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Optimal additional voluntary contribution in DC pension schemes to manage inadequacy risk

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Abstract

In defined contribution pension schemes the member bears the investment risk and her main concern is to obtain an inadequate fund at retirement. To address inadequacy risk, flexibility is often given to the member to pay additional voluntary contributions (AVCs) into the fund. In many countries the AVC schemes allow members of the workplace pension plan to increase the amount of retirement benefits by paying extra contributions. In this paper, we define a target-based optimization problem where the member of an AVC scheme can choose at any time the investment strategy and the AVCs to the fund. In setting the problem, the member faces a trade-off between the importance given to the stability of payments during the accumulation phase and the achievement of the desired annuity at retirement. We derive closed-form solutions via dynamic programming and prove that (i) the optimal fund never reaches the target final fund, (ii) the optimal amount invested in the risky asset is positive, and (iii) the optimal AVC is higher than the target one. We run numerical simulations to allow for different member's preferences, and perform sensitivity analyses to assess the controls' robustness.

Keywords Stochastic optimal control · Dynamic programming · Defined contribution pension scheme · Additional voluntary contributions · Target-based approach · Net replacement ratio · Adequacy risk

JEL Classification C61 · D81 · G11

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1 Introduction and motivation

Increasing life expectancy, ageing populations and low interest rates have led to concerns around the sustainability of traditional defined benefit (DB) pension systems (OECD 2022). To address this issue, there has been an intense pension reform activity over the past 2 decades (OECD 2021). According to Bonenkamp et al. (2017), pension reforms that are undertaken in nearly all OECD countries in different forms (reducing the benefits' generosity via stricter eligibility rules, and/or lowering the benefits' level) can all be seen as a welfare-best response to the problem of population ageing. Challenges to achieving adequate levels of retirement income have been addressed with a general shift towards individual defined contribution (DC) arrangements, which has pushed the responsibilities to make financial decisions and the risks of financing retirement onto individuals. Over the period 2001–2021 in all OECD countries occupational DC plans have been gaining prominence at the expense of DB plans, also in countries with a historically high assets share in DB plans (OECD 2023).

The difference between a DB and a DC scheme is the way the financial risk is treated: it is borne by the sponsor (the employer) in a DB scheme, while it is borne by the member in a DC scheme. The financial risk borne by the member has two components: (i) investment risk—the risk that low investment performance during the accumulation phase leads to a low investment fund at retirement, and (ii) annuity risk—the risk that retirement happens in a period of low interest rates, yielding a low conversion factor from fund into annuity (see Vigna and Haberman 2001). A common way to deal with the annuity risk is to postpone the purchase of the annuity. If the retiree chooses this possibility (that is called sometimes "income drawdown option", "phased withdrawals", "programmed withdrawals", "do-it-yourself-and-then-switch" strategy, see e.g. Milevsky (1998), Gerrard et al. (2012), and Olivieri and Pitacco (2015)), she leaves the money invested in the pension fund, withdraws periodic amounts and chooses a better annuitization time some years after retirement. By doing so, the member hopes to annuitize in a moment characterized by better interest rates than those prevailing at retirement. Bernhardt and Donnelly (2018) present an extensive review of decumulation strategies post retirement.

Due to the combined effect of investment and annuity risk, the major drawback of DC pension schemes consists in the high degree of uncertainty regarding the level of pension benefits: a system that defines a given level of contributions cannot also define or guarantee the level of benefits received, see Knox (1993). Indeed, two individuals with same work histories could receive completely different pension incomes because they retire in different years with different market conditions. As a consequence, two individuals with same work histories could reach a different replacement ratio, defined as the ratio between the initial pension income and the final wage. In fact, Antolin (2009) finds that the replacement ratio enjoyed by retirees of DC pension plans with the same working and contribution history (5% contribution for 40 years service) and retiring in years 1940–2008 varies remarkably: the range is as broad as 3–80% for Japan, and 18–55% for the U.S.A. The impact of market conditions on retirement savings accumulated in DC pension plans can be particularly dramatic in the case of bad market performance just prior to retirement. For instance, due to the 2008 market crash, the replacement ratio could have suffered an approximate 10% drop from 2007 to 2008: a retiree in U.S.A. in 2007 aged 65 could have enjoyed a replacement ratio of about 24%, while the same individual retiring in 2008 would have obtained only about 15%.

The documented large variability of replacement ratios enjoyed by members of DC schemes causes a large inequality experienced by individuals belonging to different cohorts. As a consequence, the adequacy of retirement benefits in DC schemes, as measured by the replacement ratio, is at risk. It must be specified that the concept of adequacy of pensions is complex: there is no universal consensus on exactly what an adequate retirement income is, for there are a variety of perspectives and definitions, see OECD (2013).[1] Although there are different approaches, researches conducted by international institutions in the area of pensions provide definitions of adequacy that are homogeneous, see Rosado-Cebrian et al. (2020). According to the broadest definition provided by OECD (2013)—and likely the one which is most relevant from the individual's perspective—a retirement income is adequate *"if it replaces a worker's earnings at a level which enables him or her to maintain a standard of living in retirement comparable to that enjoyed in working life"*. The retirement income mentioned in the definition is considered in its entirety, inclusive of the public pay-as-you-go (PAYG) pension and the pension income provided by funded pension funds. In this paper, we refer to adequacy of retirement benefits using the OECD (2013) broad definition (focusing on the part that is paid out by funded pension funds), and consequently choose the replacement ratio to measure it. Moreover, the income replacement ratio has historically been the most widespread metric for evaluating the adequacy of retirement savings by sponsors/employers, financial planners, pension plan advisors, academics and public policy makers with different aims: (i) comparing different plan outcomes, (ii) helping employees to understand the impacts of their savings decisions on their retirement income, (iii) assessing whether workers are able to afford retirement or if they need encouragement to build up more savings (Bajtelsmit et al. 2013; MacDonald et al. 2016; Alonso-García et al. 2018).

Four measures could be adopted in order to deliver adequate retirement savings for DC pension funds (see Antolin 2009): (i) increasing contributions, (ii) postponing retirement, (iii) adopting conservative investment strategies in the period prior to retirement, (iv) managing inflation risk in the decumulation phase with appropriate index-linked annuities. In this paper, we focus on the first measure.

The possibility of increasing periodically the contribution rate has been considered recently by the pensions regulators, under the name *additional voluntary contributions* (AVCs), also due to the higher levels of uncertainty experienced during the COVID-19 pandemics. According to OECD (2021), a number of countries allow the possibility for workers to make AVCs into the pension fund. In Australia, employees have no obligation to contribute to a plan but can make voluntary contributions on top of the employer's contributions. In New Zeland, workers in the "Kiwisaver Plan" are required to contribute with a minimum of 3% but have the option to pay a higher contribution (4%, 6%, 8% or 10% of the wage). In Poland, the "Employee Capital Plans (PPK)" requires a minimum 2% contribution for employees and 1.5% for employers,

[1] Possible definitions may be related to (i) equivalence to pre-retirement standard of living (ii) sufficiency to cover all future expected expenses or (iii) the minimum needs to alleviate poverty.

with the possibility for both of making additional contributions of up to 2.5% and 2%, respectively. The OECD analysis highlights the role of the AVCs during the pandemics, when some countries provided flexibility on mandatory contributions. Due to this flexibility, in some cases the effective contributions in 2021 have been higher than in 2020, thanks also to the additional contributions. For example, according to official data from the Inland Revenue Department of New Zeland,[2] in 2021 there was a (small) reduction in employees deductions, partially offset by a (small) increase in their voluntary contributions. Also in the UK workers are allowed to make AVCs. For example, the UK Government reports the financial position of its Civil Service Additional Voluntary Contribution Scheme (CSAVCS),[3] which provides for civil service employees to make AVCs to increase their pension entitlements or to increase life cover. Making AVCs is even proposed in the House of Commons Staff Pension Schemes as a way for workers in the Parliament to boost their pensions.[4] In Ireland, AVC to the occupational pensions is one of the possibilities for workers to set aside savings under the third pillar.[5] Nkeki (2017) highlights that in Nigeria, the Pension Reform in 2014 allows for AVC on top of the compulsory minimum contribution of 8% by workers. Ombuki and Oteki (2019) investigated the key determinants of voluntary contributions in Kenya, and the survey indicated an inverse relation between income and AVCs: members in the lowest earning brackets participated more in AVCs than those on higher earning brackets, the plausible reason being that those earning less not only contributed less but also tended to have lower contributions by their employers, and therefore felt the need to make AVCs.

Apparently, the actuarial literature on AVCs in a DC pension scheme is scarce. The typical way to deal with the investment risk of DC pension funds has been the choice of the investment strategy only. Indeed, the literature on the accumulation phase of DC pension schemes is full of examples of optimal investment strategies to attack the investment risk.[6] In those papers, the contribution paid into the fund is assumed to be a fixed proportion of the wage, in line with the essence of a DC pension scheme. In other words, in the optimization problems mainly solved in the literature, the contribution rate is not a control variable. To the best of our knowledge, the only exception is Nkeki (2017) in a jump-diffusion model with constant relative risk aversion utility function. A few papers find the optimal investment strategy with constant absolute risk aversion utility function in a DC scheme modelling the AVCs as a stochastic state variable rather than a control variable, see Akpanibah and Oghen'Oro (2018) and Chinyere

[2] See Inland Revenue NZ database at https://www.ird.govt.nz/about-us/tax-statistics/kiwisaver/datasets.

[3] See the Civil Superannuation annual report and account 2021-22 at https://www.gov.uk/government/publications/civil-superannuation-annual-account-2021-to-2022/civil-superannuation-annual-report-and-account-2021-22-html.

[4] See the House of Commons Staff Pension Plan guidebook at https://www.parliament.uk/globalassets/documents/commons-resources/Staff-handbook/chapter-27-pensions.pdf.

[5] See https://assets.gov.ie/200479/05ab1b26-da17-4e86-b1e7-653de0d4791b.pdf.

[6] See, among many others, Boulier et al. (2001), Haberman and Vigna (2002), Deelstra et al. (2003), Devolder et al. (2003), Battocchio and Menoncin (2004), Cairns et al. (2006), Xiao et al. (2007), Gao (2008), and Di Giacinto et al. (2011) for the maximization of expected utility, and He and Liang (2013), Yao et al. (2013), Yao et al. (2014), Vigna (2014), Menoncin and Vigna (2017), Menoncin and Vigna (2020), Guan and Liang (2015), and Wu et al. (2015) for mean-variance criteria.

et al. (2022). They find, intuitively, that the payment of AVCs reduces the investment in the risky asset.

It is clear that paying higher contributions into the pension fund could be a way to reduce the risk of an inadequate retirement income. The indications provided by OECD (2021) during the COVID-19 pandemics go in the same direction. Yet, the analysis of the possibility of paying additional contributions seems to be an unexplored area of research in the actuarial community.

