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Optimisation of drawdowns by generalised reinsurance in the classical risk model

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Abstract

We consider a Cramér–Lundberg model representing the surplus of an insurance company under a general reinsurance control process. We aim to minimise the expected time during which the surplus is bounded away from its own running maximum by at least $d > 0$ (discounted at a preference rate $\delta > 0$) by choosing a reinsurance strategy. By analysing the drawdown process (i.e. the absolute distance of the controlled surplus model to its maximum) directly, we prove that the value function fulfils the corresponding Hamilton–Jacobi–Bellman equation and show how one can calculate the value function and the optimal strategy. If the initial drawdown is critically large, the problem corresponds to the maximisation of the Laplace transform of a passage time. We show that a constant retention level is optimal. If the drawdown is smaller than d, the problem can be expressed as an element of a set of Gerber–Shiu optimisation problems. We show how these problems can be solved and that the optimal strategy is of feedback form. We illustrate the theory by examples of the cases of light and heavy tailed claims.

Keywords Drawdowns · General optimal reinsurance · Classical risk model · Hamilton–Jacobi–Bellman equation

JEL Classification C61 · D81 · G22

1 Introduction

The *drawdown* of the surplus of a company is the distance of the current surplus to its last record high and can therefore be interpreted as a relative loss. In the literature of financial mathematics and economics, it serves as a performance-adjusted indicator

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of risk and is also widely used in practice, e.g. to measure and evaluate the success of funds (see, for example, Goldberg and Mahmoud (2017); Burghardt et al. (2003) or the series of papers Chekhlov et al. (2005); Zabarankin et al. (2014); Ding and Uryasev (2022) on portfolio optimisation under drawdown-based constraints). Controlling drawdowns of the surplus is generally favourable as it promotes stability. Especially for insurance companies, this is a valuable aspect because establishing and signalling reliability towards potential customers, investors and regulatory institutions is the basis of operation. An important tool for insurance companies to control the surplus (and, thus, its drawdown) is reinsurance.

Based on a Cramér–Lundberg risk process, we model the *net surplus* $X^B = (X_t^B)_{t\geq 0}$ of an insurance portfolio *under a general reinsurance form* by

$$X_t^B = x_0 + \int_0^t c(B_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} r(B_{T_i-}, Y_i)\;, \qquad t \geq 0\,, \tag{1}$$

and its running maximum $M^B = (M_t^B)_{t\geq 0}$ and *(absolute) drawdown* $D^B = (D_t^B)_{t\geq 0}$ by

$$M_t^B = \max\left\{\bar{x},\ \sup_{0\leq s\leq t} X_s^B\right\}\,, \qquad D_t^B = M_t^B - X_t^B\;, \qquad t\geq 0\,.$$

Here, $x_0 \leq \bar{x}$ is a constant representing the initial capital. Correspondingly, $\bar{x} \in \mathbb{R}$ is the historical maximum at the beginning of the observation period and $x = \bar{x} - x_0$ is the initial drawdown. $N = (N_t)_{t\geq 0}$ is a Poisson process with rate $\lambda > 0$ marking the claim arrivals. The sequence $(Y_i)_{i\in\mathbb{N}}$ of iid random variables models the sizes of the respective claims and is independent of N. We assume that their distribution function G is continuous with $G(0) = 0$ and mean $\mu < \infty$. We focus on continuous claim distribution functions in order to simplify the presentation. In the following, we work on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on which the process N and the claim size sequence $(Y_i)_{i\in\mathbb{N}}$ are defined. The information available at time t is given by $\mathcal{F}_t$; $(\mathcal{F}_t)_{t\geq 0}$ is the smallest complete and right continuous filtration such that $(\sum_{k=1}^{N_t} Y_i)_{t\geq 0}$ is adapted. We express the *reinsurance strategy* in terms of the retention level process $B = (B_t)_{t\geq 0}$ that is assumed to be càdlàg and adapted with values in $[0, 1]$. By B_{t-} we denote the limit from the left of $t \mapsto B_t$. We note that, by rescaling the functions introduced below, one could also consider processes with values in a general, non-empty, compact interval $J = [\underline{b}, \overline{b}] \subseteq \overline{\mathbb{R}}_0^+$. For example, $r(b, x) = \min\{x, \tan(\{b_0 + b(1-b_0)\}\pi/2)\}$ denotes excess of loss reinsurance where the minimal allowed retention level is $\tan(b_0\pi/2)$. We call a process with these properties admissible and denote the set of admissible strategies by $\mathcal{B}$. The *retention function* $r : [0, 1] \times (0, \infty) \to \mathbb{R}_0^+$ represents the part $r(b, y)$ paid by the first insurer for a claim of size y if the retention level b is chosen. The reinsurer takes over the remaining $I(b, y) = y - r(b, y)$. We assume that $r(b, y)$ is increasing and continuous in both components with $r(1, y) = y$. This means, the larger the retention level (and the larger the risk), the more the insurer has to pay and that it is not possible to increase the risk by reinsurance. Moreover, the strategy of 'not buying reinsurance' is allowed. The *premium rate function* $c : [0, 1] \to \mathbb{R}_0^+$ represents the premium income after reinsurance and is assumed to be continuous and strictly increasing with $0 = c(0) < c(1) < \infty$. We choose the function to be strictly increasing because it is not sensible to assume that the reinsurance company would offer different levels of coverage at the same price. Moreover, we assume that reinsurance is more expensive than first insurance, i.e. full reinsurance would cause a negative premium rate. In particular, we have $\mathbb{E}[r(0, Y)] > 0$. Otherwise, selling all risk would be a trivial solution to the problem stated below. The condition $c(1) > 0$ implies that the insurer has a positive premium income before purchasing reinsurance. The condition $c(0) = 0$ implies that it is possible to use the full premium income to purchase reinsurance. Alternatively, one could require a positive net profit for the admissible strategies, i.e. $c(b) \geq \lambda \mathbb{E}[r(b, Y)]$ for all $b \in [0, 1]$, which would be a stronger condition. Another important aspect of our model assumptions is that, with $r(b, y)$ increasing and $r(1, y) = y$, $I(b, y) \geq 0$ is always fulfilled. This is a reasonable assumption because the costs of a reinsured claim should not exceed the claim settlement costs without reinsurance (otherwise, the reinsurance contract would be hard to sell). However, we do not need to assume that $I(b, y)$ is non-decreasing in y. In practice, if the latter is not true, there is a risk of moral hazard on behalf of the first insurer. In our numerical analysis and in many other realistic examples, $r(b, y)$ is thus chosen so that $I(b, y)$ increasing. With the above construction, we implicitly assume that the Cramér–Lundberg model is already adapted to inflation, such that the premium rate and the claim size distribution without reinsurance can be kept constant over time. In particular, the functions $c(b)$ and $r(b, y)$, in comparison with their real-world counterparts, include an inflation correction. Moreover, even though in the definition of the surplus process (1) it might seem as if $c(b)$ only depends on the chosen retention level, it is a general function that naturally also depends on fixed parameters like the claim size distribution, the claim intensity and costs associated to the reinsurance market and economic environment (or estimations thereof). For example, $c(b)$ can correspond to an expected value premium calculation principle (i.e. the reinsurance premium corresponds to the expected costs of accumulated claims reported by the insurer plus a safety loading).

The target of our optimisation is the weighted occupation time of the drawdown process in the critical set (d, ∞)

$$v(x) = \inf_{B \in \mathcal{B}} v^B(x) = \inf_{B \in \mathcal{B}} \mathbb{E}^x \left[\int_0^\infty \mathrm{e}^{-\delta t} \mathbb{1}_{\{D_t^B > d\}} \, \mathrm{d}t \right], \tag{2}$$

where we write $\mathbb{E}^x[\,\cdot\,] = \mathbb{E}[\,\cdot \mid D_0^B = x]$. Here, $d > 0$ is a predetermined parameter indicating the size of drawdowns acceptable for the company and $\delta > 0$ is a preference rate. The consideration of this model can be motivated as follows. Simply speaking, drawdowns of an insurer's surplus occur naturally whenever a claim follows a new record. If the size and frequency of claim payments do not surpass the estimation of the insurer, a drawdown can be unproblematic. However, there are events which lead to an unexpectedly high number of reported claims or to extraordinarily large claim settlements. Even if these payments do not force the company out of business, the drawdown in this case is a large relative loss. Moreover, because the company will not let the surplus become infinitely large (for example by paying dividends) a severe drawdown can, in some cases, be associated with undercapitalisation. Thus, a critically large drawdown sends a bad signal to other market participants and can have a negative effect on the credit rating and the share prices. The parameter d introduced above could therefore, for example, be a target solvency level in the case the company follows a barrier dividend strategy. We call (d, ∞) the 'critical' and $[0, d)$ the 'uncritical' area. We assume here that d is fixed, which leads to a discontinuous penalisation of drawdowns in line with the above application. Besides the size of the drawdown, a second influential factor is the duration of the drawdown phases. If the surplus needs a long time to 'recover' from a large drawdown, investors may lose confidence in the company. Hence, frequently occurring and prolonged drawdowns can damage the manager's and/or the company's reputation. In the value function (2), this effect is included by measuring the time with critical drawdown. For the model introduced above, it follows from the Borel–Cantelli Lemma that it is not possible to stay in the uncritical area for all times. But, in reality, one has a preference to postpone inevitable large drawdowns for as long as possible. We assume that this is reflected in an exponential preference 'discounting' of the time at rate $\delta > 0$, giving more weight to drawdowns today than to drawdowns tomorrow. Mathematically, this also ensures well-posedness of the optimisation problem.

An important strength of drawdown as a risk measure is that it depends on the history of the surplus. It is a one-sided and path-dependent (i.e. performance-adjusted) indicator of risk. In this regard, it is important to note that, in the way the optimisation problem in (2) is stated, there is only one state variable (the drawdown x in $v(x)$) instead of two state variables (the surplus x_0 and its maximum $\overline{x}$ in $v(x_0, \overline{x})$). By basing the value function on the controlled drawdown process (instead of the surplus and its running maximum), we overcome the difficulty of working with the path-dependent optimisation problem (and path-dependent strategies) directly. We note that this is possible because the surplus process can be recovered from the controlled drawdown and its local time at zero due to the uniqueness of solutions to the Skorohod problem for spectrally positive càdlàg paths (compare Chaleyat-Maurel et al. 1980).

Note that our value function, like ruin probabilities, is a technical tool for management decisions. A natural idea might be to stop at ruin of the surplus process. This (and other types of possibly finite time horizons) could lead to an optimal strategy of 'deliberately' sending the surplus into the ruin barrier in order to avoid future drawdowns. Such a strategy is not reasonable in applications of the presented problem. That means, a sensibly chosen penalty payment of Gerber–Shiu type would have to be introduced to the value function. The present model could be interpreted as a model with dividends paid according to a barrier strategy. The running maximum takes the role of the accumulated dividend payments. Thus, indirectly, ruin and the severity of ruin are included in the sense that the penalty becomes the future value of the drawdowns. Directly introducing the ruin time of X to the value function would prevent the reduction to one state variable. A mathematically more tractable alternative with implications similar to bankruptcy would be to stop at the first 'catastrophic' drawdown of size $d_\tau \gg d$. d_τ could, for example, correspond to the company's equity at the beginning of the observation. However, in reality, drawdowns can be unfavourably large in the sense that they lead to severe reputational damage without substantially threatening the financial survival of a company. In our notation, this corresponds to X_t being much larger than d. In this regard, a strength of our model is that it enables optimisation of 'soft' objectives (such as stabilising the process at a high level and, thus, reducing reputational risks). In this way, it complements known results on decision making driven by the evaluation of existential risks, e.g. survival probabilities.

