta_{k,l}, & p = 1, \dots, j, l = r(i) + j \cdot (2i - 1) + p - 1 \\ z_p \delta_{k,l}, & p = 1, \dots, j, l = r(i) + 2j \cdot i + p - 1 \\ 0, & \text{otherwise,} \end{cases}$$

and extend $C_{kl}^{(j)}$ below the diagonal to a symmetric matrix. The key property of this matrix is that if the column $\{a_{i,j}\}_{i \in \mathbb{N}}$ has infinitely many 1s, then its is unitarily equivalent to an infinite direct sum of infinitely many B_j together with the zero operator acting on some subspace (whose dimension is equal to the number of zeros in the column). In this case $\text{Sp}(C^{(j)}) = \{-1, 1, 0\}$ or $\{-1, 1\}$. On the other hand, if $\{a_{i,j}\}_{i \in \mathbb{N}}$

has finitely many 1s, then $C^{(j)}$ is unitarily equivalent to the direct sum of a finite number of B_j , the diagonal operator $\text{diag}\{z_1, \dots, z_k\}$ and the zero operator acting on some subspace. In this case $\{z_1, \dots, z_j\} \subset \text{Sp}(C^{(j)})$. Let $A = \bigoplus_{j=1}^{\infty} C^{(j)}$, then it is clear that if $\tilde{\Xi}_2(\{a_{i,j}\}) = 1$, then $\text{Sp}(A)$ is a finite set, otherwise it is the entire interval $[-1, 1]$.

Now we use the following facts for bounded self-adjoint operators A . If $\text{Sp}(A)$ is a finite set then $\Xi_{\text{cap}}(A) = 0$, whereas if $\text{Sp}(A) = [-1, 1]$ then $\Xi_{\text{cap}}(A) = 1/2$ (this can be proven easily using the minimal l^∞ norm property of monic Chebyshev polynomials). We then define $\tilde{\Gamma}_{n_2, n_1}(\{a_{i,j}\}) = \min\{\max\{1 - 2\Gamma_{n_2, n_1}(A), 0\}, 1\}$. It is clear that this defines a generalised algorithm. In particular, given N we can evaluate $\{A_{k,l} : k, l \leq N\}$ using only finitely many evaluations of $\{a_{i,j}\}$, where we can use a bijection between canonical bases of $l^2(\mathbb{N})$ and $\bigoplus_{j=1}^{\infty} X_j$ to view A as acting on $l^2(\mathbb{N})$. We also have the convergence $\lim_{n_2 \rightarrow \infty} \lim_{n_1 \rightarrow \infty} \tilde{\Gamma}_{n_2, n_1}(\{a_{i,j}\}) = \tilde{\Xi}_2(\{a_{i,j}\})$, a contradiction.

Step 4 $\{\Xi_{\text{cap}}, \Omega_D, \Lambda_2\} \notin \Delta_2^G$. This is the same argument as in step 3 of the proof of Theorem 3.3. However, we now replace A_m by $A_m = \text{diag}\{d_1, d_2, \dots, d_m\} \in \mathbb{C}^{m \times m}$, where $\{d_m\}$ is a dense subsequence of $[-1, 1]$, and use the fact that $\Xi_{\text{cap}}(B_k) = 0$. It follows that given the proposed height one tower $\{\Gamma_n\}$ and the constructed A , $\Xi_{\text{cap}}(A) = 1/2$ but $\Gamma_{n(k)}(A) < 1/4$, the required contradiction.

Step 5 $\{\Xi_{r,p}, \Omega_{\text{SA}}, \Lambda_2\} \notin \Delta_2^G$. Recall that we are given some polynomial p of degree at least two. We assume without loss of generality that the zeros of p are ± 1 and $|p(0)| > 1$ (the more general case is similar). The argument is similar to step 3 of the proof of Theorem 3.3, but we spell it out since it uses Lemma 6.3. Suppose for a contradiction that a height one tower, $\{\Gamma_n\}$, solves the problem. We will gain a contradiction by showing that $\Gamma_n(A)$ does not converge for an operator of the form,

$$A = \bigoplus_{r=1}^{\infty} B(z_1, \dots, z_{l_r}),$$

and define

$$C = \text{diag}\{z_1, z_2, \dots\} \in \Omega_B.$$

We assume that $l_r \geq r$ to ensure that the spectrum of A is equal to $\{-1, 1\}$ and hence $\Xi_{r,p}(A) = 0$. Now suppose that l_1, \dots, l_k have been chosen and consider the operator

$$B_k = B(z_1) \oplus \dots \oplus B(z_1, \dots, z_{l_k}) \oplus C.$$

The spectrum of B_k is $[-1, 1]$ so that $\Xi_{r,p}(B_k) > 1$ and hence there exists $n(k) \geq k$ such that $\Gamma_{n(k)}(B_k) > 1/4$. But $\Gamma_{n(k)}(B_k)$ can only depend on the evaluations of the matrix entries $\{B_k\}_{ij} = \langle B_k e_j, e_i \rangle$ with $i, j \leq N(B_k, n(k))$ (as well as evaluations of the function f) into account. If we choose $l_{k+1} > N(B_k, n(k))$ then by the assumptions in Definition 5.1, $\Gamma_{n(k)}(A) = \Gamma_{n(k)}(B_k) > 1/4$. But $\Gamma_n(A)$ must converge to 0, a contradiction. □

Remark 6.4 (Efficiently computing the capacity) Listing the monic polynomials with rational coefficients in the above proof is very inefficient. In practice, it is much better to split the domain of interest into intervals (or squares if in the complex plane, but we stick to the self-adjoint case in the following discussion). Suppose that each interval has dyadic endpoints and a diameter of 2^{-n_2} and that our operator is self-adjoint with known bounded dispersion. One can then apply Lemma 3.21 (denoting the index of that tower by n_1) to obtain an interval covering of the spectrum which will converge as $n_1 \rightarrow \infty$, modulo the possibility of isolated points of the spectrum located at the endpoints of the intervals. Since the capacity of a compact set is unaltered by adding finitely many points, we do not have to worry about the endpoints—the limit of the capacity of this covering as $n_1 \rightarrow \infty$ will be the capacity of a covering of the spectrum. As $n_2 \rightarrow \infty$, we can use the fact that capacity is right-continuous as a set function (for compact sets E_n, E with $E_n \downarrow E$, one has $\text{cap}(E_n) \downarrow \text{cap}(E)$) to obtain a Π_2^A algorithm. The point of this is that it reduces the computation of the resulting tower $\{\Gamma_{n_2, n_1}\}$ to computing the capacity of finite unions of disjoint closed intervals in \mathbb{R} . In our computational examples, we made use of the method in [109], which uses conformal mappings and can deal with thousands of intervals. \square

7 Proofs Concerning Essential Numerical Ranges, Essential Spectra and Spectral Pollution

Proof of Theorem 3.10 for Ξ_{we} For the lower bounds, it is enough to note that $\{\Xi_{we}, \Omega_D, \Lambda_2\} \notin \Delta_2^G$ by the same argument as step 1 of the proof of Theorem 3.6. The construction is exactly the same but yields $d_H(\Gamma_{n(k)}(A), \{0\}) \leq 1/2$, whereas $\Xi_{we}(A) = [0, 1]$. Hence, the proposed height one tower cannot converge. To construct a Π_2^A tower for general operators, we need the following Lemma:

Lemma 7.1 *Let $B \in \mathbb{C}^{n \times n}$ and $\epsilon > 0$. Then using finitely many arithmetic operations and comparisons, we can compute points $z_1, \dots, z_k \in \mathbb{Q} + i\mathbb{Q}$ such that*

$$d_H(\{z_1, \dots, z_k\}, W(B)) \leq \epsilon.$$

Proof Recall from step 1 of the proof of Theorem 3.3 that we can compute an upper bound $M \in \mathbb{Q}_+$ for $\|B\|$ in finitely many arithmetic operations and comparisons. Now choose points $x_1, \dots, x_k \in \mathbb{Q}^n$, each of norm at most 1, such that $d_H(\{x_1, \dots, x_k\}, \{x \in \mathbb{C}^n : \|x\| = 1\}) < \epsilon/(3M)$. These can be computed in finitely many arithmetic operations and comparisons using generalised polar coordinates and approximations of trigonometric identities. It follows that

$$d_H(\{\langle Bx_1, x_1 \rangle, \dots, \langle Bx_k, x_k \rangle\}, W(B)) \leq 2\epsilon/3.$$

We then let each $z_j \in \mathbb{Q} + i\mathbb{Q}$ be a $\epsilon/4$ approximation of $\langle Bx_j, x_j \rangle$, which can be computed in finitely many arithmetic operations and comparisons. \square

Remark 7.2 (Efficient computation) In practice, there are much more efficient methods of computation. For example, the method of Johnson [96], reduces the computation of $W(A)$ for $A \in \mathbb{C}^{n \times n}$ to a series of $n \times n$ Hermitian eigenvalue problems. \square

It is well known that for $A \in \Omega_B$,

$$\overline{W(P_n A|_{P_n \mathcal{H}})} \uparrow \overline{W(A)},$$

$$\overline{W((I - P_n)A|_{(I - P_n)\mathcal{H}})} \downarrow W_e(A).$$

Given A , let $\Gamma_{n_2, n_1}(A)$ be a finite collection of points produced by the algorithm in Lemma 7.1 applied to $B = (I - P_{n_2})P_{n_1+n_2+1}A|_{P_{n_1+n_2+1}(I - P_{n_2})\mathcal{H}}$ and $\epsilon = 1/n_1$. The above limits show that $\{\Gamma_{n_2, n_1}\}$ provides a Π_2^A tower for $\{\Xi_{er}, \Omega_B, \Lambda_1\}$. \square

Proof of Theorem 3.10 for $\Xi_{poll}^{\mathbb{F}}$ We will prove that $\{\Xi_{poll}^{\mathbb{R}}, \Omega_D, \Lambda_i\} \notin \Delta_3^G$ and $\{\Xi_{poll}^{\mathbb{C}}, \Omega_B, \Lambda_1\} \in \Sigma_3^A$. The construction of towers for $\Xi_{poll}^{\mathbb{R}}$ is similar, as are the arguments for lower bounds.

Step 1 $\{\Xi_{poll}^{\mathbb{C}}, \Omega_B, \Lambda_1\} \in \Sigma_3^A$. Let $\{\tilde{\Gamma}_{n_2, n_1}\}$ be the Π_2^A tower for $\{\Xi_{er}, \Omega_B, \Lambda_1\}$ constructed above. Recall the definition

$$\gamma_{n_2, n_1}(z; A) = \min\{\sigma_{\inf}(P_{n_1}(A - zI)|_{P_{n_2}\mathcal{H}}), \sigma_{\inf}(P_{n_1}(A^* - \bar{z}I)|_{P_{n_2}\mathcal{H}})\}$$

and that this can be approximated to any given accuracy in finitely many arithmetic operations and comparisons (see also ‘‘Appendix A’’). We assume that we approximate from below to an accuracy of $1/n_1$ and call this approximation $\tilde{\gamma}_{n_2, n_1}$. The function $\gamma_{n_2, n_1}(z; A)$ is Lipschitz continuous with Lipschitz constant bounded by 1. Define the set

$$V_{n_1} = \bigcup_{m=1}^{n_1} U_m,$$

where U_m are the approximations to the open set U . By taking squares of distances to ball centres, we can decide whether a point $z \in \mathbb{Q} + i\mathbb{Q}$ has $\text{dist}(z, V_{n_1}) < \eta$ for any given $\eta \in \mathbb{Q}_+$. Let $\Upsilon_{n_2, n_1}(A, U)$ be the finite collection of all $z \in \tilde{\Gamma}_{n_2, n_1}(A)$ with $\text{dist}(z, V_{n_1}) < 1/n_2 - 1/n_1$. If $\Upsilon_{n_2, n_1}(A, U)$ is empty then set $Q_{n_2, n_1}(A, U) = 0$, otherwise set

$$Q_{n_2, n_1}(A, U) := \sup_{z \in \Upsilon_{n_2, n_1}(A, U)} \tilde{\gamma}_{n_2, n_1}(z; A) - \frac{1}{n_1}.$$

The above remarks show that this can be computed using finitely many arithmetic operations and comparisons.

Let $W_{n_2} = \overline{W((I - P_{n_2})A|_{(I - P_{n_2})\mathcal{H}})}$ and $W_{n_2, n_1} = \overline{W((I - P_{n_2})P_{n_1+n_2+1}A|_{P_{n_1+n_2+1}(I - P_{n_2})\mathcal{H}})}$. We claim that the set $\Upsilon_{n_2, n_1}(A, U)$ converges to

$$\Upsilon_{n_2}(A, U) := \overline{\left\{ z \in W_{n_2} : \text{dist}(z, \bar{U}) < \frac{1}{n_2} \right\}},$$

as $n_1 \rightarrow \infty$, meaning also if $\Upsilon_{n_2}(A, U)$ is empty then $\Upsilon_{n_2, n_1}(A, U)$ is empty for large n_1 . If $z \in \Upsilon_{n_2, n_1}(A, U)$, then there exists $\hat{z} \in W_{n_2, n_1} \subset W_{n_2}$ with $|z - \hat{z}| \leq 1/n_1$. Since

$$\text{dist}(z, \bar{U}) \leq \text{dist}(z, V_{n_1}) < 1/n_2 - 1/n_1,$$

it follows that $\text{dist}(\hat{z}, \bar{U}) < 1/n_2$ and hence $\Upsilon_{n_2}(A, U)$ is non-empty. So to prove convergence, we only need to deal with the case $\Upsilon_{n_2}(A, U) \neq \emptyset$. The above argument also shows that any limit point of a subsequence $z_{m(j)} \in \Upsilon_{n_2, m(j)}(A, U)$ must lie in $\Upsilon_{n_2}(A, U)$. Hence to prove the claim, we need to only prove that for any $z \in \Upsilon_{n_2}(A, U)$, there exists z_{n_1} that are contained in $\Upsilon_{n_2, n_1}(A, U)$ for large n_1 and converge to z .

