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# Cross-section instability in financial markets: impatience, extrapolation, and switching

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## Abstract

This paper presents a stylized model of interaction among boundedly rational heterogeneous agents in a multi-asset financial market to examine how agents' impatience, extrapolation, and switching behaviors can affect cross-section market stability. Besides extrapolation and performance based switching between fundamental and extrapolative trading documented in single asset market, we show that a high degree of 'impatience' of agents who are ready to switch to more profitable trading strategy in the short run provides a further cross-section destabilizing mechanism. Though the 'fundamental' steady-state values, which reflect the standard present-value of the dividends, represent an unbiased equilibrium market outcome in the long run (to a certain extent), the price deviation from the fundamental price in one asset can spill-over to other assets, resulting in cross-section instability. Based on a (Neimark–Sacker) bifurcation analysis, we provide explicit conditions on how agents' impatience, extrapolation, and switching can destabilize the market and result in a variety of short and long-run patterns for the cross-section asset price dynamics.

**Keywords** Heterogeneous beliefs · Asset pricing · Portfolio choice · Strategy switching · Bifurcation analysis

**JEL Classification** C61 · D84 · G11 · G12

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## 1 Introduction

A well-established strand of research on financial market dynamics emphasizes the impact of behavioral heterogeneity on market stability in an evolutionary framework, in which asset prices, investors' beliefs, and trading strategies coevolve over time, via 'expectations feedback' processes (see, e.g., Hommes 2013 and the recent survey Dieci and He 2018). While large, computationally oriented agent-based models (e.g., LeBaron 2006) naturally allow for great flexibility and a rich modeling of heterogeneity at the micro-level, they necessarily rely on extensive numerical simulations. As such, they are limited in providing clear-cut and easily interpretable analytical results about the way how certain behavioral assumptions (and associated parameters) affect market dynamics and stability. On the other hand, analytical tractability of market stability requires the introduction of substantial simplifications, aimed at reducing the dimension and complexity of the resulting dynamical systems. Such simplifications consist in focusing on a stylized market with few assets, assuming few and well-identified belief-types or behavioral rules (typically, fundamental traders and trend-followers) and, in particular, relying on a limited set of parameters to capture a variety of behavioral sources of instabilities in financial markets, such as market sentiment, extrapolation and herding.[1] Among such behavioral forces, a number of papers on market stability have focused on the parameters capturing investors' tendency to extrapolate recent market trends, through adaptive expectations schemes or more sophisticated rules based on the use and comparison of moving averages, similar to real-world technical analysis (see, e.g., Chiarella et al. 2006; He and Li 2008; He and Zheng 2010). More important, a large body of literature has emphasized the role of the so-called *intensity of switching* (or *intensity of choice*), namely, investors' propensity to change their strategy depending on the strategies' relative performances, within a multinomial logit framework. As is well known, increasing the intensity of choice results in the majority of agents simultaneously switching to the rule that performed best in the past, thus destabilizing the financial market and paving the way to a 'rational route to randomness' (Brock and Hommes 1998; Hommes 2013). Furthermore, early heterogeneous-agent models usually rely on a simplified setting with one risky asset and a riskless asset. Although a number of models with multiple risky assets were also developed (see Westerhoff 2004; Westerhoff and Dieci 2006 for early examples), a theoretical analysis on cross-section market stability in a general multi-asset market setting is rather difficult and this issue has not been fully explored yet.

