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American options with acceleration clauses

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Abstract
Acceleration clauses shorten the residual life of an option when an acceleration condition is met. Acceleration clauses are frequent in warrants, American call options on traded stocks. In warrants with the acceleration clause, if an index (e.g. the average underlying stock) triggers an acceleration threshold, the American call option can be exercised on a much shorter maturity (e.g. 30 days). The actual time-to-maturity of an American option with an acceleration condition is therefore stochastic. In order to evaluate these contracts we first reduce the generic American option with stochastic time-to-maturity to a compound American option with constant maturity, and provide estimates for their prices. Finally we propose an efficient algorithm to price American call options with the acceleration clause in a binomial setting.

Keywords Optimal stopping · American options · Acceleration clauses · Warrants · Backward recursion

JEL classification G12 · G13

1 Introduction

In this paper we study American options with path-dependent acceleration clauses, that shorten the residual life of American options. Acceleration clauses have become frequent in stock warrants, that are call options on stocks issued by parent companies, often part of a financing arrangement or as an incentive to investors. Warrants' maturities can be fairly long (from 1 to 10 years). When the underlying price is high,

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warrants become expensive for the issuer. Callability features and acceleration clauses mitigate this risk, and are common in warrants (see for instance Burney and Moore 1997).

The callability feature allows the issuer to terminate the warrant contract as soon as the underlying share price exceeds a prespecified level. Their evaluation is relatively easy, as the standard callability feature turns the contract into a knock-out barrier option, some times with a rebate. Yagi and Sawaki (2010), evaluate callable warrants, when the investor (resp. the firm) can exercise (resp. call back)) the warrant at any time during the life of the contract, thus extending the work of Burney and Moore (1997), who allow the exercise of the warrant at maturity only. To mitigate potential legal issues, holders of callable warrants are typically provided with a notice period (usually 30 days) to decide whether to exercise their warrants or accept the rebate, if any (see Schultz 1993). The presence of a notice period defines an implicit acceleration clause associated to the callability feature. This acceleration clause, however, is usually neglected in the evaluation of warrants (see Burney and Moore 1997). Unlike callable features, stand-alone acceleration clauses reduce the remaining lifespan of the warrant, without terminating the contract; for this reason, warrants with acceleration clauses are more appealing to investors than callable warrants. Due to the increase of Special Purpose Acquisition Companies (SPACs), that issue warrants as part of their unit financing, warrants with stand-alone acceleration clauses have proliferated in recent years.[1] To be more precise, acceleration clauses enable the company to shorten the expiration period in the event that the company's share price or its average value over a specified number of days surpasses a predetermined threshold. Consequently, acceleration clauses significantly mitigate the risk of the warrant becoming excessively costly for the issuer, even in the absence of any callability feature. Acceleration clauses written on the average value of the underlying create a complex path dependency in the warrants payoff structure and make their valuation challenging. While constant-maturity American options have already been studied in the literature (see Yagi and Sawaki 2010), we are the first to our knowledge to formally address the problem of evaluating American options with acceleration clauses whose maturity is stochastic. In Carr (1998) the constant maturity of the American option is replaced by a random variable with a suitable distribution in order to reduce the complexity of the resulting pricing equations for several underlying's models. Importantly, Carr's randomization of the American option uses independent stopping times. On the contrary, the acceleration clause determines a stochastic maturity that is typically highly dependent on the whole history of the underlying asset.

We propose an efficient approach to solve the pricing problem of an American option with generic payoff X and an acceleration clause. When the option is accelerated at the stochastic date τ_Θ, its time to maturity shrinks. We first transform the American option with payoff X and stochastic time-to-maturity into a compound American option with modified payoff X_Θ and constant maturity T. The modified compound payoff X_Θ internalizes the stochastic time-to-maturity. We then approximate the compound payoff X_Θ to obtain an analytical approximation for the price of the American option with stochastic time-to-maturity. We finally apply our results

[1] https://cbvinstitute.com/events/valuing-warrants-with-acceleration-clause/.

in a binomial setting and develop a forward-shooting grid method to price warrants with the acceleration clause on the average underlying value. The paper is organized as follows. After the introduction (Sect. 1), we introduce in Sect. 2 the compound American call option. In Sect. 3 we introduce acceleration clauses, and rewrite the American option with the acceleration clause as a compound option with a modified payoff. This allows us to obtain an accurate lower bound for the price of the American option with the acceleration clause. In Sect. 4, we evaluate the effectiveness of acceleration clauses in mitigating the risk of high warrant payoffs when exercise is possible at maturity only. We analyze the acceleration clauses implicitly defined by the 30-day notice period of callable warrants and assess the relative premium implied by such acceleration clauses on callable warrants with no rebate. In Sect. 5 we focus on American call options with an acceleration clause written on the average price of the underlying stock, and markovianize the related pricing problem in a binomial framework. Section 6 provides an effective and intuitive implementation of the markovianized pricing problem for the American call option with the acceleration clause in a binomial framework. These results can be used for the post-issuance evaluation of warrants that display the American option feature and a path-dependent acceleration clause. Section 7 concludes.

2 The American compound call option

Consider an arbitrage-free and complete market on the triple $(\Omega, \mathcal{F}, \mathbb{Q})$. $\mathcal{F}$ denotes the filtration, i.e. the information available to investors, and $\mathbb{Q}$ is the risk neutral probability. Denote with r the constant riskless interest rate. Consider the American option with maturity T and denote its payoff at t with $X(t) \geq 0$, for all $t \in [0, T]$. We assume that X is strong Markov with respect to $\mathcal{F}$. For instance, X can be a call option on a strong Markov underlying asset S with respect to $\mathcal{F}$. The discounted no-arbitrage price of the (constant-maturity) American option at time t is

$$\widetilde{V}(t) = \sup_{t \leq \tau \leq T} \mathbb{E}_t \left[e^{-r\tau} X(\tau) \right] = \sup_{t \leq \tau \leq T} \mathbb{E}_t \left[\widetilde{X}(\tau) \right] \tag{1}$$

where the expectation is taken under the risk-neutral measure and τ denotes a generic stopping time with respect to $\mathcal{F}$.

The discounted value of the American option $\widetilde{V}$ is the smallest risk-neutral supermartingale dominating the discounted payoff $\widetilde{X}$. Consider now the American zero-strike call option on the previous American option defined in (1), i.e. the option to entry at any time from 0 to T in the American option V of above at zero price. This is an American compound call option. Its discounted no-arbitrage price at time t is

$$\widetilde{CC}(t) = \sup_{t \leq \tau \leq T} \mathbb{E}_t \left[\widetilde{V}(\tau) \right] \tag{2}$$

This zero-strike compound American call option on V has the same value as the underlying American option, as we show in the next

Lemma 1 *With respect to the previous notations,* $\widetilde{CC}(t) = \widetilde{V}(t)$

Proof From the definition of the American compound call option (2) we immediately see that $\widetilde{CC}(t) \geq \widetilde{V}(t)$. Moreover, by the properties of the Snell envelope, we know that the $\widetilde{CC}(t)$ is the smallest risk-neutral supermartingale dominating $\widetilde{V}(t)$. Since $\widetilde{V}(t)$ is itself a risk-neutral supermartingale, we have that $\widetilde{CC}(t) = \widetilde{V}(t)$. □

3 American options with acceleration clauses

Denote with τ_Θ the $\mathcal{F}-$stopping time describing the date of the acceleration clause. Acceleration occurs only once during the life of the option. As long as $\tau_\Theta > t$, the time-to-maturity of the option is the whole residual life of the option at t, i.e. $T - t$. If $\tau_\Theta = t \leq T$ then the the time-to-maturity shrinks and actual maturity becomes $\Theta = \min\left(T' + \tau_\Theta, T\right)$, with $T' < T$. Typically, T is much larger than T', with $T = 5$ or $T = 10$ years and $T' = 1$ month. The option with the acceleration clause has therefore a stochastic maturity Θ defined as follows:

$$\Theta = \begin{cases} T & \text{if } \tau_\Theta > T \\ \min\left(T, T' + \tau_\Theta\right) & \text{if } \tau_\Theta \leq T \end{cases} \tag{3}$$

with $T' < T$. We denote with $\Theta\left(\tau_\Theta\right) = \min\left(T, T' + \tau_\Theta\right)$ the maturity of the option evaluated at the acceleration date τ_Θ.

