Decisions in Economics and Finance (2023) 46:667–680
https://doi.org/10.1007/s10203-023-00404-2

Green economy with efficient public incentives

Marcello Galeotti[1] · Emanuele Vannucci[2]

Received: 26 October 2022 / Accepted: 12 June 2023 / Published online: 15 July 2023

Abstract

There is a widespread interest among institutions and economic agents for a reduction of the environmental impact of the production system. An important role seems to be played by the ability of public institutions to push the transition toward a green economy also through the application of fiscal policies that envisage a system of rewards and penalties, respectively, for those companies which adopt green strategies and those which do not. It is clear that readjusting older production systems to new pollution regulations can lead in the short term to profitability reductions for the companies implementing them, even though it is possible to assume increases in profitability over medium-long time horizons. One possible approach to this issue is the classical econometric one, which analyzes the effect of different parameters of multivariate models, that influence the level of pollution due to production systems with different propensity for environmental protection. Optimal control models have been also considered with control variables relating to the technologies of production systems and public incentive policies for the green economy: see for example (Tan et al. in J Syst Sci Inf 9(1):61–73, 2021). In recent years, many scholars have studied the relationship between environmental regulation and enterprise technological innovation using evolutionary games, involving mainly economic incentives and fiscal strategies (see see Suyong et al. in Appl Math Comput 355(15):343–355, 2019; Zhang and Li in Appl Math Model 63:577–590, 2018). In our article, we propose a dynamical model where the public administration uses pollution penalties as a control variable in order to push a production sector toward better performances concerning two targets, pollution level and profitability. To this end, we consider the effects of competitiveness among firms and technology innovation.

✉ Emanuele Vannucci
emanuele.vannucci@unipi.it

Marcello Galeotti
marcello.galeotti@unifi.it

1 University of Florence, Florence, Italy

2 University of Pisa, Pisa, Italy

Keywords Climate change · Green economy · Dynamical systems · Evolutionary games · Environmental policy · Environmental governance · Sustainability

Mathematics Subject Classification C61 · E32 · O44 · Q01 · Q56

1 Introduction

In a vast part of the geopolitical context, there is an increasing demand of production systems that reduce the environmental impact and there exist many public finance instruments, decided by national and international authorities, pushing the so-called transition to a green economy, such as the financial recovery and resilience plan at the European level for the post-pandemic restart phase.

It is clear that readjusting older generation productions to the new pollution regulations can lead in the short term to profitability reductions for the companies that implement them, even though it is possible to assume increases in profitability over medium-long time horizons. In a recent paper (Darvas and Wolff 2021) by Zsolt Darvas and Guntram Wolff presented at ECOFIN 2021, the authors highlight the importance of an international tax pact, which supports the transition to a green economy.

Analogously a European Commission publication of June 2021 (Incentives to boost the circular economy—a guide for public authorities) argues that the optimal incentives to increase the weight of the green economy are linked to strategies related to the tax system, which together with direct funding are the most common policy instruments to drive sustainable development.

In Kozhevina (2015) and Aleksejeva (2016), the role of the transformation of industrial production is underlined as a crucial factor for the achievement of a really whole green economy.

Therefore, a dynamical model addressing the problem of the transition to a greener economy, that is to an economy with a lower level of pollution than the initial one, can investigate the efficiency of public incentive policies as far as they can act on the assumed gap between profitability of a classic and a lower environmental impact production system. The use of incentive policies (and/or penalties for those who do not proceed in this direction) for the transition to a greener economy, considering aspects related to competition, which can make it not convenient to convert one's production system if competitors do not, must be such as to achieve virtuous balances of the production system, that is high profitability associated with low levels of pollution.

The work is organized as follows. The second section proposes a review of the literature, recalling the different approaches to the economic-financial evaluation of green production strategies compared to non-green ones. The third section is dedicated to the presentation of our model. In the concluding section, we propose comments and hints at possible future research.

2 Literature review

In the literature, there exist multiple quantitative approaches to the topic of green economy analysis compared to classic profitability maximization approach.

An existing field of literature has faced the problem of achieving the objectives of the successive protocols for driving a transition to less polluting production systems (Kyoto 1997, Paris 2015, Sharm el-Sheikh 2022) from the point of view of game theory, using both cooperative and non-cooperative games (see for example Wood 2011; Ciano et al. 2021).

One approach concerns the possibility of investing in financial instruments supporting the green economy, which can be reconducted to the theory of portfolio, as it offers opportunities for diversification with respect to classic financial instruments. In Cheema-Fox et al. (2021), the authors show that "decarbonizing" a portfolio of US and European-listed equities generates better returns compared to not-decarbonized portfolios in the period 2009–2018.

In Habel and Scholz (2020), the yield differential in terms of share price between companies with and without ESG purposes is analyzed, showing an over performance of no-ESG-driven companies.

In Hang et al. (2019), the crucial importance of time horizon for evaluating the profitability of companies investing in a green direction is developed, highlighting how in the short term the "green conversion" is not convenient, while it can become so over longer time horizons.

