Explicit Computation of a Galois Representation Attached to an Eigenform Over SL_3 from the $H_{\text{ét}}^2$ of a Surface

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Abstract

We describe a method to compute mod ℓ Galois representations contained in the $H_{\text{ét}}^2$ of surfaces. We apply this method to the case of a representation with values in $GL_3(\mathbb{F}_9)$ attached to an eigenform over a congruence subgroup of SL_3 . We obtain, in particular, a polynomial with Galois group isomorphic to the simple group $PSU_3(\mathbb{F}_9)$ and ramified at 2 and 3 only.

Keywords Galois representation · Algorithm · Étale cohomology · Dévissage · Surface · Automorphic form · GL_3 · Unitary group

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1 Introduction

Several techniques (such as [3, 15, 16]) have recently been developed to compute explicitly mod ℓ Galois representations appearing in the torsion of Jacobians of curves. However, many “interesting” representations (e.g. in the Langlands program) are not naturally found in Jacobians, but rather in higher étale cohomology spaces of higher-dimensional varieties. They are thus inaccessible to the aforementioned methods, and, to the author’s knowledge, computational methods to deal with these representations have not yet been developed apart from the case of modular forms over GL_2 .

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The purpose of this article is to sketch such a method, thus answering the conjecture made in the second-to-last point of the epilogue of [3]. Although our method is still at an experimental stage, it is already sufficiently advanced for us to be able to prove our concept by giving a concrete example of application, which we also present in this article.

Remark 1 Very general algorithms to compute with étale cohomology are presented in [18,21] (cf. also [9] for a more explicit approach in the case of curves). However, these algorithms are described with a high level of abstraction, no focus on practical efficiency (cf. [18, 0.10]), and have never led to an implementation as far as we know. Our purpose is to give a detailed presentation of a method which is completely explicit and really practical on a moderately simple case, namely that of the $H_{\text{ét}}^2$ of surfaces, and which we expect to be generalizable to other similar cases.

The concrete example that we have chosen comes from [5], where B. van Geemen and J. Top give evidence towards a conjecture of L. Clozel's. They exhibit a Hecke eigenform u over a congruence subgroup of $\text{SL}_3(\mathbb{Z})$ of level $128 = 2^7$ whose Hecke eigenvalues lie in $\mathbb{Z}[2\sqrt{-1}]$, and an algebraic surface S over $\mathbb{Q}$ equipped with an automorphism ϕ_S of order 4 defined over $\mathbb{Q}$, such that for all primes $\ell \in \mathbb{N}$, the ℓ -adic H^2 of S equipped with the $\mathbb{Q}_\ell(\sqrt{-1})$ -vector space structure induced by ϕ_S contains a Galois-submodule affording the quadratic twist by -2 of the ℓ -adic representation

$$\tilde{\rho}_{u,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_3(\mathbb{Q}_\ell(\sqrt{-1}))$$

attached to u .

Remark 2 At the time when [5] was published, representations attached to this kind of modular forms were not even known to exist, especially when, like $\tilde{\rho}_{u,\ell}$, they are not self-dual. The existence of these representations was established recently independently by [6,22], and the fact that $\tilde{\rho}_{u,\ell}$ is indeed afforded by the ℓ -adic H^2 of S is proved in [7].

According to [5, 2.4] and to the first paragraph of Sect. 4 of [7], the ℓ -adic representation $\tilde{\rho}_{u,\ell}$ is unramified away from¹ 2 and ℓ , and for each unramified prime $p \in \mathbb{N}$, the characteristic polynomial of $\tilde{\rho}_{u,\ell}(\text{Frob}_p)$ is

$$\chi_p(x) = x^3 - a_p x^2 + p \overline{a_p} x - p^3 \in \mathbb{Q}_\ell(\sqrt{-1})[x], \quad (1)$$

where $a_p \in \mathbb{Z}[2\sqrt{-1}]$ is the corresponding Hecke eigenvalue of u and $\overline{a_p}$ is the image of a_p under complex conjugation. In particular, the determinant of this representation is the cube of the ℓ -adic cyclotomic character, and the value of a_p can be recovered as the trace of the Frobenius.

Remark 3 Let $\rho_{u,\ell}$ be the semisimplification of the mod ℓ reduction of $\tilde{\rho}_{u,\ell}$. Since a_p lies in $\mathbb{Z}[2\sqrt{-1}]$ for all p , the characteristic polynomial $\chi_p(x)$ is always congruent

¹ This also follows of course from the fact that S has good reduction away from 2.

to $(x-1)^3 \bmod 2$, which shows that the mod 2 representation $\rho_{u,2}$ is trivial. Therefore, we have chosen to consider the more interesting (and challenging) case $\ell = 3$.

The purpose of this article is thus to explain how we have almost certainly succeeded to compute explicitly the mod 3 representation

$$\rho_{u,3} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_3(\mathbb{F}_9)$$

found up to twist in $H_{\text{et}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/3\mathbb{Z})$. The code used for this computation is available in the GitHub repository [13].

Here and in the rest of the article, by *computing explicitly a mod ℓ Galois representation*, we mean computing a polynomial whose roots are permuted by Galois in the same way as the vectors in the space of the representation (so that its splitting field agrees with the field cut out by the representation), as well as extra data making it possible to determine for any unramified prime $p \in \mathbb{N}$ the image of the Frobenius at p up to conjugacy. In our case, this implies in particular that our computations allow us to determine mod 3 the eigenvalue a_p of u for all $p \notin \{2, 3\}$. For instance, we can compute in four seconds that if $p = 10^{1000} + 453$ is the first prime after 10^{1000} , then $a_p \equiv -1 \bmod 3\mathbb{Z}[\sqrt{-1}]$. As a bonus, we obtain a polynomial whose Galois group over $\mathbb{Q}$, which is by construction the image of $\rho_{u,3}$, turns out to be a particularly interesting subgroup of $\text{GL}_3(\mathbb{F}_9)$ (cf. Sect. 5).

Remark 4 Unfortunately, because of the reason given in Remark 17, we are unable to certify rigorously that the results of our computations are correct. However, the fact that we are eventually able to recover the values of the $a_p \bmod 3$ from the representation (cf. Remark 23) shows that our results are correct beyond reasonable doubt. Besides, the author has plans to implement a method to rigorously certify *a posteriori* the output of such Galois representation computations; this method is sketched at the end of Remark 17.

The central idea making this computation possible, which we owe to J.-M. Couveignes, is a method to construct by *dévisage* a curve C (depending on ℓ) such that $\rho_{u,\ell}$ is contained (up to semisimplification) in the ℓ -torsion of the Jacobian of C . The point is that once we have obtained an explicit model for C , we are able (at least in theory) to compute the representation, thanks to our technique presented in [16].

Remark 5 In principle, this *dévisage* technique could be iterated to construct a curve whose Jacobian contains a given representation found in the H_{et}^d of a variety of dimension d ; however, although we have not checked the details, we are afraid that for $d \geq 3$, the genus of this curve will be so high that practical computations with it will not be amenable. For instance, Remark 11 demonstrates that even for $d = 2$, in the particular case under consideration in this article, considerable difficulties arise for $\ell \geq 5$.

We will sketch this *dévisage* construction in Sect. 2, after which we will apply it to $\rho_{u,3}$ in Sect. 3. Next, we will explain in Sect. 4 how we used the curve thus obtained to compute a polynomial corresponding to $\rho_{u,3}$, and we will then use this polynomial to determine the image of $\rho_{u,3}$ in Sect. 5. Finally, we will show in Sect. 6 how to compute the image of Frobenius elements.

Remark 6 This computation is made possible by the fact that the surface S is an elliptic surface. In fact, the method presented in the article extends smoothly to any elliptic surface; moreover, we believe that it can actually be generalized to any surface, cf. the discussion at the end of Sect. 2 for details.

2 Dévissage

Suppose we are given a surface S defined over $\mathbb{Q}$ as well as a prime $\ell \in \mathbb{N}$ such that $H_{\text{ét}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$ contains a Galois-submodule affording a mod ℓ Galois representation ρ that we wish to compute. We are going to show how to construct a curve C whose Jacobian will also contain ρ (up to twist) in its ℓ -torsion. Of course, this curve C will depend on ℓ . This construction is an example of the *dévissage* method summarized in [23, 3.4].

Let μ_ℓ be the Galois module formed by the ℓ th roots of unity. Given a Galois module M and an integer $n \in \mathbb{Z}$, we will denote by $M(n)$ the twist of M by the n th power of the mod ℓ cyclotomic character. Thus $\mu_\ell = (\mathbb{Z}/\ell\mathbb{Z})(1)$ for instance. We will sometimes write $\mu_\ell^\vee$ for $(\mathbb{Z}/\ell\mathbb{Z})(-1)$. Finally, we will also denote by μ_ℓ and $(\mathbb{Z}/\ell\mathbb{Z})(n)$ the corresponding constant sheaves on the étale site of a variety.