This paper builds a general framework, yet still analytically tractable, to deal with the joint destabilizing effect of such forces in a multi-asset market. In particular, this paper seeks to describe and investigate in greater detail the impact of the tendency of investors to revise their strategies based on their relative performances. While models based on the logit dynamics rely on a unique parameter (the intensity of switching) which, in its infinite limit, allows *the whole* population of traders to switch across competing strategies in one time step, here we adopt a more realistic view that some market participants are 'patient', or 'confident' about their strategy and never change

[1] See, e.g. Lux (1998), Bischi et al. (2006), Boswijk et al. (2007), He and Li (2007) and the surveys Hommes (2006), Chiarella et al. (2009).

it over time, while the others are 'impatient' and have performance-based adaptive behavior, governed by a discrete-choice logit model based on observed profitability in the short run. Therefore, while the latter behave as short-term speculators, the former may be regarded as long-term investors. The intensity of choice parameter thus governs the share of adaptive, switching investors, while two additional parameters govern the size and the composition of the exogenous proportion of confident, long-term investors. Dieci et al. (2006) first developed and investigated this idea in a framework with a single risky asset. By extending this idea to a multi-asset framework with possibly heterogeneous and time-varying parameters governing investors' first and second moment beliefs, we explore additional destabilizing mechanisms in cross-section market. Although well known equilibrium relationships (resulting in standard 'fundamental prices') characterize the model's steady state, which represents the long-run outcome of the economy under certain conditions, a loss of asymptotic stability of this 'fundamental steady state' may occur via the joint effect of a variety of parameters, among which extrapolation, intensity of switching and the share of (im)patient agents play a prominent role. Initial price deviations from equilibrium resulting from excess demand/supply tend to be amplified by market forces in this case. Moreover, due to investors' updating rules for their second-moment beliefs (including asset covariances) and the resulting portfolio adjustments, price fluctuations arising for one asset can spill over to other assets, too. Despite the richness of the resulting scenarios, we are able to obtain significant simplifications and clear-cut analytical results.

The rest of the paper is organized as follows. In Sect. 2 we first introduce the basic notations and represent excess demand and price adjustment of each asset as a function of the shares of patient and impatient fundamental traders and trend followers, then derive investors' optimal asset holdings in a mean-variance framework and specify how investors form and update their first and second moment beliefs on asset prices. We further describe the way strategy shares evolve based on observed relative profitability and connect the model building blocks into a high-dimensional recursive system and characterize its (fundamental) steady state. In Sect. 3 we provide analytical results about the local asymptotic stability of the steady state in the parameter space and discuss their economic implications and the joint role of different behavioral forces in bringing about a loss of stability and self-sustained cross-section price fluctuations. In Sect. 4 we illustrate and support our analytical results by means of numerical and graphical examples showing the impact of extrapolation, intensity of choice and impatience. Section 5 summarizes and draws some key conclusions and possible avenues for future research.

## 2 Model

Below we first introduce the basic ideas and notations and illustrate the main building blocks of a financial market model characterized by multiple risky assets (and one risk-free asset) and investors' mean variance portfolio choice under two types of beliefs that result in fundamental-based and trend-follower trading strategies. We then introduce an impatience parameter to differentiate short-term investors from long-term investors, and adaptive switching behavior between the two strategies among the short-term investors. Finally, we derive a high-dimensional and quite general dynamical system that governs the time evolution of prices, investors' beliefs and strategy shares and characterize its steady-state behavior. The model builds on and generalizes both the single-risky asset model of Dieci et al. (2006) and the baseline multi-asset framework developed by Chiarella et al. (2007).

### 2.1 Impatience and short/long-term investment

Our multi-asset model is based on the interplay of fundamental traders (type $f$) and trend followers (type $c$), who rely on fundamental and extrapolative expectations, respectively, resulting in different trading strategies.[2] There are $n$ risky assets and one risk-free asset. At time $t$, let the population fractions of investors who use the fundamental and trend-following strategies be $q_t^f$ and $q_t^c$, respectively. Over the time period from $t$ to $t+1$, market price $p_{i,t+1}$ of risky asset $i$ $(i = 1, 2, ..., n)$ is adjusted to the excess demand, so that the price change is expressed as:

$$p_{i,t+1} - p_{i,t} = \mu \left( q_t^f z_{i,t}^f + q_t^c z_{i,t}^c - z_i^s \right), \quad \mu > 0, \tag{1}$$

where $z_{i,t}^j$, $j \in \{f, c\}$, represents the optimal demand on risky asset $i$ of agent $j$ in group $j$ (to be determined later), $z_i^s$ is the available quantity (in terms of supply per investor) of asset $i$, $i = 1, 2, ..., n$, and $\mu$ measures the speed of price adjustment.[3]