Drawdowns and their distributional properties have been extensively studied in the literature of financial mathematics and stochastic processes (see, for example, Mijatović and Pistorius 2012 and Landriault et al. 2017). In recent years, stochastic control of drawdowns has become increasingly popular. In Bäuerle and Bayraktar (2014), a generalised stochastic control problem in a diffusion model is analysed by stochastic ordering techniques. As a result, the authors derive that the probability of a large 'proportional' drawdown (i.e. events of the type $\{X_t < \alpha M_t\}$ for $\alpha \in [0, 1)$) occurring in finite time is minimised by the same constant strategy as the ruin probability. In particular, they show for different types of controls (e.g. proportional reinsurance, excess of loss reinsurance, investments in an independent asset and combinations thereof) that it is optimal to maximise the ratio of drift to volatility squared. The minimisation of the probability of a large proportional drawdown of an investment portfolio in a Black Scholes market is analysed in Angoshtari et al. (2016a). In Chen et al. (2015) and Angoshtari et al. (2016b), variants of the investment problem are considered with an exponential lifetime and consumption. In Angoshtari et al. (2015), the authors consider optimal investment strategies to minimise the expected (exponential) lifetime spent with proportional drawdown in a Black Scholes model. The authors prove that the optimal strategy is composed of the optimal strategy one uses to minimise the probability of a large drawdown (when currently in the state $X_t \geq \alpha M_t$) and the optimal strategy used to minimise the expected occupation time (when in the unfavourable state $X_t < \alpha M_t$). The optimisation of reinsurance strategies to minimise the probability of large proportional drawdowns was studied in Han et al. (2018) (diffusion approximation of a model with thinning dependence structure) and Han et al. (2019) (diffusion approximation of a model with common shock dependence). In both articles, the reinsurance cover is assumed to be proportional. Among other results, the authors derive conditions under which large proportional drawdowns can be prevented, in which case the optimal strategy is again the one minimising the ruin probability. In Arun (2012), Angoshtari et al. (2019) and very recently in Albrecher et al. (2023), dividend maximisation problems are studied, where drawdown constraints are imposed on the payout strategies (i.e. the rate of consumption may decrease but only below a certain fraction of its maximum rate).

Brinker (2021), Brinker and Schmidli (2022) and Brinker and Schmidli (2021) deal with the minimisation of absolute drawdowns for a diffusion risk model. As an absolute drawdown occurs in finite time with probability 1, it is not sensible to consider drawdown probabilities in this context. Instead, variants and extensions of the diffusion-analogue to (2) are solved. As an alternative concept to stopping the observation after an exponentially distributed time as in Angoshtari et al. (2015), the authors propose an infinite time horizon with exponential preference. This means, the time in drawdown is weighted in the sense explained above. The approach to solving these problems is based on the Hamilton–Jacobi–Bellman (HJB) equations. In the cases of optimal reinsurance treated in Brinker and Schmidli (2022) and Brinker and Schmidli (2021), the optimal controls are calculated explicitly. In particular, the optimal strategies turn out to be strictly increasing with the drawdown in the uncritical area, so that the methods of Bäuerle and Bayraktar (2014) do not apply. Similarities to the cases of proportional drawdowns exist in the sense that, under some conditions, growth of the running maximum is 'sacrificed' for the sake of controlling drawdowns (compare, e.g. Angoshtari et al. 2016a and Brinker and Schmidli 2022) and that, in the critical area, the optimal control is to re-enter the uncritical area as fast as possible (compare, e.g. Brinker 2021 and Angoshtari et al. 2015).

In comparison with the literature on the optimisation of drawdown-related quantities for diffusion models, articles with jump surplus models (such as the Cramér–Lundberg or Lévy risk process) are scarce. Exceptions include Wang and Xu (2021) (maximal dividends with transaction costs, stopped at a generalised drawdown time) and, recently, Azcue et al. (2022). In Azcue et al. (2022), the problem of finding an optimal reinsurance strategy to minimise the probability of a large proportional drawdown is analysed for a Cramér–Lundberg model with a Mean-Variance Premium.

This paper contributes to the existing literature by introducing an optimisation problem with the target of minimising the duration of absolute, critical drawdowns. Different from the minimisation of drawdown probabilities, this approach takes the (absolute and lasting) severity of drawdowns into account. In contrast to Brinker and Schmidli (2022) and Brinker and Schmidli (2021), we consider a model with jumps and generalised expressions for the retention level and reinsurance premium. This changes the mathematical techniques and results. Parts of this paper are included in the PhD thesis 'Stochastic Optimisation of Drawdowns via Dynamic Reinsurance Controls' of the first author.

The remainder of this paper is organised as follows. In Sect. 2, we prove a verification theorem based on the HJB equation. In particular, we derive general conditions for existence of the drawdown process under a feedback control induced by a solution to the equation. A dynamic programming principle shows that the problem can be split at the critical line into the problems of

i) minimising the weighted time in critical drawdown
ii) maximising the weighted time in the uncritical area with a penalty for the overshoot at the exit time.

In Sect. 3, we solve subproblem i) explicitly in terms of the value at the boundary $v(d)$. We show that the optimal strategy is constant up to the first passage of the drawdown through d and that it minimises a quantity related to the Lundberg coefficient. We use these results to construct a set of Gerber–Shiu-type optimisation problems containing ii). We solve this general version of the problem and reconnect solutions to characterise the minimal expected time in critical drawdown in (2). In Sect. 4, we consider a discrete version of the problem that can be solved numerically. We illustrate the theory with examples of light and heavy tailed claims for proportional and excess of loss reinsurance. In these examples, we compare the performance of the optimal strategy to certain alternative strategies and comment on the influence of the reinsurance premium. We finish with concluding remarks on economic implications, applicability and possible extensions in Sect. 5.

2 General results

We denote the occurrence times of the claims by T_i, $i \in \mathbb{N}$, and write $\ell_Y(s) = \mathbb{E}[e^{-sY_1}]$ for the Laplace-transform of the claim size distribution. As we will see, the solution depends on the times at which the drawdown process crosses the boundary between the critical and uncritical area.

Notation 1 *For a strategy $B \in \mathcal{B}$ and $y \geq 0$, we define the stopping times*

$$\tau_y(B) = \inf\{t \geq 0 : D_t^B \leq y\}\ , \quad \tau^y(B) = \inf\{t \geq 0 : D_t^B > y\}\ ,$$

and, for the special case $y = d$,

$$\tau(B) = \max\{\tau_d(B), \tau^d(B)\}.$$

Note that $\min\{\tau_d(B), \tau^d(B)\} = 0$. *For $b \in [0, 1]$, we write $\Psi_b(z) := c(b)z - \lambda(1 - \ell_{r(b,Y)}(z))$. For $b > 0$, we denote by $\gamma(b) > 0$ the unique non-negative solution $z = \gamma(b)$ to*

$$\Psi_b(z) = \delta. \tag{3}$$

That there exists, indeed, a unique non-negative solution to Eq. (3) is a consequence of the properties of the Laplace transform. In particular, $z \mapsto \Psi_b(z)$ is defined on $[0, \infty)$ and strictly convex with $\Psi_b(0) = 0$ and $\lim_{z\to\infty} \Psi_b(z) = \infty$ by $c(b) > 0$ for $b > 0$.

Lemma 1 *The function v is increasing with $0 \leq v(x) \leq \delta^{-1}$ for all $x \in [0, \infty)$, fulfils $\lim_{x\to\infty} v(x) = \delta^{-1}$ and is Lipschitz continuous with*

$$|v(x) - v(y)| \leq 2\frac{\lambda + \delta}{\delta c(1)}|x - y|\ .$$

In particular, v is absolutely continuous and differentiable almost everywhere.

Proof The boundedness is clear because the integrand always is an element of $\{0, e^{-\delta t}\}$. One can show that the function increases by choosing the same strategy for different initial drawdowns. Let $0 \leq y < x$. Let $h = (x - y)/c(1)$ and choose $\epsilon > 0$. By the definition of $v(y)$, there exists a strategy $\tilde{B}$ such that $v^{\tilde{B}}(y) < v(y) + \epsilon$. Now we consider a compound strategy for initial drawdown x chosen in the following way. We define $B_t = 1 \cdot \mathbb{1}_{\{T_1 \wedge t < h\}} + \tilde{B}_{t-h}\mathbb{1}_{\{T_1 \wedge t \geq h\}}$ for $t \geq 0$. For $D_0 = x$ this means, we start with a strategy that is constant and equal to one up to the time $h \wedge T_1$, i.e. until y is reached or the first claim occurs. The set $\{T_1 < h\}$ and its complement are $\mathcal{F}_h$-measurable. On the set $\{T_1 \geq h\}$, we have $D_h^B = y$ almost surely and we continue from this new starting point with the strategy $\tilde{B}$. In the definition of B, $\tilde{B}$ is therefore shifted by the deterministic time h. If the first jump is observed before time h, the constant strategy is kept. In particular, this means that if no claim occurs in $(0, h)$, one uses the strategy $\tilde{B}$ from time h, and the strategy of 'no reinsurance' in all other cases. Then

$$\begin{aligned} v(x) - v(y) &\le v^B(x) - v^{\tilde{B}}(y) + \epsilon \\ &\le \int_0^h \mathrm{e}^{-\delta t}\,\mathrm{d}t - (1 - \mathrm{e}^{-(\lambda+\delta)h}) v^{\tilde{B}}(y) + (1 - \mathrm{e}^{-\lambda h})\delta^{-1} + \epsilon \\ &\le 2(1 - \mathrm{e}^{-\lambda h})\delta^{-1} + \epsilon \le 2(\lambda + \delta)h/\delta + \epsilon = 2\frac{\lambda + \delta}{\delta c(1)}(x - y) + \epsilon, \end{aligned}$$

where we used $\mathbb{1}_{\{D_t^B > d\}} \le 1$ and $v(D_{T_1}^B) \le \delta^{-1}$ in the case $T_1 < h$. Because the left-hand side does not depend on ϵ, we can let $\epsilon \to 0$ and Lipschitz continuity follows.

For $x > d$, the time to reach d is at least $(x - d)/c(1)$, yielding $v(x) > \delta^{-1}(1 - \exp\{-\delta(x - d)/c(1)\}$. Therefore, $\lim_{x\to\infty} v(x) = \delta^{-1}$. □

Lipschitz continuity of the value function enables us to choose 'universally' ϵ-optimal strategies. This will enable us to prove the dynamic programming equation (4).

Lemma 2 *For every $\varepsilon > 0$, a strategy $B^\epsilon \in \mathcal{B}$ can be chosen such that $v^{B^\varepsilon}(x) < v(x) + \varepsilon$ holds for all $x \ge 0$ and that the choice is measurable in x.*

Proof Choose $\varepsilon > 0$. Because $v(x)$ is Lipschitz continuous, we have $|v(x) - v(y)| < \varepsilon/2$ for any $x, y \ge 0$ with $|x - y| < \varepsilon/(2L)$, where L denotes the Lipschitz constant of Lemma 1. Let $x_k = k\varepsilon/(2L)$. For each $k \in \mathbb{N}$, we can choose a strategy $B^{\varepsilon,k} \in \mathcal{B}$ such that $v^{B^{\varepsilon,k}}(x_k) < v(x_k) + \varepsilon/2$. In general, this strategy depends on ε and k. Letting $B_t^\epsilon = B_t^{\epsilon,0}\mathbb{1}_{\{x=0\}} + \sum_{k=1}^\infty B_t^{\epsilon,k}\mathbb{1}_{\{x\in(x_{k-1},x_k]\}}$ for all $t \ge 0$ defines a strategy for initial capital x. Note that this choice is measurable and that B^ε is again adapted and càdlàg with values in $[0, 1]$ and, hence, admissible. For $x \in (x_{k-1}, x_k]$, we have

$$v^{B^\epsilon}(x) \le v^{B^\epsilon}(x_k) < v(x_k) + \frac{\varepsilon}{2} - v(x) + v(x) < v(x) + \varepsilon\ .$$

This means, the strategy B^ϵ as defined above is ϵ-optimal for all $x > 0$. For $x = 0$, the inequality holds by the definition of $B^{\epsilon,0}$. □

We next show that we can split the problem at $x = d$.