Let $z \in W_{n_2}$ with $\text{dist}(z, \bar{U}) < 1/n_2$, then there exists $\epsilon > 0$ and $j > 0$ such that $\text{dist}(z, U_j) < 1/n_2 - \epsilon$. There also exists $z_{n_1} \in \tilde{\Gamma}_{n_2, n_1}(A)$ with $z_{n_1} \rightarrow z$. It must hold for $n_1 > j$ that

$$\begin{aligned} \text{dist}(z_{n_1}, V_{n_1}) &\leq \text{dist}(z_{n_1}, V_j) \leq |z_{n_1} - z| + \text{dist}(z, U_j) \\ &< |z_{n_1} - z| + \frac{1}{n_2} - \epsilon. \end{aligned}$$

This last quantity is smaller than $1/n_2 - 1/n_1$ for large n_1 and hence $z_{n_1} \in \Upsilon_{n_2, n_1}(A, U)$ for large n_1 . It follows for any $z \in \Upsilon_{n_2}(A, U)$, there exists z_{n_1} that are contained in $\Upsilon_{n_2, n_1}(A, U)$ for large n_1 and converge to z .

Define

$$Q_{n_2}(A, U) := \sup_{z \in \Upsilon_{n_2}(A, U)} \gamma_{n_2}(z; A),$$

where we recall that $\gamma_{n_2}(z; A) = \min\{\sigma_{\text{inf}}((A - zI)|_{P_{n_2}\mathcal{H}}), \sigma_{\text{inf}}((A^* - \bar{z}I)|_{P_{n_2}\mathcal{H}})\}$. If $z \in \Upsilon_{n_2, n_1}(A, U)$, then the above shows that there exists $\hat{z} \in \Upsilon_{n_2}(A, U)$ with $|z - \hat{z}| \leq 1/n_1$. It follows that

$$\begin{aligned} \tilde{\gamma}_{n_2, n_1}(z; A) - \frac{1}{n_1} &\leq \gamma_{n_2, n_1}(z; A) - \frac{1}{n_1} \\ &\leq \gamma_{n_2, n_1}(\hat{z}; A) \leq \gamma_{n_2}(z; A), \end{aligned}$$

where we have used the bound on the Lipschitz constant and the fact that $\tilde{\gamma}_{n_2, n_1}$ converge up to γ_{n_2} (and uniformly on compact subsets of \mathbb{C}). It follows that $Q_{n_2, n_1}(A, U) \leq Q_{n_2}(A, U)$ and this also covers the case that $\Upsilon_{n_2}(A, U) = \emptyset$ if we define the supremum over the empty set to be 0. The set convergence proven above and uniform convergence of $\tilde{\gamma}_{n_2, n_1}$ implies that $Q_{n_2, n_1}(A, U)$ converges to $Q_{n_2}(A, U)$. It is also clear that the $\Upsilon_{n_2}(A, U)$ are nested and converge down to $W_e(A) \cap \bar{U}$ since W_{n_2} converges down to $W_e(A)$. The functions γ_{n_2} also converge down to

$$\gamma(z; A) = \|R(z, A)\|^{-1}$$

uniformly on compact subsets of \mathbb{C} and hence $Q_{n_2}(A, U)$ converges down to

$$Q(A, U) = \sup_{z \in W_e(A) \cap \bar{U}} \|R(z, A)\|^{-1}.$$

Define

$$\Gamma_{n_3, n_2, n_1}(A, U) = 1 - \chi_{[0, 1/n_3]}(Q_{n_2, n_1}(A, U)) \in \{0, 1\}.$$

The above show that

$$\lim_{n_1 \rightarrow \infty} \Gamma_{n_3, n_2, n_1}(A, U) = 1 - \chi_{[0, 1/n_3]}(Q_{n_2}(A, U)) =: \Gamma_{n_3, n_2}(A, U).$$

Since $\chi_{[0, 1/n_3]}$ has right limits and $Q_{n_2}(A, U)$ are non-increasing,

$$\lim_{n_2 \rightarrow \infty} \Gamma_{n_3, n_2}(A, U) = 1 - \chi_{[0, 1/n_3]}(Q(A, U) \pm) =: \Gamma_{n_3}(A, U),$$

where \pm denotes one of the right or left limits (it is possible to have either). Now if $\Xi_{poll}^{\mathbb{C}}(A, U) = 0$, then $\Gamma_{n_3}(A, U) = 0$ for all n_3 . But if $\Xi_{poll}^{\mathbb{C}}(A, U) = 1$, then for large n_3 , $\Gamma_{n_3}(A, U) = 1$. Moreover, in this latter case, $\Gamma_{n_3}(A, U) = 1$ signifies the existence of $z \in W_e(A) \cap \bar{U}$ with $\gamma(z; A) > 0$ and hence $z \notin \text{Sp}(A)$. Hence, $\{\Gamma_{n_3, n_2, n_1}\}$ provides a Σ_3^A tower.

Step 2 $\{\Xi_{poll}^{\mathbb{R}}, \Omega_D, \Lambda_2\} \notin \Delta_3^G$. We will argue for the case that $U = U_1 = \mathbb{R}$ and the restricted case is similar. Assume for a contradiction that this is false and that $\{\widehat{\Gamma}_{n_2, n_1}\}$ is a general height two tower for $\{\Xi_{poll}^{\mathbb{R}}, \Omega_D, \Lambda_2\}$. We follow the same strategy as the proof of Theorem 3.3 step 4 (recall also the results of Sect. 5). Let (\mathcal{M}, d) be discrete space $\{0, 1\}$ and $\tilde{\Omega}$ denote the collection of all infinite matrices $\{a_{i, j}\}_{i, j \in \mathbb{N}}$ with entries $a_{i, j} \in \{0, 1\}$ and consider the problem function

$\tilde{\Xi}_1(\{a_{i, j}\})$: Does $\{a_{i, j}\}$ have a column containing infinitely many nonzero entries?

For $j \in \mathbb{N}$, let $\{b_{i, j}\}_{i \in \mathbb{N}}$ be a dense subset of $I_j := [1 - 1/2^{2j-1}, 1 - 1/2^{2j}]$. Given a matrix $\{a_{i, j}\}_{i, j \in \mathbb{N}} \in \tilde{\Omega}$, construct a matrix $\{c_{i, j}\}_{i, j \in \mathbb{N}}$ by letting $c_{i, j} = a_{i, j} b_{r(i, j), j}$ where

$$r(i, j) = \max \left\{ 1, \sum_{k=1}^i a_{k, j} \right\}.$$

Now consider any bijection $\phi : \mathbb{N} \rightarrow \mathbb{N}^2$ and define the diagonal operator

$$A = \text{diag}(c_{\phi(1)}, c_{\phi(2)}, c_{\phi(3)}, \dots).$$

The algorithm $\widehat{\Gamma}_{n_2, n_1}$ thus translates to an algorithm Γ'_{n_2, n_1} for $\{\tilde{\Xi}_1, \tilde{\Omega}\}$. Namely, set $\Gamma'_{n_2, n_1}(\{a_{i, j}\}_{i \in \mathbb{N}}) = \widehat{\Gamma}_{n_2, n_1}(A)$. The fact that ϕ is a bijection shows that the lowest

level Γ'_{n_2, n_1} are generalised algorithms (and are consistent). In particular, given N , we can find $\{A_{i,j} : i, j \leq N\}$ using finitely many evaluations of the matrix values $\{c_{k,l}\}$ (the same is true for A^*A and AA^* since the operator is diagonal). But for any given $c_{k,l}$ we can evaluate this entry using only finitely many evaluations of the matrix values $\{a_{m,n}\}$ by the construction of r . Finally, note that

$$\text{Sp}(A) = \{1\} \cup \left(\bigcup_{j: \{a_{i,j}\}_{i \in \mathbb{N}} \text{ has infinitely many 1s}} I_j \right) \cup Q,$$

where Q lies in the discrete spectrum. The intervals I_j are also separated. It follows that there is a gap in the essential spectrum if and only if there exists a column $\{a_{i,j}\}_{i \in \mathbb{N}}$ with infinitely many 1s. Otherwise the essential spectrum is $\{1\}$. It follows that $\tilde{\Xi}(\{a_{i,j}\}) = \Xi_{\text{poll}}^{\mathbb{R}}(A, \mathbb{R})$, and hence, we get a contradiction. \square

7.1 Essential Numerical Range for Unbounded Operators

The essential numerical range (see (3.1)) was first introduced for a bounded operator A in [152], as the closure of the numerical range of the image of A in the Calkin algebra:

$$W_e(A) = \bigcap_{B \in \Omega_K} \overline{W(A + B)}.$$

Other equivalent characterisations were then given in [82]. The unbounded case is significantly different from the bounded case, and definitions that are equivalent in the bounded case may yield very different sets in the unbounded case. The definition for unbounded operators appeared in [34] and required the development of several new ideas and tools. In this section, we let Ω_C denote the set of closed operators T with domain $\mathcal{D}(T) \subset l^2(\mathbb{N})$ such that the linear span of the canonical basis forms a core of T . This latter condition ensures that we can use the usual matrix representation of the operator T and hence the evaluation functions Λ_1 . We follow [34] and define

$$W_e(T) = \left\{ \lambda \in \mathbb{C} : \exists \{x_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(T), \|x_n\| = 1, x_n \xrightarrow{w} 0, \lim_{n \rightarrow \infty} \langle Tx_n, x_n \rangle = \lambda \right\}. \tag{7.1}$$

In [34], it was shown that for any $T \in \Omega_C$, $W_e(T)$ consists precisely of the essential spectrum of T together with all possible spectral pollution that may arise by applying projection methods to find the spectrum of T numerically. This result therefore generalises Theorems 3.8 and 3.9. The set $W_e(T)$ is closed and convex, but, unlike the case when T is bounded, $W_e(T)$ may be empty. We first need two simple lemmas.

Lemma 7.3 *Let $T \in \Omega_C$, then $W(P_n T|_{P_n \mathcal{H}}) \uparrow \overline{W(T)}$ in the Attouch–Wets topology as $n \rightarrow \infty$.*

Proof It is clear that

$$W(P_n T|_{P_n \mathcal{H}}) \subset W(T) := \{\langle Tx, x \rangle : x \in \mathcal{D}(T), \|x\| = 1\},$$

and that the sets $W(P_n T|_{P_n \mathcal{H}})$ are increasing with n . Now let $\lambda \in \overline{W(T)}$ be arbitrary. It is enough to show that there exists $\lambda_n \in W(P_n T|_{P_n \mathcal{H}})$ such that $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. By assumption, there exists $x_n \in \mathcal{D}(T)$ such that $\|x_n\| = 1$ and $\lim_{n \rightarrow \infty} \langle Tx_n, x_n \rangle = \lambda$. Since the linear span of the canonical basis forms a core of T , we can assume without loss of generality that each x_n has finite support with respect to the canonical basis. By taking subsequences if necessary, we may assume that $P_n x_n = x_n$ and hence $\langle Tx_n, x_n \rangle \in W(P_n T|_{P_n \mathcal{H}})$. The result now follows. \square

Lemma 7.4 *Let $T \in \Omega_{\mathcal{C}}$. If $W_e(T) \neq \emptyset$, then $\overline{W((I - P_n)T|_{(I - P_n)\mathcal{H}})} \downarrow W_e(T)$ in the Attouch–Wets topology as $n \rightarrow \infty$. If $W_e(T) = \emptyset$, then for any compact set K , $K \cap \overline{W((I - P_n)T|_{(I - P_n)\mathcal{H}})} = \emptyset$ for large n .*

Proof We clearly have that $\overline{W((I - P_n)T|_{(I - P_n)\mathcal{H}})}$ are non-empty and decreasing in n . It is enough to show the following two results:

- (1) If $\lambda \in W_e(T)$, then $\lambda \in \overline{W((I - P_n)T|_{(I - P_n)\mathcal{H}})}$ for all n .
- (2) If $\lambda \notin W_e(T)$, then $\liminf_{n \rightarrow \infty} \text{dist}(\lambda, \overline{W((I - P_n)T|_{(I - P_n)\mathcal{H}})}) > 0$.

We first prove (1), so assume that $\lambda \in W_e(T)$. Then, since the linear span of the canonical basis functions form a core of T , we can assume that there exists x_n with $\|x_n\| = 1$ such that each x_n has finite support with respect to the canonical basis, $x_n \xrightarrow{w} 0$ and $\lim_{n \rightarrow \infty} \langle Tx_n, x_n \rangle = \lambda$. It follows that for any fixed m , $\lim_{n \rightarrow \infty} P_m x_n = 0$ and hence $\lambda \in \overline{W((I - P_m)T|_{(I - P_m)\mathcal{H}})}$.

Finally, to see (2), suppose that this were false for some $\lambda \notin W_e(T)$. We may then choose $\lambda_n \in \overline{W((I - P_n)T|_{(I - P_n)\mathcal{H}})}$ such that $\liminf_{n \rightarrow \infty} |\lambda - \lambda_n| = 0$. By taking subsequences if necessary, we may assume that $\lambda_n \rightarrow \lambda$ and that there exists x_n with $\|x_n\| = 1$, $P_n x_n = 0$ and $|\langle Tx_n, x_n \rangle - \lambda_n| \rightarrow 0$. But this implies that $x_n \xrightarrow{w} 0$ and $\lim_{n \rightarrow \infty} \langle Tx_n, x_n \rangle = \lambda$. Therefore, $\lambda \in W_e(T)$, the required contradiction. \square

We have the following corollary, which shows that the SCI classification of computing $W_e(T)$ for $T \in \Omega_{\mathcal{C}}$ remains Π_2^A (one can make this precise by adding the empty set to the Attouch–Wets topology, but we omit the details).