We now introduce 'impatience', a key feature of the model. Within each type of investors, some are long-term investors who are patient (or confident) and do not change their trading strategies over time; they correspond to fixed shares $n^f$ and $n^c$ of the total number of investors, respectively. The others are short-term investors who are impatient (or less confident) and they are constantly ready to switch to better performed strategy over time. They correspond to proportions $h_t^f$ and $h_t^c = 1 - h_t^f$ (of the residual population share $1 - n^f - n^c$), respectively. Broadly speaking, they may also be regarded as short-term speculators, in contrast to long-term investors. Besides generalizing the basic framework developed by Dieci et al. (2006) directly, the model is also close in spirit to Palczewski et al. (2016) where, in an evolutionary finance framework, investors can shift (only part of) their funds between competing portfolio managers with different investment styles, and the total amount of freely flowing capital is treated as an exogenous parameter, broadly corresponding to the proportion $1 - n^f - n^c$ of *impatient* short-term speculators in our setup.[4]

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[2] They can also be treated as different fund styles offered to investors.

[3] The price setting rule in (1) captures disequilibrium price adjustment to order imbalances. It characterizes a 'market maker' inventory mechanism that is commonly used in the literature, in which the adjustment parameter $\mu$ may reflect the adverse selection, market risk tolerance and liquidity. It can be thought of as a convenient, tractable way to model asset prices in a market setting in which investors' (excess) demand is matched by underlying changes in the market maker's inventory position.

[4] Note that Taylor and Allen (1992) provide empirical evidence that some market participants always stick to their (technical or fundamental) rules and regard them as mutually exclusive tools of analysis. Note also that other ways have been proposed in the literature to introduce constraints to agents' switching. These include the so called 'asynchronous updating' of strategies (Diks and van der Weide 2005; Hommes et al. 2005), by which a fraction of traders stick to their choice *in the previous period*. Although this mechanism

It follows that the actual shares of each group in time period $t$ are given by:

$$q_t^f = n^f + (1 - n^f - n^c)h_t^f, \quad q_t^c = n^c + (1 - n^f - n^c)h_t^c = 1 - q_t^f.$$

Population shares $q_t^f$ and $q_t^c$ can be rewritten in a way that better emphasizes the role played by the exogenous fixed share of confident investors and the composition (or strategy mix) of the investors who are impatient and adaptively rational. This can be done by defining new parameters and variables, as follows:

$$n_0 = n^f + n^c, \quad m_0 = \frac{n^f - n^c}{n_0} = \frac{n^f - n^c}{n^f + n^c},$$
$$m_t = h_t^f - h_t^c = 2h_t^f - 1,$$

from which $h_t^f = (1 + m_t)/2$, $h_t^c = (1 - m_t)/2$, $n^f = n_0(1 + m_0)/2$, $n^c = n_0(1 - m_0)/2$ and finally:

$$\begin{cases} q_t^f = \frac{1}{2}\left[n_0\left(1 + m_0\right) + \left(1 - n_0\right)\left(1 + m_t\right)\right], \\ q_t^c = \frac{1}{2}\left[n_0\left(1 - m_0\right) + \left(1 - n_0\right)\left(1 - m_t\right)\right]. \end{cases}$$

Parameter $n_0$ captures the magnitude of the population of patient, long-term investors. Conversely, $1 - n_0$ can be interpreted as the degree of impatience or short-term investment. Intuitively, a high degree of impatience (a low $n_0$) means less long-term and more short-term investment, which would destabilize market price from its long-run equilibrium. Parameter $m_0 \in [-1, 1]$ measures the relative difference among patient and long-term investors who use fundamental strategy comparing with trend following strategy. Given the stabilizing role of fundamental trading, a high $m_0$ would stabilize the market price to its long-run equilibrium. While variable $m_t \in [-1, 1]$ captures the varying *composition* of the population of impatient switching investors.[5] Intuitively, when the switching is significant, market price fluctuations are reinforced, which would destabilize the price. We will show that our results are in line with these intuitions.