Another approach is the classic econometric one, which analyzes the effect of different parameters of multivariate models, influencing the measurement of the level of pollution for production systems with different propensity to environmental protection. For example, in Gawrycka and Szymczak (2021) the impact of the green transition on the Gross Domestic Product of several national economies is studied.

Optimal control models have been also considered, with control variables related to technologies of production systems and green economy public incentive policies: see, for example, Tan et al. (2021) and the explanatory flowchart of the production cycle, object of these authors' investigation.

Moreover, green economy can be considered as playing a role in mitigating environmental damages, especially the impact of climate change on economic system and human society. The magnitude of the climate change problem is emphasized by several analyses and forecasts, and specific plans involving financial support have been discussed.

Finally, in recent years, many scholars have studied the relationship between environmental regulation and enterprise technological innovation using evolutionary game models, which involve mainly economic incentives and fiscal strategies (see Suyong et al. 2019; Zhang and Li 2018; Liang and Weijun 2020).

3 A model of investment return and pollution

This section is devoted to describing our dynamical model and how its equilibria can depend on different public incentives policies for green economy.

We consider two variables:

- x is the current total return of a given industrial sector, measured by a normalized numerical scale, with minimum value 0;
- p is the current total pollution produced by a given industrial sector, measured by a normalized numerical scale, with minimum value 0.

Hence, we set $p = 0$ to indicate the minimal polluting contribution attainable by that sector.

We will consider a continuous time dynamics. On one hand, that allows to transform any time horizon T into infinity, through the change of the time variable $t' = \frac{t}{T-t}$ (see Hartman 1982). On the other hand, we can assume the adjustments of x and p to be instantaneous, which can be justified by a high frequency data pattern of observations.

The dynamical system is defined in the non-negative quadrant $\overline{Q} = \{x, p \geq 0\}$, while we pose $Q = \{x, p > 0\}$. We assume x to increase when x and p stay below certain thresholds, whereas p increases if x and p are sufficiently high (in particular, when x is high, we can think of a flux of investments in the sector which, with the given technology, will make its contribution to pollution grow). However, we assume that, when p is possibly high but x is very low, the investments in that sector will fall, thus leading to a lower pollution.

Translating the above considerations into a formal dynamical system, we require, first of all, $x \geq 0$ and $p \geq 0$. Then, the condition that x increases when both x and p stay under a certain threshold is formalized by requiring that a suitable linear combination of x and p is lower than a quantity h.

Next, the evolution of the pollution p is considered. Here we assume that p increases when a combination of x and px is higher than a quantity k: in fact, as we observed, one can argue that when x is very low the investment in the sector falls, leading, therefore, to a decrease in the pollution produced. Finally, we propose an economic interpretation of all the coefficients involved. Hence, the system we choose as a description of the evolution of x and p is the following:

$$\begin{cases} \dot{x} = x\,(-ax - bp + h) \\ \dot{p} = p\,(cx + dpx - k) \end{cases} \tag{1}$$

where the parameters $a, b, c, d, h, k > 0$ depend both on technical (industrial and financial) and political (e.g., fiscal) factors. In fact, we can interpret the system parameters as follows.

For the first equation, e.g., return dynamics, we observe that:

- a denotes the rate of contribution to return decreasing, in the productive sector, of the current level of return: the higher is a, the more competition contributes to return decreasing.
- b can be considered the unitary penalty for the pollution generated by the sector: the higher is the pollution, the more the return decreases.
- h is the upper limit (when $x, p \to 0$) of the return increasing rate.

  For the second equation, pollution dynamics, we observe that:

- c measures the contribution to pollution increasing due to the profitability of the sector: the higher is the return, the more numerous will be the firms operating in the sector, making the pollution increase.
- d measures the contribution to pollution increasing due to the combined effect of the current levels of return and pollution (the effect of such a combination is small if either x or p is sufficiently small).
- k is the upper limit (when $x, p \to 0$) of the pollution decreasing rate.

As we have said, the technology is assumed not to change in any considered period of time (i.e., in $(0, +\infty)$ in a continuous model).

The first observation concerns the equilibria (or fixed points) of the system, which are of two types:

- boundary equilibria, lying on $\{x = 0\} \cup \{p = 0\}$
- internal equilibria, lying in the positive quadrant $Q = \{x, p > 0\}$.

The former are two: $E_0 = (0, 0)$ and $E_1 = \left(\frac{h}{a}, 0\right)$. It is easily checked that E_0 is a saddle, while E_1 can be either a saddle or a sink (i.e., an attractor). On the other hand, the internal equilibria can be one, zero or two. In fact, by suitable choices of the parameters, several dynamical scenarios can occur.

3.1 Scenario 1

Let us consider the condition $\frac{k}{c} < \frac{h}{a}$, where

- $\frac{k}{c}$ can be interpreted as a ratio between technical improvements for pollution reduction and the return appetite due to the current level of return, which implies more pollution;
- $\frac{h}{a}$ can be interpreted as a ratio between cost rationalization for increasing returns and economic competition in the sector, which implies return reduction.

So, the condition can be interpreted in this way: effects on returns are "more reactive" than ones on pollution.