Lemma 3 (Dynamic Programming). *The value function $v(x)$ fulfils*

$$v(x) = \begin{cases} \inf\limits_{B\in\mathcal{B}} \mathbb{E}^x\big[\mathrm{e}^{-\delta\tau^d(B)} v(D_{\tau^d(B)}^B)\big], & x \in [0, d]\,, \\ \delta^{-1} - (\delta^{-1} - v(d)) \sup\limits_{B\in\mathcal{B}} \mathbb{E}^x[\mathrm{e}^{-\delta\tau_d(B)}]\,, & x > d. \end{cases} \tag{4}$$

Proof Considering an arbitrary strategy B for initial drawdown x, we write $\tilde{B}$ for the strategy $\tilde{B}_t = B_{\tau(B)+t}$. This is the strategy B after time $\tau(B)$ and thus a strategy for initial drawdown $D_{\tau(B)}^B$ conditional on $\mathcal{F}_{\tau(B)}$. We have

$$
\begin{aligned}
v^B(x) &= \mathbb{E}^x\Big[\frac{\mathbb{1}_{\{x>d\}}(1-\mathrm{e}^{-\delta\tau(B)})}{\delta} + \mathrm{e}^{-\delta\tau(B)} v^{\tilde{B}}(D^B_{\tau(B)})\Big] \\
&\geq \mathbb{E}^x\Big[\frac{\mathbb{1}_{\{x>d\}}(1-\mathrm{e}^{-\delta\tau(B)})}{\delta} + \mathrm{e}^{-\delta\tau(B)} v(D^B_{\tau(B)})\Big]
\end{aligned}
$$

and taking the infimum over all admissible strategies B yields

$$
v(x) \geq \inf_{B\in\mathcal{B}} \mathbb{E}^x\left[\frac{\mathbb{1}_{\{x>d\}}(1-\mathrm{e}^{-\delta\tau(B)})}{\delta} + \mathrm{e}^{-\delta\tau(B)} v(D^B_{\tau(B)})\right]. \tag{5}
$$

Now consider a strategy that is arbitrary up to time $\tau(B)$ and then ϵ-optimally chosen as in Lemma 2. Similarly as above, we have

$$
v(x) \leq v^B(x) \leq \mathbb{E}^x\Big[\frac{\mathbb{1}_{\{x>d\}}(1-\mathrm{e}^{-\delta\tau(B)})}{\delta} + \mathrm{e}^{-\delta\tau(B)} v(D^B_{\tau(B)})\Big] + \epsilon.
$$

Taking the infimum over all strategies B, the right-hand side becomes independent of ϵ. Thus, we can let $\epsilon \to 0$,

$$
v(x) \leq \inf_{B\in\mathcal{B}} \mathbb{E}^x\left[\frac{\mathbb{1}_{\{x>d\}}(1-\mathrm{e}^{-\delta\tau(B)})}{\delta} + \mathrm{e}^{-\delta\tau(B)} v(D^B_{\tau(B)})\right]. \tag{6}
$$

Combining (5) and (6) and noting that $D^B_{\tau(B)} = d$ for $x > d$, we obtain the dynamic programming equation. □

The dynamic programming lemma implies that we can split the problem in two subproblems. We deal with each case separately in the next two subsections. The following verification theorem will help to 'reconnect' the cases thereafter. The verification theorem uses the Hamilton–Jacobi–Bellman equation

$$
\inf_{b\in[0,1]} \mathcal{A}^b f(x) = -\mathbb{1}_{\{x>d\}}\,, \tag{7}
$$

where we write

$$
\mathcal{A}^b f(x) = -c(b) f'(x) - (\lambda+\delta) f(x) + \lambda \int_0^\infty f(x + r(b,y))\, \mathrm{d}G(y)\,,
$$

for an absolutely continuous function f with density f'. On $(0,\infty)$, note that $\mathcal{A}^b$ corresponds to the generator of the process $(t, D^B_t)_{t\geq 0}$ (following a constant strategy $B \equiv b \in [0,1]$) applied to a function of the form $\mathrm{e}^{-\delta t} f(x)$. Intuitively, the first term originates from the drawdown process moving into the opposite direction as the surplus with premium income function c. Additionally, λ is the rate at which upward jumps of this process occur, where G denotes the claim size distribution. In the appendix, we provide a detailed motivation of this equation. Related to any solution to this equation is a feedback strategy, given by the respective pointwise minimiser of the equation. We firstly state conditions for the existence of the drawdown process with this type of feedback control. In particular, we characterise the controlled drawdown process as a piecewise deterministic Markov process, see Davis (1984), only using the properties of the integro-differential equation. An important part is the existence of the deterministic integral curves, which depend on the measurability and structure of the pointwise optimiser.

Lemma 4 *Let $f : [0, \infty) \to \mathbb{R}$ be an absolutely continuous, bounded and increasing solution to Eq. (7) with density f'. Denote by $b(x) : [0, \infty) \to [0, 1]$ the pointwise minimiser. There exists a measurable version of $b(x)$ such that the drawdown process D^* under the feedback strategy B^* given by $B^*_t = b(D^*_t)$, $t \ge 0$, exists and B^* is admissible. Moreover, f is strictly increasing.*

Proof Since [0, 1] is compact and the left-hand side of (7) is continuous, there is a function $b(x)$ such that $\mathcal{A}^{b(x)} f(x) = \inf_{b\in[0,1]} \mathcal{A}^b f(x)$. By Wagner (1977, Thm. 7.4), we can choose the minimiser $b(x)$ in a measurable way. We first show that $f(x)$ is strictly increasing. Suppose that $f(a_2) = f(\bar{a}_2)$ for some $a_2 < \bar{a}_2$. We can assume that we have chosen the interval maximally. In particular, $d \notin (a_2, \bar{a}_2)$ by (7). Then $f'(x) = 0$ and hence $b(x) = 0$ can be chosen for any $x \in (a_2, \bar{a}_2)$. This implies

$$\int_0^\infty f(x + r(0, y))\, \mathrm{d}G(y) = \int_0^\infty f(\bar{a}_2 + r(0, y))\, \mathrm{d}G(y).$$

Since $f(z)$ is increasing, $f(x + r(0, y)) = f(\bar{a}_2 + r(0, y))$ at all points of increase of $G(y)$. In particular, this would imply $\bar{a}_2 = \infty$, which is not possible. Now we prove existence of the process. Firstly, we assume that the starting point x of the drawdown process lies within $[0, d]$. A necessary condition for $b(x) = 0$ is $H_0(x) = 0$ with

$$H_0(x) = -(\delta + \lambda) f(x) + \lambda \int_0^\infty f(x + r(0, y))\, \mathrm{d}G(y). \tag{8}$$

On the other hand, (7) implies that $H_0(x) \ge 0$. It is no loss of generality to choose $b(x) = 0$ whenever $H_0(x) = 0$. Note that the set $\{x : H_0(x) = 0\}$ is measurable because f and thus H_0 are continuous.

If $b(x) = 0$, the process exists and is constant until the next jump. By continuity of H_0, there exists for any x_1 with $H_0(x_1) > 0$ an open interval $I_{x_1} = (a_1, \bar{a}_1)$ with $H_0(x) > 0$ for all $x \in I_{x_1}$. In particular, we have $b(x) > 0$ for all $x \in I_{x_1}$. We can choose this interval maximally, such that $H_0(a_1) = 0$ or $a_1 = 0$ is fulfilled.

If $H_0(d) > 0$, we analogously find a half-open interval $(a_1, d]$. Now for a fixed x_1, the function

$$\xi(u) = \int_{x_1}^u \frac{1}{-c(b(s))}\, \mathrm{d}s\,,$$

defined for $u \in I_{x_1}$, is absolutely continuous with a strictly negative density and fulfils $\xi(x_1) = 0$. Hence, there exists a unique, absolutely continuous and decreasing inverse $g(t)$ defined on $(\xi(\bar{a}_1), \xi(a_1))$ with values in I_{x_1}. $g(t)$ fulfils $g(0) = x_1$ and

$\lim_{t\to\xi(a_1)} g(t) = a_1$. $g(t)$ exists for all $t \in [0, \xi(a))$, that is, until $g(t)$ reaches the next point a_1 with $b(a_1) = 0$. In particular, by differentiating $t = \xi(g(t))$, we get that the density of g is given by $g'(t) = -c(b(g(t)))$ for almost all $t \in (0, \xi(a))$. This means, the path of the controlled drawdown process starting from x_1 with $H_0(x_1) > 0$ is (up to the next jump or until the next point a_1 with $H_0(a_1) = 0$ is reached) uniquely defined as solution to the equation

$$g(t) = x_1 - \int_0^t c(b(g(s)))\,\mathrm{d}s.$$

One may treat the cases of $x > d$ with $H_0(x) = -1$ and $H_0(x) > -1$ using the same arguments. If $H_0(x) > -1$ holds on an interval (d, x_1), the above integral equation determines the behaviour of the paths up to the point in time when it reaches d. Clearly the feedback strategy B^* is adapted because $b(x)$ is measurable. Moreover, by $c(b) \geq 0$ for all $b \in [0, 1]$, the critical area is entered by a jump, so $t \mapsto B_t^*$ is càdlàg and, thus, B^* is admissible. □

Remark On every interval (a, b) with $b(x) > 0$ for any $x \in (a, b)$, we get the representation

$$f'(x) = \inf_{\beta\in(0,1]} \frac{\mathbb{1}_{\{x>d\}} + \lambda \int_0^\infty f(x + r(\beta, y))\,\mathrm{d}G(y) - (\delta + \lambda) f(x)}{c(\beta)}. \tag{9}$$

As the term we minimise is continuous for (β, x) in $[0, 1]] \times ([0, \infty)\backslash\{d\})$ and $[0, 1]$ is compact, $f'(x)$ is continuous for $x \in (a, b)$ if $(a, b) \subset [0, d]$ or $(a, b) \subset (d, \infty)$. If the pointwise minimiser of the HJB-equation is unique for every $x \in (a, b)$, one can use continuity of $f'(x)$ to prove that $b(x)$ is continuous for $x \in (a, b)$ as well. Note, however, that the general formulation of Lemma 4 does not require that the arg inf is unique.

Theorem 1 (Verification for the Optimal Control Problem) *Let $f : [0, \infty) \to \mathbb{R}$ be an increasing, absolutely continuous and bounded solution to* (7) *with density f'. Denote by $b(x) : [0, \infty) \to [0, 1]$ the measurable pointwise optimiser chosen as in Lemma* 4. *Assume that either $b(0) = 0$ or $f'(0) = 0$. Then $f(x) = v^{B^*}(x) = v(x)$ for all $x \geq 0$. That is, f is the value function and B^* is an optimal strategy.*

Proof We have seen in Lemma 4 that the process D^* following the feedback strategy exists. For every strategy $B \in \mathcal{B}$, the process

$$\Big(\mathrm{e}^{-\delta t} f(D_t^B) - f(D_0^B) - \int_0^t \mathrm{e}^{-\delta s} f'(0)\,\mathrm{d}M_s^B - \int_0^t \mathrm{e}^{-\delta s} \mathcal{A}^{B_s} f(D_s^B)\,\mathrm{d}s\Big)_{t\geq 0} \tag{10}$$

is a martingale by Dynkin's theorem (Theorem 4.6.1 of Jacobsen (2006)). This means that

$$\begin{aligned} f(x) &= \mathbb{E}^x\big[\mathrm{e}^{-\delta t} f(D_t^B)\big] - \mathbb{E}^x\Big[\int_0^t \mathrm{e}^{-\delta s} f'(0)\,\mathrm{d}M_s^B + \int_0^t \mathrm{e}^{-\delta s} \mathcal{A}^{B_s} f(D_s^B)\,\mathrm{d}s\Big] \\ &\leq \mathbb{E}^x\big[\mathrm{e}^{-\delta t} f(D_t^B)\big] + \mathbb{E}^x\Big[\int_0^t \mathrm{e}^{-\delta s} \mathbb{1}_{\{D_s^B>d\}}\,\mathrm{d}s\Big]. \end{aligned}$$

Letting $t \to \infty$, the first term goes to zero by the boundedness of f. The second term converges to $v^B(x)$. Thus, taking the infimum, we get $f(x) \le v(x)$. On the other hand, under the strategy B^* we get equality and by the conditions given, the integral with respect to the running maximum disappears, implying that $f(x) = v^{B^*}(x)$. Since B^* is an admissible strategy, the assertion follows. □

3 The optimal solution

Theorem 1 implies uniqueness of a bounded increasing solution to the HJB-equation. Our next goal is therefore to prove existence of a solution. In view of Lemma 3, we distinguish the cases of large initial drawdown, that is $x > d$, and small initial drawdown, that is $x \le d$.

3.1 Minimising the time in the critical area

We look for a solution for large initial drawdown, $x > d$. That is, we have to find

$$V(x) = \sup_{B \in \mathcal{B}} \mathbb{E}^x[\mathrm{e}^{-\delta \tau_d(B)}].$$

Since $x \mapsto \mathrm{e}^{-\delta x}$ is decreasing, one could interpret this as minimising the weighted time until (re-)entering the uncritical area. We note that the set $\mathcal{B}$ remains the same as for the original problem. Strategies that are equal up to time $\tau_d(B)$ lead to the same return. We start with some heuristics. Because all the jumps of the drawdown process are upwards, it has to pass through every $y \in (d, x)$ before reaching d. The function V thus fulfils $V(x) = V(d+x-y)V(y)$. This means that V is an exponential function. Splitting the interval (d, x) in 2^n equal parts, we observe that the problem is the same as maximising $\mathbb{E}^{d+(x-d)2^{-n}}[\mathrm{e}^{-\delta \tau_d(B)}]$ in each of the intervals. For this reason, also the same strategy is optimal in each of the intervals. This holds true for every $n \in \mathbb{N}$, which implies that the optimal strategy in (d, ∞) must be constant. For a strategy constant and equal to $b > 0$, the Laplace transform of the passage time is an exponential function depending on $\gamma(b)$ as defined in Notation 1. It is well known that $\mathbb{E}^x[\mathrm{e}^{-\delta \tau_d(B)}] = \mathrm{e}^{-\gamma(b)(x-d)}$ for a constant strategy $B_t = b$ for $t < \tau_d(B)$, see for example Rolski et al. (1999) or Schmidli (2017). That means, we expect the optimal strategy to minimise the coefficient $\gamma(b)$. In the following, we give a rigorous proof.