To fill in this gap in the literature, in this paper we solve an optimization problem in a DC pension scheme with two control variables: the investment strategy and the AVC paid by the member on top of the employer's contribution. The optimization problem is defined reflecting two conflicting needs of the member: reaching an adequate replacement ratio at retirement coupled with the desire for stable contribution payments over time. We solve the problem via dynamic programming and provide closed-form solutions. Numerical simulations show that the desired net replacement ratio can be approached very closely by paying appropriate AVCs into the pension fund, and that the contributions' stability can be achieved too, but at the price of a lower net replacement ratio. As expected, a longer accumulation phase as well as higher contributions paid by the employer help to better manage the trade-off faced by the worker.

The remainder of the paper is structured as follows. In Sect. 2, the financial market and the evolution of the fund are defined, as well as the two control variables—the investment strategy and the AVC. In Sect. 3, we specify the target-based preferences of the member and define the optimization problem. In particular, the member aims at minimizing both the adequacy risk and the contribution risk, by minimizing simultaneously (i) the square of the distance of the final fund from a targeted wealth (which makes the member able to purchase the desired annuity) and (ii) the square of the distance of the contribution paid from the target level of contribution rate. Sections 4 and 5 contain the theoretical results of the paper. In Sect. 4, we derive the analytical solution using dynamic programming techniques and provide the optimal investment strategy and the optimal AVC in closed-form, while in Sect. 5 we prove some additional theoretical results: (i) the optimal amount invested in the risky asset is positive, (ii) the optimal contribution paid is higher than the target one, (iii) the final fund is lower than the target one. Section 6 is devoted to numerical simulations: we start with a base case scenario for the value of the parameters, adopting first the optimal unconstrained controls derived in Sect. 4, and then suboptimal clipped controls where borrowing and short-selling of the risky asset are forbidden; stress tests in the financial constrained case only are then carried out, by changing the value of a variety of parameters. Finally, Sect. 7 concludes and outlines future research.

2 The model

The financial market is described by a Black and Scholes framework with a riskless asset and a risky asset. The risk is described by a Brownian Motion $W(t)$ defined on a complete filtered-probability space $\{\Omega, \mathcal{F}(t), \mathcal{P}\}$, with $\mathcal{F}(t)$ being the filtration generated by $W(t)$. The price of the riskless asset $G(t)$ evolves according to the following deterministic ordinary differential (ODE) equation:

$$dG(t) = rG(t)dt \tag{1}$$

where $r \geq 0$. The price of the risky asset $S(t)$ evolves according to a Geometric Brownian Motion with constant drift $\lambda > r$ and volatility $\sigma > 0$:

$$dS(t) = \lambda S(t)dt + \sigma S(t)dW(t) \tag{2}$$

The member joins the fund at time 0 aged χ with a transfer value of $x_0 \geq 0$, works for the next T years and then retires at time T aged $\chi + T$. There is no risk of unemployment, and her wage $w(t)$ grows exponentially at a constant rate $g \geq 0$:

$$dw(t) = gw(t)dt \tag{3}$$

starting from an initial salary $w(0) = w_0$. The exponential assumption for the salary growth is in line with previous literature on DC pension schemes: exponential deterministic or stochastic salary growth is considered e.g. in Boulier et al. (2001), Deelstra et al. (2003), Battocchio and Menoncin (2004), and Menoncin and Vigna (2017), to name a few.

At any time $t \in [0, T]$, the member chooses (i) the proportion $y(t)$ of the portfolio wealth invested in the risky asset, and (ii) the additional voluntary contribution (AVC) $c(t)$. The employer contributes to the fund with a fixed proportion $\gamma \in (0, 1)$ of the wage $w(t)$, while the only contribution paid by the member is the AVC. For simplicity, taxation, commission expenses and transaction costs are ignored.

Therefore, the value of the fund $X(t)$ evolves according to the following stochastic differential equation (SDE), with initial condition x_0:

$$\begin{cases} dX(t) = \{X(t)\left[y(t)(\lambda - r) + r\right] + \gamma w(t) + c(t)\}\, dt + X(t)y(t)\sigma dW(t) \\ X(0) = x_0 \end{cases} \tag{4}$$

The control process $u(t)$ consists therefore in a couple of stochastic processes:

$$\{u(t)\}_{t\in[0,T]} = \left(\{y(t)\}_{t\in[0,T]}, \{c(t)\}_{t\in[0,T]}\right) \tag{5}$$

The set of admissible strategies $\mathcal{U}$ is defined as the set of $\mathbb{R}^2$-valued stochastic processes $u = \{u(t)\}_{t\in[0,T]}$ that are Markov control processes, $\mathcal{F}_t$-adapted and such that the SDE (4) has a unique strong solution.

3 The optimization problem

The preferences of the member are driven by two conflicting objectives. On the one hand, the member desires stability for the contribution inflow, i.e. she has a target for the ideal AVC, say a fraction $\eta \in (0, 1)$ of the wage, and would like to pay AVCs that do not deviate too much from η times the wage.[7] On the other hand, she pays additional contributions because she aims at achieving a given targeted pension rate at retirement upon conversion of the final fund $X(T)$ into lifetime annuity. The targeted pension rate b_T at retirement time T is defined as a fraction $\alpha \in (0, 1)$ of the last wage, in order to obtain a net replacement ratio equal to α. The loss experienced by the member is composed by two parts: an immediate, continuous loss experienced at t whenever the voluntary contribution paid $c(t)$ is different from $\eta w(t)$, plus a final loss at time T, if the income provided by the annuity that can be purchased with the final fund $X(T)$ differs from b_T. Therefore, the continuously experienced running loss $L(t, c(t))$ and the final loss $\Phi(T, X(T))$ are defined, respectively, as:

$$L(t, c(t)) = v\,(\eta w(t) - c(t))^2 \tag{6}$$

$$\Phi(T, X(T)) = \left(b_T a_{\chi+T} - X(T)\right)^2 \tag{7}$$

where $b_T = \alpha w(T)$ is the initial pension income that the member wishes to achieve at retirement, $a_{\chi+T}$ is the price of the constant real unitary lifetime annuity for a policyholder aged $\chi + T$ and $v > 0$ is a weight that gives relative importance to the running loss with respect to the terminal one (see Remark 1). The total expected loss at time 0 with initial fund x_0 is:

$$\mathbb{E}_{0,x_0}\left[\int_0^T e^{-\rho s} L(s, c(s))ds + e^{-\rho T}\Phi(T, X(T))\right] \tag{8}$$

where $\rho > 0$ is the subjective inter-temporal discount rate.

Remark 1 Notice that the weight $v > 0$ plays a crucial role in displaying the member's preferences. Higher values of v mean that the member is more willing to keep the additional contributions close to the target AVC, i.e. the need for stability in the contributions payment is higher, and the importance given to the achievement of the desired net replacement ratio is lower. Vice versa, when v is lower, the importance given to the achievement of the desired annuity is higher, and the desire for contributions' stability is lower. The trade-off between the two conflicting objectives of contributions' stability and adequate net replacement ratio is solved by an appropriate choice of v, which is then crucial for the optimization outcomes, as we will see in Sect. 6. In the remainder of the paper, we will refer to v as the weight given to the contributions' stability. It is worth noting that, in the context of notional defined contribution pension systems, Devolder et al. (2021) also consider a trade-off by defining an optimization problem with the two different objectives of social adequacy and contribution rate sustainability, each weighted according to its relative importance.

Remark 2 In setting this problem, the annuity price $a_{\chi+T}$ has been fixed at initial time 0, when the member joins the scheme. In other words, we assume that a guaranteed

[7] We see from (4) that the member pays only the AVC into the fund. However, in Sect. 5 we prove that the optimal AVC is larger than the target one, so the minimum contribution paid by the member is effectively a percentage η of the salary.

annuitization rate (GAR) is applied, rather than a current annuitization rate (CAR), see Olivieri and Pitacco (2015). However, in reality the price of the lifetime annuity for a retiree aged $\chi + T$ at retirement time T will depend on both the prevailing yield rates and the current demographic assumptions at that time, meaning that a CAR is commonly applied rather than a GAR. The CAR is clearly an unknown quantity at time 0. Therefore, if the annuity price at time T turns out to be higher than the one originally stated, it might well happen that the pension income received by the pensioner is inadequate even if the targeted wealth has been reached. This is exactly the annuity risk already mentioned above, which Boulier et al. (2001) suggest to treat by adding a protection on the annuity with the guarantee of a minimal annuity. The price of this protection would consist in an extra-contribution paid by the member into the fund. In this paper, we do not explore this possibility, because we focus only on the investment risk borne during the accumulation phase.

The member wishes to minimize the total expected loss (8) from 0 to T, choosing amongst all possible investment and AVC strategies in the set of admissible strategies $\mathcal{U}$.

The problem is solved using the dynamic programming principle, hence we define the criterion function $J_{t,x}(u(.))\colon \mathcal{U} \to \mathbb{R}$:

$$J_{t,x}(u(.)) = J_{t,x}(y(.), c(.)) = \mathbb{E}_{t,x}\left[\int_t^T e^{-\rho s} L(s, c(s))ds + e^{-\rho T}\Phi(T, X(T))\right] \tag{9}$$

and define the value function $V(t, x)$ as:

$$V(t, x) = \inf_{u(.)\in\mathcal{U}} J_{t,x}\,(u(.)) \tag{10}$$

We aim to find $V(t, x)$ and the associated optimal control $\{u^*(t)\} = (\{y^*(t), c^*(t)\})$ such that:

$$V(t, x) = J_{t,x}(u^*(.)) = J_{t,x}(y^*(.), c^*(.)) \tag{11}$$

Regarding the formulation of the problem, we notice the following:

- The use of quadratic loss functions is quite common in the literature on pension funds: in the context of DB pension funds some examples include Haberman and Sung (1994); Owadally and Haberman (2004); Josa-Fombellida and Rincón-Zapatero (2008, 2010, 2012). In the context of DC schemes, see, among others, Cairns (2000), Vigna and Haberman (2001), Haberman and Vigna (2002), Gerrard et al. (2004), Gerrard et al. (2006), Emms (2010), and Gerrard et al. (2012). Moreover, the quadratic target-based loss function has been shown to be a particular case of the celebrated mean-variance approach, see Zhou and Li (2000).
- Also the objective of achieving the contributions' stability is not new in the literature on pension plans. The importance of reducing the contributions' volatility for the U.S.A. public sector defined benefit pension plans has been stressed by Clair

and Guzman (2018), who underline that high fluctuations in the required pension contributions pose a challenge to state budgeting, and all states should address the issue of fluctuating contributions. Smoothness of contributions is a desirable objective also in DB pension schemes as a way to minimize the variability of the funding level or of the unfunded liability in the plan (Owadally and Haberman 2004). Hence, different approaches to reduce the contribution rate volatility in DB plans are introduced in many papers in the actuarial literature: Owadally and Haberman (2004), Haberman and Sung (2005), Huang and Cairns (2006), and Josa-Fombellida and Rincón-Zapatero (2008), to name a few. In most of these papers, the authors minimize a linear combination of the contribution rate risk and the solvency risk, the latter being a typical source of risk of DB plans.