Theorem 1 *If* $\frac{k}{c} < \frac{h}{a}$, *system* 1 *exhibits exactly one internal equilibrium,* $E_2 = (x_2, p_2)$, $x_2 < x_1, 0 < p_2 < \frac{h}{b}$. *Precisely,* E_1 *is a saddle, while* E_2 *is an attractor (sink) if* $dp_2 < a$, *or is a repellor (source) surrounded by an attracting periodic orbit (limit cycle) if* $dp_2 > a$. *In any case, all the internal orbits (that is, those starting in the positive quadrant* Q*) converge, as* $t \to +\infty$, *to the internal attractor (equilibrium or limit cycle).*[1]

Proof See "Appendix A".

Figure 1 a provides a graphical representation of the isoclines of the system for Scenario 1, while the next figures illustrate the convergence to E_2 or to a limit cycle around it.

[1] The above stability change is known as Hopf bifurcation (see Guckenheimer and Holmes 1997). Its consequence is that all the orbits lying in a certain region (the basin of attraction of the limit cycle) will have an oscillating (asymptotically periodic) behavior.

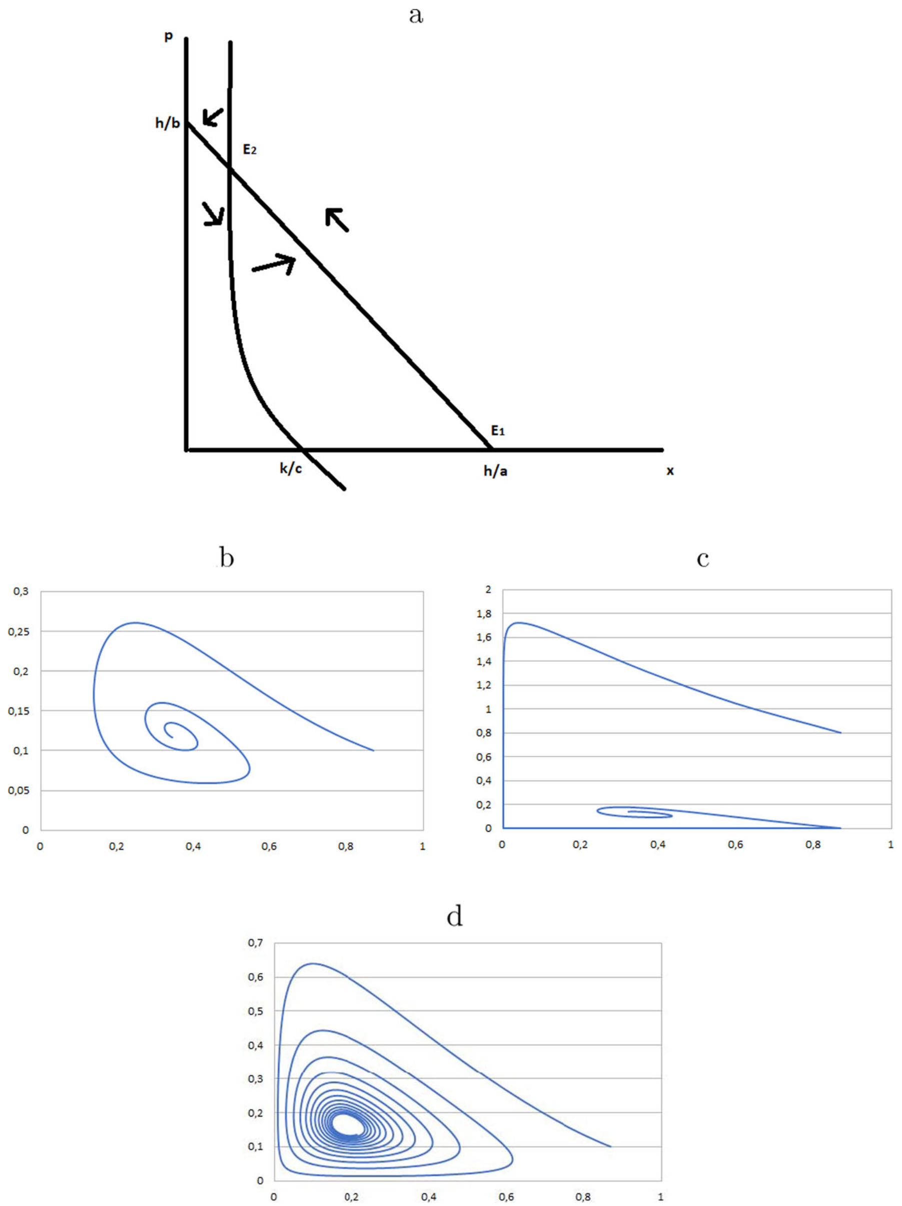

Fig. 1 **a**: Scenario 1 isoclines **b**: Convergence to the poverty trap from starting point (h/a, 0.1) **c**: Convergence to the poverty trap from starting point (h/a, 0.1) **d** Convergence to a limit cycle around the poverty trap