Lemma 5 *There exist constants* $\tilde{\gamma} > 0$ *and* $\tilde{b} \in (0, 1]$ *satisfying*

$$c(\tilde{b})\tilde{\gamma} - \lambda\big(1 - \mathbb{E}^x\big[\mathrm{e}^{-\tilde{\gamma} r(\tilde{\gamma}, Y_1)}\big]\big) = \max_{b \in [0,1]} \Big(c(b)\tilde{\gamma} - \lambda\big(1 - \mathbb{E}^x\big[\mathrm{e}^{-\tilde{\gamma} r(\tilde{\gamma}, Y_1)}\big]\big)\Big) = \delta. \tag{11}$$

$\tilde{\gamma}$ *is the unique positive value fulfilling the last equality. Moreover,* $\tilde{b}$ *minimises* $\gamma(b)$ *over* $(0, 1]$.

Proof Since $\Psi_b(z)$ is convex, also $\overline{\Psi}(z) = \max_{b\in[0,1]} \Psi_b(z)$ is convex and therefore continuous, with $\overline{\Psi}(0) = 0$ and $\lim_{x\to\infty} \overline{\Psi}(z) = \infty$. Hence, there must be a positive solution $\tilde{\gamma}$ to $\overline{\Psi}(\tilde{\gamma}) = \delta$. Uniqueness follows from the convexity. Because $r(b, y)$ and $c(b)$ are continuous, also $\Psi_b(\tilde{\gamma})$ is continuous in b. Because $[0, 1]$ is compact, $\tilde{b}$ exists such that $\Psi_{\tilde{b}}(\tilde{\gamma}) = \delta$. Because for any b, $\Psi_b(\tilde{\gamma}) \le \delta$, we find $\gamma(b) \ge \tilde{\gamma} = \gamma(\tilde{b})$. Since $\Psi_0(x) < 0$ for all $x > 0$, $\tilde{b} = 0$ is not possible. □

Remark As stated in Lemma 5, $\tilde{\gamma}$ is uniquely defined by Eq. (11). In many practical examples, the optimiser $\tilde{b}$ is also unique. For example, if $c(b)$ is convex, then there exists at most one solution $b \in [0, 1]$ with $\Psi_b(z) = \delta$ for every $z > 0$ for which $f_1(b) = \lambda(1 - \mathbb{E}[e^{-r(b,Y_1)z}])$ is concave (by $f_1(0) > 0 > c(b)z - \delta$). In particular, this is fulfilled if $c(b)$ and $r(b, y)$ are linear functions of b and for the example of the expected value premium presented in the remark below Proposition 1. However, the general conditions imposed on $c(b)$, $r(b, y)$ and the claim distribution function $G(y)$ also allow for cases in which $\Psi_b(z) = \delta$ has more than one solution $b \in [0, 1]$ for a fixed $z > 0$. For example, we assume $c(b) = C_1 b$ and $r(b, y) = [(1 - b_0)b^2 + b_0]y$ (for some $C_1 > 0$, $b_0 \in (0, 1)$). This choice of a convex $b \mapsto r(b, y)$ and linear $c(b)$ can be interpreted in the way that increases in the reinsurance cover become cheaper if more reinsurance is bought. For the claim distribution $G(y) = \int_0^y [w\mathbb{1}_{[0,1]}(w) + (2 - w)\mathbb{1}_{(1,2]}(w)]\,dw$, the parameter set $C_1 = 1.7$, $b_0 = 0.1$, $\lambda = 1.7$, $\delta = 0.1$ produces $\Psi_{0.52}(0.3) = \Psi_{0.81}(0.3) = \delta$.

Proposition 1 *For $\tilde{\gamma}$ and $\tilde{b}$ defined as in Lemma 5, we have $V(x) = e^{-(x-d)\tilde{\gamma}}$ and the value function fulfils*

$$v(x) = \delta^{-1} - (\delta^{-1} - v(d))e^{-(x-d)\tilde{\gamma}}\,, \qquad x > d. \tag{12}$$

The optimal strategy in both cases is constant and equal to $\tilde{b}$ up to the first passage through d.

Proof The process $(Q_t^B)_{t\ge0}$ with

$$Q_t^B = \sum_{k=1}^{N_{t\wedge\tau_d}} (e^{-\tilde{\gamma} D^B_{T_i}} - e^{-\tilde{\gamma} D^B_{T_i-}})e^{-\delta T_i} - \int_0^{t\wedge\tau_d} \lambda e^{-\tilde{\gamma} D^B_s - \delta s}(\ell_{r(B_s,Y)}(\tilde{\gamma}) - 1)\,ds$$

is a martingale. From Lemma 5, it follows that

$$e^{-\tilde{\gamma} D^B_{t\wedge\tau_d} - \delta(t\wedge\tau_d)} = e^{-\tilde{\gamma}x} + Q^B_{t\wedge\tau_d} + \int_0^{t\wedge\tau_d} [\Psi_{B_s}(\tilde{\gamma}) - \delta]e^{-\tilde{\gamma} D^B_s - \delta s}\,ds \le e^{-\tilde{\gamma}x} + Q^B_{t\wedge\tau_d}.$$

Taking expectation and letting $t \to \infty$ shows $\mathbb{E}[e^{-\tilde{\gamma} D^B_{\tau_d} - \delta\tau_d}] \le e^{-\tilde{\gamma}x}$. With the constant strategy $b_t = \tilde{b}$, we obtain equality. The last statement follows from Lemma 3. □

Remark If the premium rate functions of the insurer and reinsurer are calculated via the expected value principle with parameters η and θ, respectively, Proposition 1 implies that the optimal strategy is the one of maximal drift (and maximal risk) $B \equiv 1$. Indeed, in order to maximise $\mathrm{e}^{-(x-d)\gamma(b)}$, we have to minimise $\gamma(b)$ over the set of b with $c(b) > 0$. We denote the mean claim size (before reinsurance) by $\mu = \mathbb{E}[Y_1]$ and note that

$$\begin{aligned}\Psi_b(z) &= \lambda(1+\theta)\mathbb{E}[r(b, Y_1)]z + \lambda\mu(\eta-\theta)z - \lambda(1-\mathbb{E}[\mathrm{e}^{-r(b,Y_1)z}]) \\ &= \lambda(\mathbb{E}[r(b, Y_1)z + \mathrm{e}^{-r(b,Y_1)z}]) + \lambda\theta\mathbb{E}[r(b, Y_1)]z + \lambda\mu(\eta-\theta)z - \lambda \quad (13)\end{aligned}$$

which is increasing in $b \in (0, 1]$ for $z \geq 0$, because $b \mapsto r(b, Y)$ is non-negative and increasing and $x \mapsto xz + \mathrm{e}^{-xz}$ is increasing. Thus, $b \mapsto \gamma(b)$ must be decreasing in b, meaning that it is minimised for $\tilde{b} = 1$ at $\tilde{\gamma} = \gamma(1)$.

3.2 Maximising the time to entering the critical area with a penalty

Next, we consider the expected time in drawdown for an initial drawdown below the critical value. In view of Lemma 3, this function depends on the time of exiting the uncritical area and the overshoot at that time. The overshoot is penalised with the function given in Eq. (12) of Proposition 1 for initial drawdowns larger than d. However, $v(x)$ for $x > d$ is given in terms of the unknown value $v(d)$. To overcome this problem, we define a broader class of related optimal control problems. We again minimise over the set $\mathcal{B}$ of admissible strategies and let for $C \in [0, \delta^{-1}]$

$$v_C(x) = \inf_{B\in\mathcal{B}} \mathbb{E}^x\left[\mathrm{e}^{-\delta\tau(B)} p_C(D^B_{\tau(B)})\right], \qquad x \in [0, d], \tag{14}$$

where $p_C : [d, \infty) \to [0, \delta^{-1}]$ is given by

$$p_C(x) = \frac{1}{\delta} - \Big(\frac{1}{\delta} - C\Big)\mathrm{e}^{-(x-d)\tilde{\gamma}}.$$

This means that p_C serves as a penalty function in the definition of (14), weighing the size of the overshoot. We first observe that the statement of Lemma 1 remains true for the function v_C on $[0, d]$ with a modified Lipschitz constant. For this reason, we also know that v_C is differentiable almost everywhere. We have $v_{v(d)}(x) = v(x)$ for $x \leq d$ and $p_{v(d)} = v(x)$ for $x > d$. In particular, the composition of these functions should be continuous at $x = d$. Similarly as in the proof of Proposition 1, one can show that p_C fulfils $\inf_{b\in[0,1]} \mathcal{A}^b p_C(x) = -1$ for $x \geq d$ with $p_C(d) = C$. Hence, the aim of this subsection is to derive that v_C solves a modified version of Eq. (7) and that there exists a unique, ‘appropriate’ constant C_d such that v_{C_d} solves (7).

Remark Considering the original process X with a dividend barrier strategy, initial capital $d - x$ and dividend barrier d, our problem is equivalent to minimising the Gerber–Shiu function $\mathbb{E}[\{\delta^{-1} - (\delta^{-1} - C)\mathrm{e}^{\tilde{\gamma}X_{\tilde{\tau}}}\}\mathrm{e}^{-\delta\tilde{\tau}}]$, where $\tilde{\tau}$ denotes the time of ruin. Gerber–Shiu functions under a dividend barrier strategy are discussed in Lin et al. (2003) and Gerber et al. (2006).

Lemma 6 *There exists $C_d \in (0, \delta^{-1})$ such that $v_{C_d}(d) = C_d$, $v_C(d) \le C$ for all $C \ge C_d$ and $v_C(d) \ge C$ for all $C \le C_d$. Moreover, the function $w_C(x) = v_C(x)\mathbb{1}_{\{x\le d\}} + p_C(x)\mathbb{1}_{\{x>d\}}$ is increasing in x for all $C \ge C_d$.*

Remark From Proposition 3, we can conclude that C_d is unique, $v(d) = C_d$ and $v(x) = v_{C_d}(x)$. The assumptions of the proposition will be proven in the following.

Proof That $w_C(x)$ is increasing in x follows readily. For $C^{(1)} < C^{(2)}$ and arbitrary $B \in \mathcal{B}$, we have

$$v^B_{C^{(1)}}(x) - C^{(1)} - (v^B_{C^{(2)}}(x) - C^{(2)}) = \Big(1 - \mathbb{E}^x\big[\mathrm{e}^{-\delta\tau^d(B)}\mathrm{e}^{-(D^B_{\tau^d(B)}-d)\tilde{\gamma}}\big]\Big)(C^{(2)} - C^{(1)}) \,.$$

The right-hand side is positive which means that $C \mapsto v^B_C(x) - C$ is decreasing in C for every $B \in \mathcal{B}$. Hence, $C \mapsto v_C(x) - C$ is also decreasing. Moreover, the functions $\{C \mapsto v^B_C(x) - C\}_{B\in\mathcal{B}}$ are Lipschitz continuous with a common Lipschitz constant L. Now, let $B^{\epsilon,2}$ be an ϵ-optimal strategy, such that $v^{B^{\epsilon,2}}_{C^{(2)}}(x) < v_{C^{(2)}}(x) + \epsilon$. We get

$$\begin{aligned} v_{C^{(1)}}(x) - C^{(1)} - (v_{C^{(2)}}(x) - C^{(2)}) &\le v_{C^{(1)}}(x) - C^{(1)} - (v^{B^{\epsilon,2}}_{C^{(2)}}(x) - C^{(2)}) + \epsilon \\ &< L(C^{(2)} - C^{(1)}) + \epsilon \,. \end{aligned}$$

By letting $\epsilon \to 0$, we conclude that $C \mapsto v_C(x) - C$ is Lipschitz continuous. Lastly, we consider the values $C = 0$ and $C = \delta^{-1}$ on the boundary. We have

$$v_0(x) - 0 = \delta^{-1} \inf_{B\in\mathcal{B}} \mathbb{E}^x\big[\mathrm{e}^{-\delta\tau^d(B)}(1 - \mathrm{e}^{-(D^B_{\tau^d(B)}-d)\tilde{\gamma}})\big] > 0\,,$$

because $D_{\tau^d(B)} > d$ almost surely and $\tau^d(B) < \infty$ with positive probability. The latter holds because, by $r(0, y) > 0$ for all $y > 0$, there is a strictly positive probability that $\sum_{k=1}^{N_1} r(0, Y_k) > d + c(1)$ and, thus, by the Borel–Cantelli theorem $\{\sum_{k=N_{n-1}}^{N_n} r(0, Y_k) > d + c(1)\}$ happens infinitely often. On the other hand,

$$v_{\delta^{-1}}(x) - \delta^{-1} = \delta^{-1} \inf_{B\in\mathcal{B}} \mathbb{E}^x\big[\mathrm{e}^{-\delta\tau^d(B)}\big] - \delta^{-1} < 0\,,$$

because $\tau^d(B) \ge T_1$ almost surely. There exists at least one C_x, so that $v_{C_x}(x) - C_x = 0$ by the continuity of $v_C(x) - C$. Since $C \mapsto v_C(x) - C$ is decreasing in C, the assertion follows. □

We write

$$y_b(x) = \inf\{y \ge 0 : x + r(b, y) > d\}$$

with the convention that $\inf\emptyset = \infty$ and for an absolutely continuous function $f : [0, d] \to \mathbb{R}$ with density f'

$$\begin{aligned} \mathcal{A}^b_C f(x) = -(\delta + \lambda) f(x) - c(b) f'(x) + \lambda \int_0^{y_b(x)} f(x + r(b, y))\,\mathrm{d}G(y) \\ + \lambda \int_{y_b(x)}^{\infty} p_C(x + r(b, y))\,\mathrm{d}G(y). \end{aligned}$$

We note that for $f = v_C$, the last two terms add up to the integral with integrand $w_C(x + r(b, y))$. Next, we show in several steps that v_C is a solution to

$$\inf_{b\in[0,1]} \mathcal{A}_C^b v_C(x) = 0\,, \tag{15}$$

(Lemmata 7 and 8 and Proposition 2). We further give conditions to distinguish v_C from other possible solutions, similarly as in Theorem 1 (see Proposition 3). We start with the initial condition.