4 The analytical solution

The problem of minimizing (9) is a stochastic optimal control problem, which we solve using dynamic programming, see for instance Yong and Zhou (1999); Björk (1998).

The value function $V(t, x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) Equation:

$$\inf_{u(.)\in\mathcal{U}} \left[e^{-\rho t} L(t, c(t)) + \mathcal{L}^u V(t, x)\right] = 0 \tag{12}$$

with the boundary condition:

$$V(T, x) = e^{-\rho T}\Phi(T, x) \tag{13}$$

where $\mathcal{L}^u V(t, x)$ is the linear infinitesimal operator applied to $V(t, x)$, given by:

$$\mathcal{L}^u V(t, x) = V_t + \left[x\,(y(\lambda - r) + r) + \gamma w(t) + c\right] V_x + \frac{1}{2}x^2 y^2 \sigma^2 V_{xx} \tag{14}$$

where V_t, V_x and V_{xx} stand for $\frac{\partial V(t,x)}{\partial t}$, $\frac{\partial V(t,x)}{\partial x}$ and $\frac{\partial^2 V(t,x)}{\partial x^2}$ respectively.

To solve Equation (12), we first fix an arbitrary point $(t, x) \in [0, T] \times \mathbb{R}$ and see it as a static optimization problem in $\mathbb{R}^2$. We lighten notation in (12) by denoting the term in square brackets by $\Psi(t, x, y, c)$, i.e.:

$$\begin{aligned}\Psi(t, x, y, c) = {} & e^{-\rho t} v\,(\eta w(t) - c)^2 + V_t + \left[x\,(y(\lambda - r) + r) + \gamma w(t) + c\right] V_x \\ & + \frac{1}{2}x^2 y^2 \sigma^2 V_{xx}\end{aligned} \tag{15}$$

so that the minimization, in y and c, is written as:

$$\inf_{(y,c)} \Psi(t, x, y, c) = 0 \tag{16}$$

We find the minimizers y^* and c^* in Equation (16) using necessary First Order Conditions (FOCs):

$$\frac{\partial \Psi(t,x,y^*,c)}{\partial y} = 0 \Rightarrow \quad y^* = -\left(\frac{\lambda - r}{x\sigma^2}\right)\frac{V_x}{V_{xx}} \tag{17}$$

$$\frac{\partial \Psi(t,x,y,c^*)}{\partial c} = 0 \Rightarrow \quad c^* = \eta w(t) - \frac{1}{2v}e^{\rho t}V_x \tag{18}$$

which depend on the partial derivatives of $V(t,x)$. The sufficient Second Order Condition (SOC) for the Hessian matrix is checked afterwards.

Replacing the expressions of y^* and c^* given by (17) and (18) back into the HJB Equation (12) leads to the following partial differential equation (PDE):

$$V_t + \left[xr + (\gamma + \eta)w(t)\right]V_x - \left[\frac{e^{\rho t}}{4v} + \frac{\beta^2}{2V_{xx}}\right]V_x^2 = 0 \tag{19}$$

where β is the Sharpe Ratio:

$$\beta = \frac{\lambda - r}{\sigma} \tag{20}$$

To solve the PDE we use an *ansatz*. Since the loss function is quadratic, it is reasonable to assume that $V(t,x)$ is in the form:

$$V(t,x) = e^{-\rho t}[A(t)x^2 + B(t)x + C(t)] \tag{21}$$

In such case, the boundary condition of Equation (13) becomes:

$$V(T, X(T)) = e^{-\rho T}\Phi(T, X(T)) = e^{-\rho T}[b_T \mathrm{a}_{\chi+T} - X(T)]^2$$

that implies

$$A(T) = 1 \qquad B(T) = -2b_T\mathrm{a}_{\chi+T} \qquad C(T) = b_T^2\mathrm{a}_{\chi+T}^2 \tag{22}$$

The partial derivatives of the *ansatz* are given by:

$$V_t = e^{-\rho t}\left[\left(A'(t) - \rho A(t)\right)x^2 + \left(B'(t) - \rho B(t)\right)x + \left(C'(t) - \rho C(t)\right)\right] \tag{23}$$

$$V_x = e^{-\rho t}\left[2A(t)x + B(t)\right] \tag{24}$$

$$V_{xx} = 2e^{-\rho t}A(t) \tag{25}$$

Replacing the partial derivatives (23), (24) and (25) into the PDE (19) yields

$$e^{-\rho t}\left[\left(A'(t) - \rho A(t)\right)x^2 + \left(B'(t) - \rho B(t)\right)x + \left(C'(t) - \rho C(t)\right)\right]$$
$$+\left[xr + (\gamma + \eta)w(t)\right]e^{-\rho t}\left[2A(t)x + B(t)\right]$$

$$-\left[\frac{e^{\rho t}}{4v}+\frac{\beta^2}{2\left(2e^{-\rho t}A(t)\right)}\right]\left[e^{-\rho t}\left(2A(t)x+B(t)\right)\right]^2=0 \tag{26}$$

which holds $\forall (t,x) \in [0,T]\times\mathbb{R}$. Combining it with the boundary conditions of Equation (22), it leads to a system of three initial value problems (IVPs) that needs to be verified if the *ansatz* formulation is to work:

$$\begin{cases} A'(t)=\left(\rho-2r+\beta^2\right)A(t)+\dfrac{A^2(t)}{v} \\ A(T)=1 \end{cases} \tag{27}$$

$$\begin{cases} B'(t)=\left(\rho-r+\beta^2+\dfrac{A(t)}{v}\right)B(t)-2(\gamma+\eta)w(t)A(t) \\ B(T)=-2b_T \mathrm{a}_{\chi+T} \end{cases} \tag{28}$$

$$\begin{cases} C'(t)=\rho C(t)-(\gamma+\eta)w(t)B(t)+\left(\dfrac{1}{4v}+\dfrac{\beta^2}{4A(t)}\right)B^2(t) \\ C(T)=b_T^2\mathrm{a}_{\chi+T}^2 \end{cases} \tag{29}$$

Notice that Equations (17) and (18) for the optimal control depend only on the partial derivatives V_x and V_{xx}, given by (24) and (25), neither of which depends on $C(t)$. Therefore, in the following we provide solutions only for $A(t)$ and $B(t)$.

It turns out that a crucial quantity is δ defined as follows:

$$\delta=2r-\rho-\beta^2 \tag{30}$$

In the remaining of the paper, we will assume that $\delta \neq 0$, the case $\delta=0$ leading to similar results (see Footnote 8). The differential equation in the IVP (27) is a Bernoulli type. The solution is given by:[8]

$$A(t)=\frac{v\delta e^{\delta(T-t)}}{e^{\delta(T-t)}+v\delta-1} \tag{31}$$

It is easy to show that[9]

$$A(t)>0 \qquad \forall\delta\in\mathbb{R},\quad \forall t\in[0,T] \tag{32}$$

Noting from (3) that the wage at time t is given by

$$w(t)=w_0e^{gt}$$

[8] If $\delta=0$, $A(t)=v(v+T-t)^{-1}$. Notice that in this case $A(t)>0$. Furthermore, if $\delta=0$, Equations (34), (35), (36) and (38) still hold, while Equations (37) and (39) should be rewritten accordingly.

[9] In all cases, we have $v>0$. In the case $\delta=0$, see Footnote 8. In the case $\delta\neq 0$, if $\delta>0$ (<0), both the numerator and the denominator are >0 (<0).

the solution of the IVP in Equation (28) for $B(t)$ is:

$$B(t) = \frac{-2v\delta e^{\delta(T-t)}}{e^{\delta(T-t)} + v\delta - 1} \left[\frac{(\gamma+\eta)w_0}{g-r} \left(e^{gt} - e^{gT - r(T-t)} \right) + b_T \mathrm{a}_{\chi+T} e^{-r(T-t)} \right] \tag{33}$$

Note that $B(t)$ can be written as:

$$B(t) = -2A(t) \left[\frac{(\gamma+\eta)w_0}{g-r} \left(e^{gt} - e^{gT - r(T-t)} \right) + b_T \mathrm{a}_{\chi+T} e^{-r(T-t)} \right] \tag{34}$$

4.1 Optimal control

To find the optimal control, first simplify the expressions for y^* and c^*, given by (17) and (18), using Equations (20), (24) and (25):

$$y^*(t,x) = -\frac{\beta}{\sigma} \left[1 + \frac{B(t)}{2A(t)x} \right] = -\frac{\beta}{\sigma x} [x - h(t)] \tag{35}$$

$$c^*(t,x) = \eta w(t) - \frac{1}{2v} [2A(t)x + B(t)] = \eta w(t) - \frac{A(t)}{v} [x - h(t)] \tag{36}$$

where

$$h(t) = -\frac{B(t)}{2A(t)} = \frac{(\gamma+\eta)w_0}{g-r} \left(e^{gt} - e^{gT - r(T-t)} \right) + b_T \mathrm{a}_{\chi+T} e^{-r(T-t)} \tag{37}$$

Therefore, we obtain the expressions for the feedback control maps:

$$y^*(t,x) = -\frac{\beta}{\sigma} \left\{ 1 - \frac{1}{x} \left[\frac{(\gamma+\eta)w_0}{g-r} \left(e^{gt} - e^{gT - r(T-t)} \right) + b_T \mathrm{a}_{\chi+T} e^{-r(T-t)} \right] \right\} \tag{38}$$

$$\begin{aligned} c^*(t,x) = \eta w_0 e^{gt} &- \frac{\delta e^{\delta(T-t)}}{e^{\delta(T-t)} + v\delta - 1} \\ &\times \left[x - \frac{(\gamma+\eta)w_0}{g-r} \left(e^{gt} - e^{gT - r(T-t)} \right) - b_T \mathrm{a}_{\chi+T} e^{-r(T-t)} \right] \end{aligned} \tag{39}$$

4.2 Second order conditions

Now we check the sign of Hessian of $\Psi(t, x, y, c)$ given by Equation (15). The partial derivatives of $\Psi(y, c)$ are:

$$\frac{\partial^2 \Psi(y,c)}{\partial c \partial y} = 0 \qquad \frac{\partial^2 \Psi(y,c)}{\partial y^2} = x^2 \sigma^2 V_{xx} \qquad \frac{\partial^2 \Psi(y,c)}{\partial^2 c} = 2v e^{-\rho t}$$

Therefore, given a generic vector $u = (y, c) \in \mathbb{R}^2$, the Hessian Matrix of $\Psi(y, c)$ is:

$$H_\Psi(y, c) = \begin{bmatrix} x^2\sigma^2 V_{xx} & 0 \\ 0 & 2ve^{-\rho t} \end{bmatrix}$$

Due to (25) and (32), we have $V_{xx} > 0 \quad \forall t \in [0, T]$. Hence we see that $H_\Psi(y, c)$ is positive definite for every point $(y, c) \in \mathbb{R}^2$, which implies that $\Psi(y, c)$ is strictly convex. Therefore, the FOCs become sufficient to guarantee that the unique stationary point (y^*, c^*) is indeed the unique point of global minimum.