Recall [17, 14.2] that when X is a complete, connected and non-singular curve over $\overline{\mathbb{Q}}$, we have canonical (and hence Galois-equivariant) identifications

$$H_{\text{ét}}^r(X, \mu_\ell) \simeq \begin{cases} \mu_\ell & \text{if } r = 0, \\ J[\ell] & \text{if } r = 1, \\ \mathbb{Z}/\ell\mathbb{Z} & \text{if } r = 2, \\ 0 & \text{if } r \geq 3, \end{cases}$$

where $J = \text{Jac}(X)$ is the Jacobian of X . By [17, 14.4], twisting by μ_ℓ yields

$$H_{\text{ét}}^r(X, (\mathbb{Z}/\ell\mathbb{Z})(n)) \simeq \begin{cases} (\mathbb{Z}/\ell\mathbb{Z})(n) & \text{if } r = 0, \\ J[\ell](n-1) & \text{if } r = 1, \\ (\mathbb{Z}/\ell\mathbb{Z})(n-1) & \text{if } r = 2, \\ 0 & \text{if } r \geq 3 \end{cases} \quad (2)$$

for all $n \in \mathbb{Z}$.

Let U be X with finitely many points deleted, which is still a non-singular connected curve, but is affine instead of complete. Then, we have

$$H_{\text{ét}}^r(U, (\mathbb{Z}/\ell\mathbb{Z})(n)) \simeq \begin{cases} (\mathbb{Z}/\ell\mathbb{Z})(n), & \text{if } r = 0, \\ J[\ell](n-1) \text{ extended by copies of } (\mathbb{Z}/\ell\mathbb{Z})(n-1) & \text{if } r = 1, \\ 0 & \text{if } r \geq 2, \end{cases} \quad (3)$$

for all $n \in \mathbb{Z}$, where the first case is obvious, and the last two follow respectively from corollary 16.2 and proposition 14.12 of [17].

Suppose now that we have a proper regular surface S over $\mathbb{Q}$, equipped with a proper dominant morphism $\pi : S \rightarrow B$ to a non-singular complete curve B , with S , B , and π defined over $\mathbb{Q}$.

Let $Z \subset B$ be a Galois-stable, nonempty² finite subset containing the image of the bad fibres of π , and let $Y = \pi^{-1}(Z) \subset S$. Define $B' = B \setminus Z$, and $S' = S \setminus Y$, so that

the fibres of the induced map $\pi : S' \rightarrow B'$ are smooth proper curves. (4)

The representability of the relative Picard functor [19, 9.4.4] thus guarantees the existence of a cover $\psi : C' \rightarrow B'$ whose fibre at any $b \in B'$ is $C'_b = \text{Jac}(S_b)[\ell]$, where $S_b = S' \times_{B'} b$ is the fibre of π at b .

The closed subscheme Y of S is made up of curves, possibly with multiplicities, and intersecting in some way. Define Y' to be the scheme obtained from Y by first passing to the reduced scheme structure, and then deleting the singular points. Thus Y' is a disjoint union of smooth curves which are defined over $\mathbb{Q}$. Its geometrically irreducible components are therefore permuted by Galois; let

$$\eta = \prod_{\text{Components of } Y'} \mathbb{F}_\ell$$

be the corresponding mod ℓ permutation representation, and denote by $\eta(-1) = \eta \otimes \mu_\ell^\vee$ its twist by the inverse of the mod ℓ cyclotomic character.

With this notation, we can prove that the “interesting” Galois representations which lie in $H_{\text{et}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$ are also afforded in $H_{\text{et}}^1(C'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$:

Theorem 7 *Suppose ρ is a mod ℓ Galois representation contained in $H_{\text{et}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$ (up to semi-simplification). Assume that ρ has no Jordan-Hölder components of the form $(\mathbb{Z}/\ell\mathbb{Z})(n)$ for any $n \in \mathbb{Z}$, and no component in common with $\eta(-1)$. Then the twist of ρ by the mod ℓ cyclotomic character is also contained (up to semi-simplification) in $H_{\text{et}}^1(C'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$.*

Remark 8 The number field cut out by $\eta(-1)$ is contained in the compositum of the ℓ th cyclotomic field and of the fields of definition of the geometric components of the bad fibres of π . In general, we expect this field to be considerably smaller than that cut out by ρ if ρ is an “interesting” representation. For instance, for the surface considered in Sect. 3, the field cut out by η is merely $\mathbb{Q}(\sqrt{2})$ (cf. Remark 11). Therefore, the requirement that ρ have no common component with $\eta(-1)$ ought to be harmless for “interesting” representations ρ .

Proof Let us first show that ρ is also contained in $H_{\text{et}}^2(S'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$, so that the bad fibres of π will no longer be a nuisance. The localization exact sequence [17, 9.4] shows that the kernel of $H_{\text{et}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z}) \rightarrow H_{\text{et}}^2(S'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$ is a quotient of $H_{\text{et}, Y}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$. Étale cohomology with coefficients in $(\mathbb{Z}/\ell\mathbb{Z})(n)$ satisfies the Bloch-Ogus axioms [2],

² We insist that Z must not be empty because we will need B' to be affine later.

so in particular the Poincaré duality axiom [8, 6.1.j] shows that

$$H_{\text{ét}, Y}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z}) \simeq H_2(Y_{\overline{\mathbb{Q}}}, (\mathbb{Z}/\ell\mathbb{Z})(2)).$$

Applying [8, 6.1.f] twice shows that $H_2(Y_{\overline{\mathbb{Q}}}, (\mathbb{Z}/\ell\mathbb{Z})(2)) \simeq H_2(Y'_{\overline{\mathbb{Q}}}, (\mathbb{Z}/\ell\mathbb{Z})(2))$, and since Y' is a disjoint union of smooth curves, applying Poincaré duality [8, 6.1.j] and then (2) or (3) component-wise reveals that

$$H_2(Y'_{\overline{\mathbb{Q}}}, (\mathbb{Z}/\ell\mathbb{Z})(2)) \simeq \eta(-1).$$

Our assumptions on ρ thus show that it must be contained in $H_{\text{ét}}^2(S'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$ as claimed.

Consider now the Leray spectral sequence [17, 12.7]

$$E_2^{p,q} = H_{\text{ét}}^p(B'_{\overline{\mathbb{Q}}}, R^q\pi_*\mathbb{Z}/\ell\mathbb{Z}) \Rightarrow H_{\text{ét}}^{p+q}(S'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$$

attached to $\pi : S' \longrightarrow B'$. We know by proper base change [17, 17.7] that the stalk of the sheaf $R^q\pi_*\mathbb{Z}/\ell\mathbb{Z}$ at any $b \in B'$ is

$$(R^q\pi_*\mathbb{Z}/\ell\mathbb{Z})_b = H_{\text{ét}}^q(S_b, \mathbb{Z}/\ell\mathbb{Z}). \quad (5)$$

In particular, in view of (4) and of (2), we find that for $q = 1$, this stalk at any $b \in B'$ is

$$(R^1\pi_*\mathbb{Z}/\ell\mathbb{Z})_b = H_{\text{ét}}^1(S_b, \mathbb{Z}/\ell\mathbb{Z}) = C'_b(-1) \quad (6)$$

since by definition $C'_b = \text{Jac}(S_b)[\ell]$ for any $b \in B'$. Similarly, for $q = 2$ we have

$$R^2\pi_*\mathbb{Z}/\ell\mathbb{Z} \simeq \mu_{\ell}^{\vee}. \quad (7)$$

Besides, (4) and the fact that the base B' is a non-singular connected curve show that

$$E_2^{p,q} = 0 \text{ unless } 0 \leq p, q \leq 2.$$

Therefore $E_2^{p,q} = E_{\infty}^{p,q}$ for all p, q such that $p + q = 1$, which means that $H^2(S'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$ admits a filtration with components

- $H_{\text{ét}}^2(B'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z}) = 0$ by (3),
- $H_{\text{ét}}^0(B'_{\overline{\mathbb{Q}}}, R^2\pi_*\mathbb{Z}/\ell\mathbb{Z}) = H_{\text{ét}}^0(B'_{\overline{\mathbb{Q}}}, \mu_{\ell}^{\vee}) = \mu_{\ell}^{\vee}$ by (7) and (3),
- and $H_{\text{ét}}^1(B'_{\overline{\mathbb{Q}}}, \mathcal{F})$,

where $\mathcal{F} = R^1\pi_*\mathbb{Z}/\ell\mathbb{Z}$ is the sheaf on B' whose stalk at any $b \in B'$ is

$$\mathcal{F}_b = H_{\text{ét}}^1(S_b, \mathbb{Z}/\ell\mathbb{Z}) = C'_b(-1)$$

as shown in (6). Our assumptions on ρ thus show that it must be contained in $H_{\text{ét}}^1(B'_{\overline{\mathbb{Q}}}, \mathcal{F})$.