Lemma 7 *v_C fulfils*

$$v_C(0) = \inf_{b\in[0,1]} \frac{\lambda}{\lambda+\delta} \int_0^\infty w_C(r(b, y))\, \mathrm{d}G(y) \tag{16}$$

For $C \geq C_d$, the infimum is attained at $b = 0$.

Proof For all strategies, starting in zero the process D_t^B remains in zero until the first jump. The value is thus $(\lambda+\delta)^{-1}\lambda \int_0^\infty w_C(r(b, y))\, \mathrm{d}G(y)$, showing (16). The value for $C \geq C_d$ follows from the fact that w_C is increasing in this case. □

Remark The case $C \geq C_d$ is of particular interest as it includes our original control problem. An easy argument for optimality of 0 for $x = 0$ is that, because of the reflecting boundary, the only influential factor is the size Y_1 of the first claim, penalised by the increasing function w_C. Hence, the optimal strategy is to choose the smallest claim size possible, $r(0, Y)$. We comment on efficiency and consequences of this choice in Sect. 5.

In the case $C < C_d$, w_C is not necessarily increasing, which means that one could favour a large jump into the area above (but close to) d over a jump into the area below and close to d.

Lemma 8 *The function v_C fulfils $\inf_{b\in[0,1]} \mathcal{A}_C^b v_C(x) \geq 0$ for all $x \in (0, d]$ at which a one-sided derivative exists or the infimum is attained at $b = 0$. If at $x = 0$ either the infimum is attained at $b = 0$, or the derivative from the right exists and is equal to zero, then $\inf_{b\in[0,1]} \mathcal{A}_C^b v_C(0) = 0$.*

Proof For $x \in (0, d]$, we let $0 < h < x/c(1)$, so zero is not reached in $(0, T_1 \wedge h)$. Let $\epsilon > 0$. We choose $b \in [0, 1]$ and consider the strategy $B_t = b$ for $t < T_1 \wedge h$ and $B_t = B^\epsilon_{t-(T_1\wedge h)}(D^B_{T_1\wedge h})$ for $t \geq T_1 \wedge h$, provided $D^B_{T_1\wedge h} \leq d$. $B^\epsilon(z)$ is a strategy as defined in Lemma 2. We find

$$\begin{aligned}
v_C(x) &\leq v_C^B(x)\\
&= \mathbb{E}^x[\mathrm{e}^{-\delta h} v_C^{B^\epsilon}(x - c(b)h)\mathbb{1}_{\{T_1>h\}}] + \mathbb{E}^x[\mathrm{e}^{-\delta T_1} v_C^{B^\epsilon}(D^B_{T_1})\mathbb{1}_{\{T_1\leq h\}}\mathbb{1}_{\{D^B_{T_1}\leq d\}}]\\
&\quad + \mathbb{E}^x[\mathrm{e}^{-\delta T_1} p_C(D^B_{T_1})\mathbb{1}_{\{T_1\leq h\}}\mathbb{1}_{\{D^B_{T_1}>d\}}]\\
&\leq \mathrm{e}^{-(\delta+\lambda)h}\{v_C(x - c(b)h) + \epsilon\} + \mathbb{E}^x[\mathrm{e}^{-\delta T_1}\{w_C(D^B_{T_1}) + \epsilon\}\mathbb{1}_{\{T_1\leq h\}}]\,.
\end{aligned}$$

Letting $\varepsilon \to 0$ yields

$$v_C(x) \le \mathrm{e}^{-(\delta+\lambda)h} v_C(x - c(b)h) + \mathbb{E}^x[\mathrm{e}^{-\delta T_1} w_C(D^B_{T_1}) 1\!\!1_{\{T_1 \le h\}}] \,.$$

Consider first the case $b = 0$. We have

$$0 \le (\mathrm{e}^{-(\delta+\lambda)h} - 1) v_C(x) + \mathbb{E}^x[\mathrm{e}^{-\delta T_1} w_C(x + r(0, Y_1)) 1\!\!1_{\{T_1 \le h\}}].$$

Dividing by h and letting $h \to 0$ shows

$$0 \le -(\delta + \lambda) v_C(x) + \lambda \mathbb{E}^x[w_C(x + r(0, Y_1))],$$

which shows the desired inequality in this case.

Now suppose $b > 0$; that is, $c(b) > 0$. Then

$$\begin{aligned} 0 \le\; & v_C(x - c(b)h) - v_C(x) + (\mathrm{e}^{-(\delta+\lambda)h} - 1) v_C(x - c(b)h) \\ & + \mathbb{E}^x[\mathrm{e}^{-\delta T_1} w_C(x - c(b)T_1 + r(b, Y_1)) 1\!\!1_{\{T_1 \le h\}}]. \end{aligned}$$

Dividing by h and letting $h_n \downarrow 0$ in such a way that the limit below exists we find

$$0 \le - \lim_{n\to\infty} \frac{v_C(x) - v_C(x - c(b)h_n)}{h_n} - (\delta + \lambda) v_C(x) + \lambda \mathbb{E}[w_C(x + r(b, Y_1))] \,,$$

showing the inequality for the derivative from the left. For $\tilde{x} = x + c(b)h$ and $x \in [0, d)$, we get analogously

$$0 \le - \lim_{n\to\infty} \frac{v_C(x + c(b)h_n) - v_C(x)}{h_n} - (\delta + \lambda) v_C(x) + \lambda \mathbb{E}[w_C(x + r(b, Y_1))], \tag{17}$$

which shows the assertion for the derivative from the right. The result for $x = 0$ follows from (17) combined with (16). □

Verifying also the converse inequality, we obtain:

Proposition 2 *v_C fulfils the HJB-equation* $\inf_{b\in[0,1]} \mathcal{A}^b_C v_C(x) = 0$.

Proof Choose a strategy $B(h) = (B_t(h))_{t\ge 0}$, such that $v^{B(h)}_C(x) < v_C(x) + h^2$. Since we have chosen the filtration in a minimal way and $B(h)$ is adapted, $t \mapsto B_t(h)$ must be deterministic as long as no claim occurs. We consider $x \in (0, d]$, $h < x/c(1)$ and let $\Delta^{B(h)}_t = x - \int_0^t c(B_s(h))\,\mathrm{d}s$ denote the (deterministic) path of $D^{B(h)}_t$ on $\{T_1 > h\}$. Stopping at $T_1 \wedge h$ gives

$$\begin{aligned} v_C(x) &> v^{B(h)}_C(x) - h^2 \\ &= \mathrm{e}^{-(\lambda+\delta)h} v^{\tilde{B}(h)}_C(\Delta^{B(h)}_h) + \mathbb{E}^x[\mathrm{e}^{-\delta T_1} w^{\tilde{B}(h)}_C(D^{B(h)}_{T_1}) 1\!\!1_{T_1 \le h}] - h^2 \end{aligned}$$

$$\geq e^{-(\lambda+\delta)h} v_C(\Delta_h^{B(h)}) + \mathbb{E}^x[e^{-\delta T_1} w_C(D_{T_1}^{B(h)}) \mathbb{1}_{T_1 \leq h}] - h^2.$$

If $\Delta_h^{B(h)} = x$ on $\{T_1 > h\}$ for some $h > 0$, we find,

$$-\frac{1 - e^{-(\lambda+\delta)h}}{h} v_C(x) + h^{-1}\mathbb{E}^x[e^{-\delta T_1} w_C(D_{T_1}^{B(h)}) \mathbb{1}_{T_1 \leq h}] - h < 0.$$

For $\Delta_h^{B(h)} = x$ to hold, $B_t(h) = 0$ must be fulfilled for almost all $t \leq h$. Therefore, letting $h_n \downarrow 0$ for an appropriate subsequence gives

$$-(\lambda + \delta) v_C(x) + \lambda \mathbb{E}[w_C(x + r(0, Y_1))] \leq 0.$$

If there exists no such h, we have $\Delta_h^{B(h)} < x$ and rearranging the terms gives

$$\begin{aligned} 0 > &-\frac{v_C(x) - v_C(\Delta_h^{B(h)})}{x - \Delta_h^{B(h)}} \frac{x - \Delta_h^{B(h)}}{h} - \frac{1 - e^{-(\lambda+\delta)h}}{h} v_C(\Delta_h^{B(h)}) \\ &+ h^{-1}\mathbb{E}^x[e^{-\delta T_1} w_C(D_{T_1}^{B(h)}) \mathbb{1}_{T_1 \leq h}] - h. \end{aligned}$$

Because $c(b)$ is a bounded function, there is a subsequence $\{h_n\}$ such that $h_n^{-1}(x - \Delta_{h_n}^{B(h_n)}) = h_n^{-1} \int_0^{h_n} c(B_s(h_n)) \, ds$ converges. In particular, there is $\tilde{b} \in [0, 1]$, such that the limit is $c(\tilde{b})$. Then, also $h_n^{-1} \int_0^{h_n} B_s(h_n) \, ds$ and $B_t(h_n) \mathbb{1}_{\{t \leq h_n\}}$ converge to $\tilde{b}$. Note that $\mathbb{E}[w_C(x + r(b, Y_1))]$ is continuous in b because the claim size distribution and $y \mapsto r(b, y)$ are continuous. Letting $n \to \infty$ yields $\mathcal{A}_C^{\hat{b}} v_C(x) \leq 0$, where in the case that $v_C(x)$ is not differentiable at x, we may have to take a subsequence of $\{h_n\}$ to get convergence (which exists by the Lipschitz continuity). From Lemma 8, we conclude that $\mathcal{A}_C^{\hat{b}} v_C(x) = 0$. Because the limit does not depend on the subsequence, the assertion is shown. □

The following verification result implies uniqueness of solutions to the modified HJB-equation (15).

Proposition 3 *Let $f : [0, d] \to \mathbb{R}$ be an absolutely continuous bounded solution to the equation $\inf_{b \in [0,1]} \mathcal{A}_C^b f(x) = 0$ with $f'(0) \geq 0$. Denote by $b(x) : [0, \infty) \to [0, 1]$ the measurable pointwise optimiser chosen as in Lemma 4. Assume that either $b(0) = 0$ or $f'(0) = 0$. The drawdown process D^* under the feedback strategy $B_t^* = b(D_t^*)$ exists and $f(x) = v_C^{B^*}(x) = v_C(x)$ for all x. That is, f is equal to v_C and B^* is an optimal strategy.* □

This can be shown analogously to the proof of Theorem 1.

Remark Lemma 7 and Propositions 2 and 3 together imply that v_C defined as in (14) is completely characterised as the unique absolutely continuous and bounded solution to the equation $\inf_{b \in [0,1]} \mathcal{A}_C^b v_C(x) = 0$ for $x \in [0, d]$ fulfilling the initial condition given in (16).

By Lemma 7, the pointwise optimiser in the HJB-equation for v_C (with $C \geq C_d$) at zero is 0. This means that the running maximum of the controlled process is constant. We are now able to give the complete characterisation of the minimal expected time in drawdown v, as well.