In particular, it follows from the proof of Theorem 1 that a globally attracting limit cycle appears when, fixed $a, b, c, d, h \in (\frac{b}{d}a, \frac{b}{d}(a+c))$ and $k < (\frac{h}{a} - \frac{b}{d}(a+c))$.
We consider the following values of the parameters
$a = 0.092, b = 0.39, h = 0.08, c = 0.09, d = 0.4, k = 0.49$
where $\frac{k}{c} = 0.05444 < 0.8695 = \frac{h}{a}$
Then we can take an orbit from any point of the half-line $(\frac{h}{a}, \cdot)$ with $\cdot > 0$, for instance $(\frac{h}{a}, 0.1)$ and $(\frac{h}{a}, 0.8)$, generating, respectively, the orbits of Fig. 1b, c, converging to the attractor E_2
Vice-versa, if we vary the parameter h, so as to remain in Scenario 1 but with the condition $dp_2 > a$, e.g., setting
$a = 0.092, b = 0.39, h = 0.2, c = 0.09, d = 0.4, k = 0.49$,
an orbit starting from any point of the half-line $(\frac{h}{a}, \cdot)$ with $\cdot > 0$, for instance from $(\frac{h}{a}, 0.1)$, converges to a limit cycle around E_2, as shown by Fig. 1d. □

3.2 Scenario 2

Assume $\frac{k}{c} > \frac{h}{a}$ holds, which is the opposite condition with respect to Scenario 1, meaning that effects on returns are "less reactive" than ones on pollution.
Assume, moreover, that the curves (isoclines)

$$\gamma_1, \ p = \frac{a}{b}\left(\frac{h}{a} - x\right) \text{ and } \gamma_2, \ p = \frac{k - cx}{dx}$$

do not intersect in Q. Then the following theorem holds.

Theorem 2 *Under the above assumptions, no internal equilibrium exists. Then E_1 is an attractor and all the orbits starting in Q converge to E_1.*

Proof By writing the Jacobian matrix J_1 at E_1, E_1 is checked to be an attractor. Moreover, by Poincaré–Bendixson theorem (Hartman 1982), E_1 is the only attractor in $\overline{Q}$. Then, following the vector field $\left(\dot{x}, \dot{p}\right)$, it is easily seen that all the orbits of Q converge to E_1. □

Figure 2 a provides a graphical representation of the isoclines of the system for Scenario 2, while Fig. 2b, c shows the convergence of the internal orbits to the virtuous equilibrium E_1 respectively, from (h/a, 0.1) and (h/a, 0,8).
We consider the following values of the parameters, generating Scenario 2,
$a = 0.092, b = 0.39, h = 0.08, c = 0.09, d = 0.4, k = 0.079$
where $\frac{k}{c} = 0,8777 > 0,8695 = \frac{h}{a}$.
We can consider orbits from any point of the half-line $(\frac{h}{a}, \cdot)$ with $\cdot > 0$, for instance from $(\frac{h}{a}, 0.1)$ and $(\frac{h}{a}, 0.8)$, which are represented by orbits in Fig. 2b, c, converging to the attractor E_1

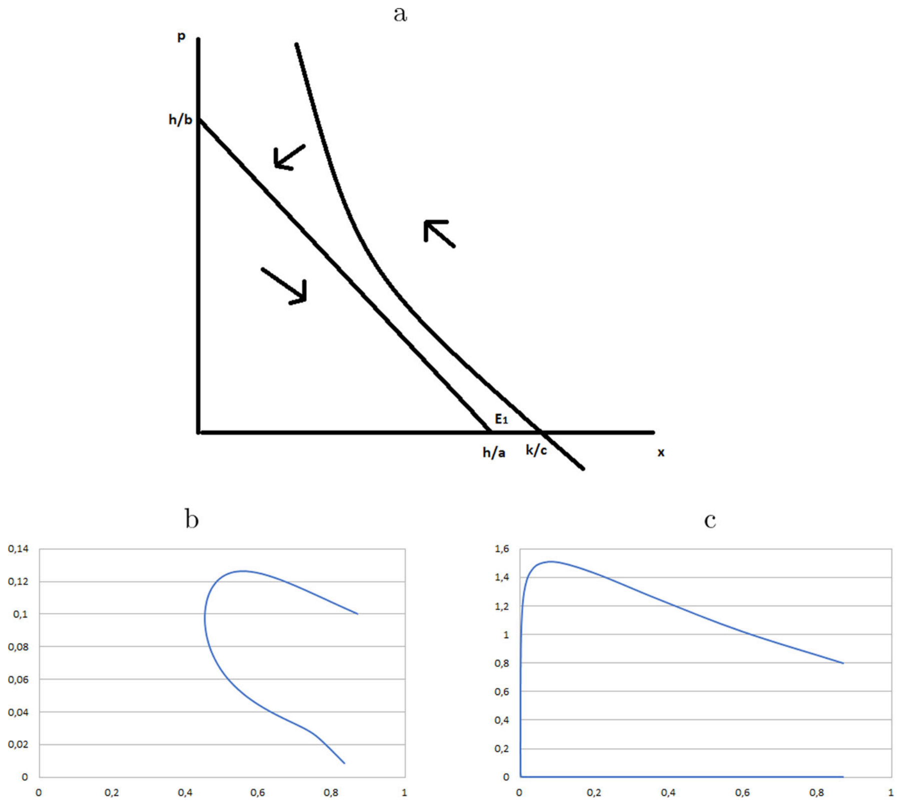

Fig. 2 **a**: Scenario 2 isoclines **b**: Convergence to the virtuous equilibrium from starting point (h/a, 0.1) **c**: Convergence to the virtuous equilibrium from starting point (h/a, 0.8)