Similarly, the Leray spectral sequence

$$H_{\text{ét}}^p(B'_{\overline{\mathbb{Q}}}, R^q \psi_* \mathbb{Z}/\ell \mathbb{Z}) \Rightarrow H_{\text{ét}}^{p+q}(C'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell \mathbb{Z})$$

attached to $\psi : C' \rightarrow B'$ shows that $H_{\text{ét}}^1(C'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell \mathbb{Z}) = H_{\text{ét}}^1(B'_{\overline{\mathbb{Q}}}, \mathcal{G})$, where $\mathcal{G}$ is the sheaf on B' whose stalk at any $b \in B'$ is

$$\mathcal{G}_b = H_{\text{ét}}^0(C'_b, \psi_* \mathbb{Z}/\ell \mathbb{Z}) = \mathbb{F}_{\ell}^{C'_b}.$$

Since $C'_b = \text{Jac}(S_b)[\ell]$ is an Abelian ℓ -torsion group, there is a natural surjection

$$\begin{aligned} \mathcal{G}_b &= \mathbb{F}_{\ell}^{C'_b} \rightarrow C'_b = \mathcal{F}_b(1) \\ \lambda &\mapsto \sum_{c \in C'_b} \lambda_c c. \end{aligned}$$

Patching these together for all $b \in B'$, and then twisting, we obtain an epimorphism $\mathcal{G}(-1) \twoheadrightarrow \mathcal{F}$. Let $\mathcal{K}$ be its kernel, so that we have the short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{G}(-1) \rightarrow \mathcal{F} \rightarrow 0$$

of sheaves on $B'_{\overline{\mathbb{Q}}}$. The associated long exact sequence contains

$$\cdots \rightarrow H_{\text{ét}}^1(B'_{\overline{\mathbb{Q}}}, \mathcal{G}(-1)) \rightarrow H_{\text{ét}}^1(B'_{\overline{\mathbb{Q}}}, \mathcal{F}) \rightarrow H_{\text{ét}}^2(B'_{\overline{\mathbb{Q}}}, \mathcal{K}) \rightarrow \cdots,$$

and as $H_{\text{ét}}^2(B'_{\overline{\mathbb{Q}}}, \mathcal{K}) = 0$ by [23, 1.3.3.6.ii], we conclude that since ρ is contained in $H_{\text{ét}}^1(B'_{\overline{\mathbb{Q}}}, \mathcal{F})$, it also appears in $H_{\text{ét}}^1(B'_{\overline{\mathbb{Q}}}, \mathcal{G}(-1)) = H_{\text{ét}}^1(C'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell \mathbb{Z})(-1)$. $\square$

The curve C' constructed in Theorem 7 will not, in general, be connected, because of the zero section $0 \in \text{Jac}(S_b)[\ell] = C'_b$. However, we can modify the definition of C' so that

$$C'_b = \text{Jac}(S_b)[\ell] \setminus \{0\}.$$

The curve thus redefined has now a good chance of being geometrically connected and will contain³ ρ in its $H_{\text{ét}}^1$. Assuming that C' is indeed connected, let C be the smooth proper model of C' over $\mathbb{Q}$; as $H_{\text{ét}}^1(C'_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell \mathbb{Z})$ is an extension of $H_{\text{ét}}^1(C_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell \mathbb{Z})$ by copies of $\mu_{\ell}^{\vee}$ according to (3), we conclude by (2) that the twist of ρ by a power of the cyclotomic character will be contained in $\text{Jac}(C)[\ell]$ up to semi-simplification.

This leads to the following plan of attack to compute $\rho \subset H_{\text{ét}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell \mathbb{Z})$:

³ Unless ρ comes from $H_{\text{ét}}^1(B_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell \mathbb{Z})$, but in this case we could compute it in the Jacobian of B in view of (2).

1. Fix a proper dominant morphism π from S to a curve B ,
2. Compute the Galois representations afforded by the ℓ -torsion of the Jacobian of the fibre S_b of π for various $b \in B$,
3. Interpolate to glue these data into an explicit model of the cover $C \rightarrow B$,
4. Catch a twist of ρ in the ℓ -torsion of the Jacobian of C .

Let us comment on these steps by increasing order of difficulty.

- The first step requires choosing a curve B and a morphism $\pi : S \rightarrow B$. While for some particular surfaces S , there may be a natural choice, for generic S , one may simply take $B = \mathbb{P}^1$, and choose a “simple” morphism π so that its fibres have reasonably low genus. For example, if S is embedded in some space, we may take π to be one of the coordinates.
- The fourth step requires one to be able to compute explicitly the representations appearing in the ℓ -torsion of the Jacobian of any smooth curve over $\mathbb{Q}$, which we can do thanks to the method presented in [16]; in fact, the present article is the reason why we invented this method in the first place. The output of the algorithm described in [16] is a polynomial in $\mathbb{Q}[x]$ whose roots in $\overline{\mathbb{Q}}$ are permuted by Galois in the same way as the ℓ -torsion points of this Jacobian.
- In all generality, the second and third steps would require generalizing this algorithm [16] to *families* of curves, that is to say to curves defined over the function field $\mathbb{Q}(B)$; the output would then be a polynomial in $\mathbb{Q}(B)[x]$, which would define C as a cover of B . The author believes that this generalization is possible and is currently working on the case of curves defined over the rational function field $\mathbb{Q}(t)$, which corresponds to $B = \mathbb{P}^1$. Fortunately, for the particular example that we have in mind in this article, the fibres S_b are elliptic curves, so this generalization is not needed, and indeed the method presented in this article applies directly to any elliptic surface.

3 Computation of a Nice Model of C

We now apply the method presented in the previous section to the case of the representation $\rho_{u,3}$ introduced in Sect. 1. According to [5, 3.10], for all ℓ , the mod ℓ representation $\rho_{u,\ell} \otimes \left(\frac{-2}{\cdot}\right)$ is contained in $H_{\text{ét}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$, where S is the minimal regular model of the projective closure of the surface over $\mathbb{Q}$ defined by the equation

$$z^2 = xy(x^2 - 1)(y^2 - 1)(x^2 - y^2 + 2xy). \quad (8)$$

The automorphism ϕ_S mentioned in Sect. 1 is that induced by $(x, y, z) \mapsto (y, -x, z)$. It has order 4, and yields a $\mathbb{F}_\ell(\sqrt{-1})$ -structure on the piece of $H_{\text{ét}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$ containing $\rho_{u,\ell} \otimes \left(\frac{-2}{\cdot}\right)$ according to [5, 3.4, 3.7].

In order to apply the *dévisage* method presented in the previous section to this surface, we need to choose a non-constant map $\pi : S \rightarrow B$, where B is a curve. We choose $B = \mathbb{P}^1$ with coordinate λ , and π the map sending (x, y, z) to $\lambda = y/x$.

Remark 9 Although π is only a rational map from the surface defined by Eq. (8) to $\mathbb{P}^1$, it becomes a morphism once we blow up this surface in order to obtain a regular model. In fact, π is the canonical map of S (cf. [5, 3.2]).

The fibres of π at λ are obtained by setting $y = \lambda x$ in (8), which yields

$$\left(\frac{z}{x^2}\right)^2 = \lambda(1 - \lambda^2 + 2\lambda)(x^2 - 1)(\lambda^2 x^2 - 1).$$

The right-hand side is a quartic in x , which admits the rational root $x = 1$ (among others), so these fibres are elliptic curves. We turn this quartic into a cubic by substituting $1/x + 1$ for x , so as to send this rational root to infinity. After that, we substitute $\frac{x/2+\lambda^2}{1-\lambda^2}$ for x , which simplifies this cubic into

$$\left(\frac{2(\lambda^2 - 1)}{(x + 2\lambda^2)^2}\right)^2 \lambda(\lambda^2 - 2\lambda - 1)(x - 2\lambda)(x + 2\lambda)(x + \lambda^2 + 1).$$

We deduce that the fibres of π are the elliptic curves

$$E_\lambda : y^2 = \lambda(\lambda^2 - 2\lambda - 1)(x - 2\lambda)(x + 2\lambda)(x + \lambda^2 + 1), \quad (9)$$

which define an elliptic curve E over $\mathbb{Q}(\lambda)$ whose locus of bad fibres and j -invariant are, respectively,

$$Z = \{0, \pm 1, 1 \pm \sqrt{2}, \infty\} \subset B = \mathbb{P}_\lambda^1, \quad j = 2^4 \frac{(\lambda^4 + 14\lambda^2 + 1)^3}{\lambda^2(\lambda^2 - 1)^4}.$$

Remark 10 Equation (9) reveals that $E[2]$ is already defined over $\mathbb{Q}(\lambda)$. This reflects the fact that the mod 2 representation $\rho_{u,2}$ attached to u is trivial, as we noted in Remark 3.

The ℓ -division polynomial $\psi_{\ell,\lambda}(x)$ of E_λ is easily computed thanks to `Pari/GP` [20]. By definition, for each λ , the Galois action on the roots of $\psi_{\ell,\lambda}(x)$ describes the Galois action on the x -coordinates of the points of $E_\lambda[\ell]$. We then compute $R_{\ell,\lambda}(y)$, the resultant in x of $\psi_{\ell,\lambda}(x)$ and of the Weierstrass Eq. (9) of E_λ , which yields a polynomial describing the Galois action on the y -coordinates of $E_\lambda[\ell]$. For $\ell = 3$, the y -coordinate happens to be injective on $E_\lambda[\ell]$ for generic λ ; indeed, $R_{3,\lambda}(y)$ is squarefree for $\lambda = 2$. We have thus computed the mod 3 Galois representation afforded by the Jacobian of the fibre of π in terms of λ . Substituting x for λ , we obtain the following rather ugly plane model for C :

$$\begin{aligned}
0 &= R_{3,x}(y) \\
&= -256x^{56} + 6144x^{55} - 62464x^{54} + 333824x^{53} - 859648x^{52} - 120832x^{51} \\
&\quad - 16046080x^{49} - 9891072x^{48} + 90136576x^{47} - 73076736x^{46} - 237805568x^{45} \\
&\quad + 420485120x^{44} + 341843968x^{43} - 1165840384x^{42} - 192667648x^{41} + 2178936320x^{40} \\
&\quad - 238563328x^{39} - 3063240704x^{38} + 639488000x^{37} + 3412593664x^{36} - 639488000x^{35} \\
&\quad - 3063240704x^{34} + 238563328x^{33} + 2178936320x^{32} + 192667648x^{31} - 1165840384x^{30} \\
&\quad - 341843968x^{29} + (-288y^4 + 420485120)x^{28} + (3456y^4 + 237805568)x^{27} \\
&\quad + (-14400y^4 - 73076736)x^{26} + (14976y^4 - 90136576)x^{25} + (56160y^4 - 9891072)x^{24} \\
&\quad + (-142848y^4 + 16046080)x^{23} + (-52992y^4 + 7252992)x^{22} + (400896y^4 + 120832)x^{21} \\
&\quad + (-55872y^4 - 859648)x^{20} + (-624384y^4 - 333824)x^{19} + (134784y^4 - 62464)x^{18} \\
&\quad + (624384y^4 - 6144)x^{17} + (-55872y^4 - 256)x^{16} + (16y^6 - 400896y^4)x^{15} \\
&\quad + (-96y^6 - 52992y^4)x^{14} + (-384y^6 + 142848y^4)x^{13} + (3232y^6 + 56160y^4)x^{12} \\
&\quad + (-5424y^6 - 14976y^4)x^{11} + (960y^6 - 14400y^4)x^{10} - 3456y^4x^9 + (960y^6 - 288y^4)x^8 \\
&\quad + 5424y^6x^7 + 3232y^6x^6 + 384y^6x^5 - 96y^6x^4 - 16y^6x^3 + 27y^8.
\end{aligned}$$

A Magma [11] session still manages to reveal in a few seconds that C is geometrically integral and has (geometric) genus $g = 7$. This is good news, as our method [16] probably cannot reasonably cope with genera beyond 20 or 30.