Theorem 2 *There is a unique $C_d \in (0, \delta^{-1})$ with $v_{C_d}(d) = p_{C_d}(d)$. It holds*

$$v(x) = \begin{cases} \inf\limits_{B \in \mathcal{B}} \mathbb{E}^x\left[\mathrm{e}^{-\delta\tau(B)} p_{C_d}(D^B_{\tau(B)})\right], & x \in [0, d], \\ \delta^{-1} - (\delta^{-1} - C_d)\mathrm{e}^{-(x-d)\tilde{\gamma}}, & x > d. \end{cases} \tag{18}$$

$v|_{[0,d]}$ is the unique bounded solution to $\inf_{b \in [0,1]} \mathcal{A}^b_{C_d} v(x) = 0$ with $v'(0) \geq 0$ and (16) for $C = C_d$. An optimal strategy B^* is of feedback form with $B^*_t = \tilde{b}\mathbb{1}_{\{D^*_t > d\}} + b(D^*_t)\mathbb{1}_{\{D^*_t \leq d\}}$, where $b : [0, d] \to [0, 1]$ denotes the pointwise minimiser in the HJB-equation $\inf_{b \in [0,1]} \mathcal{A}^b_{C_d} v(x) = 0$. Additionally, $b(0) = 0$.

Proof From Lemma 6, we know that there exists at least one $C_d \in (0, \delta^{-1})$ so that $v_{C_d}(d) = C_d$, where v_C is the function defined in Eq. (14). We define now a candidate f for the function v by setting $f(x) = v_{C_d}(x)$ for $x \in [0, d]$ and $f(x) = p_{C_d}(x)$ for $x > d$. This corresponds to the right-hand side of Eq. (18). For $x > d$, this function is continuously differentiable and solves

$$\begin{aligned} &\inf_{b \in [0,1]} \left\{-(\delta + \lambda) f(x) - c(b) f'(x) + \lambda \int_0^\infty f(x + r(b, y))\, \mathrm{d}G(y)\right\} \\ &\quad = -1 - (\delta^{-1} - C_d)\mathrm{e}^{-(x-d)\tilde{\gamma}} \max_{b \in [0,1]} \left\{c(b)\tilde{\gamma} - \lambda\big(1 - \mathbb{E}^x\left[\mathrm{e}^{-\tilde{\gamma} r(\tilde{\gamma}, Y_1)}\right]\big)\right\} = -1 \end{aligned} \tag{19}$$

by Lemma 5. By $\tilde{\gamma} > 0$, f is also bounded for $x > d$. By Proposition 2, f is absolutely continuous on $[0, d)$ (and therefore bounded everywhere) with $f(d-) = f(d+)$ and fulfils

$$\begin{aligned} &\inf_{b \in [0,1]} \left\{-(\delta + \lambda) f(x) - c(b) f'(x) + \lambda \int_0^\infty f(x + r(b, y))\, \mathrm{d}G(y)\right\} \\ &\quad = \inf_{b \in [0,1]} \left\{-(\delta + \lambda) v_{C_d}(x) - c(b) v'_{C_d}(x) + \lambda \int_0^{y_b(x)} v_{C_d}(x + r(b, y))\, \mathrm{d}G(y)\right. \\ &\qquad \left. + \lambda \int_{y_b(x)}^\infty p_{C_d}(x + r(b, y))\, \mathrm{d}G(y)\right\} = 0\,. \end{aligned} \tag{20}$$

In particular, f solves (7) with the pointwise optimiser stated above and $b(0) = 0$. By Lemma 4, the associated feedback strategy B^* is admissible. Thus, the conditions of Theorem 1 are fulfilled and $f(x) = v(x)$ for all x and B^* is optimal. Assuming there was $\tilde{C}_d \neq C_d$ leading to $v_{C_d}(d) = C_d$, repeating this argument and inspecting the case $x > d$ in (19) shows that C_d is unique. ◻

This characterisation of the value function implies that an optimal retention-level strategy can be constructed by observing the current drawdown and evaluating the feedback function induced by the equation at this point. As we have seen in Sect. 3.1, this means that the constant retention level maximising the relation of drift and jump size with regard to a quick recovery (i.e. (11)) is optimal when the current drawdown is critical. If the drawdown is uncritical, the strategy depends on its exact level in a non-trivial way, as the examples of the next section show. In particular, the shape of the feedback function depends heavily on the claim size distribution, type of reinsurance treaty and the tolerance for drawdowns. As stated in the remark following Lemma 7, the optimality of the zero-drift strategy at the maximum is a direct consequence of the choice of the value function. It should be noted that this strategy prevents the running maximum from increasing. This can be viewed as an anti-cyclic strategy which promotes stability close to the initial maximum. We comment on the economic implications in Sect. 5.

4 Numerical illustration

In the following, we compare return functions of optimal strategies to alternative strategies for exponentially and Pareto-distributed claims. For our numerical illustration, we consider a discrete version of the problem. We assume that there is a partition $(x_k)_{k=0,\dots n}$ of $[0, d]$ and write $I_k = (x_{k-1}, x_k]$. This corresponds to the cautious approach of rounding up changes in the drawdown not 'seen' by the partition or, respectively, the monetary unit. Alternatively, one could consider a general rounding mechanism casting drawdown values to a grid, see Brinker (2022). Correspondingly, we consider a discrete value function $\hat{v}$ (i.e. minimal time in drawdown casted to the grid with respect to the set of discrete admissible strategies) and a discrete feedback strategy $\hat{b}$, given in terms of the values $\{(\hat{v}_k, \hat{b}_k)\}_{k=0,\dots,n}$ at the grid points for $x \in [0, d]$. Similarly to Eqs. (4), (11) and (18), it can be derived that an optimal pair $(\hat{v}, \hat{b})$ should solve the equation system

$$\begin{aligned} \hat{v}(x) &= \inf_{b \in R} \mathbb{E}^{x_k}\Big[\mathrm{e}^{-\delta T}\, \hat{v}(D_T^b) \cdot \mathbb{1}_{\{D_T^b \le d\}} + \mathrm{e}^{-\delta T}\, p_{\hat{v}_n}(D_T^b) \cdot \mathbb{1}_{\{D_T^b > d\}}\Big] \\ &= \mathbb{E}^{x_k}\Big[\mathrm{e}^{-\delta T}\, \hat{v}(D_T^{\hat{b}_k}) \cdot \mathbb{1}_{\{D_T^{\hat{b}_k} \le d\}} + \mathrm{e}^{-\delta T}\, p_{\hat{v}_n}(D_T^{\hat{b}_k}) \cdot \mathbb{1}_{\{D_T^{\hat{b}_k} > d\}}\Big], \quad x \in I_k\,,\ k = 0, \dots, n\,, \end{aligned} \tag{21}$$

where T denotes the time of the first exit from I_k and D_T^b the drawdown at that time when following a constant strategy equal to b chosen from a finite set R of admissible retention levels. The numerical illustrations considered below are calculated by a policy-iteration-algorithm resulting in a solution to (21). It should be noted that, a priori, it is not clear that a (unique) solution to the equation system exists. In particular, this depends on the coefficients in the equation and their linear (in-)dependence. These are determined by the claim size distribution and the functions c and r. However, if there exist vectors $\hat{v}$ and $\hat{b}$ so that both equations of (21) are fulfilled, then $\hat{v}$ is the discrete return of $\hat{b}$ and $\hat{b}$ is an 'optimiser' in the second equation at every point. Then the dynamic programming principle implies that $\hat{b}$ induces an optimal feedback

Table 1 Parameters of the numerical study

δ	λ	d	η	θ
0.3	1	1.5	0.2	0.33
				1.1

strategy and $\hat{v}$ is the value function. A detailed description of the algorithm and further comments on the validity of its solution are found in the second chapter and appendix of Brinker (2022).

Remark Our 'infinitesimal' results targeting the HJB-equation (7) alternatively allow for a finite-difference-algorithm based on Eq. (9). However, the specific characteristics of the presented, drawdown-based problem lead to strategies fundamentally different from many other optimisation objectives. In particular, we showed in Lemma 7 that the strategy of zero drift (and positive claims) can be an optimal choice. In computational experiments, this caused significant problems with the convergence of the numerical schemes. Moreover, the strategies obtained in this way would have to be discretised again in order to apply them to an insurance portfolio. We comment on this issue in the last section of this article. Our approach presented above overcomes these problems.

The basic parameter set is given in Table 1. The relatively small preference factor $\delta = 0.3$ reflects a long-term oriented optimisation. $\lambda = 1$ means that we expect one claim in a unit interval. In the examples below, we consider a mean claim size $\mu = \mathbb{E}[Y_1]$ of $\mu \approx 0.5$. This means, three average sized claims in short period of time lead to a critically large drawdown $D_t > d = 1.5$. We now assume that reinsurance strategies take values in a compact interval $[\underline{b}, \overline{b}]$, where $\overline{b}$ corresponds to the strategy of 'not purchasing reinsurance' and $\underline{b}$ is determined by the zero-drift condition $c(\underline{b}) = 0$. As mentioned in the introduction, this corresponds to an appropriate rescaling of the premium and the retention function. We change here the notation for easier interpretation of the result. For example, for proportional reinsurance, B_{t-} is the actual proportional retention level at time t, and for excess-of-loss reinsurance, B_{t-} corresponds to the deductible of the first insurer at time t. Further, we assume that both premia are calculated by an expected value principle, leading to $c(b) = (1+\eta)\lambda\mu - (1+\theta)\lambda(\mathbb{E}[Y_1 - r(b, Y_1)])$, compare (13). $\eta = 0.2$ is the safety loading charged by the insurer. $\theta \in \{0.33, 1.1\}$ quotes the cost (in terms of diminished proceeds from premia) of reduced claim payments. Therefore, the size of θ is an important external factor in the trade-off between earning and keeping premia as a risk-buffer and passing on risk to the reinsurance company.

For the case of $r(b, y) = by$ (proportional reinsurance), we let $\overline{b} = 1$ and compare discrete versions of

i) the optimal strategy b^*,
ii) the 'no reinsurance' strategy $b_{\max}$ with $b_{\max}(x) = \overline{b}\mathbb{1}_{\{x \geq 0\}}$,
iii) the linearly increasing strategy b_i with $b_i(x) = [\underline{b} + (\overline{b} - \underline{b})x/d]\mathbb{1}_{\{x \leq d\}} + \overline{b}\mathbb{1}_{\{x > d\}}$,
iv) region dependent switching b_s with $b_s(x) = \underline{b}\mathbb{1}_{\{x \leq d\}} + \overline{b}\mathbb{1}_{\{x > d\}}$.

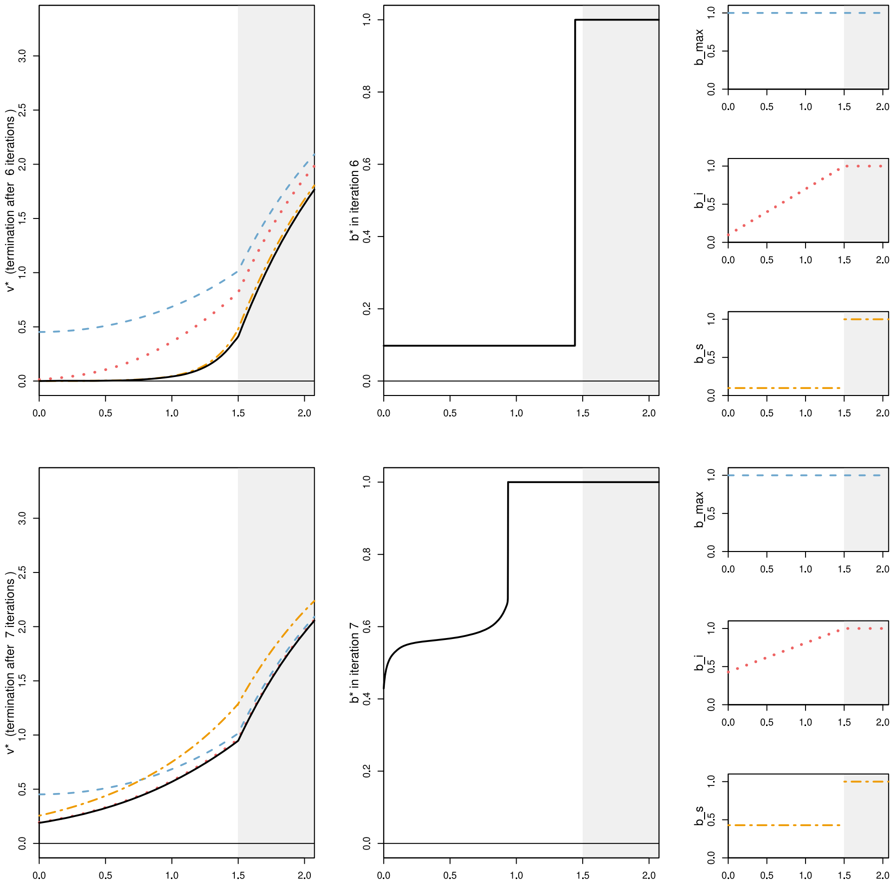

Fig. 1 Exponential claims with proportional reinsurance for $\theta = 0.33$ (top) and $\theta = 1.1$ (bottom)

For excess of loss reinsurance with $r(b, y) = \min\{b, y\}$, 'no reinsurance' corresponds to $\overline{b} = \infty$. In the examples below, we assume $\overline{b} \approx 10^{10}$. We compare the strategies i), ii) and iv) as defined above to

iii') the increasing strategy b_i with $b_i(x) = [\underline{b} + (d-x)^{-1/4} - d^{-1/4}]\mathbb{1}_{\{x \le d\}} + \overline{b}\mathbb{1}_{\{x>d\}}$.