3.3 Scenario 3

Assume $\frac{k}{c} > \frac{h}{a}$ and two internal equilibria exist, which requires the further conditions $dh > bc$ and $(dh + bc)^2 > 4abdk$. Let us denote such equilibria as $E_2 = (x_2, p_2)$ and $E_3 = (x_3, p_3)$, $x_2 < x_3$, $p_2 > p_3$. Then, the following theorem holds.

Theorem 3 *Under the above assumptions, E_1 is a sink (attractor), E_3 is a saddle and E_2 is a sink if $dp_2 < a$ (higher competition, expressed by the value of a), or is a source (repellor) surrounded by an attracting limit cycle if $dp_2 > a$ (lower competition). In any case, there exist two basins of attraction, which are separated by the stable manifold of E_3 (that is, by the union of the two orbits converging to E_3).*

Proof See "Appendix B".

In fact, it can be seen that the above *separatrix* has the shape of an *unbounded* loop (the closure of the loop is at the point $(x, p) = (+\infty, 0)$). Hence, orbits starting inside the loop converge to the internal attractor (equilibrium or limit cycle), while orbits starting in Q outside the loop converge to E_1.

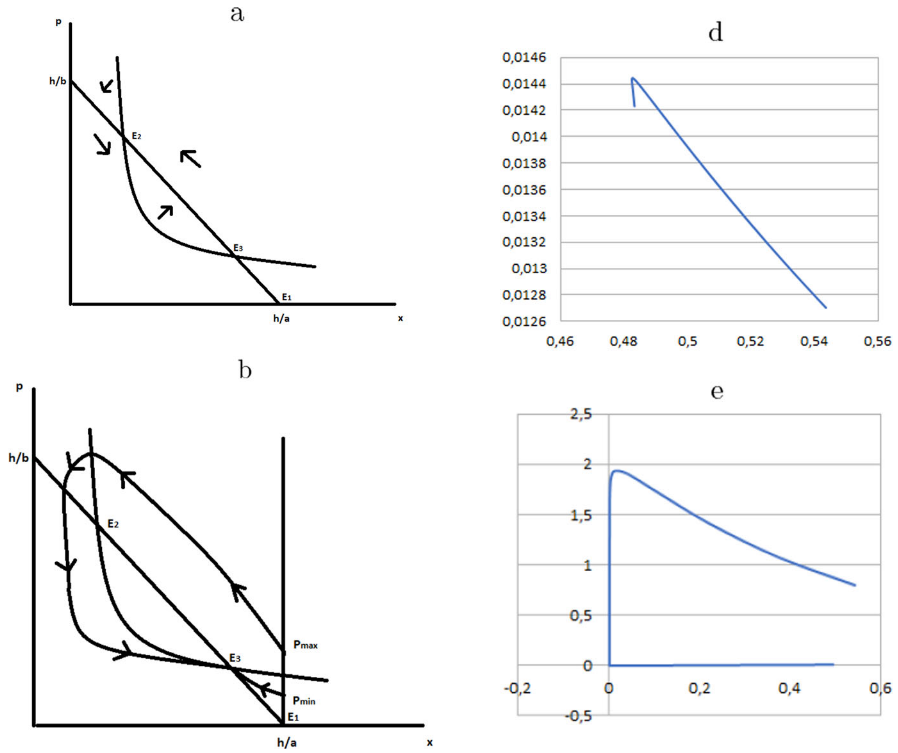

Fig. 3 **a**: Scenario 3 isoclines **b**: Poverty trap attracting basin **c**: Convergence to the poverty trap from starting point (h/a, 0.0127) **d**: Convergence to the virtuous equilibrium from starting point (h/a, 0.8)

In fact R, the region bounded by the loop, can be described as a "poverty trap" (see Antoci et al. 2011), since orbits lying in R converge to an attractor performing worse than E_1, both in terms of pollution level and profitability.

From an economic point of view, we are interested in the strip bounded by the half-lines $x = 0$ and $x = \frac{h}{a}$, which represents the maximum attainable return. Hence, in order to describe the attracting basins of Scenario 3, we should find the segment $\left(p_{\min}, p_{\max}\right)$ on $x = \frac{h}{a}$, from whose points depart the orbits lying in the poverty trap R.

In fact, it is easily observed that $p_{\min} = 0$ implies $p_{\max} = +\infty$, while, when the isocline $\dot{p} = 0$ is tangent to $ax + bp = h$, so that $(dh + bc)^2 = 4abdk$, and hence $p_{\min} = p_{\max}$.