Remark 11 We can easily do the same computation for other values of ℓ and thus get plane models of curves C that contain a twist of $\rho_{u,\ell}$ in their Jacobian. However, already for $\ell = 5$, the model of C that we get is so terrible that Magma [11] is unable to determine its genus (the computation was interrupted after 5 days, because it was using more than 400 GB of RAM).

We can still compute this genus, by exploiting the fact that $\pi : S \rightarrow B$ is an elliptic fibration. Indeed, since $B = \mathbb{P}_\lambda^1$ has genus 0, Riemann–Hurwitz tells us that if C is connected, then its genus is

$$g = 1 - d + \frac{1}{2} \sum_{c \in C} (e_c - 1),$$

where $d = \ell^2 - 1$ is the degree of the projection $\psi : C \rightarrow B$ induced by π , and the e_c are its ramification indices.

Rewrite

$$\sum_{c \in C} (e_c - 1) = \sum_{\lambda \in B} \sum_{\psi(c)=\lambda} (e_c - 1),$$

and notice that for each λ , we have

$$\sum_{\psi(c)=\lambda} (e_c - 1) = \sum_{\psi(c)=\lambda} e_c - \sum_{\psi(c)=\lambda} 1 = d - \#\psi^{-1}(\lambda).$$

Besides, the ramification of ψ can only come from the bad fibres of π , so this expression is 0 for $\lambda \notin Z$.

Our surface S is the minimal proper regular model of E/B , so we can analyse its bad fibres thanks to Tate's algorithm. It reveals that at $\lambda = 0$ and ∞ , the special fibre is of Kodaira type I_2^* , which, according to Table 6 of [1, V.10], implies that the monodromy around λ acts on the homology of the fibre of π by $T = -\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. If ℓ is an odd prime, this means that $\#\psi^{-1}(\lambda)$, which is the number of orbits of T acting on $\mathbb{F}_{\ell^2} \setminus \{0\}$, is $\frac{1}{2\ell}(1(\ell^2 - 1) + (\ell - 1)(\ell - 1)) = 2\ell - 2$ by Burnside's formula. Similarly, at $\lambda = \pm 1$, the special fibre is of type I_4 , whence $T = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ and $\#\psi^{-1}(\lambda) = 2\ell - 2$, whereas at $\lambda = 1 \pm \sqrt{2}$, the special fibre is of type I_0^* , whence $T = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\#\psi^{-1}(\lambda) = \frac{\ell^2 - 1}{2}$.

As a result, we find that if ℓ is an odd prime and if C is connected, then its genus is

$$g = \frac{3}{2}\ell^2 - 3\ell + \frac{5}{2} = 1 + \frac{3}{2}(\ell - 1)^2.$$

In particular, we recover $g = 7$ for $\ell = 3$, and we find that $g = 25$ for $\ell = 5$ and that $g = 55$ for $\ell = 7$. This means that our method [16] could probably manage to compute the mod 5 representation $\rho_{u,5}$ if we could find a decent enough model for C for $\ell = 5$ and if we were patient enough, whereas $\ell \geq 7$ seems out of our reach.

Let us get back to the case $\ell = 3$ and to our curve C of genus 7. The model that we have just obtained has degree 56, and therefore arithmetic genus 1485. We do not want to work with such a badly singular model, so we attempt to eliminate the worst of the singularities by having Magma [11] determine the canonical image of C in $\mathbb{P}^{7-1}$ and project it on a plane. This yields the already more appealing model

$$\begin{aligned} & (351y^5 - 35100y^4 + 1285362y^3 - 20961720y^2 + 148459311y - 374594220)x^4 \\ & + (-8y^8 + 864y^7 - 34064y^6 + 519536y^5 + 110400y^4 - 23713936y^3 - 2298974896y^2 \\ & + 48122966496y - 247835237752)x^2 + (208y^9 - 26624y^8 + 1124032y^7 - 779584y^6 \\ & - 1602165344y^5 + 64797194176y^4 - 1296163023168y^3 + 14839155321536y^2 \\ & - 93670614701872y + 252643199407040) = 0. \end{aligned}$$

We check that this model still has genus 7, which by Riemann–Hurwitz ensures that it is birational to the previous one, as opposed to being a quotient of it.

By projecting the canonical image onto another plane, we also find that C is a cover of degree 2 (simply given by $x \mapsto x^2$ on our new model) of a curve A of genus 1, which turns out to be an elliptic curve isomorphic to the modular curve $X_0(24)$. We learn from the LMFDB [10] that $A(\mathbb{Q}) \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has rank 0; pulling back these 8 points to C yields points at infinity and the points whose coordinates on this new model are

$$(\pm 13\sqrt{2}, 7), \quad (0, 20), \quad (\pm 13\sqrt{2}, 33).$$

Our strategy to further simplify our model for C is to perform a change of coordinates so as to make these points nicer, in the hope that the resulting model of C will also be nicer. We thus begin by replacing x with $13x$; then, observing that 7, 20, 33 is an

arithmetic progression, we replace y with $20 + 13y$. After a final rescaling of x by 3, we obtain the quite satisfying model

$$(3y^5 - 6y^3 + 3y)x^4 + (2y^8 - 8y^7 + 4y^6 + 12y^5 + 12y^3 - 4y^2 - 8y - 2)x^2 + (9y^9 - 36y^8 - 36y^7 + 36y^6 + 18y^5 - 36y^4 - 36y^3 + 36y^2 + 9y) = 0. \quad (10)$$

Remark 12 The whole process to simplify our model of C was rather rustic. Most computer algebra systems include algorithms that, given a complicated polynomial defining a number field, are able to find a much simpler polynomial defining the same field (when it exists); it would be nice to have similar algorithms for curves!

Remark 13 The automorphism ϕ_S of S naturally induces an automorphism ϕ_C of C whose order divides 4, and which in turn induces an automorphism ϕ_J on the Jacobian J of C .

Actually, ϕ_C and ϕ_J have order exactly 4, since ϕ_J defines the $\mathbb{F}_9 = \mathbb{F}_3(\sqrt{-1})$ -vector space structure on the piece of $J[3]$ that affords $\rho_{u,3}$.

Although it is not too difficult to determine that ϕ_S is given by

$$(x, y, \lambda) \mapsto \left(-2 \frac{x + 2\lambda^2 - 2\lambda + 2}{\lambda(x + 2\lambda)}, \left(\frac{2(\lambda - 1)}{\lambda^2(x + 2\lambda)} \right)^2 y, -\frac{1}{\lambda} \right)$$

on (9), it is arduous to determine explicitly the action of ϕ_C on (10) because of the method by which we have obtained this model. Fortunately, (10) is simple enough that Magma [11] is able to inform us that the automorphism group of C happens to be cyclic of order 4, and to provide us with an explicit generator, which must therefore coincide with ϕ_C or its inverse. We thus find that the action of ϕ_C on (10) is simply given by $(x, y) \mapsto (\pm x/y, -1/y)$.

We will see in Remark 19 that knowing the action of ϕ_J on J explicitly does not really help to speed up our computations. We still note that if we homogenize (10) and then dehomogenize with respect to x , then ϕ_C becomes a mere rotation of angle $\pi/2$ around the origin; it may be that for general ℓ , simple models of C could be obtained by looking for ones such that ϕ_C acts in a similarly simple way.

The arguments of the previous section show that the 3-torsion of the Jacobian J of (10) contains the representation contained in $H_{\text{ét}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/3\mathbb{Z})$ up to twist by the mod 3 cyclotomic character χ_3 , which agrees with the quadratic character $\left(\frac{-3}{\cdot}\right)$. Since this representation was already the twist of the representation $\rho_{u,3}$ we are interested in by $\left(\frac{-2}{\cdot}\right)$, this is just an extra twist.