As an example, consider the acceleration clause written on the average price of the underlying stock. Let $\tau_A = \inf\left\{t > 0 : A(t) \geq \overline{A}\right\}$ be the hitting time when $A(t)$, the current average price of the underlying, hits $\overline{A}$, the prespecified acceleration threshold, with $\overline{A} > S(0)$, the initial price of the underlying. As soon as $A(t) \geq \overline{A}$ then the acceleration clause is met and the call option must be exercised in 1 month at the most. If $A(t) < \overline{A}$ for all t, then the call option can be exercised till the maturity T. The contingent maturity of the call option is therefore

$$\Theta_A = \begin{cases} T & \tau_A > T \\ \min\left(\tau_A + \frac{1}{12}, T\right) & \text{if } \tau_A \leq T \end{cases} \tag{4}$$

Given τ_Θ, Θ as in Equation (3), the discounted no-arbitrage value of the American option with payoff X and acceleration clause Θ is

$$\widetilde{V}_\Theta(t) = \sup_{t \leq \tau \leq T} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{0 \leq \tau < \tau_\Theta} + \widetilde{X}(\tau)\mathbb{I}_{\tau_\Theta \leq \tau \leq \tau_\Theta + T'}\right] \tag{5}$$

where τ denotes a generic stopping time with respect to $\mathcal{F}$. In definition (5), if $T \leq \tau_\Theta$ (i.e. if the option is not accelerated) then the set $\tau_\Theta \leq \tau \leq \tau_\Theta + T'$ is empty, $\mathbb{I}_{\tau_\Theta \leq \tau \leq \tau_\Theta + T'} = 0$ and $\widetilde{V}_\Theta(t) = \sup_{t \leq \tau \leq T} \mathbb{E}_t\left[\widetilde{X}(\tau)\right]$. On the contrary, if $T > \tau_\Theta$ then after the acceleration date τ_Θ the residual life of the option shrinks to $\min\left(T - \tau_\Theta, T'\right)$

as $\mathbb{I}_{0\leq\tau<\tau_\Theta}=0$, and $\mathbb{I}_{\tau_\Theta\leq\tau\leq\tau_\Theta+T'}=1$ in (5). In general, as $\Theta\leq T$, the definition of (5) immediately implies that the value of the American call option with stochastic maturity Θ is always dominated by the value of the constant-maturity American option, i.e. $\widetilde{V}_\Theta(t)\leq\widetilde{V}(t)$ for all t. We also observe that after $\tau_\Theta+T'$ the payoff of $\widetilde{V}_\Theta$ is null, because of the acceleration clause.

The value function defined in Equation (5) differs from an American knock-out option, because when the event is triggered at τ_Θ the option payoff does not become null (see Vidal Nunes et al. 2020 and Detemple et al. 2020). On the contrary the payoff stays the same, but it can be exercised only within a shorter time horizon. $\widetilde{V}_\Theta$ in Equation (5) is indeed an American option with payoff X and stochastic maturity Θ.

In the next proposition we prove that the American option (5) coincides with the *compound* American option with a new payoff X_Θ. The payoff of the compound American option X_Θ coincides with the original X until τ_Θ. When the acceleration clause is triggered at $\tau_\Theta<T$, the payoff X_Θ is the value of an American option with payoff X and time-to-maturity $\min\left(T-\tau_\Theta,T'\right)$. After τ_Θ the payoff X_Θ is null. By introducing the compound option, the next proposition[2] allows to embed in our continuous-time framework the discrete Bellman principle at τ_Θ, taking the maximum between the discounted immediate payoff $\widetilde{X}(\tau_\Theta)$ and the discounted value of continuing the option over the residual life $\left[\tau_\Theta,\min\left(T,T'+\tau_\Theta\right)\right]$.

Proposition 2 *$\widetilde{V}_\Theta(t)$ defined in equation (5) can be rewritten as $\widetilde{V}_\Theta(t)=\widetilde{V}_{X_\Theta}(t)$ where*

$$\widetilde{V}_{X_\Theta}(t)=\sup_{t\leq\tau\leq T}\mathbb{E}_t\left[\widetilde{X}_\Theta(\tau)\right] \tag{6}$$

where

$$\widetilde{X}_\Theta(u)=\widetilde{X}(u)\mathbb{I}_{0\leq u<\tau_\Theta}+\widetilde{V}_{\Theta(\tau_\Theta)}(\tau_\Theta)\mathbb{I}_{u=\tau_\Theta}\ \textit{for } u\in[t,T]$$

and where $\widetilde{V}_{\Theta(\tau_\Theta)}$ denotes the discounted no-arbitrage price of the American option with payoff X whose maturity in (1) is $\Theta(\tau_\Theta)=\min\left(T,T'+\tau_\Theta\right)$, known at τ_Θ.

Proof We have to show that the option values in (5) and (6) coincide. We first observe that $\widetilde{V}_\Theta$ is a supermartingale and that it dominates $\widetilde{X}_\Theta$, the payoff of $\widetilde{V}_{X_\Theta}$. In fact, from the definitions (5) and (6) it follows that

$$\begin{aligned}\widetilde{V}_\Theta(t)&\geq\widetilde{X}(t)=\widetilde{X}(t)\mathbb{I}_{0\leq t<\tau_\Theta}=\widetilde{X}_\Theta(t)\text{ for any }0\leq t<\tau_\Theta\\ \widetilde{V}_\Theta(\tau_\Theta)&=\widetilde{X}_\Theta(\tau_\Theta)\text{ for }t=\tau_\Theta\\ \widetilde{V}_\Theta(t)&\geq\widetilde{X}(t)\geq0=\widetilde{X}_\Theta(t)\text{ for any }\tau_\Theta<t<T\end{aligned}$$

Since $\widetilde{V}_\Theta\geq\widetilde{X}_\Theta$, then $\widetilde{V}_\Theta\geq\widetilde{V}_{X_\Theta}$, as $\widetilde{V}_{X_\Theta}$ is the smallest supermartingale dominating $\widetilde{X}_\Theta$.

[2] Battauz and Pratelli (2004) employ a similar argument to extend the set of admissible values of exercise policies of American call options in continuous time when the underlying stock pays discrete dividends during the life of the option.

To prove the opposite inequality, we observe that, if $t \geq \tau_\Theta$, then $\widetilde{V}_\Theta(t) = \widetilde{V}_{X_\Theta}(t)$ by the Lemma 1. If $t < \tau_\Theta$, for any $\tau \in [t, T]$ we have that

$$\begin{aligned}
\sup_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)}\right] &= \sup_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)} \mathbb{E}_t\left[\mathbb{E}_{\tau_\Theta}\left[\widetilde{X}(\tau)\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)}\right]\right] \\
&\leq \mathbb{E}_t\left[\sup_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)} \mathbb{E}_{\tau_\Theta}\left[\widetilde{X}(\tau)\right]\right] \text{ by Fatou's lemma} \\
&= \mathbb{E}_t\left[\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)}\widetilde{V}_\Theta(\tau_\Theta)\right]
\end{aligned}$$

because $\widetilde{V}_\Theta(\tau_\Theta) = \sup_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)} \mathbb{E}_{\tau_\Theta}\left[\widetilde{X}(\tau)\right]$ and $\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)} = 1$ for $\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)$. Therefore

$$\begin{aligned}
\sup_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{\tau_\Theta \leq \tau \leq \tau_\Theta + T'}\right] &\leq \mathbb{E}_t\left[\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)}\widetilde{V}_\Theta(\tau_\Theta)\right] \\
&= \sup_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)} \mathbb{E}_t\left[\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)}\widetilde{V}_\Theta(\tau_\Theta)\right].
\end{aligned}$$

This implies that

$$\begin{aligned}
\widetilde{V}_\Theta(t) &= \sup_{t \leq \tau \leq T} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{0 \leq \tau < \tau_\Theta} + \widetilde{X}(\tau)\mathbb{I}_{\tau_\Theta \leq \tau \leq \tau_\Theta + T'}\right] \\
&= \sup\left(\sup_{t \leq \tau < \tau_\Theta} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{0 \leq \tau < \tau_\Theta} + \widetilde{X}(\tau)\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)}\right],\right. \\
&\qquad \left.\sup_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{0 \leq \tau < \tau_\Theta} + \widetilde{X}(\tau)\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)}\right]\right) \\
&= \sup\left(\sup_{t \leq \tau < \tau_\Theta} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{0 \leq \tau < \tau_\Theta}\right], \sup_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)}\right]\right) \\
&\leq \sup\left(\sup_{t \leq \tau < \tau_\Theta} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{0 \leq \tau < \tau_\Theta}\right], \sup_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)} \mathbb{E}_t\left[\widetilde{V}_\Theta(\tau_\Theta)\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)}\right]\right) \\
&= \sup_{t \leq \tau \leq T} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{0 \leq \tau < \tau_\Theta} + \widetilde{V}_\Theta(\tau_\Theta)\mathbb{I}_{\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)}\right] \\
&= \sup_{t \leq \tau \leq T} \mathbb{E}_t\left[\widetilde{X}(\tau)\mathbb{I}_{0 \leq \tau < \tau_\Theta} + \widetilde{V}_\Theta(\tau_\Theta)\mathbb{I}_{\tau_\Theta = \tau}\right] \\
&= \sup_{t \leq \tau \leq T} \mathbb{E}_t\left[\widetilde{X}_\Theta(\tau)\right] = \widetilde{V}_{X_\Theta}(t)
\end{aligned}$$

and our thesis is proved. In all inequalities above we observe that if $\tau_\Theta > T$ then $\tau \leq \min(\tau_\Theta, T)$, and the set $\tau_\Theta \leq \tau \leq \Theta(\tau_\Theta)$ is empty. □

Remark 3 The modified discounted payoff $\widetilde{X}_\Theta$ in (6) is the discounted value of the American call option at τ_Θ. Because of Lemma 1, the American option (6) at τ_Θ is just a zero-strike American call option on the underlying American option V with maturity $\min\left(T, T' + \tau_\Theta\right)$.