Corollary 7.5 *There exists a height two tower of arithmetic algorithms $\{\Gamma_{n_2, n_1}\}$, using Δ_1 (the matrix values with respect to the canonical basis) and Δ_1 -information (see Definition 5.11), such that for any $T \in \Omega_{\mathcal{C}}$, the following hold with respect to the Attouch–Wets topology:*

- $\Gamma_{n_2, n_1}(T) \uparrow \Gamma_{n_2}(T) \subset \overline{W(T)}$ as $n_1 \rightarrow \infty$.
- If $W_e(T) \neq \emptyset$, then $\Gamma_{n_2}(T) \downarrow W_e(T)$ as $n_2 \rightarrow \infty$. If $W_e(T) = \emptyset$, then for any compact set K , $K \cap \Gamma_{n_2}(T) = \emptyset$ for large n_2 .

Proof We simply let $\Gamma_{n_2, n_1}(T)$ be an approximation of

$$W\left((I - P_{n_2})P_{n_1+n_2+1}T|_{P_{n_1+n_2+1}(I - P_{n_2})\mathcal{H}}\right)$$

that can be computed in finitely many arithmetic operations and comparisons, even when using inexact input (see Definition 5.11 and Remark 5.12), using the arguments in Sect. 7. The results now follow from Lemmas 7.3 and 7.4. \square

8 Proofs Concerning Lebesgue Measure

We use the function DistSpec in ‘‘Appendix A’’. For ease of notation, we suppress the dispersion function f in calling DistSpec , but assume that we know $\{c_n\}$ with $D_{f,n}(A) \leq c_n$ and $c_n \rightarrow 0$ as $n \rightarrow \infty$. However, the proof of convergence also works when using $c_n = 0$ (which does not necessarily bound $D_{f,n}(A)$). The key observation is the following:

Observation: If $A \in \Omega_f$, then the function $F_n(z) := \text{DistSpec}(A, n, z, f(n)) + c_n$ converges uniformly to $\|R(z, A)\|^{-1}$ from above on compact subsets of \mathbb{C} . By taking successive minima, we can assume without loss of generality that F_n is non-increasing in n .

The other ingredient needed is the following proposition

Proposition 8.1 *Given a finite union of disks in the complex plane, the Lebesgue measure of their intersection with the interior of a rectangle can be computed within arbitrary precision, using finitely many arithmetical operations and comparisons on the centres and radii of the discs, as well as the position of the rectangle.*

Proof Without loss of generality, we assume that the rectangle is $\{x + iy : x, y \in [0, 1]\}$. Consider dividing the rectangle into n^2 subrectangles using the division of $[0, 1]$ into n equal intervals. Given such a subrectangle, we can easily test via a finite number of arithmetic operations and comparisons whether the centre is in the union of the circles. Let $r(n)$ denote the number of subrectangles whose centre lies in the union. Then, since the boundary of the union of the circles has measure zero, it is easy to see that $r(n)/n^2$ converges to the desired Lebesgue measure. Moreover, we can bound the number of subrectangles that intersect the boundary of any of the circles, and this can be used to obtain any desired precision. \square

Proof of Theorem 3.13 *Step 1* $\{\Xi_1^L, \Omega_f, \Lambda_i\}, \{\Xi_1^L, \Omega_D, \Lambda_i\} \in \Pi_2^A$. It is enough to consider Λ_1 . We will estimate $\text{Leb}(\text{Sp}(A))$ by estimating the Lebesgue measure of the resolvent set on the closed square $[-C, C]^2$, where $\|A\| \leq C$. We do not assume C is known. For $n_1, n_2 \in \mathbb{N}$, let

$$\text{Grid}(n_1, n_2) = \left(\frac{1}{2n_2} \mathbb{Z} + \frac{1}{2n_2} i \mathbb{Z} \right) \cap [-n_1, n_1]^2.$$

Letting $B(x, r)$ and $D(x, r)$ denote the closed and open balls of radius r around x , respectively,¹⁸ in \mathbb{C} (or \mathbb{R} where appropriate), we define

$$U(n_1, n_2, A) = [-n_1, n_1] \times [-n_1, n_1] \cap (\cup_{z \in \text{Grid}(n_1, n_2)} B(z, F_{n_1}(z))).$$

¹⁸ We set $D(x, 0) = \emptyset$.

Note that $\text{Leb}(U(n_1, n_2, A))$ can be computed up to arbitrary predetermined precision using only arithmetic operations and comparisons by Proposition 8.1. Using this, we can define

$$\Gamma_{n_2, n_1}(A) = 4n_1^2 - \text{Leb}(U(n_1, n_2, A))$$

where, without loss of generality, we assume that we have computed the exact value of the Lebesgue measure (since we can absorb this error in the first limit). Γ_{n_2, n_1} are arithmetical algorithms using the fact that DistSpec is and the above proposition. The only non-trivial part is convergence. The algorithm is summarised in the routine LebSpec in § B.3.

We now show that the algorithm LebSpec converges and realises the Π_2^A classification. There exists a compact set K such that $\|R(z, A)\|^{-1} > 1$ on K^c and without loss of generality we can make C larger, $C \in \mathbb{N}$ and take $K = [-C, C]^2$. For $n_1 \geq C$

$$U(n_1, n_2, A) = ([-C, C]^2 \cap (\cup_{z \in \text{Grid}(n_1, n_2)} B(z, F_{n_1}(z)))) \cup ([-n_1, n_1]^2 \setminus [-C, C]^2),$$

since $F_n(z) \geq \|R(z, A)\|^{-1}$. It follows that for large n_1

$$\Gamma_{n_2, n_1}(A) = 4C^2 - \text{Leb}([-C, C]^2 \cap (\cup_{z \in \text{Grid}(n_1, n_2)} B(z, F_{n_1}(z)))).$$

As $n_1 \rightarrow \infty$, $[-C, C]^2 \cap (\cup_{z \in \text{Grid}(n_1, n_2)} B(z, F_{n_1}(z)))$ converges to the closed set

$$K(n_2, A) = [-C, C]^2 \cap (\cup_{z \in \text{Grid}(C, n_2)} B(z, \|R(z, A)\|^{-1}))$$

from above and hence

$$\lim_{n_1 \rightarrow \infty} \Gamma_{n_2, n_1}(A) = 4C^2 - \text{Leb}(K(n_2, A)),$$

from below. Consider the relatively open set

$$V(n_2, A) = [-C, C]^2 \cap (\cup_{z \in \text{Grid}(C, n_2)} D(z, \|R(z, A)\|^{-1})).$$

Clearly, $\text{Leb}(K(n_2, A)) = \text{Leb}(V(n_2, A))$ since the sets differ by a finite collection of circular arcs or points (recall we defined the open ball of radius zero to be the empty set). Hence, we must show that

$$\lim_{n_2 \rightarrow \infty} \text{Leb}(V(n_2, A)) = \text{Leb}(\rho_C(A)),$$

where $\rho_C(A) = [-C, C]^2 \setminus \text{Sp}(A)$. For $z \in \rho_C(A)$,

$$\text{dist}(z, \text{Sp}(A)) \geq \|R(z, A)\|^{-1}$$

and hence we get $V(n_2, A) \subset \rho_C(A)$. Since $\rho_C(A)$ is relatively open, a simple density argument using the continuity of $\|R(z, A)\|^{-1}$ yields $V(n_2, A) \uparrow \rho_C(A)$ as $n_2 \rightarrow \infty$

since the grid refines itself. So we get

$$\text{Leb}(V(n_2, A)) \uparrow \text{Leb}(\rho_C(A)).$$

This proves the convergence and also shows that $\Gamma_{n_2}(A) \downarrow \Xi_1^L(A)$, thus yielding the Π_2^A classification. The same argument works in the one-dimensional case when considering self-adjoint operators Ω_D and $\text{Leb}_{\mathbb{R}}$. We simply restrict everything to the real line and consider the interval $[-C, C]$ rather than a square.

Step 2 $\{\Xi_1^L, \Omega_f, \Lambda_j\}, \{\Xi_1^L, \Omega_D, \Lambda_j\} \notin \Delta_2^G$. It is enough to consider Λ_2 . We will only show that $\text{SCI}(\Xi_1^L, \Omega_D, \Lambda_2)_G \geq 2$ for which we use $\text{Leb}_{\mathbb{R}}$ and the two-dimensional case is similar. Suppose for a contradiction that there exists a height one tower $\{\Gamma_n\}$, then $\Lambda_{\Gamma_n}(A)$ is finite for each $A \in \Omega_D$. Hence, for every A and n there exists a finite number $N(A, n) \in \mathbb{N}$ such that the evaluations from $\Lambda_{\Gamma_n}(A)$ only take the matrix entries $A_{ij} = \langle Ae_j, e_i \rangle$ with $i, j \leq N(A, n)$ into account.

Pick any sequence a_1, a_2, \dots that is dense in the unit interval $[0, 1]$. Consider the matrix operators $A_m = \text{diag}\{a_1, a_2, \dots, a_m\} \in \mathbb{C}^{m \times m}$, $B_m = \text{diag}\{0, 0, \dots, 0\} \in \mathbb{C}^{m \times m}$ and $C = \text{diag}\{0, 0, \dots\}$. Set $A = \bigoplus_{m=1}^{\infty} (B_{k_m} \oplus A_{k_m})$, where we choose an increasing sequence k_m inductively as follows. Set $k_1 = 1$ and suppose that k_1, \dots, k_m have been chosen. $\text{Sp}(B_{k_1} \oplus A_{k_1} \oplus \dots \oplus B_{k_m} \oplus A_{k_m} \oplus C) = \{0, a_1, a_2, \dots, a_{k_m}\}$ and hence $\text{Leb}(\text{Sp}(B_{k_1} \oplus A_{k_1} \oplus \dots \oplus B_{k_m} \oplus A_{k_m} \oplus C)) = 0$ so there exists some $n_m \geq m$ such that if $n \geq n_m$ then

$$\Gamma_n(B_{k_1} \oplus A_{k_1} \oplus \dots \oplus B_{k_m} \oplus A_{k_m} \oplus C) \leq \frac{1}{2}.$$

Now let $k_{m+1} \geq \max\{N(B_{k_1} \oplus A_{k_1} \oplus \dots \oplus B_{k_m} \oplus A_{k_m} \oplus C, n_m), k_m + 1\}$. Any evaluation function $f_{i,j} \in \Lambda$ is simply the (i, j) th matrix entry and hence by construction

$$f_{i,j}(B_{k_1} \oplus A_{k_1} \oplus \dots \oplus B_{k_m} \oplus A_{k_m} \oplus C) = f_{i,j}(A),$$

for all $f_{i,j} \in \Lambda_{\Gamma_{n_m}}(B_{k_1} \oplus A_{k_1} \oplus \dots \oplus B_{k_m} \oplus A_{k_m} \oplus C)$. By assumption (iii) in Definition 5.1, it follows that $\Lambda_{\Gamma_{n_m}}(B_{k_1} \oplus A_{k_1} \oplus \dots \oplus B_{k_m} \oplus A_{k_m} \oplus C) = \Lambda_{\Gamma_{n_m}}(A)$ and hence by assumption (ii) in the same definition that $\Gamma_{n_m}(A) = \Gamma_{n_m}(B_{k_1} \oplus A_{k_1} \oplus \dots \oplus B_{k_m} \oplus A_{k_m} \oplus C) \leq 1/2$. But $\lim_{n \rightarrow \infty} (\Gamma_n(A)) = \text{Leb}(\{0, a_1, a_2, \dots\}) = 1$, a contradiction.

Step 3 $\{\Xi_1^L, \Omega, \Lambda_1\} \in \Pi_3^A$ for $\Omega = \Omega_B, \Omega_{SA}, \Omega_N$ or Ω_g . We will deal with the case of Ω_B . The cases of Ω_N and Ω_g then follow via $\Omega_N \subset \Omega_g \subset \Omega_B$ and the one-dimensional Lebesgue measure case for Ω_{SA} is similar. A careful analysis of the proof in step 1 yields that

- $\Gamma_{n_2, n_1}(A)$ converges to $\Gamma_{n_2}(A)$ from below as $n_1 \rightarrow \infty$.
- $\Gamma_{n_2}(A)$ converges to $\text{Leb}(\text{Sp}(A))$ monotonically from above as $n_2 \rightarrow \infty$.

We can ensure that the first limit converges from below by always slightly overestimating the Lebesgue measure of $U(n_1, n_2)$ (with error converging to zero) and using Proposition 8.1. These observations will be used later to answer question 3. We do not need to know c_n for the above proof to work, but we will need it for the first of

the above facts. A slight alteration of the proof/algorithm by inserting an additional successive limit deals with the general case.