5 Three theoretical results

In this section, we prove three theoretical results that shed some light on the nature of the optimal investment in the risky asset, and on the relationships of the optimal contribution and the optimal final fund versus, respectively, the targeted contribution and the targeted final fund.

The first result is that the optimal amount invested in the risky asset is positive. The second result is that the optimal contribution paid into the fund is higher than the targeted contribution. The third result is that the final fund under optimal control is lower than the targeted final fund. These results are due to the quadratic nature of the loss functions and are in line with previous works on DC pension schemes, see, among others, Gerrard et al. (2004) and Menoncin and Vigna (2017).

The theoretical results, which are proved in Proposition 1, are based upon the following assumption.

Assumption 1 Let the financial-labour market and the member's preferences be described as in Sects. 2 and 3. Then, we say that this assumption is satisfied if

$$b_T \text{a}_{\chi+T} > x_0 e^{rT} + \int_0^T (\gamma + \eta) w(s) e^{r(T-s)} ds \tag{40}$$

Remark 3 Notice that condition (40) is satisfied if the target fund chosen (the l.h.s. of (40)) is larger than the final fund at time T that one would have by investing the initial fund and the future flows (i.e. the contributions paid by the employer and the target contributions paid by the employee) in the riskless asset (the r.h.s. of (40)). In other words, the target chosen by the member must be "large enough": if the target were equal or lower than such threshold, it would be possible to reach or exceed it at time T just by choosing the trivial strategy of investing 100% in the riskless asset without the need to take any further risk. Therefore, Assumption 1 is necessary for the optimization problem to be interesting.

We are now ready to prove the theoretical results mentioned above.

Proposition 1 *Let the financial-labour market and the member's preferences be described as in Sects. 2 and 3, the optimal controls be given as in Sect. 4.1, and let Assumption 1 hold. Then:*

1. *the optimal amount invested in the risky asset at any time* $t \in [0, T]$ *is strictly positive, i.e.* $X^*(t)y^*(t) > 0$*;*
2. *the optimal contribution paid into the fund at any time* $t \in [0, T]$ *is strictly larger than the targeted contribution, i.e.* $c^*(t) > \eta w(t)$*;*
3. *the final fund under optimal control is strictly lower than the targeted final fund, i.e.* $X^*(T) < b_T \mathrm{a}_{\chi+T}$.

Proof The proof is in Appendix A. □

6 Numerical simulations

In order to investigate the quantities of interest to the member, we have run Monte Carlo simulations (1000 simulations for each combination of parameters) with monthly discretisation for the risky asset, and in each scenario we have adopted the optimal investment and contribution strategies derived above. We have run simulations first in the unconstrained case, adopting the optimal investment and contribution strategies provided by Equations (38) and (39), where borrowing and short-selling of the assets is allowed, and then in the financial constrained case, where borrowing and short-selling is forbidden and the proportion invested in the risky asset is constrained to lie in the range [0, 1]. In the unconstrained case, we perform a sensitivity analysis with respect to the weight v given to the contributions' stability (see Remark 1), while in the financial constrained case sensitivity analyses are performed also with respect to the time horizon T, the wage growth g, the net replacement ratio target α, the contribution rate paid by the employer γ, and the interest rate to price the annuity i. In all scenarios tested (base case and sensitivity analyses), Assumption 1 is satisfied.

6.1 Base set of parameters

The parameters for the base case are

- initial fund: $x_0 = 0$
- financial market parameters: $r = 3\%$, $\lambda = 8\%$, and $\sigma = 15\%$, which imply $\beta = \frac{1}{3}$
- subjective discount rate: $\rho = 3\%$
- time horizon: $T = 30$ years
- age at retirement: 65
- unitary annuity price at age 65: $a_{65} = 16.86$, using the discount rate $i = 2\%$
- initial wage: $w_0 = 12,000$
- annual nominal wage growth: $g = 3.5\%$
- net replacement ratio target: $\alpha = 30\%$
- contribution rate paid by the employer: $\gamma = 2\%$
- additional voluntary contribution rate target: $\eta = 5\%$
- weight given to contributions' stability: $v = 10$

The choice of the base set of the parameters is motivated as follows. The initial fund is set equal to 0, because we assume that the member joins the fund with no transfer value.[10] The riskless return is proxied by the deposit facility rate (the rate the central bank remunerates overnight deposits of financial institutions) set by the European Central Bank in its monetary policy decisions as of March 2023 at 3%.[11] The risky asset return is considered as the annualization return from the 5-year return of the Euro Stoxx 50 Index as of 12th Dec 23. The volatility is chosen at 15% so that the Sharpe Ratio is $\beta = \frac{1}{3}$, that is typical value, see for instance Menoncin and Vigna (2020). The subjective discount rate is set equal to the riskless rate. The value for the nominal wage growth is in line with the exponential form and OECD statistics of the average nominal wage growth observed in Australia, U.S.A. and UK over a 30 years period.[12] The annuity price has been calculated with the Italian projected mortality table IP55 and the discount rate $i = 2\%$ to account for a technical rate equal to 3% and pension income that increases with inflation and remains constant in real terms. This is also in line with the actuarial discount rate in real terms of 2% used by OECD (2021). The retirement age of 65 is in the range of typical retirement ages (62–67) reported by OECD (2021), which reports also a career length of 40–45 years. A plausible assumption is that the worker does not join the pension fund at the beginning of the working career, but some years later, so that the length of the accumulation phase is chosen to be 30 years (contracted to 20 years for worse scenarios and extended to 40 years in a best-case scenario within the sensitivity analyses). As for the choice of the target replacement ratio, OECD (2021) reports that for the countries where the information on voluntary pension earnings is available, the net replacement ratio with voluntary schemes can range between 23–26% depending on the level of income (low, average or high income), but can be as high as 33–37% (Ireland). For the choice of the employer's and the employee's contributions, OECD (2021) finds that the total average contribution per member varies remarkably across countries, as well as minimum contribution rates set by the regulation. According to OECD (2022) for a selected sample of countries, the employer's contribution in voluntary pension funds relative to the total contribution varies considerably, ranging from circa 2–10% up to circa 80–90%. Moreover, in the Italian case according to COVIP (2022),[13] the average contribution rate paid by employers in pension funds in 2022 was about 2%. As for the weight v assigned to the contributions' stability, this is the parameter that reflects the member's preferences (see Remark 1): the sensitivity analyses on this parameter are carried out to allow for different desires and needs.

[10] In the Monte Carlo simulations, due to technical reasons, the initial wealth has been set to 1 rather than 0.

[11] See https://www.ecb.europa.eu/press/pr/date/2023/html/ecb.mp230316~aad5249f30.en.html, last access on 12th Dec 23.

[12] See https://stats.oecd.org/index.aspx?DataSetCode=AV_AN_WAGE, data downloaded on 16 May 2023.

[13] COVIP (Commissione di Vigilanza sui Fondi Pensione) is the National Authority responsible for the supervision of Italian pension funds. The report is available at https://www.covip.it/sites/default/files/relazioneannuale/covip_relazione_per_lanno_2022_20230607.pdf

6.2 Unconstrained case

As mentioned above, for the unconstrained case, we report the simulation results in the base case of Sect. 6.1, and when changing only the value of v (leaving the other parameters as in the base case). Figure 1 reports in the base case some statistics of: (i) evolution over time of the fund under optimal control and final fund, (ii) optimal investment strategy, (iii) net replacement ratio achievable at retirement, (iv) optimal AVC and (v) optimal AVC rate (intended as the ratio between the AVC and the wage). The same statistics for the case $v = 1$ are reported in Fig. 2, and for the case $v = 100$ in Fig. 3. To facilitate the comparison across different values of the weight v, the scale of the corresponding graphs remains unchanged.

We observe the following:

- As a general result, the AVC rate is slightly decreasing over time.
- As expected, the higher the weight v given to the contributions' stability is, the less dispersed is the AVC rate above the target, and the more dispersed are the distribution of the final fund and of the net replacement ratio on the left of the target. And vice versa.
- To be more precise, when $v = 10$, in 99% of the cases, the net replacement ratio obtainable at retirement is between 29% and 29.91%, against the 30% target; the AVC rate lies always between 5% and 7%, and after 10 years in 50% of the cases it lies between 5% and 6%. When $v = 1$, in 99.70% of the cases, the net replacement ratio obtainable at retirement is between 29.80% and 29.99%, the AVC rate lies always between 5% and 8%, and after 10 years in 75% of the cases it lies between 5% and 7%. Finally, when $v = 100$, in 99% of the cases, the net replacement ratio obtainable at retirement is between 28% and 29.85% and the AVC rate lies always between 5% and 5.5%.
- For all choices of v, the optimal share to be invested in the risky asset $y^*(t)$ in the first years is negative in more than 5% of the cases (see the second-line right graphs in Figs. 1, 2, and 3, which report the percentiles of the optimal investment strategy in the first 12 months), implying short-selling of the risky asset, and after a certain number of years becomes positive and above 1, implying borrowing the riskless asset to invest in the risky one.
- To be more precise, in the base case $v = 10$, after 12 months the optimal investment strategy is negative in 8.1% of the scenarios, and the 95th percentile of $y^*(t)$ remains higher than 100% for the first 148 months, arriving at 5.3% at the time of retirement. For $v = 1$, after 12 months the optimal investment strategy is negative in 7.4% of the cases, and the 95th percentile of $y^*(t)$ remains higher than 100% for the first 138 months, arriving at 0.87% at retirement. For $v = 100$, after 12 months the optimal strategy is negative in 8.4% of the cases, and the 95th percentile of $y^*(t)$ remains higher than 100% for the first 153 months, arriving at 9.6% at retirement.
- As expected from Proposition 1, the final fund is always lower than the target fund, and the optimal AVC is always larger than the target contribution.