Based on this change of parameters, price adjustment Eq. (1) can be rewritten as follows,

$$p_{i,t+1} - p_{i,t} = \frac{\mu}{2}\left\{\left(z_{i,t}^f + z_{i,t}^c\right) + [m_t + n_0(m_0 - m_t)]\left(z_{i,t}^f - z_{i,t}^c\right) - 2z_i^s\right\}, \quad (2)$$

in which,

$$s_t := m_t + n_0(m_0 - m_t) = n_0 m_0 + (1 - n_0)\, m_t = q_t^f - q_t^c,$$

Footnote 4 continued
is somehow close in spirit to our dichotomy between patient and impatient agents, it results in different dynamics, where herding behavior is also at work.

[5] Note the two extreme cases $n_0 = 1$, which describes the situation in which all agents are patient and long-term investors, and $n_0 = 0$, in which all agents are impatient and short-term investors who endogenously switch their strategies over time.

is just the difference between *total* shares of fundamental traders and trend followers at time $t$. As illustrated below, the right-hand side of Eq. (2) ultimately depends on observed price history and realized profits, through quantities $z^f_{i,t}$, $z^c_{i,t}$ and $m_t$. Before deriving a complete model for the joint dynamics of asset prices and its steady-state equilibrium, the next subsections specify how investors' beliefs, asset demands $z^j_{i,t}$, $j \in \{f, c\}$, $i = 1, 2, ..., n$ and time-varying investors' shares captured by variable $m_t$ are obtained, based on a number of assumptions which are common in the heterogeneous-agent literature.

### 2.2 Mean-variance portfolio choice of multi-assets

Investor $j$ invests in portfolios of a riskless asset (with a risk-free gross return $R = 1 + r$) and $n$ risky assets, indexed by $i = 1, \cdots, n$ (with $n \geq 1$) to maximize the expected CARA utility $u_j(w) = -e^{-\theta_j w}$ of one-period-ahead wealth $w$. Parameter $\theta_j$ denotes the constant absolute risk aversion (CARA) of investor $j$. Let $\mathbf{p}_t = (p_{1,t}, \cdots, p_{n,t})^\top$ and $\mathbf{d}_t = (d_{1,t}, \cdots, d_{n,t})^\top$ denote the vectors of prices and dividends of the $n$ risky assets at time $t$, respectively. Under the conditional normal assumption, the portfolio choice of investor $j$ is given by the standard mean-variance choice; that is, the optimal portfolio (in terms of number of shares) of investor $j$ for the risky assets is given by the following $n$-dimensional vector,

$$\mathbf{z}^j_t := (z^j_{1,t}, \ldots, z^j_{n,t})^\top = \theta_j^{-1}(\mathbf{\Omega}^j_t)^{-1}[E^j_t(\mathbf{p}_{t+1} + \mathbf{d}_{t+1}) - R\mathbf{p}_t], \tag{3}$$

where $E^j_t(\mathbf{x}_{t+1})$ and $\mathbf{\Omega}^j_t := [Cov^j_t(x_{i,t+1}, x_{k,t+1})]_{n\times n}$, $i, k = 1, 2, ..., n$, are investor $j$'s subjectively conditional expectation and variance-covariance matrix of the end-of-period payoff (vector) $\mathbf{x}_{t+1} := \mathbf{p}_{t+1} + \mathbf{d}_{t+1}$ of the risky assets. While the conditional first-moment beliefs about dividends are assumed to be homogenous (and correct) across agent-types with $E^j_t(\mathbf{d}_{t+1}) = \mathbf{d}$, we next introduce heterogenous beliefs resulting in fundamental and extrapolative trading strategies.