Remark 4 The modified discounted payoff $\widetilde{X}_\Theta$ of the above proposition is path-dependent even if the original $\widetilde{X}$ is not. Indeed, for all $u \in [t, T]$ we need to remember whether or not τ_Θ as already occurred.

The above proposition provides an implicit characterization of $\widetilde{V}_\Theta$, as in the definition of the payoff of the compound option $\widetilde{V}_{X_\Theta}$ the value of the underlying option $\widetilde{V}_{\Theta(\tau_\Theta)}$ appears in the modified payoff $\widetilde{X}_\Theta$. The characterization is however very handy, as $\widetilde{V}_\Theta$ enters the payoff definition of $\widetilde{V}_{X_\Theta}$ only when the option is accelerated. In most applications the accelerated residual life of the option is small compared to the non-accelerated residual life of the option. For this reason it is possible

- To use analytical asymptotic formulae for $\widetilde{V}_{\Theta(\tau_\Theta)}$, if the residual life of the option is small enough (see Battauz et al. (2021) and the references therein);
- To evaluate numerically $\widetilde{V}_{\Theta(\tau_\Theta)}$ via the standard backward recursion in discrete time;
- To approximate $\widetilde{V}_{\Theta(\tau_\Theta)}$ with the discounted value of the European option $\widetilde{V}_{eur,\Theta(\tau_\Theta)}$, if the early exercise premium is negligible.

In general, $\widetilde{V}_\Theta(t)$ defined in Eq. (5) has the following analytical lower bound:

Proposition 5 *A lower bound for $\widetilde{V}_\Theta(t)$ defined in equation (5) $\in$ is*

$$\widetilde{V}_{LB}(t) = \sup_{t \leq \tau \leq T} \mathbb{E}_t \left[\widetilde{X}_{LB}(\tau) \right] \tag{7}$$

where

$$\widetilde{X}_{LB}(u) = \widetilde{X}(u)\mathbb{I}_{0 \leq u < \tau_\Theta} + \widetilde{V}_{eur,\Theta(\tau_\Theta)}(\tau_\Theta)\mathbb{I}_{\tau_\Theta = u}$$

where $\widetilde{V}_{eur,\Theta(\tau_\Theta)}(\tau_\Theta)$ is the discounted value of the European option with maturity $\Theta(\tau_\Theta)$ and discounted terminal payoff $\widetilde{X}(\Theta(\tau_\Theta))$.

Proof As the European option $\widetilde{V}_{eur,\Theta(\tau_\Theta)}$ is dominated by the American one $\widetilde{V}_{X_\Theta}(\tau_\Theta)$, it follows that $\widetilde{X}_{LB}(u) \leq \widetilde{X}_\Theta(u)$, and therefore $\widetilde{V}_{LB}(t) \leq \widetilde{V}_\Theta(t)$.

The lower bound of the above proposition coincides with the value of the option when the early exercise premium over the interval (τ_Θ, T) is null. This happens for instance when the option is an American call and the underlying does not pay any dividend in the residual life of the option (τ_Θ, T).

In the Sect. 5 we apply these results to evaluate a call option with the accelerating clause presented in (4) on a binomial underlying asset S. □

4 Warrants with acceleration clauses

This section is dedicated to the analysis of warrant contracts on lognormal shares with exercise possible at T only. We first show that acceleration clauses are as effective as callbility features to reduce the risk of high payoffs for the warrant's issuer. Secondly, we quantify the premium associated to 30-days-notice clauses, that are often embedded in callable warrants. Finally, we apply our results to see how the valuation of warrants at origination is affected by the presence of the callability and the 30-days-notice clause.

We first apply risk-neutral valuation techniques to derive a simple method to evaluate warrants with acceleration clauses after origination. As mentioned in the introduction, acceleration clauses can be linked to callability features through the 30-day notice period or can exist independently.[3] The latter are becoming more frequent in warrants contracts issued by SPACs. Acceleration clauses reduce the maturity of the contract rather than terminating it, and therefore they are more palatable to the holders than callability features. Moreover, acceleration clauses are as effective as callability features in capping the warrant payoff, thus reducing the risk of high payoff for the warrant's issuer. Indeed, let us compare a warrant on a lognormal asset with volatility σ, initial value $S(0)$, interest rate r, drift μ, zero dividend yield, and maturity T. Assume that the warrant is callable at $b = 2K$, that is assume that if the underlying S reaches the threshold $b = 2K$, the warrant is terminated with the payoff $(2K - K)^+ = K$. Consider a warrant on the same asset that instead has an acceleration clause triggered when the underlying S reaches the threshold $\overline{a}$, that reduces the time-to-maturity to T'. The acceleration threshold $\overline{a} \in (K, b)$ such that the accelerated warrant payoff is smaller than the one of the warrant with the callability features with probability 99% is $\overline{a}$ such that $\mathbb{P}\left[S(t+T') \le b \middle| S(t) = \overline{a}\right] = \mathbb{P}\left[e^{\left(\mu - \frac{1}{2}\sigma^2\right)T' + \sigma\sqrt{T'}Z'} \le \frac{b}{\overline{a}}\right] = F_{\mathcal{N}(0,1)}\left(\frac{1}{\sigma\sqrt{T'}}\left(\ln\frac{b}{\overline{a}} - \left(\mu - \frac{1}{2}\sigma^2\right)T'\right)\right) = 99\%$, where $F_{\mathcal{N}(0,1)}$ denotes the cumulative distribution function of a standard normal random variable, and Z denotes a standard normal random variable. Because the 99%−quantile of the standard normal distribution is 2.3263, we get that the acceleration threshold $\overline{a}$ that ensures that the payoff with the acceleration clause is lower than the callable one with 99% probability is

$$\overline{a} = b\exp\left(-\sigma\sqrt{T'} \cdot 2.3263 - \left(\mu - \frac{1}{2}\sigma^2\right)T'\right) \tag{8}$$

When $\mu = r$ formula (8) provides the acceleration threshold $\overline{a}$ that ensures 99% risk neutral probability of having a lower terminal payoff than with the callability feature. Table 1 provides the values of the acceleration threshold $\overline{a}$ implying with 99% probability a smaller payoff than with the callability feature for the contracts of Table 1, at page 9 of Burney and Moore (1997), for different values of T', the residual time-to-maturity after acceleration. In particular, for the most common $T' = 1$ month, the acceleration threshold that delivers an accelerated payoff smaller than the redeemed one with 99% risk neutral probability is $\overline{a} = 134.72$, for $S(0) = 100$, $K = 80$, $r = 0.08$, $\sigma = 0.25$, and $b = 2K = 160$. Assuming a historical drift of $\mu = 15\%$, the acceleration threshold that delivers an accelerated payoff smaller than the redeemed one with 99% historical probability is $\overline{a} = 133.94$. The acceleration

[3] In the US market, 80% of the outstanding warrants on October, 6th 2023 exhibits callability features, with a 30 days notice period. An independent acceleration clause appears in 1% of the warrants, as reported in https://stockmarketmba.com/stockscreener.php. On the same date, in the Italian stock exchange 8% of the outstanding warrants has acceleration clauses, but no callbility features, as reported in https://www.borsaitaliana.it/borsa/azioni/warrant/lista.html.

Table 1 Values of $\overline{a}$ in (8)

Values of T'	1 week	1 month	6 months
$\overline{a} : \mathbb{Q}\left[S(t+T') \leq b \middle| S(t) = \overline{a}\right] = 99\%$	147.46	134.72	103.50
$\overline{a} : \mathbb{P}\left[S(t+T') \leq b \middle| S(t) = \overline{a}\right] = 99\%$	147.14	133.94	99.939

$S(0) = 100$, $K = 80$, $r = 0.08$, $\sigma = 0.25$, $b = 2K = 160$, $\delta = 0$, $\mu = 15\%$

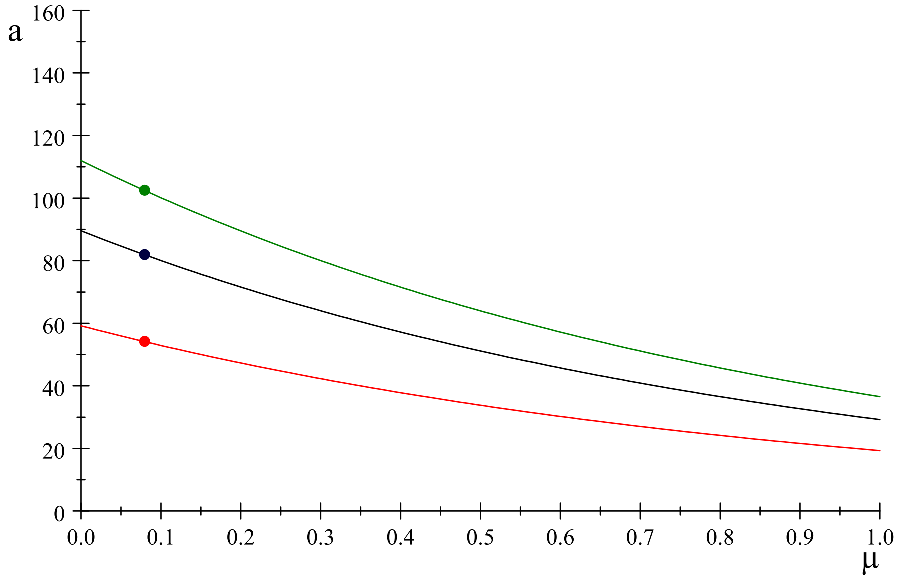

Fig. 1 The acceleration treshold $\overline{a}$ in (8) for $b = 160$, and $T' = 1$ month, as a function of μ for $\sigma = 0.15$ in green, 0.25 in black and 0.45 in red. The dots mark the 99%−risk-neutral acceleration treshold for $\mu = r$

threshold is smaller under the historical probability than under the risk-neutral one, because the historical drift $\mu = 0.15 > r = 0.08$.