Define the function

$$\gamma_{n,m}(z; A) = \min\{\sigma_{\text{inf}}(P_m(A - zI)|_{P_n\mathcal{H}}), \sigma_{\text{inf}}(P_m(A^* - \bar{z}I)|_{P_n\mathcal{H}})\},$$

where σ_{inf} denotes the injection modulus/smallest singular value (see also ‘‘Appendix A’’). One can show that $\gamma_{n,m}$ converges uniformly on compact subsets to

$$\gamma_n(z; A) = \min\{\sigma_{\text{inf}}((A - zI)|_{P_n\mathcal{H}}), \sigma_{\text{inf}}((A^* - \bar{z}I)|_{P_n\mathcal{H}})\},$$

as $m \rightarrow \infty$ and that this converges uniformly down to $\|R(z, A)\|^{-1}$ on compact subsets as $n \rightarrow \infty$ [91]. With a slight abuse of notation, we can approximate $\gamma_{n,m}(z; A)$ to within $1/m$ by $\text{DistSpec}(A, n, z, m)$ (where the spacing of the search routine is $1/m$, see also ‘‘Appendix A’’) so that this converges uniformly on compact subsets to $\gamma_n(z; A)$. In exactly the same manner as before, define

$$U(n_1, n_2, n_3, A) = [-n_2, n_2]^2 \cap (\cup_{z \in \text{Grid}(n_2, n_3)} B(z, \gamma_{n_2, n_1}(z; A))),$$

$$\Gamma_{n_3, n_2, n_1}(A) = (2n_2)^2 - \text{Leb}(U(n_1, n_2, n_3, A)).$$

The stated uniform convergence means that the argument in step 1 carries through and we have a height three tower, realising the Π_3^A classification.

Step 4 $\{\Xi_1^L, \Omega_{\text{SA}}, \Lambda_1\} \notin \Delta_3^G$. The proof is exactly the same argument as the proof of step 3 of Theorem 3.7. However, in this case to gain the contradiction, we then define $\tilde{\Gamma}_{n_2, n_1}(\{a_{i,j}\}) = \min\{\max\{1 - \Gamma_{n_2, n_1}(A)/2, 0\}, 1\}$ where $\{\Gamma_{n_2, n_1}\}$ is the supposed height two tower for $\{\Xi_1^L, \Omega_{\text{SA}}, \Lambda_1\}$.

Step 5 $\{\Xi_1^L, \Omega, \Lambda_1\} \notin \Delta_3^G$ for $\Omega = \Omega_B, \Omega_N$, or Ω_g . Since $\Omega_N \subset \Omega_g \subset \Omega_B$, we only need to deal with Ω_N . We can use a similar argument as in step 4, but now replacing each $C^{(j)}$ by

$$D^{(j)} = \bigoplus_{k=1}^j ih_k C^{(j)},$$

where h_1, h_2, \dots is a dense sequence in $[0, 1]$, and these operators act on $X_j = \bigoplus_{k=1}^j l^2(\mathbb{N})$. This ensures that the spectrum of the operator yields a positive two-dimensional Lebesgue measure if and only if $\tilde{\Xi}_2(\{a_{i,j}\}) = 0$. The rest of the argument is entirely analogous.

Step 6 $\Delta_2^G \not\cong \{\Xi_1^L, \Omega, \Lambda_2\} \in \Pi_2^A$ for $\Omega = \Omega_B, \Omega_{\text{SA}}, \Omega_N$ or Ω_g . The impossibility result follows by considering diagonal operators. For the existence of Π_2^A algorithms, we can use the construction in step 3, but the knowledge of matrix values of A^*A allows us to skip the first limit and approximate γ_n directly. \square

Proof of Theorem 3.14 Using the convergence

$$\lim_{\epsilon \downarrow 0} \text{Leb}(\widehat{\text{Sp}}_\epsilon(A)) = \text{Leb}(\text{Sp}(A)),$$

the lower bounds in Theorem 3.13 immediately imply the lower bounds in Theorem 3.14. Hence, we only need to construct the appropriate algorithms.

Step 1 $\{\Xi_2^L, \Omega_f, \Lambda_1\}, \{\Xi_2^L, \Omega_D, \Lambda_1\} \in \Sigma_1^A$. Let $A \in \Omega_f$ and

$$E_n = \frac{1}{n} (\mathbb{Z} + i\mathbb{Z}) \cap \{z \in \mathbb{C} : F_n(z) \leq \epsilon\} \cap [-n, n]^2.$$

Clearly, we can compute E_n with finitely many arithmetic operations and comparisons and we set

$$\Gamma_n(A) = \text{Leb} \left(\bigcup_{z \in E_n} D(z, \max\{0, \epsilon - F_n(z)\}) \right).$$

Proposition 8.1 shows that, without loss of generality, we can assume $\Gamma_n(A)$ can be computed exactly using finitely many arithmetic operations and comparisons. The algorithm is presented in the `LebPseudoSpec` routine in § B.3 and the following shows that this algorithm is sharp in the SCI hierarchy.

Suppose that $F_n(z) < \epsilon$ and that $|w| < \epsilon - F_n(z)$. If $z \in \text{Sp}(A)$, then clearly

$$\|R(z + w, A)\|^{-1} \leq |w| < \epsilon - F_n(z) \leq \epsilon,$$

and this holds trivially if $z + w \in \text{Sp}(A)$. So assume that neither of $z, z + w$ are in the spectrum. The resolvent identity yields

$$\|R(z + w, A)\| \geq \|R(z, A)\| - |w| \|R(z + w, A)\| \|R(z, A)\|,$$

which rearranges to

$$\|R(z + w, A)\|^{-1} \leq \|R(z, A)\|^{-1} + |w| < \epsilon.$$

It follows that $\bigcup_{z \in E_n} D(z, \max\{0, \epsilon - F_n(z)\})$ is in $\widehat{\text{Sp}}_\epsilon(A)$ and hence that $\Gamma_n(A) \leq \Xi_2^L(A)$. Without loss of generality by taking successive maxima, we can assume that $\Gamma_n(A)$ is increasing. Together, these will yield the Σ_1^A classification once convergence is shown. Using the uniform convergence of F_n and density of $1/n(\mathbb{Z} + i\mathbb{Z}) \cap [-n, n]^2$, we see that pointwise convergence holds:

$$\chi_{\bigcup_{z \in E_n} D(z, \max\{0, \epsilon - F_n(z)\})} \rightarrow \chi_{\widehat{\text{Sp}}_\epsilon(A)},$$

where χ_E denotes the indicator function of a set E . It follows by the dominated convergence theorem that $\Gamma_n(A) \rightarrow \text{Leb}(\widehat{\text{Sp}}_\epsilon(A))$. The proof for Ω_D is similar by restricting everything to the real line.

Step 2 $\{\Xi_2^L, \Omega, \Lambda_1\} \in \Sigma_2^A$ for $\Omega = \Omega_B, \Omega_{SA}, \Omega_N$ or Ω_g . To prove this, we simply replace F_{n_1} by the functions γ_{n_2, n_1} and set

$$\Gamma_{n_2, n_1}(A) = \text{Leb} \left(\bigcup_{z \in E_{n_2}} D(z, \max\{0, \epsilon - \gamma_{n_2, n_1}(z; A)\}) \right).$$

Step 3 $\{\Xi_2^L, \Omega, \Lambda_2\} \in \Sigma_1^A$ for $\Omega = \Omega_B, \Omega_{SA}, \Omega_N$ or Ω_g . The knowledge of matrix values of A^*A allows us to skip the first limit in the construction of step 2 and approximate γ_n directly. \square

Proof of Proposition 3.15 We begin with the proof of 1. Suppose $A \in \Omega_D$ has $\text{Leb}_{\mathbb{R}}(\text{Sp}(A)) = 0$ and let $A_n \in \Omega_D$ be such that $\|A - A_n\| \rightarrow 0$ as $n \rightarrow \infty$. This implies that $\text{Sp}(A_n) \rightarrow \text{Sp}(A)$ since all our operators are normal. To prove that $\text{Leb}_{\mathbb{R}}(\text{Sp}(A_n)) \rightarrow 0$, it is enough to prove that

$$\text{Leb}(F_n) \downarrow 0, \tag{8.1}$$

where $F_n = \text{Sp}(A) \cup (\cup_{m \geq n} \text{Sp}(A_m))$. But F_n decreases to $\text{Sp}(A)$ and is bounded in measure, so (8.1) holds. For the converse, let $\text{Leb}_{\mathbb{R}}(\text{Sp}(A)) > 0$. Without loss of generality, assume that all of A 's entries lie in $[0, 1]$. Let \mathbb{D}_n denote the set $\{j/2^n\}_{j=1}^n$ and consider the map $\phi_n : x \mapsto 2^{-n} \lceil x2^n \rceil$ on $[0, 1]$. Let A_n be the diagonal operator obtained by applying ϕ_n to each of A 's entries. We clearly have that $\|A - A_n\| \rightarrow 0$ as $n \rightarrow \infty$ but note that $\text{Sp}(A_n)$ is finite so has Lebesgue measure 0. Hence Ξ_1^L is discontinuous at A .

To prove 2, note that for $A \in \Omega_D$, $\text{Leb}_{\mathbb{R}}(S_\epsilon(A)) = 0$. Let $A_n \in \Omega_D$ have $\|A - A_n\| \rightarrow 0$. Then given some $0 < \delta < \epsilon$ it holds for large n that $\text{Sp}_{\epsilon-\delta}(A) \subset \text{Sp}_\epsilon(A_n) \subset \text{Sp}_{\epsilon+\delta}(A)$ and hence that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \text{Leb}_{\mathbb{R}}(\text{Sp}_\epsilon(A_n)) &\leq \text{Leb}_{\mathbb{R}}(\text{Sp}_{\epsilon+\delta}(A)) \\ \liminf_{n \rightarrow \infty} \text{Leb}_{\mathbb{R}}(\text{Sp}_\epsilon(A_n)) &\geq \text{Leb}_{\mathbb{R}}(\text{Sp}_{\epsilon-\delta}(A)). \end{aligned}$$

Now let $\delta \downarrow 0$ and use the fact that Ξ_2^L is continuous in ϵ . \square

Finally, we deal with the question of determining whether the Lebesgue measure is zero. Recall that for this problem, (\mathcal{M}, d) denotes the set $\{0, 1\}$ endowed with the discrete topology and we consider the problem function

$$\Xi_3^L(A) = \begin{cases} 0, & \text{if } \text{Leb}(\text{Sp}(A)) > 0 \\ 1, & \text{otherwise.} \end{cases}$$

Proof of Theorem 3.16 We will show that $\{\Xi_3^L, \Omega_f, \Lambda_1\} \in \Pi_3^A$ and $\{\Xi_3^L, \Omega_D, \Lambda_2\} \notin \Delta_3^G$. The analogous statements $\{\Xi_3^L, \Omega_D, \Lambda_1\} \in \Pi_3^A$ and $\{\Xi_3^L, \Omega_f, \Lambda_2\} \notin \Delta_3^G$ follow from similar arguments.

The lower bound argument can also be used when considering Λ_2 and $\Omega = \Omega_B, \Omega_{SA}, \Omega_N$ or Ω_g . We will also prove the lower bound $\{\Xi_3^L, \Omega_{SA}, \Lambda_1\} \notin \Delta_4^G$. The remaining lower bounds for Λ_1 follow from a similar argument and construction as in step 5 of the proof of Theorem 3.13 to ensure we are dealing with two-dimensional Lebesgue measure. Finally, we prove that $\{\Xi_3^L, \Omega_B, \Lambda_1\} \in \Pi_4^A$. The upper bounds for $\Omega = \Omega_{SA}, \Omega_N$ or Ω_g and Λ_1 follow an almost identical argument. When considering Λ_2 , we can collapse the first limit in the same manner as we did for solving Ξ_1^L .

Step 1 $\{\Xi_3^L, \Omega_f, \Lambda_1\} \in \Pi_3^A$. First we use the algorithm used to compute Ξ_1^L in Theorem 3.13, which we shall denote by $\tilde{\Gamma}$, to build a height 3 tower for $\{\Xi_3^L, \Omega_f\}$. As above, Ω_f denotes the set of bounded operators with the usual assumption of bounded dispersion (now with known bounds c_n). Recall that we observed

- $\tilde{\Gamma}_{n_2, n_1}(A)$ converges to $\tilde{\Gamma}_{n_2}(A)$ from below as $n_1 \rightarrow \infty$.
- $\tilde{\Gamma}_{n_2}(A)$ converges to $\text{Leb}(\text{Sp}(A))$ monotonically from above as $n_2 \rightarrow \infty$.

We can alter our algorithms, by taking maxima, so that we can assume without loss of generality that $\tilde{\Gamma}_{n_2, n_1}(A)$ converges to $\tilde{\Gamma}_{n_2}(A)$ *monotonically* from below as $n_1 \rightarrow \infty$. Now let

$$\Gamma_{n_3, n_2, n_1}(A) = \chi_{[0, 1/n_3]}(\tilde{\Gamma}_{n_2, n_1}(A)).$$

Note that $\chi_{[0, 1/n_3]}$ is left continuous on $[0, \infty)$ with right limits. Hence by the assumed monotonicity

$$\lim_{n_1 \rightarrow \infty} \Gamma_{n_3, n_2, n_1}(A) = \chi_{[0, 1/n_3]}(\tilde{\Gamma}_{n_2}(A)).$$

It follows that

$$\lim_{n_2 \rightarrow \infty} \lim_{n_1 \rightarrow \infty} \Gamma_{n_3, n_2, n_1}(A) = \chi_{[0, 1/n_3]}(\text{Leb}(\text{Sp}(A))_{\pm}),$$

where \pm denotes one of the right or left limits (it is possible to have either). It is then easy to see that

$$\lim_{n_3 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{n_1 \rightarrow \infty} \Gamma_{n_3, n_2, n_1}(A) = \Xi_3^L(A).$$

It is also clear that the answer to the question is “No” if $\Gamma_{n_3}(A) = 0$, which yields the Π_3^A classification.