In order to confirm this, we can check that the characteristic polynomials match at a few primes p . Indeed, let $\rho'_{u,3}$ be the $\text{GL}_6(\mathbb{F}_3)$ -valued representation obtained by restricting the scalars from the $\text{GL}_3(\mathbb{F}_9)$ -valued representation $\rho_{u,3}$. On the one hand, we know that for $p \neq 2, 3$, the characteristic polynomial of $\rho'_{u,3}(\text{Frob}_p)$ is the norm from $\mathbb{F}_9$ to $\mathbb{F}_3$ of the reduction mod 3 of the polynomial $\chi_p(x)$ given in (1), namely

$$\chi'_p(x) = \chi_p(x) \overline{\chi_p(x)}$$

$$= (x^3 - a_p x^2 + p \overline{a_p} x - p^3)(x^3 - \overline{a_p} x^2 + p a_p x - p^3) \in \mathbb{F}_3[x];$$

and furthermore [5, 2.5] provides us with the values of the Hecke eigenvalues a_p of u for $p \leq 67$. On the other hand, we can determine the characteristic polynomial $L_p(x) \in \mathbb{Z}[x]$ of the Frobenius at p acting on J (which is the local factor at p of its L function) by counting the $\mathbb{F}_{p^a}$ -points of C for $a \leq g$, where $g = 7$ is the genus of C ; in practice, Magma [11] can do this in reasonable time for $p \leq 19$. We then check that for $5 \leq p \leq 19$, $L_p(x) \bmod 3$ is divisible by the characteristic polynomial $\chi'_p((\frac{6}{p})x)$ of the image of Frob_p by $\rho'_{u,3} \otimes (\frac{6}{\cdot})$. This corroborates the fact that $J[3]$ contains $\rho'_{u,3} \otimes (\frac{6}{\cdot})$.

Remark 14 Since $g = 7$, the degree of $L_p(x)$ is 14. We actually observe that for all the primes $5 \leq p \leq 19$ that we can test, $L_p(x)$ is congruent mod 3 to the product

$$\chi_{A,p}(x) \chi'_p\left(\left(\frac{6}{p}\right)x\right) \chi'_p\left(\left(\frac{-2}{p}\right)x\right)$$

where the factors have respective degrees 2, 6, and 6 and are the characteristic polynomial of Frob_p for the mod 3 representation $\rho_{A,3}$ attached to the elliptic curve $A = X_0(24)$ exhibited above, the expected twist $\rho'_{u,3} \otimes (\frac{6}{\cdot})$, and the twist $\rho'_{u,3} \otimes (\frac{-2}{\cdot})$ originally contained in $H_{\text{et}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/3\mathbb{Z})$, respectively. The fact that we eventually managed to compute $\rho'_{u,3} \otimes (\frac{6}{\cdot})$ from a piece of $J[3]$, whereas we were unable to do the same for $\rho'_{u,3} \otimes (\frac{-2}{\cdot})$ (even after significantly increasing the p -adic accuracy in our computation, cf. the next section), leads us to guess that the Galois-module $J[3]$ decomposes as

$$J[3] \sim \left[\begin{array}{c|c} \rho_{A,3} & \\ \hline \rho'_{u,3} \otimes (\frac{6}{\cdot}) & * \\ \hline & \rho'_{u,3} \otimes (\frac{-2}{\cdot}) \end{array} \right],$$

where $*$ is non-trivial. In other words, the twist $\rho'_{u,3} \otimes (\frac{-2}{\cdot})$ originally contained in $H_{\text{et}}^2(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/3\mathbb{Z})$ seems to show up as *quotient* of $J[3]$; fortunately, its twist by the cyclotomic character $(\frac{-3}{\cdot})$ appears as a Galois submodule of $J[3]$ since Jacobians are self-dual via the Weil pairing.

Remark 15 The automorphism ϕ_J of J endows it with a $\mathbb{Z}[x]/(x^4 - 1)$ -structure, so the factorization

$$x^4 - 1 = (x - 1)(x + 1)(x^2 + 1)$$

over $\mathbb{Q}$ yields a decomposition of J up to isogeny, which must be reflected in the decomposition of the Galois module $J[3]$ observed in Remark 14 since ϕ_J is defined over $\mathbb{Q}$.

More specifically, again because ϕ_J defines the $\mathbb{F}_9 = \mathbb{F}_3(\sqrt{-1})$ -vector space structure on the pieces of $J[3]$ corresponding to the twists of $\rho'_{u,3}$, the isogeny factor of J corresponding to $x^2 + 1$ is not trivial and has dimension at least 6.

On the other hand, the expression of ϕ_C established in Remark 13 reveals that ϕ_C^2 is none other than the quotient map $C \rightarrow A$, $x \mapsto x^2$, so that in particular $A = C/\phi_C^2 =$

J/ϕ_J^2 . This shows that although ϕ_J has order 4 on J , it only induces an automorphism of order 2 on the component A of J , which is thus the isogeny factor corresponding to the factor $x^2 - 1$ of $x^4 - 1$. Therefore, the factor corresponding to $x^2 + 1$ has dimension exactly 6.

The automorphism ϕ_J induces an automorphism ϕ_A of $A = J/\phi_J^2$, which cannot be trivial by Galois theory applied to the function field $\mathbb{Q}(A)$. Since $A \simeq X_0(24)$ does not have CM according to the LMFDB [10], its only nontrivial automorphism is the elliptic involution. It follows that A is actually the factor of J corresponding to the factor $x + 1$ of $x^4 - 1$, and that the factor of J corresponding to $x - 1$ is trivial.

4 Computation of the Representation in J

Now that we have obtained a reasonable model for C , we are going to use our method [16] to compute the representation $\rho'_{u,3} \otimes \left(\frac{6}{\cdot}\right)$ afforded by a Galois submodule T of the 3-torsion of the Jacobian J of C . This method requires us to pick a prime $p \neq 3$ of good reduction for which the local L factor

$$L_p(x) = \det(x - \text{Frob}_p | J) \in \mathbb{Z}[x]$$

and the characteristic polynomial

$$\chi'_p\left(\left(\frac{6}{p}\right)x\right) = \det(x - \text{Frob}_p | T) \in \mathbb{F}_3[x]$$

are known, and such that $\chi'_p\left(\left(\frac{6}{p}\right)x\right)$ is coprime to its cofactor

$$\psi_p(x) = L_p(x) / \chi'_p\left(\left(\frac{6}{p}\right)x\right) \in \mathbb{F}_3[x].$$

Remark 16 In view of the decomposition of $J[3]$ observed in Remark 14, this forces in particular $p \equiv -1 \pmod{3}$.

We choose $p = 11$, which satisfies these assumptions. Then we have

$$\chi'_p\left(\left(\frac{6}{p}\right)x\right) = \sum_{k=0}^6 x^k \in \mathbb{F}_3[x], \quad (11)$$

which is irreducible; this fact will be useful in two occasions below.

In this setting, our method [16] may be summarized as follows:

1. Find $a \in \mathbb{N}$ such that the reduction mod $p = 11$ of the points of T are defined over the extension $\mathbb{F}_{p^a}$ of $\mathbb{F}_p$ of degree a , and such that $\mathbb{F}_{p^a}$ contains the ℓ th roots of 1. By (11) and proposition 6.1 of [16], we see that we can take $a = 14$.
2. Compute $\#J(\mathbb{F}_{p^a}) = \text{Res}(L_p(x), x^a - 1)$, and factor it as $\ell^v M$, where $M \in \mathbb{N}$ is coprime to $\ell = 3$.

3. Take a random point $x \in J(\mathbb{F}_{p^a})$, multiply it by M , and then repeatedly by ℓ until it has order ℓ , and then apply $\psi_p(\text{Frob}_p)$ to it so as to project it to T . Repeat this process and use the action of Frob_p until we get an $\mathbb{F}_\ell$ -basis of T .
4. Since Frob_p acts cyclically on T , we can find a point in T which generates T as an $\mathbb{F}_\ell[\text{Frob}_p]$ -module. Lift this point p -adically to the desired p -adic accuracy, and then use the action of Frob_p and the group law of T to get all the points of T at this p -adic accuracy.
5. Construct a Galois-equivariant rational map $\alpha : J \dashrightarrow \mathbb{A}^1$ as explained in [16, 2.2.3], and evaluate it at the points of T .
6. Form the polynomial

$$F(x) = \prod_{\substack{t \in T \\ t \neq 0}} (x - \alpha(t)),$$

and identify its coefficients as rational numbers.

As explained in [16], in order to perform these computations, we need to fix an effective divisor D_0 on C of degree $d_0 \geq 2g + 1 = 15$ for which we can explicitly compute the corresponding Riemann–Roch space. In order to construct the evaluation map α , we also need to pick two non-equivalent effective divisors E_1, E_2 of degree $d_0 - g$ such that we can also compute the Riemann–Roch spaces attached to $2D_0 - E_1$ and $2D_0 - E_2$. (The notations are the same as in [16, 2.2.3].)

The map α is injective on T for most choices of D_0, E_1 , and E_2 . Assuming that this is the case, we therefore obtain a polynomial $F(x) \in \mathbb{Q}[x]$ of degree $\#T - 1 = 3^6 - 1 = 728$ whose roots are permuted under Galois in the same way as the nonzero points of T .

Remark 17 We obtain the coefficients of $F(x)$ as p -adic approximations to the chosen p -adic accuracy, so this accuracy must be high enough for us to be able to identify them correctly as rational numbers.

The fact that we identify $F(x)$ from a p -adic approximation is the reason why we cannot rigorously certify that our computation results are correct. Nevertheless, we will give in Sect. 6 very strong evidence that these results are correct beyond reasonable doubt, cf. Remark 23. Also note that in principle, it should be possible to increase the p -adic accuracy so as to identify each point of T as the image of an element of $J(\mathbb{Q}[x]/F(x))[\ell]$ under an embedding of $\mathbb{Q}[x]/F(x)$ into one of its p -adic completions, and thus to rigorously certify that our results are correct; we plan to implement an algorithm certifying the output of our method [16] along these lines in the future.