In the next figure, we plot the acceleration threshold $\overline{a}$ as a function of the historical drift μ for different levels of volatility. For $b = 2K = 160$, and $T' = 1$ month, we plot the acceleration threshold as a function of μ for $\sigma = 0.15$ in green, 0.25 in black and 0.45 in red. The dots mark the acceleration threshold that ensures with 99% risk neutral probability a lower payoff with the acceleration clause.

The warrant with the acceleration clause delivers a payoff that is lower than $(\overline{a} - K)$, on the trajectories where the acceleration clause is not met because $S(t) < \overline{a}$ for all $t \in [0, T]$. On the trajectories where $S(t) = \overline{a}$ at some $t \in (0, T)$, the option is accelerated and the final payoff $\left(S(t+T') - K\right)^+ \leq (b - K)$ with probability 99%. Thus, the one-month acceleration clause is as effective as the callability feature in reducing the final payoff risk for the warrant issuer.

In the following proposition we quantify the impact of the acceleration clause implied by the 30 days notice for callable warrants at b with no rebate on a lognormal underlying that can be exercised at maturity only.

Proposition 6 (30 days notice premium for a callable warrant with no rebate). *Consider a callable warrant with strike K on a zero-dividends lognormal asset S that is knocked-out when S reaches $b > S(0)$ with no rebate. Assume that the holder is notified when the warrant becomes callable at*

$$\tau_\Theta = \inf\{t > 0 : S(t) \geq b\}$$

Then, if $\tau_\Theta < T$, the 30 days notice relative premium of the warrant is

$$\frac{A(x, r, T, \sigma, b)\, bsc\left(b, K, T', r, \sigma\right)}{W_{c,a}(x, K, r, T, \sigma, b)}$$

where

- *$bsc\left(b, K, T', r, \sigma\right)$ is the price of a European call option with maturity $T' = 30$ days, with initial stock value b, volatility σ, strike K, maturity T', interest rate r.*
- *$A(x, r, T, \sigma, b)$ is the price of a cash-at-hit contingent claim that pays 1 as soon as the barrier b is triggered before T;*
- *$W_c(x, K, r, T, \sigma, b)$ is the initial value of a knock-out call option on S with maturity T, strike K and barrier b.*

The values of the barrier derivatives are

$$\begin{aligned} A(x, r, T, \sigma, b) &= \mathbb{E}\left[\mathbb{I}_{\tau_\Theta < T} e^{-r\tau_\Theta}\right] \\ &= \left(\frac{x}{b}\right)^{\beta_-} \cdot \mathcal{N}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{x}{b}\right) - \frac{T}{2}\left(\sigma^2 + 2r\right)\right)\right) \\ &\quad + \left(\frac{x}{b}\right)^{\beta_+} \cdot \mathcal{N}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{x}{b}\right) + \frac{T}{2}\left(\sigma^2 + 2r\right)\right)\right) \end{aligned} \tag{9}$$

where

$$\beta_{-,+} = -\left(\frac{r}{\sigma^2} - \frac{1}{2}\right) \mp \frac{1}{2\sigma^2}\left(\sigma^2 + 2r\right)$$

and

$$\begin{aligned} W_c(x, K, r, T, \sigma, b) &= \mathbb{E}\left[e^{-rT}\left(S(T) - K\right)^+ \mathbb{I}_{\tau_\Theta > T}\right] \\ &= x \cdot \left[\mathcal{N}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right)\right)\right. \\ &\quad \left. - \mathcal{N}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{x}{b}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right)\right)\right] \\ &\quad - \left(\frac{b}{x}\right)^{1+\frac{2r}{\sigma^2}} \cdot \left[\mathcal{N}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{b^2}{Kx}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right)\right)\right. \\ &\quad \left. - \mathcal{N}\left(\frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{b}{x}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right)\right)\right] \end{aligned}$$

$$
\begin{aligned}
& - e^{-rT} K \cdot \left[\mathcal{N}\left(\frac{1}{\sigma\sqrt{T}} \left(\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right) T \right) \right) \right. \\
& \quad \left. - \mathcal{N}\left(\frac{1}{\sigma\sqrt{T}} \left(\ln\left(\frac{x}{b}\right) + \left(r - \frac{\sigma^2}{2}\right) T \right) \right) \right] \\
& - \left(\frac{x}{b}\right)^{1-\frac{2r}{\sigma^2}} \cdot \left[\mathcal{N}\left(\frac{1}{\sigma\sqrt{T}} \left(\ln\left(\frac{b^2}{Kx}\right) + \left(r - \frac{\sigma^2}{2}\right) T \right) \right) \right. \\
& \quad \left. - \mathcal{N}\left(\frac{1}{\sigma\sqrt{T}} \left(\ln\left(\frac{b}{x}\right) + \left(r - \frac{\sigma^2}{2}\right) T \right) \right) \right] \qquad (10)
\end{aligned}
$$

The value of the callable warrant with the 30-days-notice period is

$$W_{c,a}(x, K, r, T, \sigma, b) = W_c(x, K, r, T, \sigma, b) + A(x, r, T, \sigma, b)\, bsc\left(b, K, T', r, \sigma\right)$$

Proof At notification date $\tau_\Theta < T$ the warrant becomes a call option on S with maturity $T' = 30$ days, whose value is

$$\mathbb{E}_{\tau_\Theta}\left[e^{-rT'}\left(S\left(T' + \tau_\Theta\right) - K\right)^+\right]$$

Since the notification occurs when $S(\tau_\Theta) = b$, the markovianity of S implies that the value of this call option at notification date τ_Θ is

$$
\begin{aligned}
\mathbb{E}_{\tau_\Theta}\left[e^{-rT'}\left(S\left(T' + \tau_\Theta\right) - K\right)^+\right] &= \mathbb{E}\left[e^{-rT'}\left(S\left(T'\right) - K\right)^+ \middle| S(0) = b\right] \\
&= bsc\left(b, K, T', r, \sigma\right)
\end{aligned}
$$

where $bsc\left(b, K, T', r, \sigma\right)$ denotes the price of a call option with initial stock value b, volatility σ, strike K, maturity T', interest rate r. Right after origination at $t = 0$, the premium for the 30 days notice is therefore the value of the call option $bsc\left(b, K, T', r, \sigma\right)$ that forward-starts at the (random) date $\tau_\Theta < T$. This value is

$$
\begin{aligned}
& \mathbb{E}\left[\mathbb{I}_{\tau_\Theta < T} e^{-r\tau_\Theta} \mathbb{E}_{\tau_\Theta}\left[e^{-rT'}\left(S\left(T' + \tau_\Theta\right) - K\right)^+\right]\right] \\
& = \mathbb{E}\left[\mathbb{I}_{\tau_\Theta < T} e^{-r\tau_\Theta} bsc\left(b, K, T', r, \sigma\right)\right] \\
& = \mathbb{E}\left[\mathbb{I}_{\tau_\Theta < T} e^{-r\tau_\Theta}\right] bsc\left(b, K, T', r, \sigma\right)
\end{aligned}
$$

The first factor $\mathbb{E}\left[\mathbb{I}_{\tau_\Theta < T} e^{-r\tau_\Theta}\right]$ is the price of a cash-at-hit claim on S that pays 1 as soon as the stock S touches the barrier b before T. Its value is $A(x, r, T, \sigma, b) = \mathbb{E}\left[\mathbb{I}_{\tau_\Theta < T} e^{-r\tau_\Theta}\right]$ resulting in formula (9) (see Nelken 1996).