Actually, when $p_{\min}$ increases, $p_{\max}$ decreases. This justifies the fact that we seek an approximation of $p_{\min}$, which also allows to find an upper limit, starting from $p = 0$, for orbits from $x = \frac{h}{a}$ tending to the optimal equilibrium E_1.

In fact, in "Appendix C" a lower and an upper approximation of $p_{\min}$ are calculated.

Figure 3 a provides a graphical representation of the isoclines of the system for Scenario 3, while Fig. 3b illustrates the shape of the poverty trap, bounded by the orbits starting from $\left(\frac{h}{a}, p_{\min}\right)$ and $\left(\frac{h}{a}, p_{\max}\right)$.

The generic orbits in Q are divided in two classes: those lying in the poverty trap and those converging to the virtuous equilibrium E_1. □

We consider the following set of values of the parameters generating Scenario 3
$a = 0.092, b = 0.39, h = 0.05, c = 0.09, d = 0.8, k = 0.049$
In fact, $\frac{k}{c} = 0.05444 < 0.5434 = \frac{h}{a}$, $dh = 0.04 > 0.0351 = bc$ and $(dh + bc)^2 = 0.00564 > 0.00562 = 4abdk$.
If the orbit starts from $(\frac{h}{a}, 0.0127)$, it converges to the attractor E_2 as shown by Fig. 3c. and this implies that the point $(\frac{h}{a}, 0.0127)$ lies in $(\frac{h}{a}, p_{\min}), (\frac{h}{a}, p_{\max})$, that is within the poverty trap represented in Fig. 3b. Moreover, in this case $dp_2 < a$ and there is no limit cycle.
Vice-versa, the orbit starting from $(\frac{h}{a}, 0.8)$ converges to the attractor E_1 (see Fig. 3d). which implies $(\frac{h}{a}, 0.8)$ lying outside the poverty trap represented in Fig. 3b.

3.4 Comments

Several comments can be made, in particular concerning the equilibrium $E_2 = (x_2, p_2)$, when it exists.
We recall that, when the quantity dp_2 crosses the value a, a change of stability occurs: E_2 is an attractor as long as $dp_2 < a$, whereas when $dp_2 > a$ the attractiveness migrates from E_2 to a limit cycle surrounding the equilibrium.
So, it is interesting to investigate the role played by a, which can measure the effect of competition. In particular, assuming Scenario 3, since, when a increases, $p_2 \to p_3$, our model implies that the higher is the level of competition, in a given industrial sector, the easier is its transition toward a greener production and even, in the long-term, toward a steady profitability.

A second comment concerns the policy to be implemented in order to reduce the poverty trap. In fact, it follows from formula (2) that $p_{\min}$ is so much higher, and thus the poverty trap is so more narrow, as the absolute value of the integral in the formula 3 (see "Appendix C") is smaller. Then, increasing b—the unitary penalty for pollution—contributes to this goal.

Finally, the Scenario 3 shows an apparent paradox. In fact, with the same initial return, $\frac{h}{a}$, an orbit with a higher initial pollution can escape the poverty trap, while another one, with a lower initial pollution, cannot, as shown by Fig. 3c, d.
However, such an apparent paradox can be mathematically explained, with an important consequence from an application point of view. In fact, starting from a higher level of pollution, the penalty, given by $-bp$, produces a higher reduction of the investment profitability, so that, when the return approaches the value zero, that sector can be induced to change policy, reducing its pollution impact and therefore, in the long run, improving also its profitability. Vice-versa, when the initial point belongs to the poverty trap, that can be interpreted as the case of a productive sector which, being less penalized by its current pollution, can settle for a long-run lower profitability.
In fact, the analysis of such apparent paradox emphasizes even more the crucial role of an incentive-punishment policy (which is expressed, in particular, by the value of the parameter b).

4 Conclusions

The aim of the present work is to introduce a dynamical model that describes the problem of the transition toward a production system with lower emissions, through a system of penalties and incentives, on the basis of the current level of emissions. The dynamics is described in continuous time, thus obtaining asymptotic results which, through a discrete re-reading of the dynamics, can be transferred to any time horizon. Although it was not possible to give full confirmation of the setting of the model parameters, not having information databases for some of them, the numerical simulations allow to extrapolate some important considerations in terms of sensitivity analysis of the results, which confirm what can be observed through the analysis of the theoretical dynamical model.
One of the lines of research that we intend to carry on will be related to improve even the theoretical model, relying on parameters derived from observable data.

Funding Open access funding provided by Università di Pisa within the CRUI-CARE Agreement.

Appendices

Appendix A: Proof of Theorem 1

In order to find the local stability of E_1 and E_2, we have to examine the Jacobian matrices J_1 and J_2 of the two equilibria (see e.g., Hartman 1982). Then, it is easily calculated that $\det(J_1) < 0 < \det(J_2)$. Hence, E_1 is a saddle, while E_2 is a sinkif if $trace\,(J_2) < 0$, i.e., if $dp_2 < a$, while it is a source if $trace\,(J_2) > 0$, i.e., if $dp_2 > a$. Now, we want to show that, when $dp_2 < a$, E_2 is a global attractor for all the orbits lying in Q, whereas the global attractor in Q, when $dp_2 > a$, becomes a limit cycle surrounding E_2, generated through a Hopf supercritical bifurcation (see e.g., Guckenheimer and Holmes 1997).