We should strive to choose D_0, E_1 , and E_2 so that these three Riemann–Roch spaces are as “nice” as possible, so the values of α will have smaller arithmetic height, so that the p -adic accuracy required to identify the coefficients of $F(x)$ will be lower and the computation will be more efficient. After a bit of experimentation with Magma [11], we choose

$$d_0 = 16, \quad D_0 = 9P + 7Q, \quad E_1 = 6P + 3Q, \quad E_2 = 5P + 4Q,$$

where $P, Q \in C(\mathbb{Q})$ are points such that, in the model (10), the divisors of poles of x and y are, respectively,

$$(x)_{\infty} = 3P + Q + R + M_1 + M_2 \text{ and } (y)_{\infty} = 2P + 2Q$$

where R has degree 1 and M_1 and M_2 both have degree 2.

Remark 18 The Riemann–Roch spaces mentioned above are needed so as to describe the curve C and the map $\alpha : J \dashrightarrow \mathbb{A}^1$ in [16]. We can determine $\mathbb{Q}$ -bases of these spaces using Magma [11]. However, as our method is p -adic by nature, these bases will not be suitable if they become linearly dependent when reduced mod p . Fortunately, the occurrence of this problem is easy to detect, since functions on C are represented internally in [16] as the vector of their values at a large enough set of fixed p -adic points of C . This is also easy to fix, by Gaussian elimination: given $s_1, \dots, s_d \in \mathbb{Q}(C)$ forming the basis of a given Riemann–Roch space, if $\sum_i \lambda_i s_i \equiv 0 \pmod{p}$ for some $\lambda_i \in \mathbb{F}_p$ not all 0, it suffices to substitute $\frac{1}{p} \sum_i \tilde{\lambda}_i s_i$ for s_j , where j is such that $\lambda_j \neq 0$ and the $\tilde{\lambda}_i$ are lifts to $\mathbb{Z}$ of the λ_i . However, this has the effect of complicating the basis of the Riemann–Roch space, which in turn increases the height of the values of the evaluation map α . Fortunately, this phenomenon does not occur with our choices of D_0, E_1, E_2 and p .

After about 10 hours of CPU time, but only less than 30 minutes of real time thanks to parallelization, we obtain a polynomial $F(x)$ whose coefficients are rational numbers which all have (up to some small powers of 2 and 3) the same denominator, a 186-digit integer. The p -adic precision used was $O(11^{1024})$.

The discriminant of $F(x)$ factors into a large power of 2 times a large power of 3 times a large square, which already indicates that its coefficients have probably been correctly identified from their p -adic approximations. The fact that this discriminant is nonzero also shows that as expected, the map α that we have constructed is injective on T minus the origin, so that the roots of $F(x)$ represent faithfully the Galois action on T minus the origin, as desired.

Remark 19 Evaluating the map α at points of T with high p -adic precision is computationally costly. As explained in [16, 6.4], we save a lot of time thanks to the fact that we are able to apply explicitly the Frobenius to the points of T : not only does this allow us to generate new points of T from old ones, but it also reduces the number of evaluations of the map α , since it commutes with the Frobenius. It is tempting (and not difficult, by the same method as in [16, 2.2.5]) to use the same idea with the automorphism ϕ_J of order 4 of J ; however, while this allows us to generate even more points of T from old ones, this unfortunately does not reduce the number of evaluations of α since we do not know a formula for $\alpha \circ \phi_J$ in terms of α , and therefore only saves a marginal amount of computation time.

5 The Image of $\rho_{u,3}$

We find that the polynomial $F(x)$ computed in the previous section has three factors over $\mathbb{Q}$, of respective degrees 224, 252, and 252. This shows that the image of our representation does not act transitively on $\mathbb{F}_3^6$ minus the origin.

However, the degrees of these factors do not clearly indicate which subgroup of $\mathrm{GL}_6(\mathbb{F}_3)$ we are dealing with. In order to figure this out, we would like to determine the Galois groups of these factors; however, their degrees and heights are far too large for standard Galois group computation algorithms. We would therefore like to reduce these factors (in the sense of Remark 12), but they are actually too large even for this!

As in [14, Sect. 2], we circumvent this problem by considering the *projective* version of our representation, which has values in $\mathrm{PGL}_3(\mathbb{F}_9)$. We can obtain a polynomial corresponding to this representation by gathering the 11-adic roots $\alpha(t)$ of $F(x)$ along the $\mathbb{F}_9$ -vector lines of T in a symmetric way (e.g. by summing or multiplying them). However, this requires us to understand the $\mathbb{F}_9$ -structure of T , whereas we only know the $\mathbb{F}_3$ -structure so far.

The most direct way to determine this $\mathbb{F}_9$ -structure consists in using the action induced by the automorphism ϕ_J of order 4 of J , since it corresponds to multiplication by $i \in \mathbb{F}_9 = \mathbb{F}_3(i)$ on $T \subset J[3]$; indeed, as explained in Remark 19, it is not difficult to compute this action explicitly. However, there is also another method. Indeed, let $\Phi \in \mathrm{GL}(T)$ be the action of Frob_{11} on T . We know that $\mathbb{F}_9^\times$ acts on T by a cyclic subgroup of $\mathrm{GL}(T)$ of order 8 contained in the commutant of Φ , but luckily, Φ is cyclic, so its commutant is simply $\mathbb{F}_3[\Phi]$, which is a ring isomorphic to $\mathbb{F}_{3^6}$ since the characteristic polynomial of Φ is irreducible over $\mathbb{F}_3$. In particular, there is a unique cyclic subgroup of order 8 in $\mathbb{F}_3[\Phi]^\times$, which must thus agree with the action of $\mathbb{F}_9^\times$.

Either way, we can thus compute as above a polynomial $F_0(x)$ of degree $\frac{1}{\#\mathbb{F}_9^\times} \deg F = 91$ describing the projective representation attached to $\rho_{u,3} \otimes \left(\frac{6}{\cdot}\right)$ (which is also that attached to $\rho_{u,3}$).

We can also be a bit more subtle and consider all intermediate representations between the linear one and the projective one. Let us write $\mathbb{F}_9 = \mathbb{F}_3(i)$, where $i^2 = -1$. Then the subgroups of $\mathbb{F}_9^\times$ are, in decreasing order,

$$\mathbb{F}_9^\times \geq \{\pm 1, \pm i\} \geq \{\pm 1\} \geq \{1\}.$$

We can construct as above polynomials $F_0(x)$, $F_1(x)$, $F_2(x)$ and $F_3(x) = F(x)$ describing the corresponding quotients of $\rho_{u,3} \otimes \left(\frac{6}{\cdot}\right)$. These polynomials factor over $\mathbb{Q}$ as follows:

Quotient by	Degrees of factors
$\mathbb{F}_9^\times$	28 + 63
$\{\pm 1, \pm i\}$	56 + 63 + 63
$\{\pm 1\}$	112 + 126 + 126
$\{1\}$	224 + 252 + 252.

These degrees still do not clearly indicate what the image of the representation is. Instead, the fact that we have two factors for the projective representation that become three factors afterwards is rather mysterious.

Fortunately, one of the factors of $F_0(x)$ has degree 28, which is small enough for `Pari/GP` [20] to compute in less than three minutes a much simpler polynomial defining the same number field, namely

$$\begin{aligned} & x^{28} - 12x^{27} + 60x^{26} - 132x^{25} - 30x^{24} + 624x^{23} + 420x^{22} - 7704x^{21} \\ & + 17118x^{20} - 9504x^{19} - 14424x^{18} + 10824x^{17} + 36492x^{16} - 64992x^{15} + 19488x^{14} \\ & + 56064x^{13} - 89604x^{12} + 109296x^{11} - 88368x^{10} - 11472x^9 + 58488x^8 - 130176x^7 \\ & + 34224x^6 - 58272x^5 - 39960x^4 + 32256x^3 + 24480x^2 - 352x - 1776. \end{aligned}$$

This polynomial is good enough for `Magma` [11] to rigorously determine its Galois group in less than a minute. This group turns out to be $\mathrm{PSU}_3(\mathbb{F}_9)$ (as we could have predicted, cf. Remark 20), which explains all the observations made above!

Indeed, first of all one checks thanks to `Pari/GP` [20] that the field defined by the factor of degree 63 of $F_0(x)$ is contained in the compositum of the field defined by that of degree 28 with itself, which shows that these factors have the same splitting field. Next, since there are no nontrivial cube roots of unity in characteristic 3, the quotient $\mathrm{SU}_3(\mathbb{F}_9) \rightarrow \mathrm{PSU}_3(\mathbb{F}_9)$ is actually an isomorphism; in particular, it admits a section. This means that one twist of $\rho_{u,3}$ has image $\mathrm{SU}_3(\mathbb{F}_9)$ (and actually, this twist is the one by the mod 3 cyclotomic character $\chi_3 = \left(\frac{-3}{\cdot}\right)$ since we have seen that $\det \rho_{u,3} = \chi_3^3 = \chi_3$).