The callable warrant at b is a knock-out call option on S with maturity T and strike K whose price $W_c(x, K, r, T, \sigma, b)$ is provided in (10). The callable warrant *with* the 30-days-notice period has a complex terminal payoff. On trajectories where $\tau_\Theta > T$, the warrant is never knocked-out, and delivers the call's payoff at T. The value for this option is $W_c(x, K, r, T, \sigma, b)$. If $\tau_\Theta < T$, then the knocked-out payoff is compensated by the activated 30-days-notice period, whose value at $t = 0$

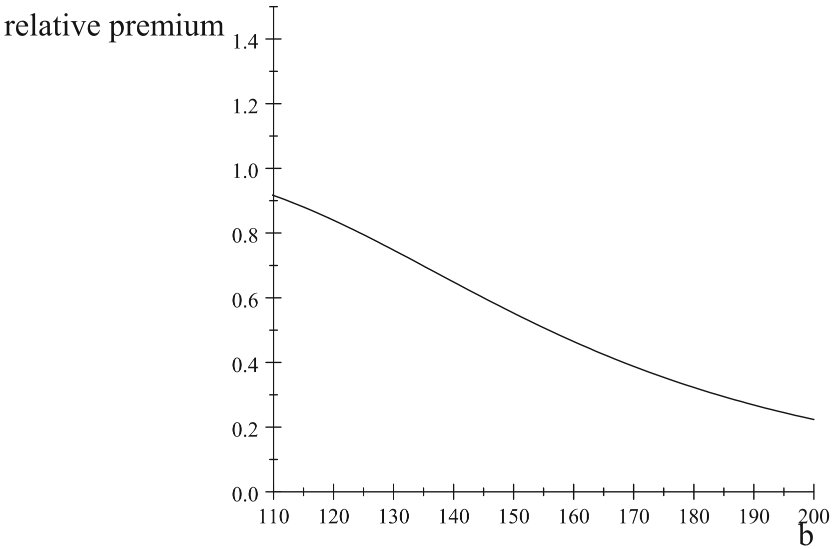

Fig. 2 30-days-notice relative premium for a callable warrant at b. Parameters values: $S\,(0) = 100$, $K = 80$, $r = 0.08$, $T = 5$ years, $\sigma = 0.25$

is $A\,(x, r, T, \sigma, b)\, bsc\,\left(b, K, T', r, \sigma\right)$. In this way, the 30-days-notice period clause effectively extends the resulting maturity of the warrant to $\tau_\Theta + T' \geq T$ if $T - T' \leq \tau_\Theta < T$. The resulting initial value of the callable warrant with the 30-days-notice period is

$$W_{c,a}\,(x, K, r, T, \sigma, b) = W_c\,(x, K, r, T, \sigma, b) + A\,(x, r, T, \sigma, b)\, bsc\,\left(b, K, T', r, \sigma\right)$$

The 30-days-notice relative premium is therefore $\frac{A(x,r,T,\sigma,b)bsc\left(b,K,T',r,\sigma\right)}{W_{c,a}(x,K,r,T,\sigma,b)}$. □

In Fig. 2, the 30-days- notice relative premium is plotted for barrier levels $b > S(0)$, with the same parameters values of Table 1, i.e. $S\,(0) = 100$, $K = 80$, $r = 0.08$, $T = 5$ years, $\sigma = 0.25$. In particular, if the warrant is called when $S(t) = b = 160$ with no rebate, the 30-days-notice premium is 9.4, that amounts to the 46% of the total value of the callable warrant with the 30-days-notice period, 20.39.

Figure 2 shows that disregarding the acceleration clause incorporated within 30-day notice periods for callable warrants can result in non-negligible errors in warrant valuation. In the remaining part of the article, we propose a simple and effective method for evaluating path-dependent acceleration clauses of American call options within a binomial setting for the underlying asset S.

Whilst it is obvious that the Black-Scholes call option formula overestimates the callable warrant, because it neglects its knock-out feature, we see in Fig. 3 that the overestimation is significant even if the callable warrant displays the 30-days-notice period that increases its value. The following figure portraits the prices of the standard Black Scholes call option bsc (green), the knock-out call option W_c, i.e. the callable warrant with no rebate (red), the callable warrant with the 30-days notice period $W_{c,a}$

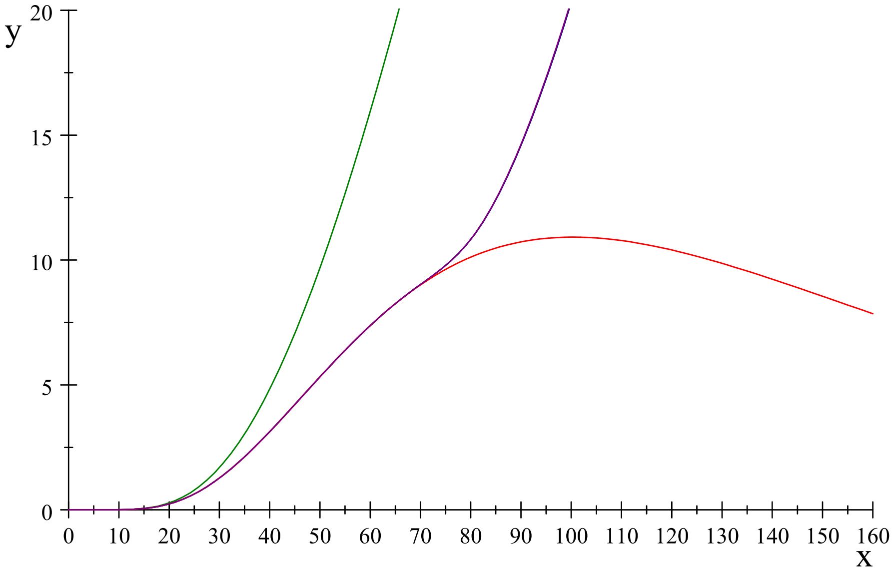

Fig. 3 $W_{c,a}(x)$ (purple), $bsc(x)$ (green), $W_c(x)$ (red). Parameters values: $S(0) = 100$, $K = 80$, $r = 0.08$, $T = 5$ years, $\sigma = 0.25$, $b = 160$

(purple) and the callable warrant with the 30 days notice period until one month from maturity (blue). The two last curves perfectly overlap.

Finally, we evaluate the impact of callability and acceleration clauses on the warrant premium at origination, when exercise is possible at T only. At origination, the evaluation of warrants takes into account the potential dilution effect of the value of the firm (see Galai and Schneller (1978) for the derivation of the warrant's and firm's value at origination as well as their potential dilution effect. See also Lim and Terry (2003) for the evaluation of crossdilution due to a series of European outstanding stock warrants and the effects of their crossdilution on the later warrant series). According to Hull (2021), at issuance, the warrant value w is the solution to a fixed point problem:

$$w = \frac{N}{N+M} \cdot bsc\left(S(0) + \frac{M}{N}w\right)$$

where N is the number of total outstanding shares, M is the number of issued warrants, $S(0)$ is the price of the underlying share and $bsc(x)$ denotes the price of an European call option on the lognormal asset whose initial value is $x = S(0) + \frac{M}{N}w$. The pricing functional for the warrant callable at b with 30-days-notice is $W_{c,a}(x) = W_{c,a}(x, K, r, T, \sigma, b)$ defined in the previous proposition. The warrant premium at origination accounting for both callability and 30-days notice is $w_{c,a}$

$$w_{c,a} = \frac{N}{N+M} \cdot W_{c,a}\left(S(0) + \frac{M}{N}w_{c,a}\right)$$

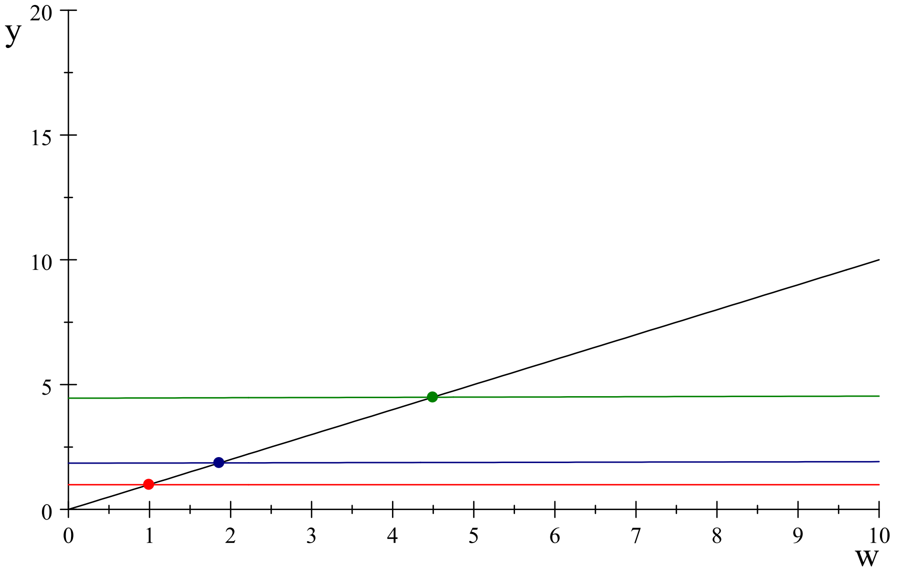

Fig. 4 $\frac{N}{N+M} \cdot W_{c,a}\left(S(0)+\frac{M}{N}w\right)$ (blue line), $\frac{N}{N+M} \cdot bsc\left(S(0)+\frac{M}{N}w\right)$ (green), $\frac{N}{N+M} \cdot W_c\left(S(0)+\frac{M}{N}w\right)$ (red). Parameters values $S(0) = 100$, $K = 80$, $r = 0.08$, $T = 5$ years, $\sigma = 0.25$, $b = 160$, $N = 10000$, $M = 1000$