Step 2 $\{\Xi_3^L, \Omega_D, \Lambda_1\} \notin \Delta_3^G$. Assume for a contradiction that this is false and $\{\hat{\Gamma}_{n_2, n_1}\}$ is a general height two tower for $\{\Xi_3^L, \Omega_D\}$. Let (\mathcal{M}, d) be discrete space $\{0, 1\}$ and $\tilde{\Omega}$ denote the collection of all infinite matrices $\{a_{i, j}\}_{i, j \in \mathbb{N}}$ with entries $a_{i, j} \in \{0, 1\}$ and consider the problem function

$\tilde{\Xi}_1(\{a_{i, j}\})$: Does $\{a_{i, j}\}$ have a column containing infinitely many nonzero entries?

For $j \in \mathbb{N}$, let $\{b_{i, j}\}_{i \in \mathbb{N}}$ be a dense subset of $I_j := [1 - 1/2^{j-1}, 1 - 1/2^j]$. Given a matrix $\{a_{i, j}\}_{i, j \in \mathbb{N}} \in \tilde{\Omega}$, construct a matrix $\{c_{i, j}\}_{i, j \in \mathbb{N}}$ by letting $c_{i, j} = a_{i, j} b_{r(i, j), j}$ where

$$r(i, j) = \max \left\{ 1, \sum_{k=1}^i a_{k, j} \right\}.$$

Now consider any bijection $\phi : \mathbb{N} \rightarrow \mathbb{N}^2$ and define the diagonal operator

$$A = \text{diag}(c_{\phi(1)}, c_{\phi(2)}, c_{\phi(3)}, \dots).$$

The algorithm $\widehat{\Gamma}_{n_2, n_1}$ thus translates to an algorithm Γ'_{n_2, n_1} for $\{\widetilde{\Xi}_1, \widetilde{\Omega}\}$. Namely, set $\Gamma'_{n_2, n_1}(\{a_{i,j}\}_{i \in \mathbb{N}}) = \widehat{\Gamma}_{n_2, n_1}(A)$. The fact that ϕ is a bijection shows that the lowest level Γ'_{n_2, n_1} are generalised algorithms (and are consistent). In particular, given N , we can find $\{A_{i,j} : i, j \leq N\}$ using finitely many evaluations of the matrix values $\{c_{k,l}\}$. But for any given $c_{k,l}$, we can evaluate this entry using only finitely many evaluations of the matrix values $\{a_{m,n}\}$ by the construction of r . Finally note that

$$\text{Sp}(A) = \left(\bigcup_{j: \sum_i a_{i,j} = \infty} I_j \right) \cup Q,$$

where Q is at most countable. Hence,

$$\text{Leb}_{\mathbb{R}}(\text{Sp}(A)) = \sum_{j: \sum_i a_{i,j} = \infty} \frac{1}{2^j}.$$

It follows that $\widetilde{\Xi}_1(\{a_{i,j}\}) = \Xi_3^L(A)$, and hence, we get a contradiction.

Step 3 $\{\Xi_3^L, \Omega_{\text{SA}}, \Lambda_1\} \notin \Delta_4^G$. Suppose for a contradiction that $\{\Gamma_{n_3, n_2, n_1}\}$ is a height three tower of general algorithms for the problem $\{\Xi_3^L, \Omega_{\text{SA}}, \Lambda_1\}$. Let (\mathcal{M}, d) be the space $\{0, 1\}$ with the discrete metric, let $\widetilde{\Omega}$ denote the collection of all infinite arrays $\{a_{m,i,j}\}_{m,i,j \in \mathbb{N}}$ with entries $a_{m,i,j} \in \{0, 1\}$ and consider the problem function

$$\begin{aligned} \widetilde{\Xi}_4(\{a_{m,i,j}\}) : & \text{For every } m, \text{ does } \{a_{m,i,j}\}_{i,j} \text{ have (only) finitely many columns} \\ & \text{with (only) finitely many 1's?} \end{aligned}$$

Recall that it is shown in Sect. 5 that $\text{SCI}(\widetilde{\Xi}_4, \widetilde{\Omega})_G = 4$. We will gain a contradiction by using the supposed height three tower to solve $\{\widetilde{\Xi}_4, \widetilde{\Omega}\}$.

The construction follows step 3 of the proof of Theorem 3.7 closely. For fixed m , recall the construction of the operator $A_m := A(\{a_{m,i,j}\}_{i,j})$ from that proof, the key property being that if $\{a_{m,i,j}\}_{i,j}$ has (only) finitely many columns with (only) finitely many 1's then $\text{Sp}(A_m)$ is a finite subset of $[-1, 1]$, otherwise it is the whole interval $[-1, 1]$. Now consider the intervals $I_m = [1 - 2^{m-1}, 1 - 2^m]$ and affine maps, α_m , that act as a bijection from $[-1, 1]$ to I_m . Without loss of generality, identify Ω_{SA} with self-adjoint operators in $\mathcal{B}(X)$ where $X = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{\infty} X_{i,j}$ in the l^2 -sense with $X_{i,j} = l^2(\mathbb{N})$. We then consider the operator

$$T(\{a_{m,i,j}\}_{m,i,j}) = \bigoplus_{m=1}^{\infty} \alpha_m(A_m).$$

The same arguments in the proof of Theorem 3.7 show that the map

$$\tilde{\Gamma}_{n_3, n_2, n_1}(\{a_{m,i,j}\}_{m,i,j}) = \Gamma_{n_3, n_2, n_1}(T(\{a_{m,i,j}\}_{m,i,j}))$$

defines a general tower using the relevant pointwise evaluation functions of the array $\{a_{m,i,j}\}_{m,i,j}$. If it holds that $\tilde{\Xi}_4(\{a_{m,i,j}\}) = 1$, then $\text{Sp}(T(\{a_{m,i,j}\}_{m,i,j}))$ is countable and hence $\Xi_3^L(T(\{a_{m,i,j}\}_{m,i,j})) = 1$. On the other hand, if $\tilde{\Xi}_4(\{a_{m,i,j}\}) = 0$, then there exists m with $\text{Sp}(A_m) = [-1, 1]$, and hence, $I_m \subset \text{Sp}(T(\{a_{m,i,j}\}_{m,i,j}))$ so that $\Xi_3^L(T(\{a_{m,i,j}\}_{m,i,j})) = 0$. It follows that $\{\tilde{\Gamma}_{n_3, n_2, n_1}\}$ provides a height three tower for $\{\tilde{\Xi}_4, \tilde{\Omega}\}$, a contradiction.

Step 4 $\{\Xi_3^L, \Omega_B, \Lambda_1\} \in \Pi_4^A$. Recall the tower of algorithms to solve $\{\Xi_1^L, \Omega_B, \Lambda_1\}$ and denote it by $\tilde{\Gamma}$. Our strategy will be the same as in step 1 but with an additional successive limit. It is easy to show that

- $\tilde{\Gamma}_{n_3, n_2, n_1}(A)$ converges to $\tilde{\Gamma}_{n_3, n_2}(A)$ from above as $n_1 \rightarrow \infty$.
- $\tilde{\Gamma}_{n_3, n_2}(A)$ converges to $\tilde{\Gamma}_{n_3}(A)$ from below as $n_2 \rightarrow \infty$.
- $\tilde{\Gamma}_{n_3}(A)$ converges to $\text{Leb}(\text{Sp}(A))$ from above as $n_3 \rightarrow \infty$.

Again, by taking successive maxima or minima where appropriate, we can assume that all of these are monotonic. Now let

$$\Gamma_{n_4, n_3, n_2, n_1}(A) = \chi_{[0, 1/n_4]}(\tilde{\Gamma}_{n_3, n_2, n_1}(A)).$$

Note that $\chi_{[0, 1/n_4]}$ is left continuous on $[0, \infty)$ with right limits. Hence by the assumed monotonicity and arguments as in step 1, it is easy to see that

$$\lim_{n_4 \rightarrow \infty} \lim_{n_3 \rightarrow \infty} \lim_{n_2 \rightarrow \infty} \lim_{n_1 \rightarrow \infty} \Gamma_{n_4, n_3, n_2, n_1}(A) = \Xi_3^L(A).$$

It is also clear that the answer to the question is “No” if $\Gamma_{n_4}(A) = 0$, which yields the Π_4^A classification. □

9 Proofs Concerning Fractal Dimensions

We begin with the box-counting dimension. For the construction of towers of algorithms, it is useful to use a slightly different but equivalent [71] definition of the upper and lower box-counting dimensions. Let $F \subset \mathbb{R}$ be bounded and $N'_\delta(F)$ denote the number of δ -mesh intervals that intersect F . A δ -mesh interval is an interval of the form $[m\delta, (m + 1)\delta]$ for $m \in \mathbb{Z}$. Then

$$\overline{\dim}_B(F) = \limsup_{\delta \downarrow 0} \frac{\log(N'_\delta(F))}{\log(1/\delta)}, \quad \underline{\dim}_B(F) = \liminf_{\delta \downarrow 0} \frac{\log(N'_\delta(F))}{\log(1/\delta)}.$$

Proof of Theorem 3.18 for box-counting dimension Since $\Omega_{BD}^D \subset \Omega_f^{BD} \subset \Omega_{SA}^{BD}$, it is enough to prove that $\{\Xi_B, \Omega_f^{BD}, \Lambda_1\} \in \Pi_2^A$, $\{\Xi_B, \Omega_{SA}^{BD}, \Lambda_2\} \in \Pi_2^A$, $\{\Xi_B, \Omega_{SA}^{BD}, \Lambda_1\} \in \Pi_3^A$, $\{\Xi_B, \Omega_{SA}^{BD}, \Lambda_1\} \notin \Delta_3^A$ and $\{\Xi_B, \Omega_D^{BD}, \Lambda_2\} \notin \Delta_2^A$.

Step 1 $\{\Xi_B, \Omega_f^{BD}, \Lambda_1\} \in \Pi_2^A$. Recall the existence of a height one tower, $\{\tilde{\Gamma}_n\}$, using Λ_1 for $\text{Sp}(A)$, $A \in \Omega_f^{BD}$ from “Appendix A”. Furthermore, $\tilde{\Gamma}_n(A)$ outputs a finite collection $\{z_{1,n}, \dots, z_{k_n,n}\} \subset \mathbb{Q}$ such that $\text{dist}(z_{j,n}, \text{Sp}(A)) \leq 2^{-n}$. Define the intervals

$$I_{j,n} = [z_{j,n} - 2^{-n}, z_{j,n} + 2^{-n}]$$

and let \mathcal{I}_m denote the collection of all 2^{-m} -mesh intervals. Let $\Upsilon_{m,n}(A)$ be any union of finitely many such mesh intervals with minimal length $|\Upsilon_{m,n}(A)|$ (“length” being the number of intervals $\in \mathcal{I}_m$ that make up $\Upsilon_{m,n}(A)$) such that

$$\Upsilon_{m,n}(A) \cap I_{j,l} \neq \emptyset, \quad \text{for } 1 \leq l \leq n, 1 \leq j \leq k_l.$$

There may be more than one such collection, so we can gain a deterministic algorithm by enumerating each \mathcal{I}_m and choosing the first such collection in this enumeration. It is then clear that $|\Upsilon_{m,n}(A)|$ is increasing in n . Furthermore, to determine $\Upsilon_{m,n}(A)$, there are only finitely many intervals in \mathcal{I}_m to consider, namely those that have non-empty intersection with at least one $I_{j,l}$ with $1 \leq l \leq n, 1 \leq j \leq k_l$. It follows that $\Upsilon_{m,n}(A)$ and hence $|\Upsilon_{m,n}(A)|$ can be computed in finitely many arithmetic operations and comparisons using Λ_1 .

Suppose that $I = [a, b] \in \mathcal{I}_m$ has $(a, b) \cap \text{Sp}(A) \neq \emptyset$. Then for large n there exists $z_{j,n} \in I$ such that $I_{j,n} \subset I$ and hence $I \subset \Upsilon_{m,n}(A)$ for large n . If $z \in \text{Sp}(A) \cap 2^{-m}\mathbb{Z}$, then a similar argument shows that $z \subset \Upsilon_{m,n}(A)$ for large n . Since $\text{Sp}(A)$ is bounded and $\text{Sp}(A) \cap 2^{-m}\mathbb{Z}$ finite, it follows that $\text{Sp}(A) \subset \Upsilon_{m,n}(A)$ for large n and hence

$$N_{2^{-m}}(\text{Sp}(A)) \leq \liminf_{n \rightarrow \infty} |\Upsilon_{m,n}(A)|.$$

Let $W_m(A)$ be the union of all intervals in \mathcal{I}_m that intersect $\text{Sp}(A)$. It is clear that $W_m(A) \cap I_{j,l} \neq \emptyset$ for $1 \leq l \leq n, 1 \leq j \leq k_l$, and hence, $|\Upsilon_{m,n}(A)| \leq N'_{2^{-m}}(\text{Sp}(A))$. It follows that $\lim_{n \rightarrow \infty} |\Upsilon_{m,n}(A)| = \delta_m(A)$ exists with

$$N_{2^{-m}}(\text{Sp}(A)) \leq \delta_m(A) \leq N'_{2^{-m}}(\text{Sp}(A)). \tag{9.1}$$

For $n_2 > n_1$ set $\Gamma_{n_2,n_1}(A) = 0$, otherwise set

$$\Gamma_{n_2,n_1}(A) = \max_{n_2 \leq k \leq n_1} \max_{1 \leq j \leq n_1} \frac{\log(|\Upsilon_{k,j}(A)|)}{k \log(2)}.$$

The above monotone convergence and (9.1) shows that

$$\begin{aligned} \lim_{n_1 \rightarrow \infty} \Gamma_{n_2,n_1}(A) &= \Gamma_{n_2}(A) = \sup_{k \geq n_2} \frac{\log(\delta_k(A))}{k \log(2)} \geq \limsup_{k \rightarrow \infty} \frac{\log(\delta_k(A))}{k \log(2)}, \\ \lim_{n_2 \rightarrow \infty} \Gamma_{n_2}(A) &= \limsup_{k \rightarrow \infty} \frac{\log(\delta_k(A))}{k \log(2)}. \end{aligned}$$

Hence, by the assumption that the box-counting dimension exists, we have constructed a Π_2^A tower.