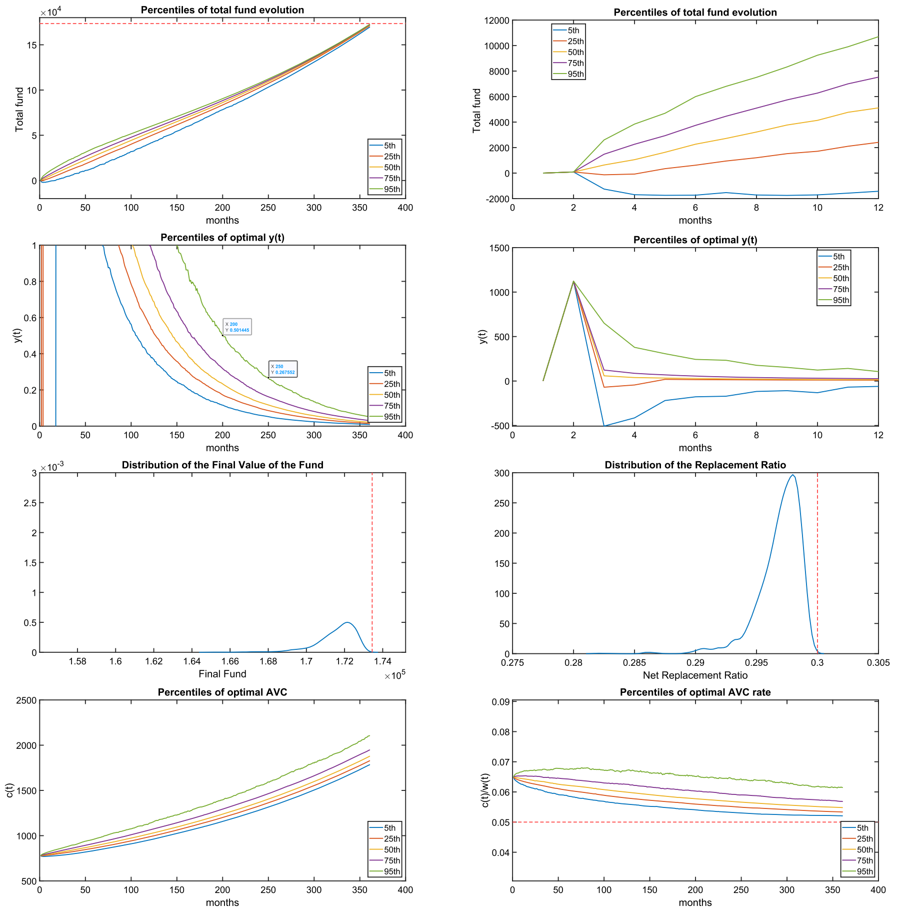

Fig. 1 Simulation results in the base case (Sect. 6.1). First-line left: percentiles of fund's evolution (red dotted line: target fund). First-line right: percentiles of fund's evolution in the first 12 months. Second-line left: percentiles of optimal investment strategy. Second-line right: percentiles of optimal investment strategy in the first 12 months. Third-line left: distribution of final wealth (red dotted line: target fund). Third-line right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution) (Color figure online)

6.3 Constrained case

The optimal investment strategy that requires both borrowing and short-selling is likely to be forbidden in practice. For this reason, we analyze the so-called *clipped* investment strategy (see Forsyth and Vetzal 2022) defined as follows

$$\hat{y}(t) = \max\{0, \min\{y^*(t), 1\}\} \tag{41}$$

Clipped investment strategies of the same type, also called "cut-shares" (see Menoncin and Vigna 2017), were applied e.g. by Forsyth and Vetzal (2022), and

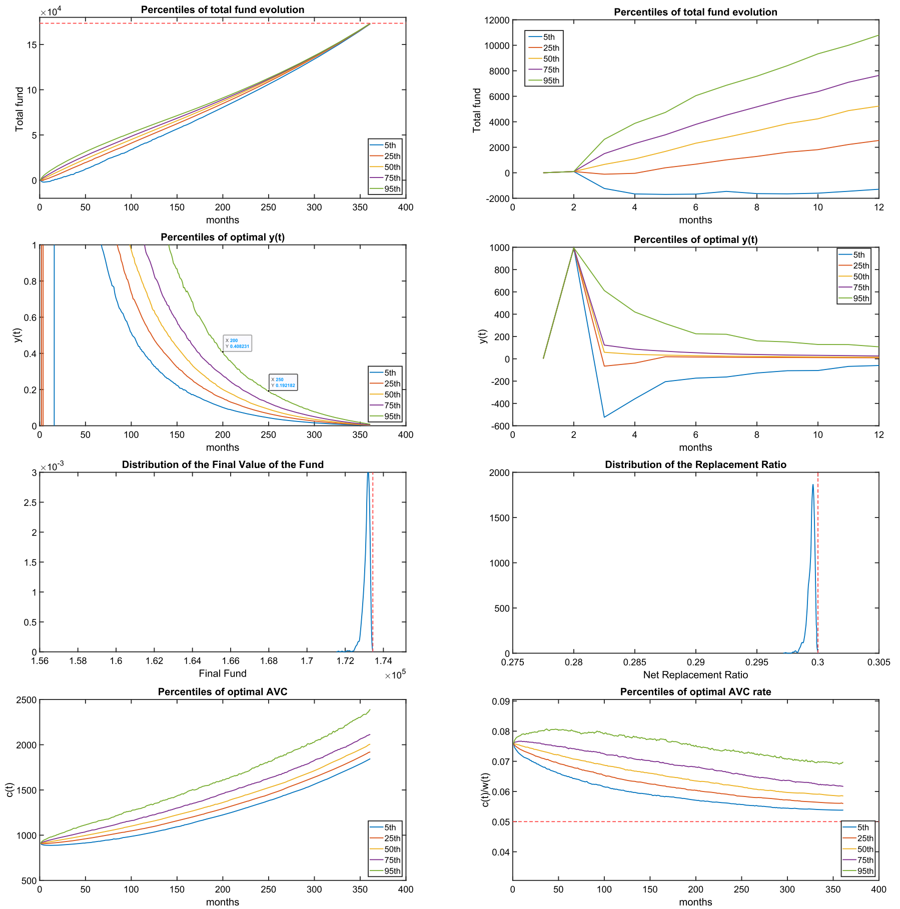

Fig. 2 Simulation results with $v = 1$ and other parameters as in Sect. 6.1. First-line left: percentiles of fund's evolution (red dotted line: target fund). First-line right: percentiles of fund's evolution in the first 12 months. Second-line left: percentiles of optimal investment strategy. Second-line right: percentiles of optimal investment strategy in the first 12 months. Third-line left: distribution of final wealth (red dotted line: target fund). Third-line right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution) (Color figure online)

by Gerrard et al. (2006) and Vigna (2014) in the context of DC pension schemes, and proved to be satisfactory: with respect to the unrestricted case the effect on the final results turned out to be negligible and the controls resulted to be more stable over time. We remark that solving the same optimization problem imposing constraints on the control variables turns out to be remarkably complex (see Di Giacinto et al. 2011).

6.3.1 Sensitivity with respect to *v*

Similarly to what has been done in the unconstrained case, Fig. 4 reports in the base case some statistics of: (i) evolution over time of the fund under optimal control and

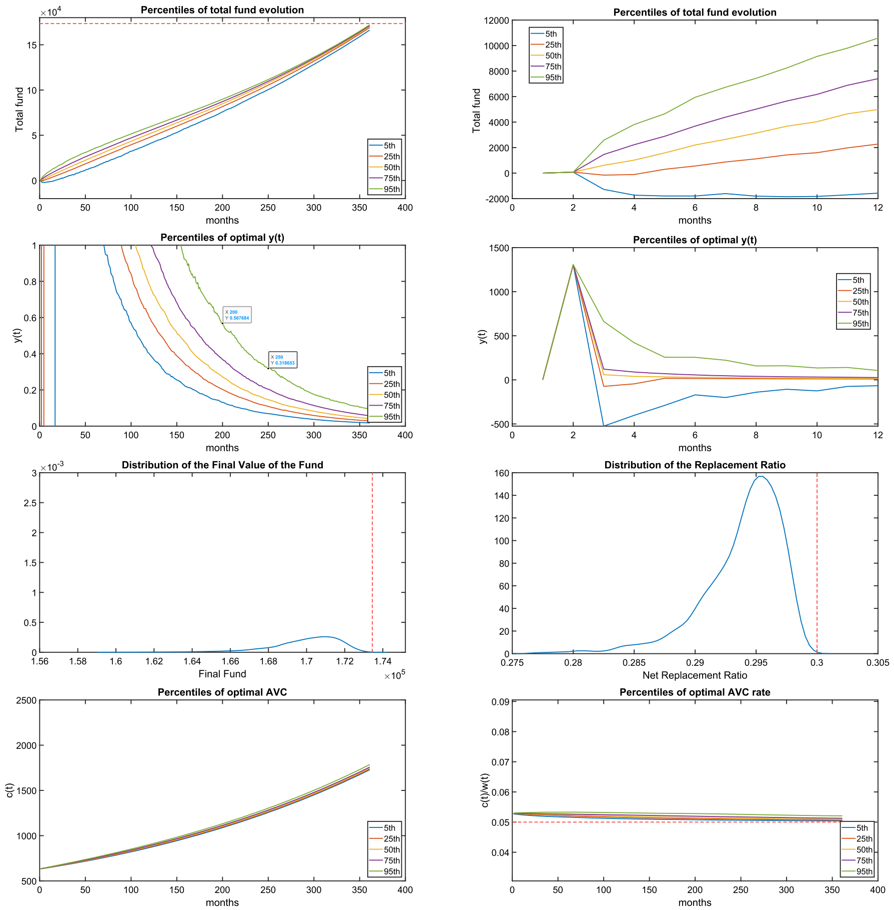

Fig. 3 Simulation results with $v = 100$ and other parameters as in Sect. 6.1. First-line left: percentiles of fund's evolution (red dotted line: target fund). First-line right: percentiles of fund's evolution in the first 12 months. Second-line left: percentiles of optimal investment strategy. Second-line right: percentiles of optimal investment strategy in the first 12 months. Third-line left: distribution of final wealth (red dotted line: target fund). Third-line right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution) (Color figure online)

final fund, (ii) optimal investment strategy, (iii) net replacement ratio achievable at retirement, (iv) optimal AVC and (v) optimal AVC rate (intended as the ratio between the AVC and the wage). The same statistics for the case $v = 1$ are reported in Fig. 5 and for the case $v = 100$ in Fig. 6. To facilitate the comparison across different values of the weight v, the scale of the corresponding graphs remains unchanged. Notice, however, that the scale is different from the unconstrained one because, as expected, the simulation results turn out to be quite different.