Because $W_{c,a}(x) < bsc(x)$, the value of a callable warrant with the 30-days notice at origination is lower than the value implied by the Black-Scholes call option formula. As an example, suppose that the number of total outstanding shares is $N = 10000$ and $M = 1000$. For the same parameters' value as above, $S(0) = 100$, $K = 80$, $r = 0.08$, $T = 5$ years, $\sigma = 0.25$ and $b = 160$, we have that $\frac{N}{N+M} \cdot W_{c,a}\left(S(0)+\frac{M}{N}w\right)$ (blue line) intersects the identity line $y = w$ at $w_{c,a} = 1.8584$. The line derived from the call option $\frac{N}{N+M} \cdot bsc\left(S(0)+\frac{M}{N}w\right)$ (green) intersects the identity line in the higher value $w = 4.4952$. In red, we plot the graph implied by the callable warrant $\frac{N}{N+M} \cdot W_c\left(S(0)+\frac{M}{N}w\right)$, whose intersection with the identity line occurs at the smallest $w_c = 0.993$. The magnitude of the difference between $w_{c,a}$ and w depend on the values of N, the number of total outstanding shares, and M, the number of issued warrants, but $w_{c,a}$ is always smaller than w.

Because there is no closed form solution for the price of an American call option on a dividend-paying stock, it is not possible to extend these results to the case of warrants that allow for exercise at any date during the life of the contract.

However, once the traded underlying price has already adjusted to warrants dilution effects, warrants must be evaluated following usual risk-neutral principles, as demonstrated by Galai and Schneller (1978), Galai (1989) and Burney and Moore (1997). In the next sections we apply risk-neutral evaluation techniques to evaluate American call options with path-dependent accelerating clauses written on a binomial share.

5 American options with accelerating clauses on the average of a binomial asset

Consider a discrete-time arbitrage-free complete market on $(\Omega, \mathcal{F}, \mathbb{Q})$ with $t = 0, \Delta t, ..., T$. In our applications $\Delta t = \frac{T}{N}$. The assets traded in the market are the riskless bond B, the risky asset S.

Given $B(0) = 1$ and $S(0) = S_0$, at time t we have riskless asset $B(t + \Delta t) = B(t)e^{r\Delta t}$ providing the riskless interest rate r and a risky stock S with risk-neutral binomial distribution:

$$S(t + \Delta t) = \begin{cases} S(t)e^{\sigma\sqrt{\Delta t}} & \text{with } q = \frac{e^{(r-\delta)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} \\ S(t)e^{-\sigma\sqrt{\Delta t}} & 1 - q \end{cases}$$

where σ is the volatility and δ is the dividend yield of the risky security S.

As $\Delta t \to 0$, the binomial process S converges to the lognormal one. See Mulinacci and Pratelli (1998) for details on the convergence of American derivatives written on S. In this section we focus on the discrete-time pricing problem of American call options with acceleration clauses.

In particular, consider the American call option on S with strike K and contingent maturity Θ defined in the acceleration clause (4). In order to obtain a computationally tractable discrete-time backward recursion for (7) we have to markovianize the problem. In fact, the American call option with acceleration clause (4) is heavily path-dependent, since clause (4) involves the arithmetic average price of S

$$A(t) = \frac{1}{n+1} \sum_{s=0}^{t} S(s) \quad \text{at } t = n\Delta t$$

and the history of A. In fact, in order to compute the payoff X_{LB} at t in (7), we must distinguish three different cases:

1. The acceleration clause (AC) has happened (strictly) before t, i.e. $\tau_A < t$. In this case the payoff $X_{LB}(t) = 0$, as the option $V_{eur,\Theta(\tau_\Theta)}(\tau_\Theta)$ has already been liquidated in the past.
2. The acceleration clause (AC) has never happened before t nor at t, i.e. $\tau_A > t$. In this case the payoff $X_{LB}(t) = (S(t) - K)^+$.
3. The acceleration clause (AC) happens at t, i.e. $\tau_A = t$. In this case $X_{LB}(t) = V_{eur,\Theta(\tau_\Theta)}(\tau_\Theta)$, the price of a European call option on S with strike K and maturity $\Theta(\tau_\Theta) = \min\left(t + \frac{1}{12}; T\right)$.

An efficient triple to markovianize this problem is $(S(t), A(t), \Upsilon(t))$ where

$$\Upsilon(t) = \begin{cases} -1 & \text{if (AC) has happened before } t \\ 0 & \text{if (AC) has never happened before nor at } t \\ 1 & \text{if (AC) happens for the first time at } t \end{cases}$$

The updating rule for the forward shooting grid on the binomial tree for $(S(t+1), A(t+1), \Upsilon(t+1))$ given $(S(t), A(t), \Upsilon(t))$ is

$$S(t+\Delta t) = \begin{cases} S(t)e^{\sigma\sqrt{\Delta t}} & \text{with } q \\ S(t)e^{-\sigma\sqrt{\Delta t}} & 1-q \end{cases}$$
$$A(t+\Delta t) = \frac{tA(t) + S(t+\Delta t)}{t+\Delta t} \tag{11}$$

and

$$\begin{aligned} &\Upsilon(t) = -1 \Rightarrow \Upsilon(t+\Delta t) = -1 \\ &\Upsilon(t) = 0 \Rightarrow \Upsilon(t+\Delta t) = \begin{cases} 0 & \text{if } A(t+\Delta t) < \overline{A} \\ 1 & \text{if } A(t+\Delta t) \geq \overline{A} \end{cases} \\ &\Upsilon(t) = 1 \Rightarrow \Upsilon(t+\Delta t) = -1 \end{aligned} \tag{12}$$

The updating rule (12) for Υ at $t+\Delta t$ depends on $A(t+\Delta t)$ and $\Upsilon(t)$. The first case $\Upsilon(t) = -1$ means that (AC) has already happened before t and (a fortiori) before $t+\Delta t$, so that $\Upsilon(t+\Delta t) = -1$.

In the second case $\Upsilon(t) = 0$ means that (AC) has never happened before or at t. Therefore (AC) may happen for the first time at $t+\Delta t$ leading to $\Upsilon(t+\Delta t) = 1$ if $A(t+\Delta t) \geq \overline{A}$ or it may not happen at $t+\Delta t$ if $A(t+\Delta t) < \overline{A}$ leading in this case to $\Upsilon(t+\Delta t) = 0$.

In the third case $\Upsilon(t) = 1$ means that (AC) happens for the first time at t. Hence (AC) has already happened before $t+\Delta t$ and $\Upsilon(t+\Delta t) = -1$

We are ready to implement a forward shooting grid on the binomial tree for the Markovian triple $(S(t), A(t), \Upsilon(t))$ with an approximated auxiliary variable A. The backward recursion for (7) in this case becomes:

Proposition 7 *In discrete time $\widetilde{V}_{LB}(0)$ defined in equation (7) can be computed by means of the following backward recursion:*

$$V_{LB}(T) = \begin{cases} (S(T)-K)^+ & \text{if } \Upsilon(T) = 0 \text{ or } \Upsilon(T) = 1 \\ 0 & \text{if } \Upsilon(t) = -1 \end{cases}$$

and for $t = T-\Delta t, ..., \Delta t, 0$

$$V_{LB}(t) = \max\left\{X_{\Theta}(t), \mathbb{E}_t^{\mathbb{Q}}\left[e^{-r\Delta t}V_{LB}(t+\Delta t)\right]\right\} \tag{13}$$

where

$$X_{\Theta}(t) = \begin{cases} 0 & \text{if } \Upsilon(t) = -1 \\ (S(t)-K)^+ & \text{if } \Upsilon(t) = 0 \\ V_{eur,\frac{1}{12}}(t) & \text{if } \Upsilon(t) = 1 \end{cases}$$

and $V_{eur,\frac{1}{12}}(t)$ *is the price at* t *of a European call option on* S *with strike* K *and maturity* $\min\left(\frac{1}{12}, T-t\right)$, *i.e.* 30 *days or the residual life of the option (if smaller than* 30 *days).*

6 An effective and intuitive numerical method

We price the American call option with the acceleration clause on the average price in the binomial setting by applying the Forward Shooting Grid (FSG) method (see Hull and White 1993, and Ritchken et al. 1993). Via the FSG method we obtain a parsimonious representation of the Markovianizing triplet $(S(t), A(t), \Upsilon(t))$ on the binomial tree. In particular, we use a vector of representative running averages spanning the full range of possible realizations of average prices $A(t)$ at each node of $S(t)$, ranging uniformly from the lowest to the highest attainable average price. More precisely, consider a **d**-dimensional representative vector $\hat{A}(t)$ and denote by $A^{\max}(t)$ and $A^{\min}(t)$ the maximum and the minimum average values achievable at a generic node of $S(t)$. Denoting with $\Delta_A = \frac{A^{\max}(t)-A^{\min}(t)}{\mathbf{d}-1}$, our vector of representative values $\hat{A}(t)$ is composed by the following components:

$$\hat{A}(t)[i] = A^{\min}(t) + i \cdot \Delta_A \; \forall i \in [0, \mathbf{d}-1].$$