### 2.3 Heterogeneous beliefs

As regards the fundamental trading strategy, investors rely on the belief that asset prices tend to mean revert toward the assets' perceived *fundamental prices* (assumed constant), $\mathbf{p}^* = (p^*_1, \ldots, p^*_n)$, while the second-moments of the payoff are assumed constant, namely, $\mathbf{\Omega}^f_t = \mathbf{\Omega}_0 := (\sigma_{ik})_{n\times n}$. More precisely, price expectations are specified as

$$E^f_t(\mathbf{p}_{t+1}) = \mathbf{p}_t + \alpha(\mathbf{p}^* - \mathbf{p}_t) = (1 - \alpha)\mathbf{p}_t + \alpha\mathbf{p}^*,$$

where $\alpha \in [0, 1]$ measures the speed of mean-reverting. The larger $\alpha$ is, the faster the expected prices $E^f_t(p_{i,t+1})$ revert to $p^*_i$, $i = 1, 2, ...n$. Alternatively, parameter $\alpha$ can be interpreted as the confidence in the fundamental prices. By allowing different mean-reverting speed across different assets, we have

$$E_t^f(\mathbf{p}_{t+1}) = \mathbf{p}_t + \mathbf{A}(\mathbf{p}^* - \mathbf{p}_t),$$

where $\mathbf{A} := diag[\alpha_1, \ldots, \alpha_n]$, with $\alpha_i \in [0, 1]$, $i = 1, 2, ..., n$. For the moment, we treat fundamental prices $\mathbf{p}^*$ as an exogenous vector of parameters. We will see in Sect. 2.6 that it can be determined by the standard discounted dividend, adjusted by investors' risk tolerance and fundamental volatility, playing the role of an unbiased long-run equilibrium for all investors.

As to the trend following strategy, investors believe that the price trend based on the moving average of history prices will continue. The moving averages of the first and second moments, $\mathbf{u}_t$ and $\mathbf{V}_t$, are updated recursively according to

$$\mathbf{u}_t = \delta \mathbf{u}_{t-1} + (1-\delta)\mathbf{p}_t, \quad \mathbf{V}_t = \delta \mathbf{V}_{t-1} + \delta(1-\delta)(\mathbf{p}_t - \mathbf{u}_{t-1})(\mathbf{p}_t - \mathbf{u}_{t-1})^\top. \quad (4)$$

As discussed in Chiarella et al. (2005, 2007), such updating rules (4) are equivalent to time averages with geometrically declining weights, where parameter $\delta \in (0, 1)$ is directly related to the weight of historical information.[6] In the learning literature, this also refers to constant-gain learning, which has been evidenced in experimental studies. Accordingly, by allowing different extrapolation parameters for different assets, investors update their expected prices and conditional covariance of the future payoffs by

$$E_t^c(\mathbf{p}_{t+1}) = \mathbf{p}_t + \mathbf{\Gamma}(\mathbf{p}_t - \mathbf{u}_t), \qquad \mathbf{\Omega}_t^c = \mathbf{\Omega}_0 + \lambda \mathbf{V}_t,$$

where $\mathbf{u}_t$ and $\mathbf{V}_t$ are defined in (4), $\mathbf{\Gamma} = diag[\gamma_1, \cdots, \gamma_n]$, $\gamma_i > 0$ $(i = 1, 2, ..., n)$ measures the 'strength' of trend extrapolation on the price deviation from its moving average for asset $i$, and $\lambda > 0$ measures the sensitivity of the second-moment estimate to the sample variance-covariance matrix.[7] Together with (4), this describes the conditional first and second moment of the trend following strategy. Our assumptions about second-moment beliefs have a common part $\mathbf{\Omega}_0$, which may be broadly interpreted as variance/covariance beliefs about the fundamentals and dividends of the risky assets.