In order to develop our argument, let us start from the limit case $p_2 = 0$, when $\frac{k}{c} = \frac{h}{a}$, E_2 being the only intersection of the isoclines in $\overline{Q}$. Then, following the oriented directions of the vector field $\left(\dot{x}, \dot{p}\right)$, it is easily checked that E_2 is a global attractor for the orbits in Q, and so it remains when k lies in a sufficiently small left neighborhood of $k_2 = \frac{hc}{a}$. In fact, choosing k as bifurcation parameter, we can consider $k \in [k_1, k_2]$, where $k_1 = \left(\frac{h}{a} - \frac{b}{d}\right)(a + c)$corresponds, as it is easily checked, to the bifurcation condition $dp_2 = a$. If E_2 were no more globally attracting in Q for $k \in \left[k_{1,}\overline{k}\right]$, $\overline{k} < k_2$, then there would exist (at least) two limit cycles for $k \in \left(k_{1,}\overline{k}\right)$, an exterior one attracting and an internal one repelling. In fact, the inner (or innest) one would shrink, and in particular, when E_2 changes stability, as $dp_2 = a$, the repelling cycle should collapse onto the equilibrium point. Actually, that corresponds to a Hopf subcritical bifurcation (see Guckenheimer and Holmes 1997).

However, the nature of the Hopf bifurcation can be established through straightforward (although lengthy) computations. So, we limit to summarize the main steps of the computation. First of all, consider the change of coordinates $u = \ln x$, $v = \ln p$. Then, for $k = k_1$, the system becomes

$$\begin{cases} \dot{u} = -ae^u - be^v + h \\ \dot{v} = ce^u + de^{u+v} - k_1 \end{cases}$$

and we can consider the Taylor developments of e^u and e^v. Next, we move the origin of the coordinates in E_2, through the translation $u' = u - \ln\left(\frac{h}{a} - \frac{b}{d}\right)$, $v' = v - \ln\left(\frac{a}{d}\right)$. Finally, by rescaling and rotating, we can express the system in coordinates z, w such that

$$\begin{cases} \dot{z} = -\lambda w + f(z, w) \\ \dot{w} = \lambda z + g(z, w) \end{cases} \tag{2}$$

where $\lambda > 0$. Then the nature of the Hopf bifurcation is (generically) decided by the sign of a quantity α, which is an expression of second and third partial derivatives of the functions f and g in $(0, 0)$. Namely (Guckenheimer and Holmes 1997),

$$\begin{aligned} \alpha = & \frac{1}{16}\left(f_{zzz} + f_{zww} + g_{zzw} + g_{www}\right) + \\ & + \frac{1}{16\lambda}\left(f_{zw}\left(f_{zz} + f_{ww}\right) - g_{zw}\left(g_{zz} + g_{ww}\right) - f_{zz}g_{zz} + f_{ww}g_{ww}]\right) \end{aligned}$$

Then, the Hopf bifurcation is supercritical (an attracting limit cycle is generated) if $\alpha < 0$, while it is subcritical (a repelling limit cycle disappears) if $\alpha > 0$. In fact, computations show that in our case $\alpha < 0$, which proves, by our previous arguments, that no limit cycle exists around E_2 when E_2 is a sink. Hence, E_2 is a global attractor in Q for $dp_2 < a$, whereas for $dp_2 > a$ the global attractor in Q becomes the limit cycle surrounding E_2.

Appendix B: Proof of Theorem 3

The stability of the three equilibrium points E_1, E_2, E_3 is determined by the Jacobian matrices J_1, J_2, J_3. In fact, it is easily calculated that E_1 is a sink ($trace(J_1) < 0 < \det(J_1)$), while $\det(J_3) < 0 < \det(J_2)$. Hence, E_3 is a saddle, whereas, analogously to what happened in Theorem 1, E_2 is a sink, if $dp_2 < a$, or a source, if $dp_2 > a$.

In any case, the union of the orbits converging to E_3, which are seen, through observation of the vector field, to originate from $x = +\infty$ and $p = 0$ (in other words, they tend to $(+\infty, 0)$ as $t \to -\infty$), separates the basin of attraction of E_1 from a region, having the shape of an unbounded loop, where the orbits converge to some sub-optimal attractor (point or cycle).

As is the case in macroeconomic literature (see e.g., Antoci et al. 2011), we denote such a region as a poverty trap. In order to investigate the dynamical behavior inside the poverty trap, we proceed similarly to the proof of Theorem 1. Namely, we can choose k as the bifurcation parameter and let k vary in an interval $[k_1, k_2]$, where k_1 corresponds to the change of stability of E_2, i.e., $dp_2 = a$, while for $k = k_2$ we assume the two isoclines to be tangent, which yields $k_2 = \frac{(dh+bc)^2}{4abd}$ Hence, if $k_1 < k_2$, when $k = k_2$ the equilibria E_2 and E_3 coincide in $\overline{E} = (\overline{x}, \overline{p})$, satisfying $d\overline{p} < a$. This way $\overline{E}$ is a saddle-node and, taking k in a sufficiently small left neighborhood of k_2, E_2 is a global attractor in the poverty trap. Then, arguments similar to those employed in the proof of Theorem 1 allow to conclude that all the orbits lying in the poverty trap converge to E_2 when $dp_2 < a$ and to a limit cycle surrounding E_2 when $dp_2 > a$.