Once all the representative triples $(S(t), A(t), \Upsilon(t))$ are generated at each node of the binomial-tree, we can easily implement the backward recursion of Proposition 7. In particular, we start by computing the value of the American call option at maturity T. At maturity, the American call option is exercised in all the nodes in which it is in-the-money, provided that the acceleration clause has not occurred in the past; the resulting payoff is the value of the American call option at T. At any other generic date $t \in [0, T]$ and for any representative triple $(S(t), A(t), \Upsilon(t))$, we compute the immediate payoff $X_\Theta(t)$ and we compare it to the continuation value of the American call option. The updating rule (11) -(12) allows to identify for every value of $(S(t), A(t), \Upsilon(t))$ its successor at $t+\Delta t$. When an exact match for $A(t+\Delta t)$ is not possible due to the discretization error, we take the closest from above among the representative values found for $A(t+\Delta t)$. The value of the American call option at any node of the tree for any date t is then given by formula (13). At the final step, we get the unique price of the American call option at $t=0$. Uniqueness is given by the fact that at time $t=0$ only the value $\Upsilon(0)=0$ is possible, since $S(0)=A(0)<\overline{A}$.

In what follows we report the output of the implementation of the forward-shooting grid method with dimension **d**=3, as it delivers the same numerical results as **d**=7. The robustness with respect to the dimension **d** of the auxiliary variable A is very helpful, because it reduces the computational burden of the algorithm and makes its implementation easier. Moreover, the code execution is also very quick. On a standard laptop with Intel Core i5 the VBA code output was generated in 17 seconds for **d**=3 and 21 seconds for **d**=7.

We now use our algorithm to analyze the impact of the acceleration clause on the American call option. We first investigate the relationship between the initial price

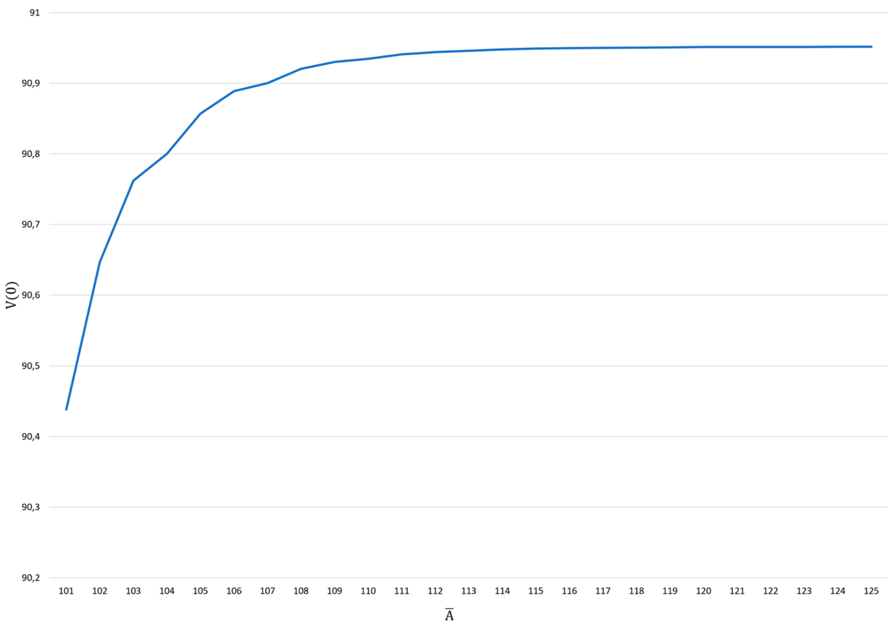

Fig. 5 $V(0)$ as a function of acceleration threshold $\overline{A}$. As $\overline{A}$ increases, $V(0)$ tends to the price of the American call $V_{Amer}(0)$, where $K = 10$, $S_0 = 100$, $\sigma = 0.3$, $T = 1y$, $r = 0.1$, $\delta = 0$, $N = 125$, $V_{Amer}(0) = 90.9516$

$V(0) = V_{LB}(0)$ and the Acceleration threshold $\overline{A}$. An increasing value of $\overline{A}$ results in a lower probability of triggering the acceleration clause during the life of the American call option. Therefore, the greater $\overline{A}$ is, the greater the *time value* of the American call option, ultimately leading to a higher initial price of the American call option. If $\overline{A}$ is too high, the acceleration clause is never activated during the life of the American call option, and its value coincides with the value of a standard American call option on S with same maturity and strike price, which we denote by $V_{Amer}(0)$. Figure 5 portraits this argument. The graph shows that as $\overline{A}$ increases, the initial American call option's price increases, tending to $V_{Amer}(0)$, the initial value of a standard American call option on S.

We now analyze the optimal exercise policies of the American call option with the acceleration clause.

If the barrier $\overline{A}$ is not too high compared to the initial value of the stock and its risk-neutral drift, then the acceleration clause determines on the binomial tree a *pure* acceleration region, where the American call option is always accelerated. In particular, V_{LB} in (13) is a function of $(t, S(t), A(t), \Upsilon(t))$. If S increases a sufficient number of times, the average A is definitely above $\overline{A}$ and the American call option is definitely accelerated. We define the *acceleration boundary* as the smallest value of S on the binomial tree such that for any larger (or equal) value of S at t the American call option is always accelerated:

$$ab(t) = \min\{S \text{ at } t : \Upsilon(t) = 1 \text{ for all } A(t) \text{ compatible with } S \text{ at } t\} \tag{14}$$

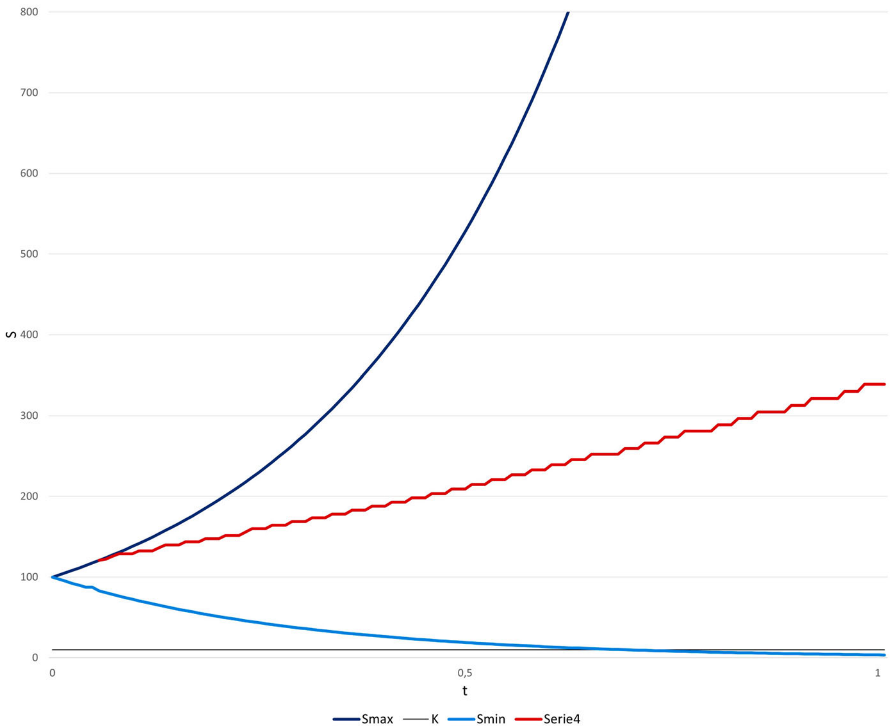

Fig. 6 Acceleration boundary, case $\delta = 0$ $K = 10$, $S_0 = 100$, $\sigma = 0.3$, $T = 1y$, $r = 0.1$, $\bar{A} = 110$, $N = 125$, $V_{Amer}(0) = 90.9516$

Whenever the stock is above this boundary, the American call option is accelerated and its value is $V_{eur,\frac{1}{12}}(t)$ as in (13).

What happens below the acceleration boundary depends on the dividend yield. If $\delta = 0$, it is never optimal to exercise early an American call option. However, the acceleration clause triggers exercise in one month whenever $\Upsilon(t) = 1$. This case is portrayed in Fig. 6, where the dividend yield is $\delta = 0$. The red line represents the values of the acceleration boundary ab above which the acceleration condition is always activated, as a function of t. The oscillating behavior of ab is due to discrete binomial realizations of the underlying S.