We observe the following:

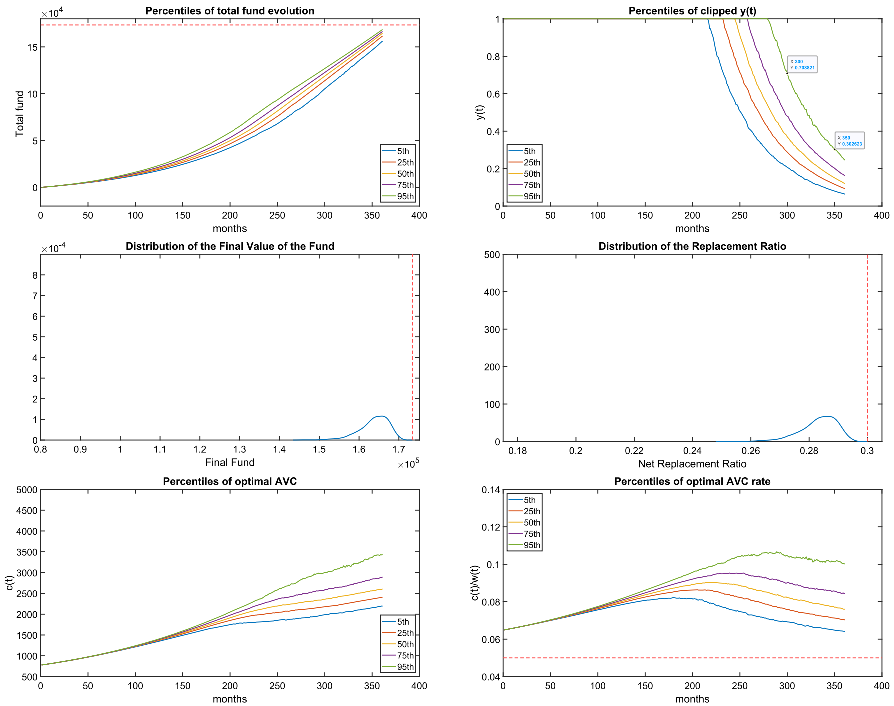

Fig. 4 Simulation results in the base case (Sect. 6.1). Top-left: percentiles of fund's evolution (red dotted line: target fund). Top-right: percentiles of clipped investment strategy. Middle-left: distribution of final wealth (red dotted line: target fund). Middle-right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution) (Color figure online)

- The AVC rate is increasing in the first 15–20 years, while it is decreasing in the remaining years, with the increasing and decreasing trends being more pronounced with small values of v.
- As in the unconstrained case and as expected, the higher the weight v given to the contributions' stability is, the more concentrated is the AVC rate close to the target, and the more dispersed are the distribution of the final fund and of the net replacement ratio on the left of the target. And vice versa. The difference with the unconstrained case consists in the level of AVC, final fund and net replacement ratio, being clearly more unfavourable for the member in the clipped strategy case.
- To be more precise, when $v = 10$, in 99% of the cases, the net replacement ratio obtainable at retirement is between 26% and 29.40%; the AVC rate lies always between 6% and 11%. When $v = 1$, in 99.80% of the cases, the net replacement ratio obtainable at retirement is between 29.25% and 29.93% and the AVC rate lies always between 7% and 14%, and in more than 50% of the cases it lies above 9% after the fifth year and almost until retirement time T. Finally, when $v = 100$, in 99% of the cases, the net replacement ratio obtainable at retirement is between 19% and 28.30% and the AVC rate lies always between 5.5% and 6.5%.

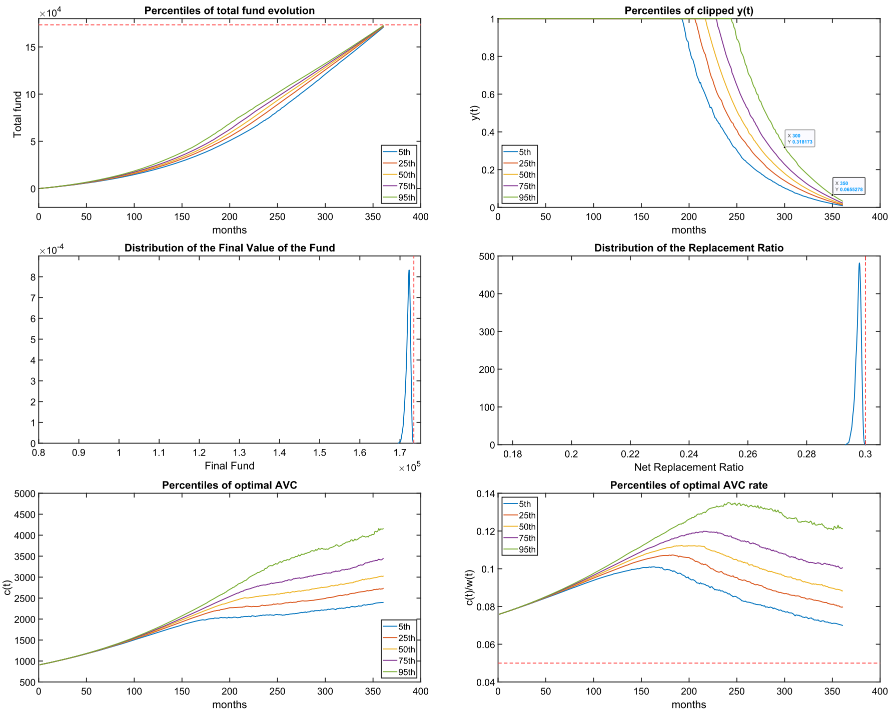

Fig. 5 Simulation results with $\upsilon = 1$ and other parameters as in Sect. 6.1. Top-left: percentiles of fund's evolution (red dotted line: target fund). Top-right: percentiles of clipped investment strategy. Middle-left: distribution of final wealth (red dotted line: target fund). Middle-right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution) (Color figure online)

- For $\upsilon = 1$ the percentiles of the investment strategy are closer to each other with respect to the base case, indicating a lower variability of the investment strategy. Conversely, for $\upsilon = 100$ the percentiles become wider with respect to the case $\upsilon = 10$, showing a higher variability of the clipped strategy.
- Clearly, at any time $t \in [0, T]$ the clipped proportion to be invested in the risky asset $\hat{y}(t)$ is bounded between 0 and 1. In all cases, it is equal to 1 for 20–25 years, and then it gradually reduces towards lower values. When υ is lower, the investment strategy is less risky, and vice versa. This is due to the fact that with a low value of υ, less weight is given to the contributions' stability, and therefore the payment of higher contributions compensates the investment in the risky asset. In contrast, when υ is higher, the optimal contribution keeps close to the target and it is necessary to adopt more aggressive investment strategies to reach the replacement ratio target. These results are in line with those found by Akpanibah and Oghen'Oro (2018) and Chinyere et al. (2022).
- To be more precise, in the base case $\upsilon = 10$, the 95th percentile of the clipped investment strategy $\hat{y}(t)$ remains capped at 100% for the first 279 months, decreasing to 24.8% at the time of retirement. For $\upsilon = 1$, the 95th percentile of the clipped

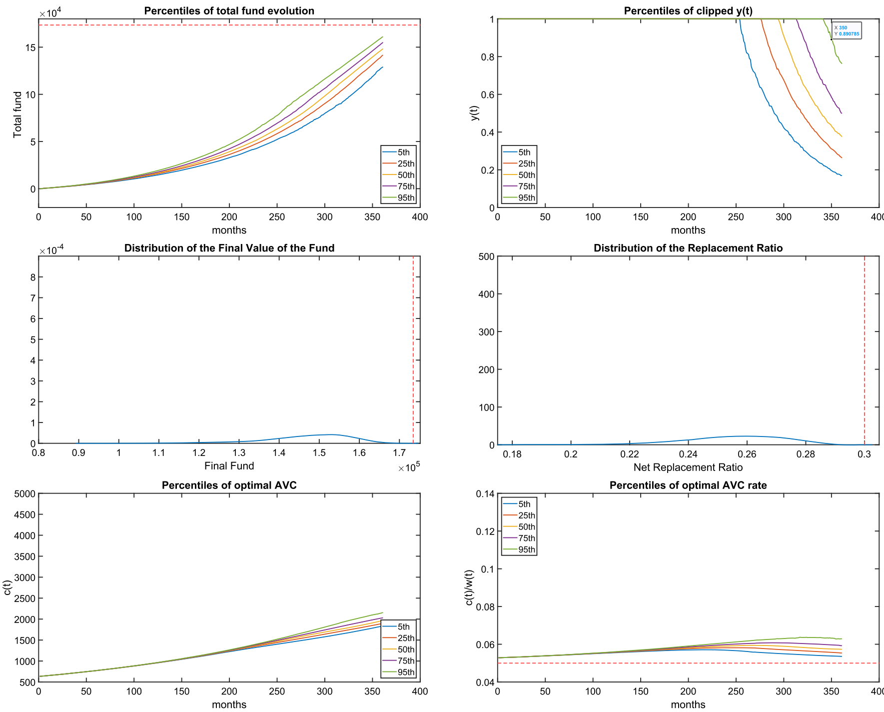

Fig. 6 Simulation results with $v = 100$ and other parameters as in Sect. 6.1. Top-left: percentiles of fund's evolution (red dotted line: target fund). Top-right: percentiles of clipped investment strategy. Middle-left: distribution of final wealth (red dotted line: target fund). Middle-right: distribution of net replacement ratio (red dotted line: target net replacement ratio). Bottom-left: percentiles of optimal AVC. Bottom-right: percentiles of optimal AVC rate (red dotted line: target contribution) (Color figure online)

strategy $\hat{y}(t)$ remains capped at 100% for the first 244 months, decreasing to 3.2% at retirement. For $v = 100$, the 95th percentile of the clipped strategy $\hat{y}(t)$ remains capped at 100% for the first 341 months, decreasing to 76.2% at retirement.

6.3.2 Further sensitivity analyses

We have performed several sensitivity analyses with respect to other parameters, changing only one parameter at a time, and leaving all the others as in the base case (Sect. 6.1). The parameters stressed are: (i) the time horizon T, considering $T = 20$ and $T = 40$, reported in Fig. 7 of Appendix B; (ii) the wage growth g, considering $g = 2\%$ and $g = 5\%$, reported in Fig. 8 of Appendix B; (iii) the net replacement ratio target α, considering $\alpha = 20\%$ and $\alpha = 40\%$, reported in Fig. 9 of Appendix B; (iv) the contribution rate paid by the employer γ, considering $\gamma = 1\%$ and $\gamma = 3\%$, reported in Fig. 10 of Appendix B; (v) the interest rate i used to price the annuity $a_{\chi+T}$, considering $i = 1\%$ and $i = 3\%$, reported in Fig. 11 of Appendix B. For each case, we report only the distribution of the net replacement ratio (on the left in each figure) and the percentiles of the AVC rate (on the right in each figure), that are the quantities of main interest to the pension fund's member (the other statistics are available upon request).