## 2.4 Speculative switching

To characterize the speculation among short-term agents, we model the dynamics of time-varying shares $h_t^f$ and $h_t^c$ (and their difference $m_t := h_t^f - h_t^c$) following the discrete-choice switching model adopted in a large body of literature on heterogeneous-agent models (see, e.g. Hommes 2013). The proportion $h_t^j$ of agents

[6] Note that, with the memory parameter $\delta$ and geometric decayed weights, (4) provides a consistent estimate of the first and second moments. See also Bretschneider (1986) and Chiarella et al. (2006) for related 'exponential smoothing' models of mean and variance estimations.

[7] Different from fundamental trading strategy, price extrapolation can increase market price risk. By taking $\mathbf{V}_t$ into account, trend followers adjust their demand based on the perceived payoff risk accordingly.

of type $j \in \{f, c\}$ in period $t$ depends on the success of their strategy in the past, as captured by a suitable measure of relative fitness, $v_t^j$. More precisely, we assume:

$$h_t^j = \frac{e^{\beta v_t^j}}{Z_t}, \quad Z_t = \sum_{j \in \{f,c\}} e^{\beta v_t^j},$$

which results in

$$m_t = h_t^f - h_t^c = \frac{1 - \exp(-\beta(v_t^f - v_t^c))}{1 + \exp(-\beta(v_t^f - v_t^c))} = \tanh\left(\frac{\beta}{2}(v_t^f - v_t^c)\right), \tag{5}$$

where parameter $\beta > 0$ is the intensity of choice, or intensity of switching, and $v_t^j$ is the fitness measure of strategy $j = f, c$.

Following Brock and Hommes (1998), we specify the fitness measure as the realized profits of the portfolio in the previous period, namely:

$$v_t^j := \pi_t^j - C_j, \quad \pi_t^j := \left(\mathbf{z}_{t-1}^j\right)^\top (\mathbf{p}_t + \mathbf{d}_t - R\mathbf{p}_{t-1}), \tag{6}$$

where $\pi_t^j$ measures an investor's realized excess profit/loss from holding the risky asset portfolio $\mathbf{z}_{t-1}^j$ (vs. investing all available wealth in the risk-free asset) and $C_f$, $C_c \geq 0$ are interpreted as (information) costs of the fundamental-based and trend following strategy.[8] Although here we assume $C_f > C_c \geq 0$, which captures the idea that fundamental analysis is more costly, parameters $C_f$ and $C_c$ may also be reinterpreted in terms of predisposition effects toward one or the other strategy (see, e.g., Franke and Westerhoff 2012).

### 2.5 Complete dynamic model

Although actual dividends $\mathbf{d}_t$ do affect realized profits and the evolution of strategy shares through Eqs. (5) and (6)—and regardless of other possible realistic sources of noise affecting price evolution (1)—here we focus on the complete dynamic model in a deterministic setting, by assuming constant dividends, $\mathbf{d}_t = \mathbf{d}$. Based on the assumptions and notation introduced above, the nonlinear multi-asset heterogeneous-agent model results in the following recursive equation for the asset price vector $\mathbf{p}_t$

[8] One may replace $\pi_t^j$ with more general profitability measures, such as the 'risk-adjusted' profits introduced by Hommes (2001) [and further generalized by Chiarella et al. (2013)], defined as:

$$\tilde{\pi}_t^j := \left(\mathbf{z}_{t-1}^j\right)^\top (\mathbf{p}_t + \mathbf{d}_t - R\mathbf{p}_{t-1}) - \frac{\theta_j}{2}\left(\mathbf{z}_{t-1}^j\right)^\top \mathbf{\Omega}_{t-1}^j \mathbf{z}_{t-1}^j,$$

or weighted time averages of past profits (see Hommes et al. 2012). This generally results in additional higher-order terms and has limited effect on the local stability, though the global dynamics can be greatly affected.