Step 2 $\{\Xi_B, \Omega_{SA}^{BD}, \Lambda_2\} \in \Pi_2^A$ and $\{\Xi_B, \Omega_{SA}^{BD}, \Lambda_1\} \in \Pi_3^A$. The first of these is exactly as in step 1, using Λ_2 to construct the relevant Σ_1^A tower for the spectrum. The proof that $\{\Xi_B, \Omega_{SA}^{BD}, \Lambda_1\} \in \Pi_3^A$ uses a height two tower, $\{\tilde{\Gamma}_{n_2, n_1}\}$, using Λ_1 for $\text{Sp}(A)$, $A \in \Omega_{SA}^{BD}$ (or any self-adjoint A) constructed in [18]. This tower has the property that each $\tilde{\Gamma}_{n_2, n_1}(A)$ is a finite subset of \mathbb{Q} and, for fixed n_2 , is constant for large n_1 . Moreover, if $z \in \lim_{n_1 \rightarrow \infty} \tilde{\Gamma}_{n_2, n_1}(A)$, then $\text{dist}(z, \text{Sp}(A)) \leq 2^{-n_2}$. It follows that we can use the same construction as step 1 with an additional limit at the start to reach the finite set $\lim_{n_1 \rightarrow \infty} \tilde{\Gamma}_{n_2, n_1}(A)$.

Step 3 $\{\Xi_B, \Omega_D^{BD}, \Lambda_2\} \notin \Delta_2^A$. This is exactly the same argument as step 2 of the proof of Theorem 3.13 with Lebesgue measure replaced by box-counting dimension.

Step 4 $\{\Xi_B, \Omega_{SA}^{BD}, \Lambda_1\} \notin \Delta_3^A$. This is exactly the same argument as step 4 of the proof of Theorem 3.13 with Lebesgue measure replaced by box-counting dimension. □

We now turn to the Hausdorff dimension. Recall Lemma 3.21 on the problem of determining whether $\text{Sp}(A) \cap (a, b) \neq \emptyset$.

Proof of Lemma 3.21 We start with the class $\Omega_f \cap \Omega_{SA}$. We can interpret this problem as a decision problem and the following algorithm as one that halts on output “Yes”. Let $c = (a + b)/2$ and $\delta = (b - a)/2$, then the idea is to simply test whether $\text{DistSpec}(A, n, c, f(n)) + c_n < \delta$. If the answer is yes, then we output “Yes”, otherwise we output “No” and increase n by one. Note that $\text{Sp}(A) \cap (a, b) \neq \emptyset$ if and only if $\|R(c, A)\|^{-1} < \delta$ and hence as $\text{DistSpec}(A, n, c, f(n)) + c_n$ converges down to $\|R(c, A)\|^{-1}$ we see that this provides a convergent algorithm. For Ω_{SA} , we require an additional successive limit by replacing $\text{DistSpec}(A, n, c, f(n)) + c_n$ with the function $\gamma_{n_2, n_1}(z; A)$. If we have access to Λ_2 , then this can be avoided in the usual way. □

To build our algorithm for the Hausdorff dimension, we use an alternative, equivalent definition for compact sets. We consider the case of subsets of \mathbb{R} . Let ρ_k denote the set of all closed binary intervals of the form $[2^{-k}m, 2^{-k}(m + 1)]$, $m \in \mathbb{Z}$. Set

$$\mathcal{A}_k(F) = \left\{ \{U_i\}_{i \in I} : I \text{ is finite, } F \subset \cup_{i \in I} U_i, U_i \in \cup_{l \geq k} \rho_l \right\}$$

and define

$$\tilde{\mathcal{H}}_k^d(F) = \inf \left\{ \sum_i \text{diam}(U_i)^d : \{U_i\}_{i \in I} \in \mathcal{A}_k(F) \right\}, \quad \tilde{\mathcal{H}}^d(F) = \lim_{k \rightarrow \infty} \tilde{\mathcal{H}}_k^d(F).$$

The following can be found in [81] (Theorem 3.13):

Theorem 9.1 *Let F be a bounded subset of \mathbb{R} . Then, there exists a unique $d' = \dim_{H'}(F)$ such that $\tilde{\mathcal{H}}^d(F) = 0$ for $d > d'$ and $\tilde{\mathcal{H}}^d(F) = \infty$ for $d < d'$. Furthermore, $d' = \dim_H(\bar{F})$.*

Denoting the dyadic rationals by \mathbb{D} , we shall compute $\dim_H(\text{Sp}(A))$ via approximating the above applied to $F = \text{Sp}(A) \cap \mathbb{D}^c$ and using Lemma 3.21.

Proof of Theorem 3.18 for Hausdorff dimension It is enough to prove the lower bounds $\{\Xi_H, \Omega_D, \Lambda_2\} \notin \Delta_3^G$, $\{\Xi_H, \Omega_{SA}, \Lambda_1\} \notin \Delta_4^G$ and construct the towers of algorithms for the inclusions $\{\Xi_H, \Omega_f \cap \Omega_{SA}, \Lambda_1\} \in \Sigma_3^A$, $\{\Xi_H, \Omega_{SA}, \Lambda_1\} \in \Sigma_4^A$ and $\{\Xi_H, \Omega_{SA}, \Lambda_2\} \in \Sigma_3^A$.

Step 1 $\{\Xi_H, \Omega_D, \Lambda_2\} \notin \Delta_3^G$. Suppose for a contradiction that a height two tower, $\{\Gamma_{n_2, n_1}\}$, exists for $\{\Xi_H, \Omega_D\}$ (taking values in $[0, 1]$ without loss of generality). We repeat the argument in the proof of Theorem 3.16. Consider the same problem

$\tilde{\Xi}_1(\{a_{i,j}\})$: Does $\{a_{i,j}\}$ have a column containing infinitely many nonzero entries?

However, now we consider the above mapping to $[0, 1]$ with the usual metric. We consider the same operator $A = \text{diag}(c_{\phi(1)}, c_{\phi(2)}, c_{\phi(3)}, \dots)$ with

$$\text{Sp}(A) = \left(\bigcup_{j: \sum_i a_{i,j} = \infty} I_j \right) \cup Q,$$

where Q is at most countable. We use the fact that the Hausdorff dimension satisfies

$$\dim_H(\bigcup_{j=1}^{\infty} X_j) = \sup_{j \in \mathbb{N}} \dim_H(X_j)$$

and that $\dim_H(Q) = 0$ for any countable Q to note that $\Xi_H(A) = \tilde{\Xi}_1(\{a_{i,j}\})$. We set $\tilde{\Gamma}_{n_2, n_1}(\{a_{i,j}\}_{i,j}) = \Gamma_{n_2, n_1}(A)$ to provide a height two tower for $\tilde{\Xi}_1$. But this contradicts Theorem 5.19.

Step 2 $\{\Xi_H, \Omega_{SA}, \Lambda_1\} \notin \Delta_4^G$. Suppose for a contradiction that $\{\Gamma_{n_3, n_2, n_1}\}$ is a height three tower of general algorithms for the problem $\{\Xi_H, \Omega_{SA}, \Lambda_1\}$ (taking values in $[0, 1]$ without loss of generality). Let (\mathcal{M}, d) be the space $[0, 1]$ with the usual metric, let $\tilde{\Omega}$ denote the collection of all infinite arrays $\{a_{m,i,j}\}_{m,i,j \in \mathbb{N}}$ with entries $a_{m,i,j} \in \{0, 1\}$ and consider the problem function

$\tilde{\Xi}_4(\{a_{m,i,j}\})$: For every m , does $\{a_{m,i,j}\}_{i,j}$ have (only) finitely many columns with (only) finitely many 1's?

Recall that it is shown in Sect. 5 that $\text{SCI}(\tilde{\Xi}_4, \tilde{\Omega})_G = 4$. We will gain a contradiction by using the supposed height three tower to solve $\{\tilde{\Xi}_4, \tilde{\Omega}\}$. We use the same construction as in step 3 of the proof of Theorem 3.16. If $\tilde{\Xi}_4(\{a_{m,i,j}\}) = 1$, then $\text{Sp}(T(\{a_{m,i,j}\}_{m,i,j}))$ is countable, and hence, $\Xi_H(T(\{a_{m,i,j}\}_{m,i,j})) = 0$. On the other hand, if $\tilde{\Xi}_4(\{a_{m,i,j}\}) = 0$, then there exists m with $\text{Sp}(A_m) = [-1, 1]$ and hence $I_m \subset \text{Sp}(T(\{a_{m,i,j}\}_{m,i,j}))$ so that $\Xi_H(T(\{a_{m,i,j}\}_{m,i,j})) = 1$. It follows that $\tilde{\Gamma}_{n_3, n_2, n_1}(\{a_{m,i,j}\}_{m,i,j}) = 1 - \Gamma_{n_3, n_2, n_1}(T(\{a_{m,i,j}\}_{m,i,j}))$ provides a height three tower for $\{\tilde{\Xi}_4, \tilde{\Omega}\}$, a contradiction.

Step 3 $\{\Xi_H, \Omega_f \cap \Omega_{SA}, \Lambda_1\} \in \Sigma_3^A$. To construct a height three tower for $A \in \Omega_f \cap \Omega_{SA}$, if $n_2 < n_3$ set $\Gamma_{n_3, n_2, n_1}(A) = 0$. Otherwise, consider the set

$$\mathcal{A}_{n_3, n_2, n_1}(A) = \left\{ \{U_i\}_{i \in I} : I \text{ is finite, } S_{n_1, n_2}(A) \subset \cup_{i \in I} U_i, U_i \in \cup_{n_3 \leq l \leq n_2} \rho_l \right\}$$

where $S_{n_1, n_2}(A)$ is the union of all $S \in \rho_{n_2}$ with $S \subset [-n_1, n_1]$ and such that the algorithm discussed in Lemma 3.21 outputs “Yes” for the interior of S and input parameter n_1 . We then define

$$h_{n_3, n_2, n_1}(A, d) = \inf \left\{ \sum_i \text{diam}(U_i)^d : \{U_i\} \in \mathcal{A}_{n_3, n_2, n_1}(A) \right\}.$$

If $S_{n_1, n_2}(A)$ is empty, then we interpret the infimum as 0. There are only finitely many sets to check and hence the infimum is a minimisation problem over finitely many coverings (see § B.4 for a discussion of efficient implementation). It follows that $h_{n_3, n_2, n_1}(A, d)$ defines a general algorithm computable in finitely many arithmetic operations and comparisons. Furthermore, it is easy to see that

$$\lim_{n_1 \rightarrow \infty} h_{n_3, n_2, n_1}(A, d) = \inf \left\{ \sum_i \text{diam}(U_i)^d : \{U_i\} \in \mathcal{C}_{n_3, n_2}(A) \right\} =: h_{n_3, n_2}(A, d)$$

from below (since we are covering larger sets as n_1 increases). Here,

$$\mathcal{C}_{n_3, n_2}(A) = \left\{ \{U_i\}_{i \in I} : I \text{ is finite, } \text{Sp}(A) \cap \mathbb{D}_{n_2}^c \subset \cup_{i \in I} U_i, U_i \in \cup_{n_3 \leq l \leq n_2} \rho_l \right\}$$

and $\mathbb{D}_k := 1/2^k \cdot \mathbb{Z}$ denotes the dyadic rationals of resolution k . We now use the property that $\mathcal{A}_k(F)$ consists of collections of finite coverings. As $n_2 \rightarrow \infty$, $h_{n_3, n_2}(A, d)$ is non-increasing (since we take infimum over a larger class of coverings and the sets $\text{Sp}(A) \cap \mathbb{D}_{n_2}^c$ decrease) and hence converges to some number. Clearly

$$\lim_{n_2 \rightarrow \infty} h_{n_3, n_2}(A, d) =: h_{n_3}(A, d) \geq \tilde{\mathcal{H}}_{n_3}^d(\text{Sp}(A) \cap \mathbb{D}^c).$$

For $\epsilon > 0$, let $l \in \mathbb{N}$ and $\{U_i\} \in \mathcal{A}_{n_3}(\text{Sp}(A) \cap \mathbb{D}_l^c)$ with

$$\sum_i \text{diam}(U_i)^d \leq \epsilon + \tilde{\mathcal{H}}_{n_3}^d(\text{Sp}(A) \cap \mathbb{D}_l^c).$$

For large enough n_2 , $\{U_i\} \in \mathcal{C}_{n_3, n_2}(A)$ and hence since $\epsilon > 0$ was arbitrary,

$$h_{n_3}(A, d) \leq \tilde{\mathcal{H}}_{n_3}^d(\text{Sp}(A) \cap \mathbb{D}_l^c)$$

for all l . For a fixed A and d , $h_{n_3}(A, d)$ is non-decreasing in n_3 and hence converges to a function of d , $h(A, d)$ (possibly taking infinite values). Furthermore,

$$\tilde{\mathcal{H}}^d(\text{Sp}(A) \cap \mathbb{D}^c) \leq h(A, d) \leq \tilde{\mathcal{H}}^d(\text{Sp}(A) \cap \mathbb{D}_l^c).$$

Since the set $\text{Sp}(A) \cap \mathbb{D}$ is countable, its Hausdorff dimension is zero. Using sub-additivity of Hausdorff dimension and Theorem 9.1,

$$\begin{aligned} \dim_H(\text{Sp}(A)) &\leq \dim_H(\text{Sp}(A) \cap \mathbb{D}^c) \\ &\leq \dim_H(\overline{\text{Sp}(A) \cap \mathbb{D}^c}) = \dim_{H'}(\text{Sp}(A) \cap \mathbb{D}^c) \\ &\leq \dim_H(\overline{\text{Sp}(A) \cap \mathbb{D}_f^c}) = \dim_{H'}(\text{Sp}(A) \cap \mathbb{D}_f^c) \\ &\leq \dim_H(\text{Sp}(A)). \end{aligned}$$

It follows that $h(A, d) = 0$ if $d > \dim_H(\text{Sp}(A))$ and that $h(A, d) = \infty$ if $d < \dim_H(\text{Sp}(A))$. Define

$$\Gamma_{n_3, n_2, n_1}(A) = \sup_{j=1, \dots, 2^{n_3}} \left\{ \frac{j}{2^{n_3}} : h_{n_3, n_2, n_1}(A, k/2^{n_3}) + \frac{1}{n_2} > \frac{1}{2} \text{ for } k = 1, \dots, j \right\},$$

where in this case we define the maximum over the empty set to be 0.