We observe the following:

- When the time horizon decreases ($T = 20$), the shorter period makes more difficult the achievement of the targets. As expected, the comparison of Fig. 4 versus Fig. 7 shows that the optimal AVC rate increases, ranging between 11% and 18%, and the net replacement ratio achievable worsens, lying mainly between 22% and 28%; in addition, the clipped investment strategy becomes more aggressive. In contrast, when the time horizon increases ($T = 40$) there is more time to achieve the targets. As a consequence, the optimal AVC rate decreases considerably, ranging between 5% and 6%, and the net replacement ratio achievable improves remarkably, lying mainly between 29% and 30%; in addition, the investment strategy becomes less aggressive. This result shows the great importance of the time horizon in the achievement of the desired net replacement ratio at retirement, coupled with a significant stability of the additional contributions to be paid by the employee during the accumulation phase.
- When the wage growth decreases ($g = 2\%$), the fund target to be achieved decreases, while the financial market parameters remain unchanged. This makes easier the achievement of the target while maintaining stability of the contributions. Therefore, as expected, the comparison of Fig. 4 versus Fig. 8 shows that the optimal AVC rate decreases, ranging between 5% and 8%, and the distribution of the net replacement ratio achievable improves, lying mainly between 28% and 30%; in addition, the investment strategy becomes less aggressive. In contrast, when the wage growth increases ($g = 5\%$), the optimal AVC rate increases (ranging between 7% and 13%), and the distribution of the net replacement ratio achievable worsens, lying mainly between 26% and 29%; in addition, the investment strategy becomes more aggressive. This result is in line with OECD (2021), Table 4.4, which finds that *"For low earners (with half of average worker earnings), the average net replacement rate across OECD countries is* 74.4% *while it is* 54.9% *for high earners* (200% *of average worker earnings)"*.
- When the net replacement ratio target α decreases ($\alpha = 20\%$), the situation is similar to that of the reduced wage growth: the target fund decreases, everything else remaining the same. Therefore, as expected, the comparison of Fig. 4 versus Fig. 9 shows that the optimal AVC rate decreases, ranging between 5% and 6.5%, the distribution of the net replacement ratio achievable improves, lying mainly between 19% and 20%, and the investment strategy becomes less aggressive. In contrast, when the net replacement ratio increases ($\alpha = 40\%$), the optimal AVC rate increases (ranging between 7% and 15%), the distribution of the net replacement ratio achievable worsens, lying mainly between 35% and 39%, and the investment strategy becomes more aggressive.
- When the contribution rate paid by the employer reduces ($\gamma = 1\%$), it is slightly more difficult to reach the target. Therefore, as expected, the comparison of Fig. 4 versus Fig. 10 shows that the optimal AVC rate slightly increases, ranging between 7% and 11%, and the investment strategy becomes slightly more aggressive. On the other hand, the distribution of the net replacement ratio achievable remains almost

unchanged with respect to the base case. Therefore, the final target seems to be achieved as in the base case, but at the price of paying higher additional contribution and investing slightly more aggressively to compensate for the reduction in the contribution paid by the employer. In contrast, when the employer's contribution increases ($\gamma = 3\%$), the optimal AVC rate slightly decreases, the distribution of the net replacement ratio achievable remains almost unchanged (with just a slight improvement) and the investment strategy becomes slightly less aggressive.

- When the interest rate used to price the final annuity reduces ($i = 1\%$), meaning that the annuity price increases, the situation is similar to that of the reduced contribution rate paid by the employer: it is more difficult to reach the target. Therefore, as expected, the comparison of Fig. 4 versus Fig. 11 shows that the optimal AVC rate slightly increases, ranging between 7% and 12% and the investment strategy becomes more aggressive. On the other hand, the distribution of the net replacement ratio achievable remains almost unchanged with respect to the base case. Therefore, the final target seems to be achieved as in the base case, but at the price of paying higher additional contributions and investing more aggressively to compensate for the increased price. In contrast, when the interest rate used to price the final annuity increases ($i = 3\%$) meaning that the annuity price decreases, the optimal AVC rate slightly decreases, the distribution of the net replacement ratio achievable remains almost unchanged (with just a slight improvement) and the investment strategy becomes less aggressive.

6.4 Discussion

The simulation results found in Sect. 6.3 show that the joint application of the clipped investment strategy (41) and the optimal AVC (39) provide outcomes that are very sensitive to the choice of the parameters. The main conclusions are:

- The choice of the weight v given to the contributions' stability is crucial for the outcome distribution: a high value of v, that reflects the desire of contributions that remain close to the target contribution rate of 5% over time, results in AVC rates that lie always very close to each other and very close to the 5% target, at the price of (i) remarkably aggressive and variable investment strategies (total investment in the risky asset for more than 20 years followed by a gradual shift to the riskless asset, but with a substantial share invested in the risky asset until retirement) (ii) a very dispersed net replacement ratio well below the desired one. Vice versa, a low value of v gives high priority to the achievement of the desired replacement ratio, which is almost reached in nearly all cases with less aggressive and less variable investment strategies, at the cost of paying considerably variable and significantly high AVC rates, that are far from the desired 5% level. An intermediate value of v provides intermediate results for the investment strategy, the optimal AVC rate and the net replacement ratio achieved.
- Keeping the value of v fixed, increasing the time horizon T dramatically improves the situation, and provides higher net replacement ratios achieved, lower and more stable AVC rates and less aggressive and less volatile investment strategies, that

is an obvious result. The extent of improvement is considerable when the time horizon is increased by one third (in the case considered, 10 years out of 30).
- Also a reduction of the annual wage growth g results in an improvement of the situation: the net replacement ratio can be almost achieved in nearly all cases without needing to make the investment strategy more risky or to increase the AVC rates. This would be perhaps the case of low earners, who would probably benefit more than high earners from the application of this model.
- As we would expect, also a reduction of the net replacement ratio target implies a remarkable improvement of the outcomes, with final net replacement ratios rather close to the desired one, stable AVC rates quite close to the target and investment strategies close to the lifestyle strategy largely adopted in the UK in DC pension schemes (whereby the fund is entirely invested into equities until 10 years prior to retirement, and is then gradually switched into riskless assets, by moving every year 10% from equities into bonds, see Cairns et al. 2006).
- Clearly, an increase of the percentage γ of wage to be paid into the fund by the employer improves the situation under all points of view, by reducing the AVCs paid by the member, improving the net replacement ratio achievable and making the investment strategy less risky.
- Finally, an increase of the interest rate i used to calculate the annuity price obviously improves the situation under all points of view, by reducing the AVCs paid by the member, improving the net replacement ratio achievable and making the investment strategy less risky.

7 Concluding remarks

In this paper, we provide a flexible analytical optimization tool that could help the member of a DC pension scheme in making her decisions during the accumulation phase of the fund. In particular, she can first set a desired replacement ratio which can be obtained at retirement via the achievement of a target annuity, and then determine throughout the investment period her optimal investment strategy and optimal additional voluntary contributions. For the latter, she defines a contribution rate target to be pursuit at each point in time, which is driven by her budgetary conditions and reflects her desire for contributions' stability. The problem has been tackled and solved using the tools of stochastic optimal control theory, in a Black and Scholes financial market. Closed-form solutions have been derived for the optimal investment strategy and the optimal AVC. We prove that the optimal fund never reaches the target final fund, the optimal amount invested in the risky asset is positive, and the optimal AVC is higher than the target one. Numerical Monte Carlo simulations using the unconstrained optimal investment strategy and a clipped constrained suboptimal investment strategy have been performed, both in a base case scenario and by performing sensitivity analyses on the key parameters.

From the numerical analysis, it turns out that, as dictated by the theory, the chosen annuity target is never reached, but the replacement ratio obtained can approach very closely the desired one, and the additional voluntary contributions play an important role in doing so. We find that the key parameter is the weight given to the contributions' stability. The higher this weight is, the closer are the additional voluntary contributions to the desired contribution rate, at the expense of (i) reaching a lower net replacement ratio at retirement, (ii) needing to take a more aggressive and more volatile investment strategy across all the investment phase. Conversely, the lower this weight is, the higher is the net replacement ratio achievable and the lower is the variability of the investment strategy, at the price of larger deviations from the targeted contribution in the accumulation phase. The numerical analysis shows that a 5% contribution rate target is appropriate to reach a 30% net replacement ratio target over a 30 years period. Furthermore, the sensitivity analyses show that a longer accumulation phase, a lower wage growth and/or a higher contribution rate paid into the fund by the employer make it easier for the member to reach a satisfactory annuity. These are expected and intuitive results.

The main messages of this paper are (i) the trade-off between two conflicting desires of the member—i.e. to respect her current budgetary conditions via contributions' stability and to reach an adequate annuity level at retirement—can be tackled by choosing an appropriate weight given to the contributions' stability in the implementation of this model; (ii) the joint achievement of both desires can be facilitated by a longer membership in the pension fund and by a higher contribution rate paid by the employer; (iii) low earners characterized by a lower annual wage growth seem to benefit by the implementation of this optimization tool better than high earners with a higher wage growth. This last result is in line with OECD (2021).

This study provides strong evidence that paying AVCs into the pension fund is a way to deliver adequate retirement income. The analytical tool here provided could facilitate the implementation of this crucial policy recommendation, and be of practical help to investment managers of DC pension schemes and to pension and financial advisors.

The paper could be extended in several directions, for instance by addressing some of the limitations imposed by the simplifying assumptions. More realistic features, such as taxation, commission expenses and transaction costs, could be incorporated into the model. The assumption of a guaranteed annuitization rate (GAR) could be relaxed to account for a current annuitization rate (CAR). A more ambitious agenda would include the solution of the optimization problem imposing constraints on the control variables, rather than using suboptimal clipped controls. Of course, this is a hard problem.

Appendix A

Proof of Proposition 1 Due to (32), (35) and (36), for claims 1. and 2. it suffices to show that

$$X^*(t) - h(t) < 0 \tag{1}$$

where $h(t)$ is given by (37). Let us define the new process $Z(t)$ as:

$$Z(t) = X^*(t) - h(t) \tag{2}$$

It is useful to rewrite $h(t)$ as follows:

$$h(t) = Ee^{gt} + He^{-r(T-t)} \tag{3}$$

where

$$E = \frac{(\eta + \gamma)w_0}{g - r} \qquad F = b_T \mathrm{a}_{\chi+T} \qquad H = F - Ee^{gT} \tag{4}$$

so that

$$h'(t) = Ege^{gt} + Hre^{-r(T-t)} \tag{5}$$

It turns out that

$$dZ(t) = dX^*(t) - h'(t)dt = \mu_Z(t, \cdot)dt + \sigma_Z(t, \cdot)dW(t) \tag{6}$$

where it remains to calculate the drift $\mu_Z(t, \cdot)$ and the volatility $\sigma_Z(t, \cdot)$.[14] Due to (35), from (4) and (6) we can easily find the volatility

$$\sigma_Z(t, \cdot) = xy^*\sigma = -\beta\,[x - h(t)] = -\beta Z(t) \tag{7}$$

Using (35), (36), (3), (5) and (4), the drift is given by

$$\begin{aligned}
\mu_Z(t, \cdot) = xy^*(\lambda - r) + rx + \gamma w(t) + c^* - h'(t) = \\
&= -\beta^2\,[x - h(t)] + rx + \gamma w(t) + \eta w(t) \\
&\quad - \frac{A(t)}{v}\,[x - h(t)] - Ege^{gt} - Hre^{-r(T-t)} = \\
&= [x - h(t)]\left(-\beta^2 - \frac{A(t)}{v}\right) + rx \\
&\quad + (\gamma + \eta)w_0 e^{gt} - Ege^{gt} - Hre^{-r(T-t)} = \\
&= [x - h(t)]\left(-\beta^2 - \frac{A(t)}{v}\right) + rx - Hre^{-r(T-t)} \\
&\quad + e^{gt}\left[(\gamma + \eta)w_0 - \frac{(\gamma + \eta)w_0 g}{g - r}\right] =
\end{aligned}$$

[14] In what follows, we will sometimes write x in the place of $X^*(t)$, y^* in the place of $y^*(t, X^*(t))$ and c^* in the place of $c^*(t, X^*(t))$.