Appendix C: Approximation of $p_{\min}$

Consider the intersection of the isocline $\dot{p} = 0$ with $x = \frac{h}{a}$. We find the point $\widetilde{Q} = \left(\frac{h}{a}, \widetilde{p}\right)$, $\widetilde{p} = \frac{ka-ch}{dh}$. It is easily checked that, along the arc $E_3\widetilde{Q}$ of $\dot{p} = 0$, $sgn\left(\ddot{p}\right) = sgn\left(\dot{x}\right)$ is negative, so that $p\,(t)$ has a maximum and the relative orbit converges to E_1Hence, $\widetilde{p} < p_{\min}$.

Now, setting $Q = \left(\frac{h}{a}, p_{\min}\right)$, let us approximately calculate $p_3 - p_{\min}$ along the orbit joining E_3 and Q, which bounds from below the poverty trap. Posing $u = \ln x$ and $v = \ln p$ and inverting the time, we get

$$\ln p_{\min} \; - \ln p_3 = \int_{u_3}^{\ln \frac{h}{a}} \frac{-ce^u - de^{u+v} + k}{ae^u + be^v - h} du \tag{3}$$

Since it is easily calculated that the negative integrand function decreases with e^v, it follows that, replacing in the integral e^v by p_3, we find a further lower approximation of $p_{\min}$, say $p^1_{\min}$, whereas, replacing e^v by $\widetilde{p}$ we find an upper approximation, say $p^2_{\min}$.

From there on, we can improve our approximations and construct an algorithm converging to the actual $p_{\min}$. In fact, consider the segment joining E_3 with $\left(\frac{h}{a}, p^2_{\min}\right)$ and along it the value $p\,(u)$, $u = \ln x$. Then, replacing in the above integral e^v by $p(u)$, we find a lower approximation of $p_{\min}$, $p^3_{\min} > p^1_{\min}$, whereas, replacing e^v by $p^1_{\min}$, we find an upper approximation, $p^4_{\min} < p^2_{\min}$. This way we can continue as long as we want.

References

Aleksejeva, L.: Country's competitiveness and sustainability: higher education impact. J. Secur. Sustain. Issues **5**(3), 355–363 (2016)

Antoci, A., Galeotti, M., Russu, P.: Poverty trap and global indeterminacy in a growth model with open-access natural resources. J. Econ. Theory **146**(2), 569–591 (2011)

Cheema-Fox, A., Realmuto LaPerla, B., Serafeim, G., Turkington, D., Wang, H.: Decarbonization factors. J. Impact ESG Invest. Fall **2**(1), 47–73 (2021)

Ciano, T., Ferrara, M., Gangemi, M., Merenda, D.S., Pansera, B.A.: Climate change sustainability: from bargaining to cooperative balanced approach. Games **12**(2), 1–13 (2021)

Darvas Z., Wolff G.: A green fiscal pact: climate investment in times of budget consolidation. Policy Contribution Bruegel Issue, No. 18 (2021)

Gawrycka, M., Szymczak, A.: A panel analysis of the impact of green transformation and globalization on the labor share in the national income. Energies **14**(21), 6967 (2021)

Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, New York (1997)

Habel, B., Scholz, H.: Integrating sustainability risks in asset management: the role of ESG exposures and ESG ratings. J. Asset Manag. **21**(1), 52–69 (2020)

Hang, M., Geyer-Klingeberg, J., Rathgeber, A.W.: It is merely a matter of time: a meta-analysis of the causality between environmental performance and financial performance. Bus. Strateg. Environ. **28**(2), 257–273 (2019)

Hartman P.: Ordinary Differential Equations. Society for Industrial and Applied Mathematics (1982)

Kozhevina, O.: Sustainable development and green growth in the agro industrial regions. Bus. Strateg 3, 14–20 (2015)

Liang, Y., Weijun, H.: Transboundary water sharing problem: a theoretical analysis using evolutionary game and system dynamics. J. Hydrol. **582**(3), 124521 (2020)

Suyong, Z., Chuanxu, W., Chao, Y.: The evolutionary game analysis and simulation with system dynamics of manufacturer's emissions abatement behavior under cap-and-trade regulation. Appl. Math. Comput. **355**(15), 343–355 (2019)

Tan, X., Sun, Z., Zho, C., Zhi, C.: Analysis of green technology upgrading strategy based on collaborative incentive of environmental policy and green finance. J. Syst. Sci. Inf. **9**(1), 61–73 (2021)

Wood, P.J.: Climate change and game theory. Ecol. Econ. Rev. **1219**(1), 153–170 (2011)

Zhang, Z., Li, H.: New evolutionary game model of the regional governance of haze pollution in China. Appl. Math. Model. **63**, 577–590 (2018)