Let now H be a non-degenerate Hermitian form on the space $\mathbb{F}_{q^2}^n$, where q is a prime power and $n \in \mathbb{N}$, and let A_n (respectively B_n) be the number of elements $t \in \mathbb{F}_{q^2}^n$ such that $H(t) = 1$ (respectively, such that $H(t) = 0$). Since the norm between finite fields is surjective, A_n is also the number of elements $t \in \mathbb{F}_{q^2}^n$ such that $H(t)$ has prescribed value $y \in \mathbb{F}_q^\times$. From this fact, one easily determines a crossed recurrence relation satisfied by A_n and B_n , from which one deduces that

$$A_n = q^{2n-1} + (-q)^{n-1}, \quad B_n = q^{2n-1} - (q-1)(-q)^{n-1}.$$

For $n = 3$ and $q = 3$, one finds $A_n = 252$ and $B_n = 225$, which explains the shape $224 + 252 + 252$ of the factorization of $F_3(x)$: the first factor corresponds to the nonzero isotropic $t \in T$, and the other two correspond to the t such that $H(t) = 1$ (respectively, such that $H(t) = -1$).

Similarly, the Galois group of the factor of degree 28 of $F_0(x)$ is permutation-isomorphic to $\mathrm{PSU}_3(\mathbb{F}_9)$ acting on the isotropic lines of $\mathbb{F}_9^3$, whereas that of the factor of degree 63 corresponds to the action of $\mathrm{PSU}_3(\mathbb{F}_9)$ on non-isotropic lines.

Finally, the fact that the value of H is not well defined at a non-isotropic t known up to scaling by $\mathbb{F}_9^\times$, but becomes well-defined if we know t up to scaling by

$$\{\pm 1, \pm i\} = \mathrm{Ker} \mathrm{Norm} : \mathbb{F}_9^\times \rightarrow \mathbb{F}_3^\times,$$

explains why the factor of degree 63 of $F_0(x)$ yields two factors of degree 63 of $F_1(x)$ instead of one of degree 126.

Remark 20 As pointed out by an anonymous referee on a previous version of this article, the fact that the image of the representation is unitary is not a coincidence. Indeed, let $\ell \equiv -1 \pmod{4}$ be a prime, identify $\mathbb{F}_{\ell^2}$ with $\mathbb{F}_{\ell}(i)$, $i^2 = -1$, and write H^r for $H^r_{\text{et}}(S_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell\mathbb{Z})$ for brevity. The trace map [17, 24.1.a] and the properness of S over $\mathbb{Q}$ provide us with a canonical isomorphism $H^4 \simeq \mathbb{Z}/\ell\mathbb{Z}(-2)$, so in particular ϕ_S must act trivially on H^4 . The cup product thus defines an alternating bilinear form

$$E : H^2 \wedge H^2 \longrightarrow H^4 \simeq \mathbb{F}_{\ell}(-2)$$

satisfying

$$E(\phi_S u, \phi_S v) = E(u, v)$$

for all $u, v \in H^2$, which may be extended into the bilinear form

$$\begin{aligned} H : H^2 \times H^2 &\longrightarrow \mathbb{F}_{\ell}(i)(-2) \\ (u, v) &\longmapsto E(\phi_S u, v) + i E(u, v) \end{aligned}$$

which satisfies $\Im H = E$ and is Hermitian with respect to ϕ_S (exactly as a Riemann form corresponds to a Hermitian form). Therefore, Galois acts on the Hermitian space H^2 by similarities of ratio given by the -2^{nd} power of the mod ℓ cyclotomic character, i.e.

$$H(\sigma u, \sigma v) = \chi_{\ell}(\sigma)^{-2} H(u, v)$$

for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $u, v \in H^2$. In particular, for $\ell = 3$ the image of our Galois representation must be contained in the unitary group since $\chi_3^{-2} = 1$.

The author had overlooked this when he computed $\rho_{u,3}$, which is why the fact that its image is unitary came to him as a surprise. However, knowing this fact in advance would not have simplified the computation of $\rho_{u,3}$ by the method presented in this article.

Remark 21 We have also obtained a simpler polynomial of degree 63 defining the same number field as the factor of degree 63 of $F_0(x)$. This polynomial is available on the author's web page [12] and the repository [13], and we do not reproduce it here. The polynomial of degree 28 displayed above and this polynomial of degree 63 thus solve the inverse Galois problem for the standard actions of the simple group $\text{PSU}_3(\mathbb{F}_9)$ in respective degrees 28 and 63, and with controlled ramification (only at 2 and 3) to boost! The respective discriminants and signatures of the corresponding number fields are as follows:

Degree	Discriminant	Signature
28	$2^{76} 3^{48}$	(4, 12)
63	$2^{166} 3^{108}$	(7, 28).

Remark 22 Since factors of the $F_i(x)$, $0 \leq i \leq 3$ correspond to towers of quadratic extensions, we can use the techniques presented in [14, Sect. 2] to compute nice polynomials defining the same number fields as the factors of $F_3(x)$. This technique has

the advantage of naturally producing even polynomials such that the field automorphism induced by $x \mapsto -x$ corresponds to the action of $-1 \in \mathbb{F}_9^\times$. This means that for all $D \in \mathbb{Q}^\times$, given such a polynomial $f(x^2)$, the polynomial $f(Dx^2)$ corresponds to the twist of the representation by the quadratic character $(\frac{D}{\cdot})$. By taking $D = -3$, we obtain polynomials corresponding to the representation $\rho_{u,3} \otimes (\frac{-3}{\cdot})$ whose image is the simple group $\mathrm{SU}_3(\mathbb{F}_9) \simeq \mathrm{PSU}_3(\mathbb{F}_9)$, thus solving the inverse Galois problem for the natural permutation representations of this group. These polynomials are also available on the author's web page [12] and the repository [13].

6 Computation of the Image of Frob_p

Let L be a Galois number field, given as the splitting field of an irreducible polynomial $f(x) \in \mathbb{Q}[x]$. In [4], the Dokchitsers show that if the action of $G = \mathrm{Gal}(L/\mathbb{Q})$ on the roots of $f(x)$ in L is known explicitly, then one can compute pairwise coprime resolvents $\Gamma_C(x) \in \mathbb{Q}[x]$ indexed by the conjugacy classes C of G , such that if the image of Frob_p in G lies in C , then the corresponding resolvent $\Gamma_C(x)$ vanishes at $x_p = \mathrm{Tr}_{\mathbb{F}_p}^{A_p}(a^p h(a)) \in \mathbb{F}_p$, where $A_p = \mathbb{F}_p[x]/f(x)$, a is the image of x in A_p , and $h(x) \in \mathbb{Z}[x]$ is a fixed parameter on which the $\Gamma_C(x)$ depend.

The point is that since the $\Gamma_C(x)$ are coprime, they remain coprime mod p for almost all p , so only one of them can vanish at x_p and we can tell in which class C the image of Frob_p lies. The finitely many p for which this is no longer true are usually quite small, and for these p we get not one but several C that may contain Frob_p ; if the conjugacy class of Frob_p is really wanted for such a p , one should recompute the resolvents with another value of the parameter $h(x)$.

Since the quotient $\mathrm{SU}_3(\mathbb{F}_9) \rightarrow \mathrm{PSU}_3(\mathbb{F}_9)$ is actually an isomorphism, and as $\det \rho_{u,3} = (\frac{-3}{\cdot})$ is known explicitly, for each prime p we can recover the image of Frob_p by $\rho_{u,3}$ from its image by the projective version of this representation. Namely, if the image of Frob_p by the projective representation is conjugate to $\overline{M} \in \mathrm{PSU}_3(\mathbb{F}_9)$, then $\rho_{u,3}(\mathrm{Frob}_p)$ is conjugate to $(\frac{-3}{p})M$ in $\mathrm{U}_3(\mathbb{F}_9)$, where $M \in \mathrm{SU}_3(\mathbb{F}_9)$ is the image of $\overline{M}$ by the inverse of this isomorphism.

We thus apply the Dokchitsers' method to the case where $f(x)$ is the polynomial of degree 28 displayed in the previous section. This polynomial has Galois group $\mathrm{PSU}_3(\mathbb{F}_9)$, and its roots are indexed by the lines of $\mathbb{F}_9^3$ that are isotropic with respect to a certain Hermitian form H . We can determine H from the fact that it is preserved by the action of Frob_{11} , and after a change of basis of $\mathbb{F}_9^3$ we can assume that H is the standard Hermitian form.

The group $\mathrm{PSU}_3(\mathbb{F}_9)$ has order 6048, which is small enough that Magma [11] can effortlessly decompose it explicitly into conjugacy classes, which is all we need to compute the resolvents $\Gamma_C(x)$.