$$
\begin{aligned}
&= [x - h(t)]\left(-\beta^2 - \frac{A(t)}{v}\right) + rx - Hre^{-r(T-t)} + e^{gt}\left[-r\frac{(\gamma+\eta)w_0}{g-r}\right] = \\
&= [x - h(t)]\left(-\beta^2 - \frac{A(t)}{v}\right) + rx - Hre^{-r(T-t)} - rEe^{gt} = \\
&= [x - h(t)]\left(-\beta^2 - \frac{A(t)}{v}\right) + r\left(x - He^{-r(T-t)} - Ee^{gt}\right) = \\
&= [x - h(t)]\left(-\beta^2 - \frac{A(t)}{v}\right) + r\left[x - h(t)\right] = \\
&= [x - h(t)]\left(r - \beta^2 - \frac{A(t)}{v}\right) \qquad (8)
\end{aligned}
$$

Therefore the process $Z(t)$ follows the dynamics

$$
dZ(t) = \left(r - \beta^2 - \frac{A(t)}{v}\right) Z(t)dt - \beta Z(t)dW(t) \qquad (9)
$$

yielding[15]

$$
Z(t) = Z(0)\exp\left[\int_0^t \left(r - \frac{3}{2}\beta^2 - \frac{A(s)}{v}\right)ds - \int_0^t \beta dW(s)\right] \qquad (10)
$$

Observe now that

$$
\begin{aligned}
Z(0) &= x_0 - h(0) = x_0 - (\gamma+\eta)w_0\frac{1 - e^{(g-r)T}}{g-r} - b_T \mathrm{a}_{\chi+T}e^{-rT} \\
&= x_0 + \int_0^T (\gamma+\eta)w_0 e^{(g-r)s}ds - b_T \mathrm{a}_{\chi+T}e^{-rT} \\
&= x_0 + \int_0^T (\gamma+\eta)w(s)e^{-rs}ds - b_T \mathrm{a}_{\chi+T}e^{-rT} < 0 \qquad (11)
\end{aligned}
$$

where the inequality is due to Assumption 1. Inequality (11), coupled with (10), implies

$$
Z(t) < 0 \qquad \forall t \in [0, T] \qquad (12)
$$

that proves claims 1. and 2. As for claim 3., notice that from (12) we have

$$
0 > Z(T) = X^*(T) - h(T) = X^*(T) - b_T \mathrm{a}_{\chi+T} \;\Rightarrow\; X^*(T) < b_T \mathrm{a}_{\chi+T} \qquad (13)
$$

that is what we needed to prove. □

[15] By application of Ito's lemma to $f(t, z) = \ln z$.

Appendix B

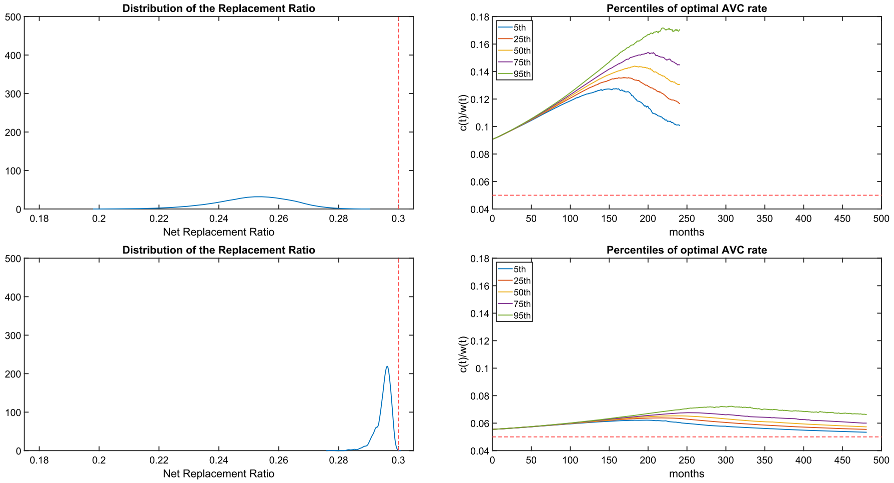

Fig. 7 Simulation results with $T = 20, 40$ and other parameters as in Sect. 6.1. Top graphs: $T = 20$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate). Bottom graphs: $T = 40$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate) (Color figure online)

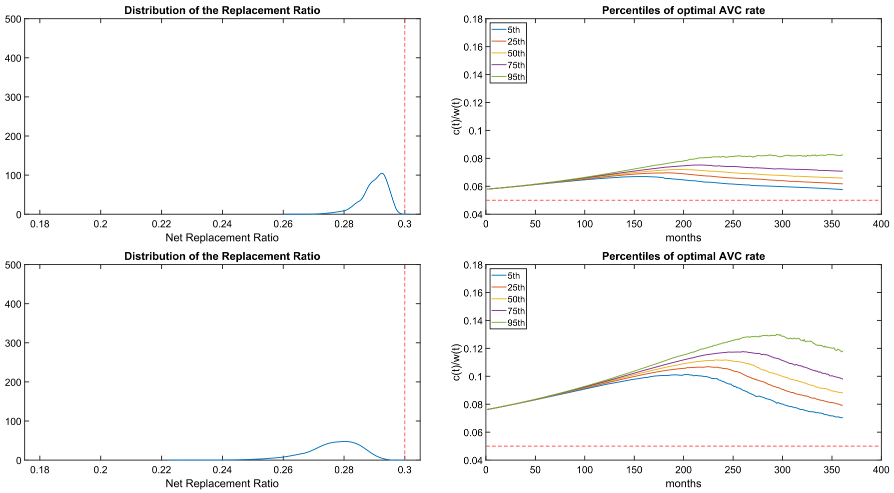

Fig. 8 Simulation results with $g = 2\%, 5\%$ and other parameters as in Sect. 6.1. Top graphs: $g = 2\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate). Bottom graphs: $g = 5\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate) (Color figure online)

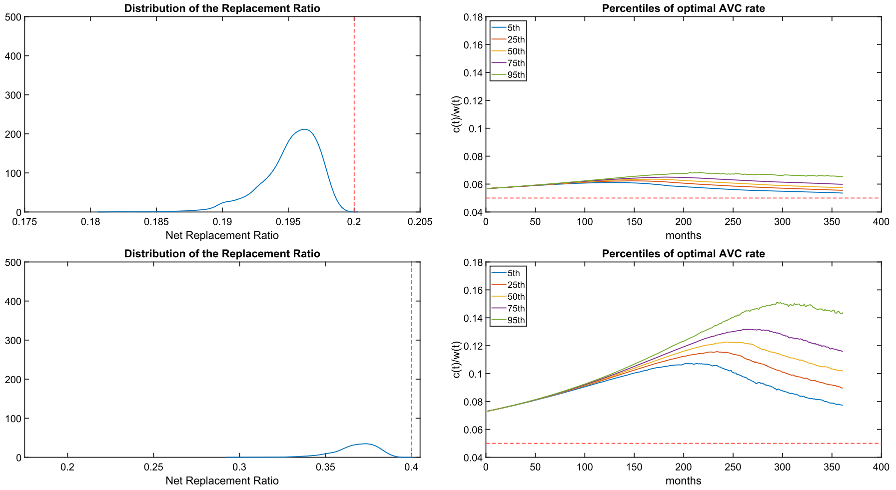

Fig. 9 Simulation results with $\alpha = 20\%, 40\%$ and other parameters as in Sect. 6.1. Top graphs: $\alpha = 20\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate). Bottom graphs: $\alpha = 40\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate) (Color figure online)

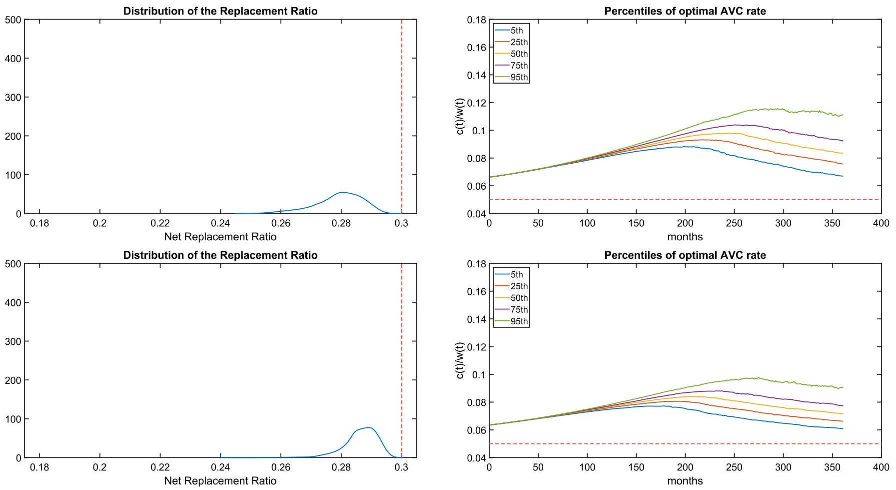

Fig. 10 Simulation results with $\gamma = 1\%, 3\%$ and other parameters as in Sect. 6.1. Top graphs: $\gamma = 1\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate). Bottom graphs: $\gamma = 3\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate) (Color figure online)

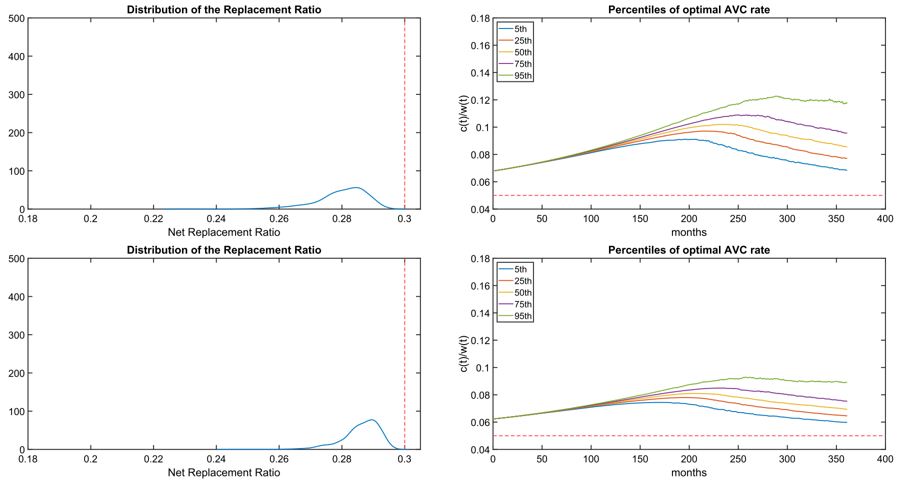

Fig. 11 Simulation results with $i = 1\%$, 3% and other parameters as in Sect. 6.1. Top graphs: $i = 1\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate). Bottom graphs: $i = 3\%$ (left: distribution of net replacement ratio; right: percentiles of optimal AVC rate) (Color figure online)

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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