If the stock pays a positive dividend yield, the optimal exercise policies of the American call option are more complex. Without acceleration clauses, the American call option is optimally early exercised whenever the stock is above the standard free boundary. Since the acceleration clause lowers the continuation value of the American call option, the stock value that triggers optimal early exercise (provided that the American call option has not been accelerated yet) is now lower. More precisely, we define the free-boundary fb for the American call option with the acceleration clause as

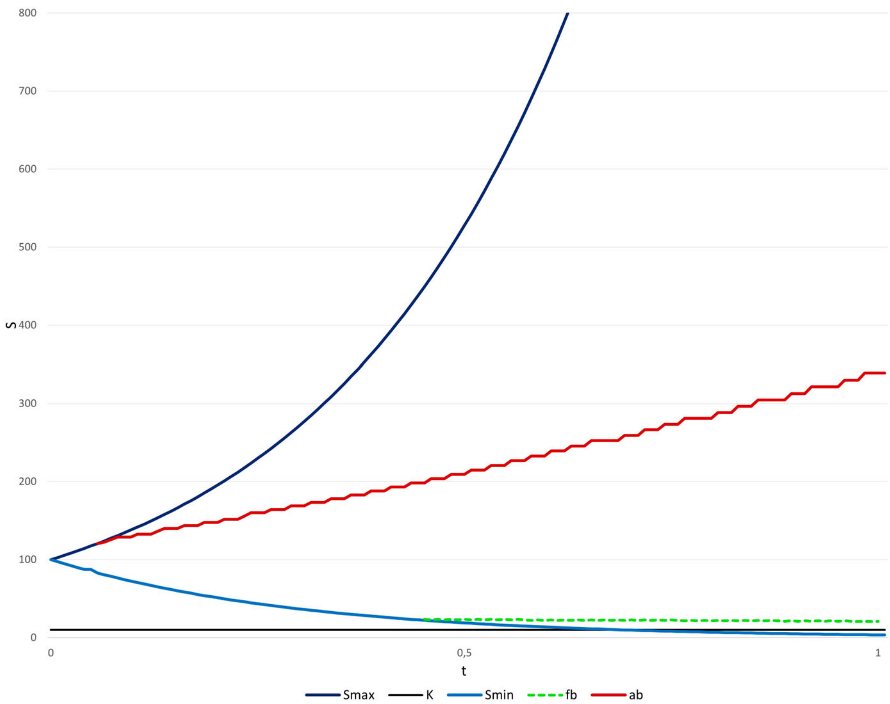

Fig. 7 Acceleration boundary and free boundary, case $\delta = 0.05$ $K = 10$, $S_0 = 100$, $\sigma = 0.3$, $T = 1y$, $r = 0.1$, $\bar{A} = 110$, $N = 125$, $V_{Amer}(0) = 90.9516$

$$fb\left(t\right)\big|_{\Upsilon(t)=0} = \min\left\{\begin{array}{l} S \text{ at } t: \quad V_{LB}(t, S\left(t\right), A\left(t\right), 0) = \left(S\left(t\right) - K\right)^{+} \\ \text{for all } A\left(t\right) \text{ compatible with } S \text{ at } t \end{array}\right\} \tag{15a}$$

The free-boundary $fb\left(t\right)\big|_{\Upsilon(t)=0}$ in Fig. 7 is portrayed with a green dashed line. With $\Upsilon\left(t\right) = 0$, the American call option is optimally continued below the green dashed line and optimally exercised above it. We thus have three regions in the plane (t, S). There is the pure acceleration region for the highest values of the stock, above the acceleration boundary plotted in red (as in Fig. 6). There is an intermediate region where acceleration and early exercise coexist, depending on $\Upsilon\left(t\right)$. In fact, if $\Upsilon\left(t\right) = 0$ and $S(t) \geq fb\left(t\right)\big|_{\Upsilon(t)=0}$, then it is optimal to exercise the American call option, realizing immediately $\left(S\left(t\right) - K\right)^{+}$. If $\Upsilon\left(t\right) = 1$ the American call option is accelerated for the first time and its value coincides with the option expiring in one month (or less, if the time to maturity is shorter than one-month). The case $\Upsilon\left(t\right) = -1$, i.e. acceleration already triggered, is also possible. Below the free-boundary, there is the region where neither early exercise of the American call option is optimal nor acceleration occurs.

The free-boundary $fb\left(t\right)\big|_{\Upsilon(t)=0}$ depicts an almost flat line of values of S, decreasing to K at maturity T.

We now analyze how the boundaries change with different values of $\overline{A}$.

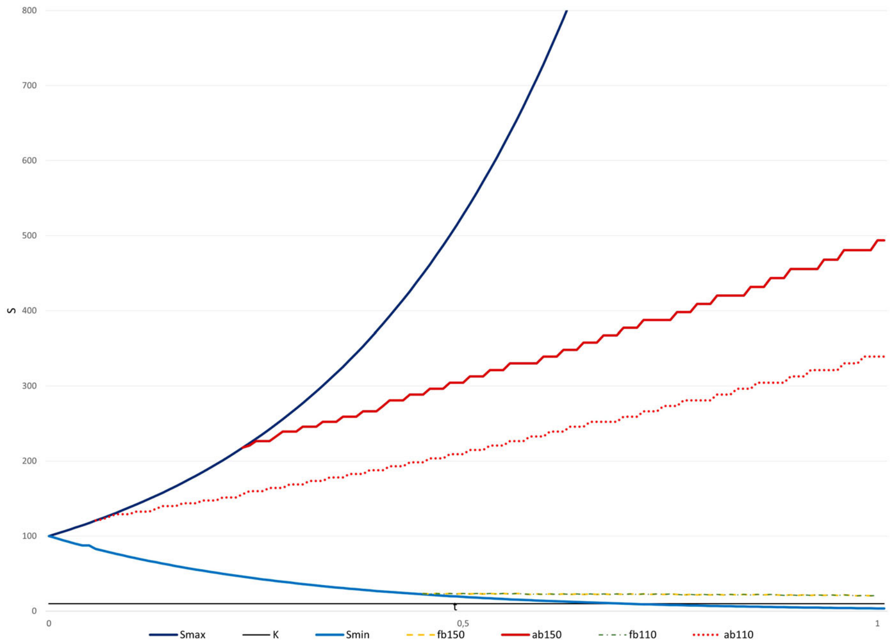

Fig. 8 Boundaries for different values of $\bar{A}$ $K = 10$, $S_0 = 100$, $\sigma = 0.3$, $T = 1y$, $r = 0.1$, $\delta = 0.05$, $N = 125$, $V_{Amer}(0) = 90.9516$

In Fig. 8, we show that the acceleration boundary is very sensitive with respect to the barrier values $\overline{A}$, but the free boundary is not. Indeed, we take two distinct values of $\overline{A} = 25$ and $\overline{A} = 30$ and plot with a green dashed line the free-boundary for $\overline{A} = 25$, and with a yellow dashed line the free-boundary for $\overline{A} = 30$. The two dashed lines almost perfectly overlap within the scale of our current plot. On the contrary, the acceleration boundary (plotted in red, dotted for $\overline{A} = 25$ and solid for $\overline{A} = 30$) moves significantly upward as the value of $\overline{A}$ increases, because the values of S that activate the acceleration condition are larger, other parameters being equal.

The discretization error due to the approximated updating rule for A is also negligible. In fact, computations with the closest-from-below approximating rule for A coincide with the ones obtained with the closest-from-above rule for A. This is due to the fact that the approximation rule does not have a direct sizeable impact on the option's payoff, but only on the option's life. In Equation (6) as soon as the barrier on A is triggered, the payoff of the American call option is not killed to zero (as in knock-out barrier options), but becomes instead $\widetilde{V}_{\Theta(\tau_\Theta)}$, the value of an American call option with reduced maturity $\Theta\,(\tau_\Theta)$.

Finally, Table 2 shows the mild entity of the under-estimation produced by algorithm (13) in the evaluation of our American call option with the acceleration clause. Indeed the relative errors in approximating the one-month-maturity American option values via the European one are around 0.02%.

Table 2 1-month early exercise premium of call options over the initial American option price

Values of $S(0)$	70	75	80	85	90	95	100
Relative Error	0.0226%	0.0227%	0.0228%	0.0229%	0.0229%	0.023%	0.023%

Parameters value are $K = 1$, $\sigma = 0.02$, $T = 1m$, $r = 0.01$, $\delta = 0.002$, $N = 30$

7 Conclusions

In this paper we study American options with acceleration clauses, that shorten the residual life of the option when an acceleration condition is met. Acceleration clauses are frequent within warrants, whose maturity shrinks typically below one month once they are accelerated. The acceleration clause is often embedded in callable warrants that guarantee a 30-days-notice period to holders. The clause is often neglected in their evaluation, possibly resulting in a significant underevaluation of the contracts. We first prove that American options with acceleration clauses, whose maturity is actually stochastic, have the same value of a compound American option with constant maturity. The key idea is to modify the instantaneous payoff of the American compound option, so that as soon as the acceleration condition is met, the payoff of the compound option becomes the value of the American option with accelerated (i.e. reduced) maturity. This allows to provide a lower bound for the American option with the acceleration clause, approximating the value of the American option with accelerated maturity with the European one. Finally, we propose a simple, efficient and robust algorithm to price warrants with the acceleration condition in a binomial setting. Exploring how the market responds to this mispricing of warrants' acceleration clauses presents an intriguing direction for future research.

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