Consider $n_2 \geq n_3$. Since $h_{n_3, n_2, n_1}(A, d) \uparrow h_{n_3, n_2}(A, d)$, it is clear that

$$\begin{aligned} \lim_{n_1 \rightarrow \infty} \Gamma_{n_3, n_2, n_1}(A) &= \sup_{j=1, \dots, 2^{n_3}} \left\{ \frac{j}{2^{n_3}} : h_{n_3, n_2}(A, k/2^{n_3}) + \frac{1}{n_2} > \frac{1}{2} \text{ for } k = 1, \dots, j \right\} \\ &=: \Gamma_{n_3, n_2}(A). \end{aligned}$$

If $h_{n_3}(A, d) \geq 1/2$, then $h_{n_3, n_2}(A, d) + 1/n_2 > 1/2$ for all n_2 otherwise $h_{n_3, n_2}(A, d) + 1/n_2 < 1/2$ eventually. Hence,

$$\lim_{n_2 \rightarrow \infty} \Gamma_{n_3, n_2}(A) = \sup_{j=1, \dots, 2^{n_3}} \left\{ \frac{j}{2^{n_3}} : h_{n_3}(A, k/2^{n_3}) \geq \frac{1}{2} \text{ for } k = 1, \dots, j \right\} =: \Gamma_{n_3}(A).$$

Using the monotonicity of $h_{n_3}(A, d)$ in d and the proven properties of the limit function h , it follows that

$$\lim_{n_3 \rightarrow \infty} \Gamma_{n_3}(A) = \dim_H(\text{Sp}(A)).$$

The fact that h_{n_3} is non-decreasing in n_3 , the set $\{1/2^{n_3}, 2/2^{n_3}, \dots, 1\}$ refines itself, and the stated monotonicity collectively shows that convergence is monotonic from below, and hence, we get the Σ_3^A classification.

Step 4 $\{\Xi_H, \Omega_{SA}, \Lambda_1\} \in \Sigma_4^A$ and $\{\Xi_H, \Omega_{SA}, \Lambda_2\} \in \Sigma_3^A$. The first of these can be proven as in step 3 by replacing (n_1, n_2, n_3) by (n_2, n_3, n_4) and the set $S_{n_2, n_1}(A)$ by the set $S_{n_3, n_2, n_1}(A)$ given by the union of all $S \in \rho_{n_3}$ with $S \subset [-n_2, n_2]$ and such that the Σ_2^A tower of algorithms discussed in Lemma 3.21 outputs ‘‘Yes’’ for the interior of S and input parameters (n_2, n_1) . To prove $\{\Xi_H, \Omega_{SA}, \Lambda_2\} \in \Sigma_3^A$, we use exactly the same construction as in step 3 now using the Σ_1^A algorithm (which uses Λ_2) given by Lemma 3.21. □

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Appendix A. Routines for Computing Spectra

We describe the SCI-sharp Σ_1^A algorithms in [64] and [60], that are used in some of our proofs. In this section, we consider the problem functions $\Xi_1(A) = \text{Sp}(A)$ and $\Xi_2(A) = \text{Sp}_\epsilon(A)$, taking values in the space of non-empty compact subsets of \mathbb{C} equipped with Hausdorff metric. The definitions of the classes Ω_g and Ω_f can be found in Sect. 2. As written, the outputs of the algorithms below may be empty for small n (and hence not lie in the correct metric space). This does not affect the classifications and can be avoided by computing successive $\Gamma_n(A)$ and outputting $\Gamma_{m(n)}(A)$ where $m(n) \geq n$ is minimal with $\Gamma_{m(n)}(A) \neq \emptyset$.

The methods in [64] and [60] use the function f to approximate the function

$$\gamma_n(z; A) = \min\{\sigma_{\text{inf}}((A - zI)|_{P_n\mathcal{H}}), \sigma_{\text{inf}}((A^* - \bar{z}I)|_{P_n\mathcal{H}})\}, \tag{A.1}$$

where P_m denotes the orthogonal projection onto the linear span of the first m basis vectors and σ_{inf} denotes the injection modulus. As $n \rightarrow \infty$, the functions γ_n converge uniformly on compact subsets down to the continuous function $\gamma(z; A) = \|R(z, A)\|^{-1}$, which we interpret as zero if the resolvent $R(z, A) = (A - zI)^{-1}$ does not exist as a bounded operator. The function f and sequence $\{c_n\}$ allow us to approximate γ_n to any given precision. To use this to compute the spectrum, we need some control on how the resolvent norm diverges near the spectrum and this is provided by the function g satisfying (2.2). At various points in this paper, we have also made use of the related functions

$$\gamma_{n,m}(z; A) = \min\{\sigma_{\text{inf}}(P_m(A - zI)|_{P_n\mathcal{H}}), \sigma_{\text{inf}}(P_m(A^* - \bar{z}I)|_{P_n\mathcal{H}})\}. \tag{A.2}$$

These can be computed from the rectangular matrices $P_m(A - zI)P_n$, $P_m(A - zI)^*P_n$ and converge uniformly on compact subsets of \mathbb{C} to γ_n as $m \rightarrow \infty$.

Algorithm 1: The subroutine `IsPosDef` checks whether a matrix is positive definite and is a standard routine that can be implemented in a myriad of ways. In practice, the while loop in `DistSpec` is replaced by a much more efficient interval bisection method. An alternative method for sparse matrices (which, however, does not rigorously guarantee an error bound on the smallest singular values) is to compute the smallest singular values of the rectangular matrices using iterative methods. See the supplementary material of [64] for further discussion on efficient numerical computation. Note also that when evaluating `DistSpec` for different z , the computation can be done in parallel.

Function `DistSpec` ($A, n, z, f(n)$)

Input : $n \in \mathbb{N}$, $f(n) \in \mathbb{N}$, matrix A , $z \in \mathbb{C}$

Output: $y \in \mathbb{R}_+$, an approximation to the function $z \mapsto \|R(z, A)\|^{-1}$

$B = (A - zI)(1 : f(n), 1 : n)$; $C = (A - zI)^*(1 : f(n), 1 : n)$

$S = B^*B$; $T = C^*C$

$v = 1, l = 0$

while $v = 1$ **do**

$l = l + 1$

$p = \text{IsPosDef}(S - \frac{l^2}{n^2})$; $q = \text{IsPosDef}(T - \frac{l^2}{n^2})$

$v = \min(p, q)$

end

$y = \frac{l}{n}$

end

Throughout, we use that `DistSpec` requires only finitely many arithmetic operations and comparisons, as proven in [60] (one can perform the `IsPosDef` routine using incomplete Cholesky decompositions). Furthermore, as outlined in Remark 5.12, we can make all of the algorithms in this paper and those in this appendix work using Δ_1 -information and restricting to arithmetical operations over the rationals.

Algorithm 2: The routine `CompSpec` computes spectra of bounded operators (see [60] for extensions to unbounded operators) on $l^2(\mathbb{N})$ (or, more generally, graphs) using the subroutines `CompInvg` and `DistSpec` described above, and provides Σ_1^A error control (without loss of generality by taking subsequences until the computed error is below a user specified tolerance).

```

Function CompInvg ( $n, y, g$ )
    Input :  $n \in \mathbb{N}$ ,  $y \in \mathbb{R}_+$ ,  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ 
    Output:  $m \in \mathbb{R}_+$ , an approximation to  $g^{-1}(y)$ 
     $m = \min\{k/n : k \in \mathbb{N}, g(k/n) > y\}$ 
end

Function CompSpec ( $A, n, g, f(n), c_n$ )
    Input :  $n \in \mathbb{N}$ ,  $f(n) \in \mathbb{N}$ ,  $c_n \in \mathbb{R}_+$  (bound on dispersion),  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  $A \in \Omega_f \cap \Omega_g$ 
    Output:  $\Gamma_n(A) \subset \mathbb{C}$ , an approximation to  $\text{Sp}(A)$ ,  $E_n(A) \in \mathbb{R}_+$ , the error estimate
     $G = \frac{1}{n}(\mathbb{Z} + i\mathbb{Z}) \cap B_n(0)$ 
    for  $z \in G$  do
         $F(z) = \text{DistSpec}(A, n, z, f(n))$ 
        if  $F(z) \leq (|z|^2 + 1)^{-1}$  then
            for  $w_j \in B_{\text{CompInvg}(n, F(z), g)}(z) \cap G = \{w_1, \dots, w_k\}$  do
                 $F_j = \text{DistSpec}(A, n, w_j, f(n))$ 
            end
             $M_z = \{w_j : F_j = \min_q \{F_q\}\}$ 
        else
             $M_z = \emptyset$ 
        end
    end
     $\Gamma_n(A) = \bigcup_{z \in G} M_z$ 
     $E_n(A) = \max_{z \in \Gamma_n(A)} \{\text{CompInvg}(n, \text{DistSpec}(A, n, z, f(n)) + c_n, g)\}$ 
end
    
```

Algorithm 3: `PseudoSpec` computes $\Gamma_n(A) \subset \text{Sp}_\epsilon(A)$ with $\lim_{n \rightarrow \infty} \Gamma_n(A) = \text{Sp}_\epsilon(A)$.

```

Function PseudoSpec ( $A, n, f(n), c_n, \epsilon$ )
    Input :  $n, f(n) \in \mathbb{N}$ ,  $c_n \in \mathbb{R}_+^{\mathbb{N}}$ ,  $A \in \Omega_f$ ,  $\epsilon > 0$ 
    Output:  $\Gamma \subset \mathbb{C}$ , an approximation to  $\text{Sp}_\epsilon(A)$ 
     $G = \text{Grid}(n)$ 
     $m = \min\{k \geq n \mid c_k < \epsilon\}$ 
    for  $z \in G$  do
         $B = (A - zI)(1 : f(m), 1 : m)$ ;  $C = (A - zI)^*(1 : f(m), 1 : m)$ 
         $S = B^*B$ ;  $T = C^*C$ 
         $p = \text{IsPosDef}(S - (\epsilon - c_m)^2)$ ;  $q = \text{IsPosDef}(T - (\epsilon - c_m)^2)$ 
         $v(z) = \min(p, q)$ 
    end
     $\Gamma = \bigcup \{z \in G \mid v(z) = 0\}$ 
end
    
```

Appendix B. Examples of Computational Routines

We provide short and simplified routines for some of the algorithms in this paper. For example, we have ignored issues like the rigorous approximation of the function $\gamma_{n,m}$ in (A.2) using arithmetical operations. For brevity, we stick to one domain Ω and the evaluation set Λ_1 (matrix values) for each problem function Ξ . In each case, we have chosen the non-trivial Ω with the simplest algorithm. For the different algorithms for different classes of operators, see the proofs. In general, different classes of operators and evaluation sets have different SCI classifications and different algorithms for the same problem function.

B.1. Spectral Radii, Capacity and Operator Norms

For the problem functions in Sects. 3.1–3.3, we consider Ω_f (see (2.1)) and $\Omega_f \cap \Omega_{SA}$ for computing the capacity of the spectrum.

Algorithm 4: `SpecRad` computes the spectral radius of operators in Ω_f using the algorithm for computing pseudospectra, `PseudoSpec`, which is parallelisable and provides Σ_1^A error control.

Function `SpecRad` ($n_1, n_2, f(n_1), c_{n_1}, A$)
Input : $n_1, n_2, f(n_1) \in \mathbb{N}, c_{n_1} \in \mathbb{R}_+, A \in \Omega_f$
Output: $\Gamma_{n_2, n_1}(A)$, a Π_2^A approximation of $r(A)$
 $S = \text{PseudoSpec}(A, n_1, f(n_1), c_{n_1}, n_2^{-1}) = \{z_1, \dots, z_m\}$
 $\Gamma_{n_2, n_1}(A) = \sup_{1 \leq j \leq m} |z_j|$
end

Algorithm 5: `EssSpecRad` computes the essential spectral radius of operators in Ω_f using the algorithm for computing essential spectra, `EssSpec`, from [18].

Function `EssSpecRad` ($n_1, n_2, f(n_1), c_{n_1}, A$)
Input : $n_1, n_2, f(n_1) \in \mathbb{N}, c_{n_1} \in \mathbb{R}_+, A \in \Omega_f$
Output: $\Gamma_{n_2, n_1}(A)$, a Π_2^A approximation of $\Xi_{er}(A)$
 $S = \text{EssSpec}(A, n_1, n_2, f(n_1), c_{n_1}) = \cup_{j=1}^m R_j$
 NB: R_j are rectangles with complex rational vertices.
 $\Gamma_{n_2, n_1}(A) = \frac{1}{2^{n_2}} + \sup_{1 \leq j \leq m} \max_{z \in R_j} |z|$
end

Algorithm 6: PolyNorm computes the operator norm of $p(A)$ for operators $A \in \Omega_f$ and polynomials p . The powers of A can be computed through “lazy evaluation” (when one computes with infinite data structures, but defers the use of the information until needed) and the function f .