The two rows of Figs. 1, 2 and 4, 5 correspond to the different values of θ, with $\theta = 0.33$ in the top and $\theta = 1.1$ in the bottom row. The respective leftmost graph of every row displays the optimised expected time in drawdown (solid line) in comparison with the return functions of the three alternative strategies $b_{\max}$ (dashed line), b_i (dotted line) and b_s (dash-dotted line), displayed on the right. The middle graph shows the optimal strategy.

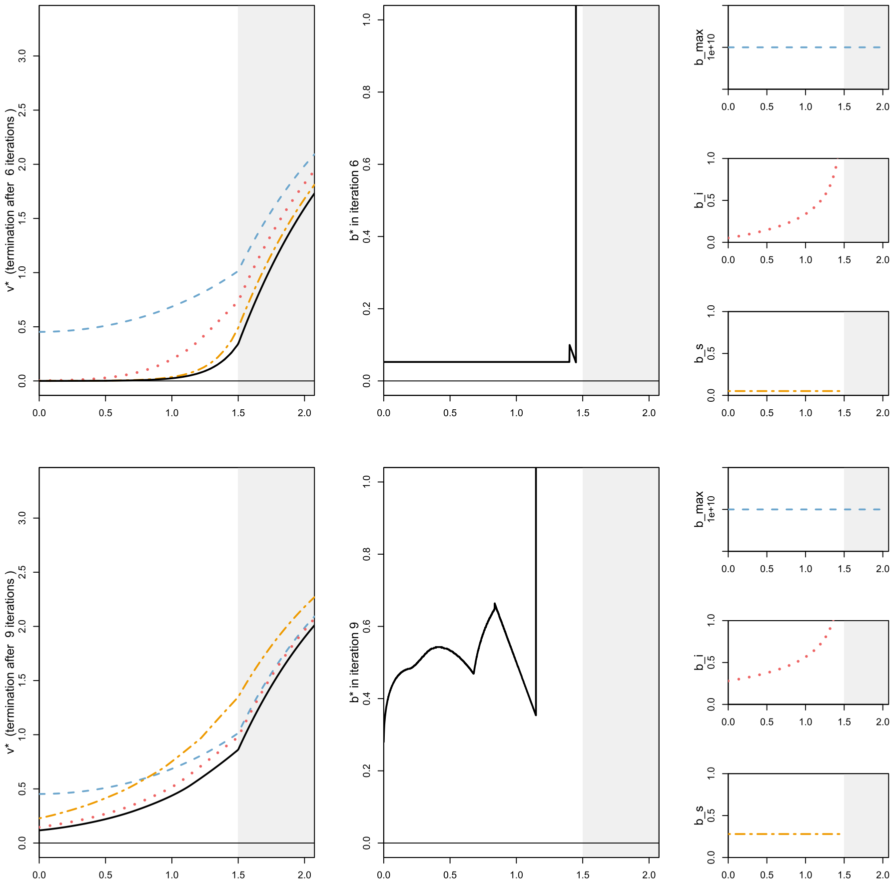

Fig. 2 Exponential claims with excess of loss reinsurance for $\theta = 0.33$ (top) and $\theta = 1.1$ (bottom)

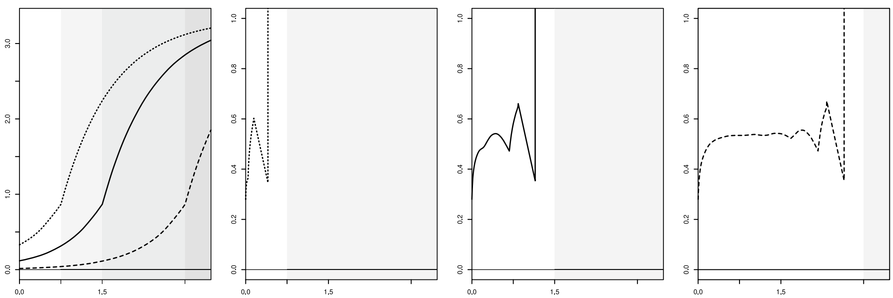

Fig. 3 Excess of loss reinsurance for exponential claims and different values of $d \in \{0.75, 1.5, 3\}$

4.1 Exponentially distributed claim sizes

We assume that claims are exp(α)-distributed with $\alpha = 2$. The parameter set given in Table 1 leads to $\tilde{\gamma} \approx 1.0868$. Figure 1 displays the case of proportional reinsurance.

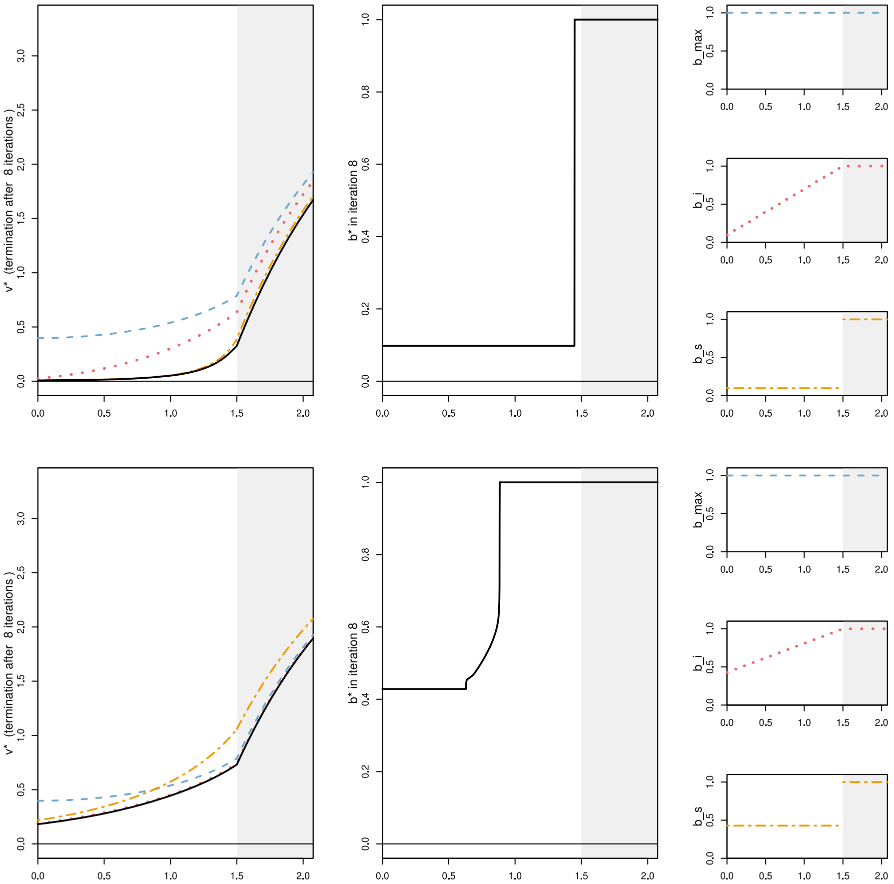

Fig. 4 Pareto distributed claims with proportional reinsurance for $\theta = 0.33$ (top) and $\theta = 1.1$ (bottom)

For $\theta = 0.33$, the numerical approximation indicates that the optimal retention level is close to $\underline{b}$ when the drawdown is small and then shoots up towards 1 when it approaches the boundary d of the uncritical area. Out of the considered alternative strategies, the switching strategy resembles this behaviour the most. Its return function lies closest to the return of the optimal strategy. Moreover, both functions are very steep in this area. This means that the cases of small initial drawdown and 'almost critical' initial drawdown differ significantly. This allows the interpretation that for 'almost critical' drawdown, the strategy of (approximately) maximal drift is chosen in order to quickly reduce the drawdown again because the area close to the boundary is too dangerous. For the case of $\theta = 1.1$, the general size of the return and value functions increases (except for the return in the absence of reinsurance plotted as the dashed line). One reason for this is that the same proportional claim reduction now leads to a larger drift of the drawdown process. Another reason is that the set of policies to choose from is narrowed because $\underline{b} = (1+\theta)^{-1}(\theta - \eta)$ increases. Moreover, we observe that the area where large retention levels are chosen grows. Figure 2 displays the case of excess of

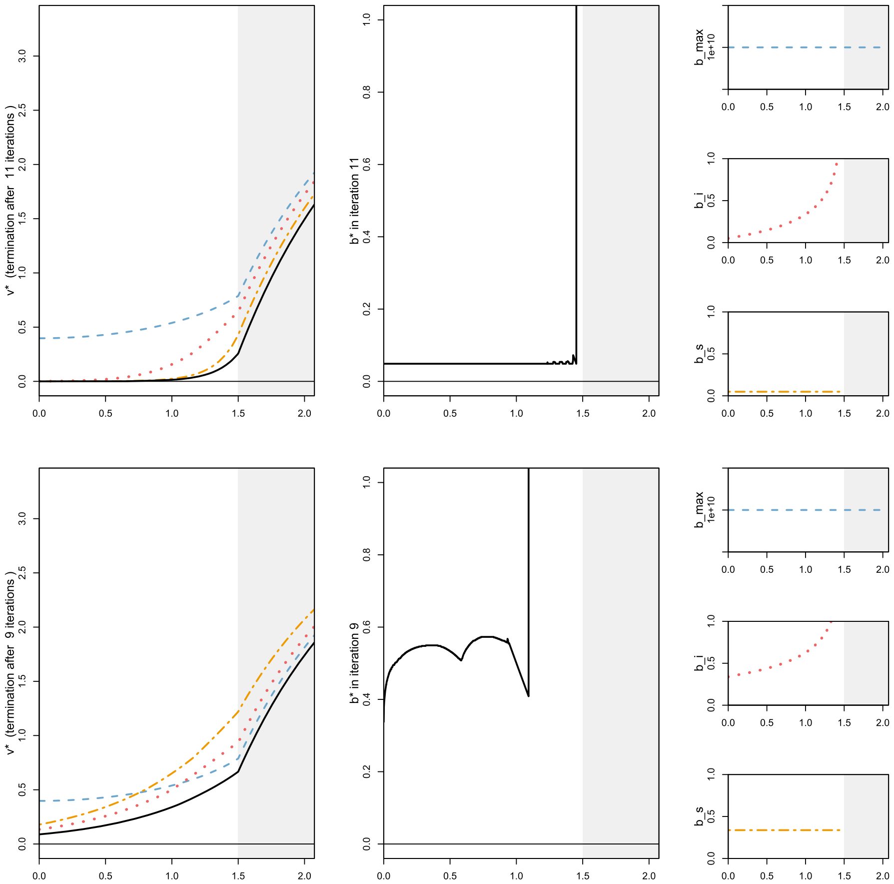

Fig. 5 Pareto distributed claims with $\theta = 0.33$ (top), $\theta = 1.1$ (bottom) and excess of loss reinsurance

loss reinsurance. Considering the value and return functions, we notice that the dashed line, i.e. the return of operating without reinsurance, remains the same as in the graphs of Fig. 1. Using this function as a reference, we can observe that the minimised expected time in drawdown (solid line) is smaller than in the case of proportional reinsurance. Thus, excess of loss reinsurance is more effective at reducing the time in drawdown in this example. This is achieved by alternating lower and larger retention levels (as seen in the middle graphs of Fig. 2). Instead of an increasing optimal strategy as in Fig. 1, the interval $[0, d]$ is divided into 'risk bands', where the optimal strategy is either to remain in the current band (i.e. to choose a lower retention level) or to aim at reaching the band below before a claim occurs (larger retention level). For example, in the case $\theta = 0.33$, it is optimal not to buy reinsurance close to d in order to quickly leave the critical boundary. For $x \approx 1.45$, b^* has a peak, which can be interpreted as trying to reach the area below 1.45, where one can guarantee that the drawdown process will not exit the uncritical area at the next jump by choosing the retention level accordingly. The influence of the critical size of drawdowns d is illustrated in Fig. 3. The graphs show value functions (leftmost graph) and optimising functions for the parameter set of Table 1 with $\theta = 1.1$ and three different values $d = 0.75$ (dotted), $d = 1.5$ (solid) and $d = 3$ (dashed). Comparing the value functions in the left graph, we observe that the smaller d, the larger the time in critical drawdown (the dotted line corresponding to $d = 0.75$ lies above the solid line for $d = 1.5$ which lies above the dashed line for $d = 3$). This is not surprising because if the uncritical area is smaller, all admissible strategies lead to a longer time spent in the critical area. This effect is particularly strong for small values of x. This can be interpreted as a result of the discounting $e^{-\delta t}$: the value function puts more weight on early critical drawdowns. For small values of x, there is a significant chance to stay in the uncritical area for $d = 3$ with the first jump. To stay below $d = 0.75$ is less likely, leading to a higher value $v(0)$. This 'benefit' of the higher tolerance for drawdowns disappears for larger initial values $x \leq d$ — in the sense that 'early' exits happen in any case if the drawdown is already almost critical. In particular, in all cases of the example, the respective values $v(d)$ on the critical lines are approximately equal to 0.86. For $x > d$, the value functions converge to $\delta^{-1} \approx 3.3333$. Regarding the optimal strategies, we see that the general structure of peaks and troughs of the optimisers b^* is similar in the three cases. As stated above, this corresponds to controlling (via excess of loss reinsurance) in such a way, that either the surplus cannot exit the uncritical area or increases quickly to a higher level. In the case $d = 0.75$, the highpoint of the 'spike' is approximately at $(0.15, 0.6)$, lower than in the cases $d = 1.5, 3$, in which it goes up to 0.67. With the lower retention level in the case $d = 0.75$, a part of the drift is 'sacrificed' in order to guarantee that an immediate claim would not cause a critical drawdown.

