Correction To: The Complexity of Factors of Multivariate Polynomials

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Correction To: Found Comput Math

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Vladimir Lysikov kindly pointed out an error in the proof of Theorem 5.7. We provide here a corrected statement and its proof.

Theorem 5.7 *For polynomials f over an algebraically closed field k , we have $\underline{L}_q(f) \leq 2\underline{L}(f)$ with $q \leq 2^{\underline{L}(f)^2}$.*

Proof We proceed as in Lehmkuhl and Lickteig [2], who proved a similar bound on the order of approximation for border rank (approximative bilinear complexity).

The proof is based on the following geometric description of the set $\{f \in A_n \mid L(f) \leq r\}$. The field k is assumed to be algebraically closed. A straight-line program Γ is a description for a computation of a polynomial from constants $z_1, \dots, z_m$ and variables $X_1, \dots, X_n$ (recall that we do not allow divisions). Let $\phi_\Gamma(z)$ denote the polynomial in $A_n := k[X_1, \dots, X_n]$ computed by Γ from the list of constants $z \in k^m$. Let r_* denote the number of multiplication instructions of Γ . Then, we have

$$\phi_\Gamma(z) = \sum_{\mu} \phi_{\Gamma, \mu}(z) X^\mu,$$

where the sum runs over all $\mu \in \mathbb{N}^n$ with $\mu_1 + \dots + \mu_n \leq 2^{r_*}$. Moreover, the coefficient polynomials $\phi_{\Gamma, \mu}(z)$ have degree at most 2^{r_*} . We interpret ϕ_Γ as a morphism $k^m \rightarrow \{f \in A_n \mid \deg f \leq 2^{r_*}\}$ of affine varieties. Applying [1, Theorem 8.48] to

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the polynomial map $z \mapsto (z, \phi_\Gamma(z))$, we see that $\deg \text{graph}(\phi_\Gamma) \leq (2^{r_*})^m =: D$. The image $\mathcal{C}_\Gamma$ of ϕ_Γ is an irreducible, constructible set. We have for fixed r that

$$\{f \in A_n \mid L(f) \leq r\} = \bigcup_{\Gamma} \mathcal{C}_\Gamma,$$

where the union is over all straight-line programs Γ of length r .

Assume now that f is in the Zariski-closure of the set on the left-hand side. Then, we have $f \in \overline{\mathcal{C}_\Gamma}$ for some Γ . (We remark that in the case $k = \mathbb{C}$ the Zariski-closure of constructible sets coincides with the closure with respect to the Euclidean topology (cf. [3, Theorem 2.33]).

We apply now two results proven in Lehmkuhl and Lickteig [2] to the morphism ϕ_Γ . Proposition 1 of [2] claims that there exists an irreducible curve $C \subseteq k^m$ such that $f \in \phi_\Gamma(C)$ and $\deg C \leq \deg \text{graph}(\phi_\Gamma)$. The Corollary to Proposition 3 in [2] states that there exists a point $\zeta = (\zeta_1, \dots, \zeta_m) \in k((\epsilon))^m$ such that $F := \phi_\Gamma(\zeta)$ is defined over $k[[\epsilon]]$, satisfies $F_{\epsilon=0} = f$ and such that all formal Laurent series ζ_i have order at least $-\deg C$. We conclude with Lemma 5.6(2) that $L(F) \leq r$ and hence $\underline{L}(f) \leq r$, which proves the nontrivial direction of Theorem 2.4. Moreover, we have shown that there is a straight-line program of length r , which computes F in $k((\epsilon))[X]$ from the X -variables and constants ζ_i having order at least $-\deg C \geq -D$. By a similar reasoning as in the proof of Lemma 5.6(1), we can construct from this a straight-line program of length at most $2r$, which computes in $k[[\epsilon]][X]$ an element of the form $\epsilon^q f + \epsilon^{q+1} f'$ with $q \leq 2^{r_*} D = 2^{(m+1)r_*}$. We therefore have $\underline{L}_q(f) \leq 2r$. To complete the proof, we note that $(m+1)r_* \leq r^2$, unless $m = r$ and $r = r_*$. However, in this case, the components of ϕ_Γ have degree at most 1 and we get $q \leq 2^{r_*}$ since $\deg \text{graph} \phi_\Gamma \leq 1$. $\square$

By tracing the proofs of the above results, it is straightforward to show the following statement.

Remark 5.8 By counting only nonscalar multiplications, one can introduce the notions $\underline{L}^{ns}$, $\underline{L}_q^{ns}$ in an analogous way. We then have $\underline{L}^{ns} = \underline{L}_\infty^{ns} = \underline{L}_q^{ns}$.

References

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3. D. Mumford. *Algebraic Geometry I: Complex Projective Varieties*. Springer, 1976.

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