Function PolyNorm($p, n, f(n), c_n, A$)

Input : polynomial $p, n, f(n) \in \mathbb{N}, c_n \in \mathbb{R}_+, A \in \Omega_f$

Output: $\Gamma_n(A)$, a Σ_1^A approximation of $\|p(A)\|$

Compute $\hat{B}_n \approx B_n = P_n p(A) P_n \in \mathbb{C}^{n \times n}$ using f to compute matrix entries of powers of A .

Compute an upper bound δ_n of $\|\hat{B}_n - B_n\|$.

(Do the above so that δ_n is bounded by a null sequence.)

$\Gamma_n(A) = \|\hat{B}_n\| - \delta_n$

end

Algorithm 7: CapSpec computes $\text{cap}(\text{Sp}(A))$ for operators $A \in \Omega_f \cap \Omega_{\text{SA}}$. The capacity of a finite union of intervals can be computed using conformal mappings. The computation of \mathcal{I}_{n_1, n_2} requires applications of DistSpec which can be performed in parallel.

Function CapSpec($n_1, n_2, f(n_1), c_{n_1}, A$)

Input : $n_1, n_2, f(n_1) \in \mathbb{N}, c_{n_1} \in \mathbb{R}_+, A \in \Omega_f \cap \Omega_{\text{SA}}$

Output: $\Gamma_{n_2, n_1}(A)$, a Π_2^A approximation of the capacity, cap , of $\text{Sp}(A)$

Form a disjoint covering of $[-n_1, n_1]$ into intervals $I_j^{n_1, n_2}, j = 1, \dots, n_1 2^{n_2+1}$, of length 2^{-n_2}

Use Lemma 3.21 with $n = n_1$ to compute $\mathcal{I}_{n_1, n_2} \uparrow \{j : \text{interior}(I_j^{n_1, n_2}) \cap \text{Sp}(A) \neq \emptyset\}$

$\Gamma_{n_2, n_1}(A) = \text{cap}\left(\bigcup_{j \in \mathcal{I}_{n_1, n_2}} \overline{I_j^{n_1, n_2}}\right)$

end

B.2. Essential Numerical Range, Gaps in Essential Spectra and Detecting Algorithm Failure for Finite Section

For the problems in Sect. 3.4, we consider Ω_B .

Algorithm 8: `EssNumRange` computes the essential numerical range for operators $A \in \Omega_B$ (see §7.1 for unbounded operators). The numerical range of a finite square matrix can be approximated to arbitrary accuracy using finitely many arithmetic operations and comparisons. In practice, one can use the method of Johnson [96], which reduces the computation of $\partial W(B)$ for $B \in \mathbb{C}^{n \times n}$ to a series of $n \times n$ Hermitian (extremal) eigenvalue problems.

Function `EssNumRange` (n_1, n_2, A)

Input : $n_1, n_2 \in \mathbb{N}, A \in \Omega_B$

Output: $\Gamma_{n_2, n_1}(A)$, a Π_2^A approximation of $W_e(A)$

$B_{n_2, n_1} = (I - P_{n_2})P_{n_1+n_2}A|_{P_{n_1+n_2}(I - P_{n_2})\mathcal{H}} \in \mathbb{C}^{n_1 \times n_1}$

$\Gamma_{n_2, n_1}(A) = W(B_{n_2, n_1})$

end

Algorithm 9: `SpecPoll` computes $\Xi_{poll}^{\mathbb{C}}(A, U)$ for operators $A \in \Omega_B$ and open sets U (given as a, possibly countably infinite, union of open balls $\{U_m\}$ with rational radii and centres). The function γ_{n_2, n_1} is the same as in (A.2).

Function `SpecPoll` (n_1, n_2, n_3, A, U)

Input : $n_1, n_2, n_3 \in \mathbb{N}, A \in \Omega_B$, open set U

Output: $\Gamma_{n_3, n_2, n_1}(A, U)$, a Σ_3^A approximation of $\Xi_{poll}^{\mathbb{C}}(A, U)$

$S_{n_2, n_1} = \text{EssNumRange}(n_1, n_2, A) = \{z_1, \dots, z_m\}$

NB: We use the version of `EssNumRange` that outputs a finite collection of points.

$V_{n_1} = \bigcup_{j=1}^{n_1} U_j$

$\Upsilon_{n_2, n_1} = \{z \in S_{n_2, n_1} : \text{dist}(z, V_{n_1}) < n_2^{-1} - n_1^{-1}\}$

if $\Upsilon_{n_2, n_1} \neq \emptyset$ **then**

$Q_{n_2, n_1} = \max_{z \in \Upsilon_{n_2, n_1}} \gamma_{n_2, n_1}(z; A) - n_1^{-1}$

else

$Q_{n_2, n_1} = 0$

end

if $Q_{n_2, n_1} \leq n_3^{-1}$ **then**

$\Gamma_{n_3, n_2, n_1}(A, U) = 0$

else

$\Gamma_{n_3, n_2, n_1}(A, U) = 1$

end

end

B.3. Lebesgue Measure

For the problems in Sect. 3.5, we consider Ω_f .

Algorithm 10: `LebSpec` computes $\text{Leb}(\text{Sp}(A))$ for operators $A \in \Omega_f$. It can be easily adapted to self-adjoint operators and computing the Lebesgue measure of the spectrum as a subset of the real line, by restricting the rectangles and balls to intervals. Again, the computation of `DistSpec` can be performed in parallel.

```

Function LebSpec ( $n_1, n_2, f(n_1), c_{n_1}, A$ )
  Input :  $n_1, n_2, f(n_1) \in \mathbb{N}, c_{n_1} \in \mathbb{R}_+, A \in \Omega_f$ 
  Output:  $\Gamma_{n_2, n_1}(A)$ , a  $\Pi_2^A$  approximation of  $\text{Leb}(\text{Sp}(A))$ 

   $G = \frac{1}{2^{n_2}} (\mathbb{Z} + i\mathbb{Z}) \cap [-n_1, n_1]^2 = \{z_1, \dots, z_m\}$ 
  for  $z \in G$  do
    |  $F_{n_1}(z) = \text{DistSpec}(A, n_1, z, f(n_1)) + c_{n_1}$ 
  end
  NB: WLOG we adapt  $F_{n_1}$  to be non-increasing in  $n_1$ .
   $U(n_2, n_1, A) = [-n_1, n_1]^2 \cap (\cup_{j=1}^m B(z_j, F_{n_1}(z_j)))$ 
   $\Gamma_{n_2, n_1}(A) = 4n_1^2 - \text{Leb}(U(n_2, n_1, A))$ 
end
    
```

Algorithm 11: `LebPseudoSpec` computes $\text{Leb}(\text{Sp}_\epsilon(A))$ for operators $A \in \Omega_f$. It can be easily adapted to self-adjoint operators and computing the Lebesgue measure of the pseudospectrum restricted to the real line, by restricting the rectangles and balls to intervals. Again, the computation of `DistSpec` can be performed in parallel.

```

Function LebPseudoSpec ( $n, A, \epsilon$ )
  Input :  $n \in \mathbb{N}, A \in \Omega_\epsilon^L, \epsilon > 0$ 
  Output:  $\Gamma_n(A)$ , a  $\Sigma_1^A$  approximation of  $\text{Leb}(\text{Sp}_\epsilon(A))$ 

   $G = \frac{1}{n} (\mathbb{Z} + i\mathbb{Z}) \cap [-n, n]^2 = \{z_1, \dots, z_m\}$ 
  for  $z \in G$  do
    |  $F_n(z) = \text{DistSpec}(A, n, z, f(n)) + c_n$ 
  end
  NB: WLOG we adapt  $F_n$  to be non-increasing in  $n$ .
   $S = \{z \in G : F_n(z) \leq \epsilon\}$ 
   $\Gamma_n(A) = \text{Leb}(\cup_{z \in S} D(z, \max\{0, \epsilon - F_n(z)\}))$ 
end
    
```

Algorithm 12: NullLebSpec computes $\Xi_L^3(A)$ (“Is $\text{Leb}(\text{Sp}(A)) = 0$?”) for operators $A \in \Omega_f$. It can be easily adapted to self-adjoint operators, where the Lebesgue measure corresponds to that of the real line, by using the relevant adaptation of LebSpec.

```

Function NullLebSpec ( $n_1, n_2, n_3, f(n_1), c_{n_1}, A$ )
  Input :  $n_1, n_2, n_3, f(n_1) \in \mathbb{N}, c_{n_1} \in \mathbb{R}_+, A \in \Omega_f$ 
  Output:  $\Gamma_{n_3, n_2, n_1}(A)$ , a  $\Pi_3^A$  approximation of  $\Xi_L^3(A)$ 

  for  $j = 1, \dots, n_1$  do
    |  $t_j = \text{LebSpec}(j, n_2, f(j), c_j, A)$ 
  end
  if  $\max_{1 \leq j \leq n_1} t_j \leq n_3^{-1}$  then
    |  $\Gamma_{n_3, n_2, n_1}(A) = 1$ 
  else
    |  $\Gamma_{n_3, n_2, n_1}(A) = 0$ 
  end
end
    
```

B.4. Fractal Dimensions

For the problems in Sect. 3.6, we consider Ω_f^{BD} for the box-counting dimension and $\Omega_f \cap \Omega_{SA}$ for the Hausdorff dimension.

Algorithm 13: `BoxDimSpec` computes the box-counting dimension of the spectrum for operators $A \in \Omega_f^{BD}$. If we enlarge the class to $\Omega_f \cap \Omega_{SA}$, the result is a tower of algorithms that converges to a quantity $\Gamma(A)$ with $\underline{\dim}_B(\text{Sp}(A)) \leq \Gamma(A) \leq \overline{\dim}_B(\text{Sp}(A))$.

```

Function BoxDim( $n_1, n_2, f(n_1), c_{n_1}, A$ )
  Input :  $n_1, n_2, f(n_1) \in \mathbb{N}, c_{n_1} \in \mathbb{R}_+, A \in \Omega_f^{BD}$ 
  Output:  $\Gamma_{n_2, n_1}(A)$ , a  $\Pi_2^A$  approximation of  $\dim_B(\text{Sp}(A))$ 
  if  $n_1 \geq n_2$  then
    for  $l = 1, \dots, n_1$  do
       $S_l = \text{CompSpec}(A, l, f(l), c_l, g : x \mapsto x) = \{z_{1,l}, \dots, z_{k_l,l}\}$ 
      NB: WLOG we assume that  $\text{dist}(z_{j,l}, \text{Sp}(A)) \leq 2^{-l}$ .
      for  $j = 1, \dots, k_l$  do
         $I_{j,l} = [z_{j,l} - 2^{-l}, z_{j,l} + 2^{-l}]$ 
      end
    end
    for  $k \in \{n_2, n_2 + 1, \dots, n_1\}, j \in \{1, 2, \dots, n_1\}$  do
      Let  $\Upsilon_{k,j}$  be any union of  $2^{-k}$ -mesh intervals of minimal length  $|\Upsilon_{k,j}|$  (where length is the number of mesh intervals that make up the union) such that
       $\Upsilon_{k,j} \cap I_{p,q} \neq \emptyset, 1 \leq q \leq j, 1 \leq p \leq k_q$ .
      Set  $a_{k,j} = \frac{\log(|\Upsilon_{k,j}(A)|)}{k \log(2)}$ 
    end
     $\Gamma_{n_2, n_1}(A) = \max\{a_{k,j} : n_2 \leq k \leq n_1, 1 \leq j \leq n_1\}$  (max over empty set is zero).
  else
     $\Gamma_{n_2, n_1}(A) = 0$ 
  end
end
  
```

Algorithm 14: `HausDimSpec` computes the Hausdorff dimension of the spectrum for operators $A \in \Omega_f \cap \Omega_{SA}$. An efficient way to compute the minimal covering is to use binary trees [153].

```

Function HausDimSpec( $n_1, n_2, n_3, A$ )
  Input :  $n_1, n_2, n_3 \in \mathbb{N}, A \in \Omega_f \cap \Omega_{SA}$ 
  Output:  $\Gamma_{n_3, n_2, n_1}(A)$ , a  $\Sigma_3^A$  approximation of  $\dim_H(\text{Sp}(A))$ 
  Notation:  $\rho_k$  denotes set of all closed intervals of form  $[2^{-k}m, 2^{-k}(m+1)]$ ,  $m \in \mathbb{Z}$ 
   $S_{n_1, n_2}$  = union of all  $S \in \rho_{n_2}$  with  $S \subset [-n_1, n_1]$  and such that the algorithm discussed in Lemma 3.21 outputs “Yes” for the interior of  $S$  and input parameter  $n_1$ .
   $\mathcal{A}_{n_3, n_2, n_1} = \{\{U_i\}_{i \in I} : I \text{ is finite}, S_{n_1, n_2} \subset \cup_{i \in I} U_i, U_i \in \cup_{n_3 \leq l \leq n_2} \rho_l\}$ 
  for  $m \in \{1, \dots, 2^{n_3}\}$  do
     $b_m = \inf \left\{ \sum_i \text{diam}(U_i)^{m/2^{n_3}} : \{U_i\} \in \mathcal{A}_{n_3, n_2, n_1} \right\} + n_2^{-1}$ 
  end
   $\Gamma_{n_3, n_2, n_1}(A) = \max\{m/2^{n_3} : b_j > 1/2 \text{ for } j = 1, \dots, m\}$  (max over empty set is zero).
end
  
```

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