4.2 Pareto distributed claim sizes

We assume that the claim size distribution is given by $G(y) = 1 - \beta^\alpha(\beta + y)^{-\alpha}$ for $y > 0$ for $\alpha = 2$ and $\beta = 0.45$. With the parameters of Table 1, one can numerically calculate $\tilde{\gamma} \approx 1.0304$. α and β are chosen such that the expected claim size $\mu = \frac{\beta}{\alpha-1}$ is slightly smaller than in the previous example. However, $\alpha = 2$ indicates that the variance is infinite.

Figure 4 shows the case of proportional reinsurance and Fig. 5 corresponds to the case of excess of loss reinsurance. Again the dashed line representing the value without reinsurance is not affected by changes of θ or the reinsurance type. In Fig. 4, the graphs of the function $b^*(x)$ indicate that the retention level is chosen as low as possible if the drawdown is small. Moreover, there is an area where the drawdown is uncritical, but in order to push the drawdown down, almost no reinsurance is bought and the largest part of the premium payments is withheld in the company. A 'ramp' connects the low and the high retention levels, such that the reinsurance level rapidly decreases as the drawdown grows larger. This effect intensifies if the price of reinsurance is raised.

Comparing Figs. 4 and 5, we see that excess of loss reinsurance is again more effective than proportional reinsurance, especially in the case of $\theta = 1.1$. The strategy is again to alternate lower and higher retention levels but the shape of the optimiser differs from the case of exponential claims.

5 Concluding remarks

We analysed the optimisation problem of minimising the expected time during which the drawdown is critically large discounted by a preference factor with respect to dynamic, general reinsurance. A drawdown of zero corresponds to a surplus at the maximum or, if one considers the surplus ex-dividend for a barrier dividend strategy, to paying dividends. For different reasons, for example the reputational risk connected to prolonged drawdown phases leading to operational losses, insurers have a natural interest in minimising the time with a large drawdown. Our approach to the optimisation problem was to examine the cases of critical and uncritical initial drawdown separately. In the former case, we calculated the value function explicitly. As it turned out, the optimal strategy is the one minimising the coefficient $\gamma(b)$ associated to the Laplace transform of the upper exit time of a classical risk model. If the drawdown starts below the critical line, reflection at the boundary and dependence on the overshoot complicate the problem. Using the explicit solutions from the other subproblem, we solved this by constructing a set of optimisation problems of Gerber–Shiu type containing our original problem. We proved that the value functions are uniquely characterised as solutions to the connected HJB-equations with certain properties, thus, also obtaining the corresponding result for the expected minimal time in drawdown. Moreover, we have seen that the optimal strategy is of feedback form whenever the drawdown is uncritical.

As we have seen in our numerical examples, the shape of the optimal strategy for uncritical initial drawdown is highly influenced by the type of reinsurance contract and the choice of the claim distribution. As expected, the results on the optimiser in the uncritical area differ strongly from those in Brinker and Schmidli (2022), where a continuous diffusion model is considered. For example, if claims follow a Pareto distribution, a single claim is more likely to cause a large drawdown than a single, exponentially distributed claim. One therefore will prefer smaller jumps over a larger drift. If the claim size is bounded or with excess of loss reinsurance is available, the insurer can sensitively control the maximal size of the next claim. This leads to an optimal strategy of alternating small and large retention levels. We see in all of the numerical examples that one tries to leave the critical line rather quickly, in particular, if reinsurance is expensive.

Our analytical and numerical approaches to the problem showed that if the drawdown is equal to zero, the retention level which eliminates the drift is optimal. This fits the intuition since the reflection barrier absorbs the drift and claim payments are the only decision-making parameter. This policy stabilises the process close to its maximum. This increased stability is favourable for insurers, especially in an uncertain economic setting. However, it also prevents growth of the maximum of the surplus process. This can be seen as a drawback of the optimisation criterion (when used on its own) because it impedes the insurer from increasing profits. These results complement previous, similar findings for the diffusion model (e.g. Brinker and Schmidli 2022 and Angoshtari et al. 2016a). However, in reality, a company cannot increase its surplus unboundedly. If the maximum surplus is large, shareholders are going to demand dividends, customers to participate in profits and taxation makes infinitely large surpluses unfavourable. Thus, we expect that drawdown minimising strategies work in most realistic scenarios, when applied to ex-dividend surpluses (as a secondary optimisation criterion) or if growth is naturally bounded (for example due to statutory regulations).

A ‘quick fix’ to implement growth of the running maximum in our setting is to define a lower limit for the rentention level and thus to define a lower bound for the increase of the running maximum. Additionally, one could use the inclusion of ‘incentives to grow’ as a starting point to construct interesting extensions to the problem. We discuss some possibilities in Brinker and Schmidli (2022). An approach preserving the properties of pure drawdown control would be to consider the optimisation of general drawdown-based Gerber–Shiu functions (i.e. a discounted penalty of the first critical drawdown). These can be analysed similarly as in Sect. 3.2. Another aspect worth examining is the applicability of optimal strategies: assuming that the insurer can adjust the retention level continuously in time is unrealistic. As an extension, one could therefore introduce discrete (random or deterministic) decision times leading to piecewise constant strategies or transaction costs associated with changes of the insurance coverage.

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Appendix

In the following, we show a heuristic derivation of the HJB-equation (7). By the same arguments as in the proof of Lemma 3, one can prove a dynamic programming equation

$$v(x) = \inf_{B\in\mathcal{B}} \mathbb{E}^x \left[\int_0^T \mathrm{e}^{-\delta t} \mathbb{1}_{\{D_t^B > d\}} \, \mathrm{d}t + \mathrm{e}^{-\delta T} v(D_T^B) \right] \tag{22}$$

for an arbitrary stopping time T. We consider the stopping time $\tilde{\tau}_h = h \wedge T_1$ for $h > 0$. If $x > d$, we assume that h is sufficiently small such that d is not reached before time h, i.e. $h \in (0, |x-d|[2c(1)]^{-1})$. Note that we have $\mathbb{1}_{\{D_t^B > d\}} = 1$ for all $t \le \tilde{\tau}_h$ if $x > d$ and $\mathbb{1}_{\{D_t^B > d\}} = 0$ for all $t \le \tilde{\tau}_h$ if $x \le d$, independent of the strategy chosen. Then, dividing by h and rearranging the terms of (22), we find

$$\begin{aligned} 0 &= \inf_{B\in\mathcal{B}} \mathbb{E}^x \left[h^{-1} \int_0^{\tilde{\tau}_h} \mathrm{e}^{-\delta t} \mathbb{1}_{\{D_t^B > d\}} \, \mathrm{d}t \right] + \mathbb{E}^x \left[\frac{\mathrm{e}^{-\delta \tilde{\tau}_h} v(D_{\tilde{\tau}_h}^B) - v(x)}{h} \right] \\ &= \mathbb{E}^x \left[h^{-1} \delta^{-1} (1 - \mathrm{e}^{-\delta \tilde{\tau}_h}) \right] \mathbb{1}_{\{x > d\}} + \inf_{B\in\mathcal{B}} \mathbb{E}^x \left[\frac{\mathrm{e}^{-\delta \tilde{\tau}_h} v(D_{\tilde{\tau}_h}^B) - v(x)}{h} \right] \end{aligned}$$

The first term fulfils

$$\begin{aligned} \mathbb{E}^x \left[h^{-1} \delta^{-1} (1 - \mathrm{e}^{-\delta \tilde{\tau}_h}) \right] \mathbb{1}_{\{x > d\}} &= h^{-1} \delta^{-1} \mathrm{e}^{-\lambda h} (1 - \mathrm{e}^{-\delta h}) \mathbb{1}_{\{x > d\}} \\ &\quad + \left(h^{-1} \delta^{-1} \int_0^h \lambda \mathrm{e}^{-\lambda t} (1 - \mathrm{e}^{-\delta t}) \, \mathrm{d}t \right) \mathbb{1}_{\{x > d\}} \\ &\rightarrow \quad \mathbb{1}_{\{x > d\}} , \end{aligned}$$

as $h \to 0$. For an optimal solution, if it exists, we expect the controlled drawdown process to be a piecewise deterministic Markov process. That is, the set $\mathcal{B}$ can be restricted to the set $\tilde{\mathcal{B}}$ of adapted, càdlàg feedback controls of the form $B_t = b(D_t^B)$, $t \ge 0$, for a measurable function b such that D^B exists. In Sects. 2 and 3 we prove that this is, indeed, the case. For arbitrary $B \in \tilde{\mathcal{B}}$, Theorem 11.1.3 combined with Corollary 11.2.1 of Rolski et al. (1999) implies that

$$\begin{aligned} &\mathbb{E}^x \left[\frac{\mathrm{e}^{-\delta \tilde{\tau}_h} v(D_{\tilde{\tau}_h}^B) - v(x)}{h} \right] \\ &\rightarrow \quad -\delta v(x) - c(b(x)) \mathbb{1}_{\{x > 0\}} v'(x) + \lambda \int_0^\infty [v(x + r(b(x), y)) - v(x)] \, \mathrm{d}G(y) , \end{aligned}$$

as $h \to 0$, provided that $h(t, x) = \mathrm{e}^{-\delta t} v(x)$ is in the extended domain of the generator of the process $(t, D_t^B)_{t \ge 0}$. Here, v' denotes a version of the density of v. The occurrence of an indicator function $\mathbb{1}_{\{x > 0\}}$ is caused by the absorption of the drift of the drawdown process on the t-axis. That the function is in the domain of this generator is shown in Sect. 2, where we prove that v is bounded and absolutely continuous. That means, we obtain for all possible choices $b(x) \in [0, 1]$:

$$-\delta v(x) - c(b(x))\mathbb{1}_{\{x>0\}}v'(x) + \lambda \int_0^\infty [v(x + r(b(x), y)) - v(x)]\, \mathrm{d}G(y) \geq -\mathbb{1}_{\{x>d\}}.$$

For an optimal strategy, we expect equality. This implies

$$\inf_{b\in[0,1]} \left\{-\delta v(x) - c(b)\mathbb{1}_{\{x>0\}}v'(x) + \lambda \int_0^\infty [v(x + r(b, y)) - v(x)]\, \mathrm{d}G(y)\right\} = -\mathbb{1}_{\{x>d\}}. \tag{23}$$

From these heuristics, we expect that v is a solution to (7), that also fulfils the initial condition

$$\inf_{b\in[0,1]} \left\{-\delta v(x) + \lambda \int_0^\infty [v(x + r(b, y)) - v(x)]\, \mathrm{d}G(y)\right\} = 0.$$

The latter implies that the integrands of the last two integrals of the martingale in (10) (defined with v instead of f) combined yield the term to minimise in (23). To see this, note that $\mathrm{d}M_t^B = c(b(0))\, \mathrm{d}t$ for the piecewise deterministic D^B. In Theorem 3, we state conditions under which a solution to (7) is equal to v. In Sect. 3, we prove that there exists such a solution.

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