Thanks to these resolvents, we can now determine the image of Frob_p by $\rho_{u,3}$ for almost all p . Let us start by the primes between 5 and 67, for which the value of the Hecke eigenvalue $a_p \in \mathbb{Z}[i]$ is given in [5, 2.5].

p	$\rho_{u,3}(\text{Frob}_p)$	a_p from [5]
5	3 possibilities	$-4i - 1$
7	$+\begin{bmatrix} 0 & i+1 & i-1 \\ 0 & i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$4i + 1$
11	$-\begin{bmatrix} 0 & i+1 & i-1 \\ 0 & i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$-10i - 7$
13	$+\begin{bmatrix} 0 & i+1 & i+1 \\ 0 & i-1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$4i - 1$
17	$-\begin{bmatrix} 1 & 0 & 0 \\ 0 & i-1 & i-1 \\ 0 & i+1 & -i-1 \end{bmatrix}$	7
19	$+\begin{bmatrix} 0 & i+1 & i-1 \\ 0 & i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$-14i + 1$
23	$-\begin{bmatrix} 0 & i+1 & i-1 \\ 0 & i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$-4i + 17$
29	$-\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$-12i - 9$
31	3 possibilities	1
37	2 possibilities	$28i - 25$
41	$-\begin{bmatrix} 1 & 0 & 0 \\ 0 & i-1 & i-1 \\ 0 & i+1 & -i-1 \end{bmatrix}$	-5
43	$+\begin{bmatrix} 1 & 0 & 0 \\ 0 & i-1 & i-1 \\ 0 & i+1 & -i-1 \end{bmatrix}$	$30i - 7$
47	$-\begin{bmatrix} 0 & i+1 & -i-1 \\ 0 & -i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$40i + 17$
53	2 possibilities	$-20i + 23$
59	$-\begin{bmatrix} 0 & 0 & -i \\ 0 & -i & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$22i - 39$
61	$+\begin{bmatrix} 0 & 0 & -i \\ 0 & -i & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$20i + 63$
67	$+\begin{bmatrix} 0 & i+1 & -i+1 \\ 0 & -i-1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$-22i + 65$

With our choice our $h(x)$, the resolvents $\Gamma_C(x)$ do not remain coprime mod p for $p \in \{5, 31, 37, 53\}$, so for these p we cannot determine the image of Frob_p without recomputing the resolvents with another choice of $h(x)$. For the other p , we can compute the conjugacy class $C \subset \text{PSU}_3(\mathbb{F}_9)$ containing Frob_p , and we display the image of Frob_p by $\rho_{u,3}$ as

$$\left(\frac{-3}{p}\right) M \in \text{U}_3(\mathbb{F}_9),$$

where M is a fixed representative of its conjugacy class in $\text{SU}_3(\mathbb{F}_9) \simeq \text{PSU}_3(\mathbb{F}_9)$ that we have arbitrarily chosen because many of its coefficients were 0.

Remark 23 The fact that the trace agrees with the reduction mod 3 of the value of a_p given in [5] is convincing evidence that we have correctly computed $\rho_{u,3}$.

Next, we do the same thing for the first twenty primes above 10^{1000} . As one would expect, the $\Gamma_C(x)$ remain coprime mod p for such large p , so we find unambiguously the conjugacy class of $\rho_{u,3}(\text{Frob}_p)$. By looking at the trace, we deduce the value of a_p mod 3. The results are displayed in the table below.

Remark 24 It takes about 100 seconds for `Pari/GP` [20] to certify the primality of such a large prime, but only 4 seconds to compute the conjugacy class of $\rho_{u,3}(\text{Frob}_p)$, almost all of which are spent computing $\text{Tr}_{\mathbb{F}_p}^{A_p}(a^p h(a)) \in \mathbb{F}_p$.

The resolvents $\Gamma_C(x)$ are available on the author's web page [12] and the repository [13].

p	$\rho_{u,3}(\text{Frob}_p)$	$a_p \bmod 3\mathbb{Z}[i]$
$10^{1000} + 453$	$+\begin{bmatrix} 1 & 0 & 0 \\ 0 & i-1 & i-1 \\ 0 & i+1 & -i-1 \end{bmatrix}$	-1
$10^{1000} + 1357$	$-\begin{bmatrix} 0 & 0 & i \\ 0 & i & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$-i$
$10^{1000} + 2713$	$-\begin{bmatrix} 0 & 0 & -i \\ 0 & -i & 0 \\ 1 & 0 & 0 \end{bmatrix}$	i
$10^{1000} + 4351$	$-\begin{bmatrix} 0 & i+1 & -i-1 \\ 0 & -i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$i-1$
$10^{1000} + 5733$	$+\begin{bmatrix} 0 & i+1 & -i+1 \\ 0 & -i-1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$-i-1$
$10^{1000} + 7383$	$+\begin{bmatrix} 0 & 0 & -i \\ 0 & -i & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$-i$
$10^{1000} + 10401$	$+\begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$	1
$10^{1000} + 11979$	$+\begin{bmatrix} 0 & i+1 & i+1 \\ 0 & i-1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$i-1$
$10^{1000} + 17557$	$-\begin{bmatrix} 1 & 0 & 0 \\ 0 & i-1 & i-1 \\ 0 & i+1 & -i-1 \end{bmatrix}$	1
$10^{1000} + 21567$	$+\begin{bmatrix} 1 & 0 & 0 \\ 0 & i-1 & i-1 \\ 0 & i+1 & -i-1 \end{bmatrix}$	-1
$10^{1000} + 22273$	$-\begin{bmatrix} 0 & i+1 & -i-1 \\ 0 & -i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$i-1$
$10^{1000} + 24493$	$-\begin{bmatrix} 0 & i+1 & -i-1 \\ 0 & -i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$i-1$
$10^{1000} + 25947$	$+\begin{bmatrix} 0 & i+1 & i-1 \\ 0 & i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$i+1$
$10^{1000} + 27057$	$+\begin{bmatrix} 0 & i+1 & -i+1 \\ 0 & -i-1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$-i-1$
$10^{1000} + 29737$	$-\begin{bmatrix} 0 & i+1 & -i+1 \\ 0 & -i-1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$i+1$
$10^{1000} + 41599$	$-\begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$	-1
$10^{1000} + 43789$	$-\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	0
$10^{1000} + 46227$	$+\begin{bmatrix} 0 & i+1 & -i-1 \\ 0 & -i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$-i+1$
$10^{1000} + 46339$	$-\begin{bmatrix} 0 & 0 & i \\ 0 & i & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$-i$
$10^{1000} + 52423$	$-\begin{bmatrix} 0 & i+1 & i-1 \\ 0 & i+1 & -i+1 \\ 1 & 0 & 0 \end{bmatrix}$	$-i-1$

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References

1. W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, Compact complex surfaces, Second edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete, Volume 4*. Springer, Berlin, 2004. ISBN: 3-540-00832-2.
2. S. Bloch and A. Ogus, Gersten’s conjecture and the homology of schemes, *Ann. Sci. École Norm. Sup.* (4) 7 (1974), 181–201.
3. Computational aspects of modular forms and Galois representations, Edited by J.-M. Couveignes and B. Edixhoven, with contributions by J. Bosman, J.-M. Couveignes, B. Edixhoven, R. de Jong, and F. Merkl. *Ann. of Math. Stud.*, 176, Princeton Univ. Press, Princeton, NJ, 2011.
4. T. Dokchitser and V. Dokchitser, Identifying Frobenius elements in Galois groups, *Algebra & Number Theory*, Volume 7, Number 6 (2013), 1325–1352.
5. B. van Geemen and J. Top, A non-selfdual automorphic representation of GL_3 and a Galois representation, *Invent. Math.* 117 (1994), no. 3, 391–401.
6. M. Harris, K.-W. Lan, R. Taylor, and J. Thorne, On the rigid cohomology of certain Shimura varieties, *Res. Math. Sci.* 3 (2016), Paper No. 37, 308 pp.
7. T. Ito, T. Koshikawa, and Y. Mieda, Galois representations associated with a non-selfdual automorphic representation of $GL(3)$, arXiv preprint <https://arxiv.org/abs/1811.11544>.
8. U. Jannsen, Mixed motives and algebraic K-theory, *Lecture Notes in Mathematics*, 1400. Springer, Berlin, 1990. xiv+246 pp. ISBN: 3-540-52260-3.
9. J. Jin, Computation of étale cohomology on curves in single exponential time, *Journal de Théorie des Nombres de Bordeaux*, Tome 32 (2020) no. 2, 311–354.
10. The LMFDB Collaboration, The L-functions and Modular Forms Database, home page of the elliptic curve with LMFDB label 24.a4 (Cremona label 24a1), <https://www.lmfdb.org/EllipticCurve/Q/24/a/4>.
11. W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* 24 (1997), 235–265.
12. N. Mascot, Personal web page, <https://www.maths.tcd.ie/~mascotn/>.
13. N. Mascot, GitHub repository, <https://github.com/nmascot/H2>.
14. N. Mascot, Certification of modular Galois representations, *Mathematics of Computation* 87 (2018), 381–423.
15. N. Mascot, Companion forms and explicit computation of PGL_2 number fields with very little ramification, *Journal of Algebra*, Volume 509, 1 (2018), 476–506.
16. N. Mascot, Hensel-lifting torsion points on Jacobians and Galois representations, *Math. Comp.* 89 (2020), 1417–1455.

17. J. S. Milne, Lectures on Etale cohomology, version 2.21, <https://www.jmilne.org/math/>
18. D. A. Madore and F. Orgogozo, Calculabilité de la cohomologie étale modulo ℓ , Algebra Number Theory 9 (2015), no. 7, 1647–1739.
19. S. Bosch, W. Lütkebohmert, and M. Raynaud, Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer, Berlin, 1990. x+325 pp. ISBN: 3-540-50587-3.
20. The PARI Group, PARI/GP development version 2.12.0, Bordeaux, 2018, <http://pari.math.u-bordeaux.fr/>.
21. B. Poonen, D. Testa, and R. van Luijk, Computing Néron-Severi groups and cycle class groups, Compos. Math. 151 (2015), no. 4, 713–734.
22. P. Scholze, On torsion in the cohomology of locally symmetric varieties, Ann. of Math. (2) 182 (2015), no. 3, 945–1066.
23. P. Deligne, Cohomologie étale, Séminaire de géométrie algébrique du Bois-Marie (SGA 4 $\frac{1}{2}$), Lecture Notes in Mathematics, 569, Springer, Berlin, 1977.

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