(where $\mathbf{z}^s = (z_1^s, \cdots, z_n^s)^\top$ denotes the constant supply vector):

$$\mathbf{p}_{t+1} = \mathbf{p}_t + \frac{\mu}{2}\left\{\left(\mathbf{z}_t^f + \mathbf{z}_t^c\right) + [(1-n_0)m_t + n_0 m_0]\left(\mathbf{z}_t^f - \mathbf{z}_t^c\right) - 2\mathbf{z}^s\right\} \tag{7}$$

where:

$$\begin{aligned}\mathbf{z}_t^f &= \theta_f^{-1}\boldsymbol{\Omega}_0^{-1}\left[\mathbf{A}(\mathbf{p}^* - \mathbf{p}_t) + \mathbf{d} - r\mathbf{p}_t\right],\\ \mathbf{z}_t^c &= \theta_c^{-1}(\boldsymbol{\Omega}_0 + \lambda\mathbf{V}_t)^{-1}\left[\boldsymbol{\Gamma}(\mathbf{p}_t - \mathbf{u}_t) + \mathbf{d} - r\mathbf{p}_t\right]\end{aligned} \tag{8}$$

whereas $\mathbf{u}_t$ and $\mathbf{V}_t$ are updated recursively according to (4). Moreover, from (5) and (6), variable $m_t$ can be expressed as

$$m_t = \tanh\left(\frac{\beta}{2}(\pi_t^\Delta - \Delta C)\right), \tag{9}$$

where

$$\pi_t^\Delta := \pi_t^f - \pi_t^c = \left(\mathbf{z}_{t-1}^f - \mathbf{z}_{t-1}^c\right)^\top (\mathbf{p}_t + \mathbf{d} - R\mathbf{p}_{t-1}), \quad \Delta C := C_f - C_c > 0. \tag{10}$$

Based on price Eq. (7) and demand functions (8), and by reformulating Eqs. (4), (9) and (10) for time step $t+1$, one can represent the complete dynamic model through a map by which the state of the system at time $t+1$, identified by $(\mathbf{p}_{t+1}, \mathbf{u}_{t+1}, \mathbf{V}_{t+1}, m_{t+1})$, depends on the state at time $t$, $(\mathbf{p}_t, \mathbf{u}_t, \mathbf{V}_t, m_t)$.

### 2.6 Fundamental steady-state price

Despite the large dimension of the dynamical system in the general case of $n$ assets, insightful analytical results about the steady state solution and its stability properties are possible. Below we illustrate some general properties of the model's steady state and show how a fundamental steady-state (FSS) price can be derived.

Note first that, in steady-state, denoted using an overbar, $\bar{\boldsymbol{p}} = \bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{V}} = \mathbf{0}$, by which optimal portfolios at the steady-state read

$$\bar{z}^f = \theta_f^{-1}\boldsymbol{\Omega}_0^{-1}\left[\mathbf{d} - r\bar{\boldsymbol{p}} + \mathbf{A}(\mathbf{p}^* - \bar{\boldsymbol{p}})\right], \quad \bar{z}^c = \theta_c^{-1}\boldsymbol{\Omega}_0^{-1}(\mathbf{d} - r\bar{\boldsymbol{p}}), \tag{11}$$

while the steady-state differential between the investors' (varying) shares becomes

$$\bar{m} = \tanh\left(\frac{\beta}{2}(\bar{\pi}^\Delta - \Delta C)\right), \tag{12}$$

and

$$\bar{\pi}^\Delta = \bar{\pi}^f - \bar{\pi}^c = \left(\bar{z}^f - \bar{z}^c\right)^\top (\mathbf{d} - r\bar{\boldsymbol{p}}). \tag{13}$$

Finally, based on (7), the steady-state prices are implicitly defined by market clearing condition of zero excess demand, which takes the following general form:

$$\left(\bar{z}^f + \bar{z}^c\right) + [(1-n_0)\bar{m} + n_0 m_0]\left(\bar{z}^f - \bar{z}^c\right) = 2\mathbf{z}^s. \tag{14}$$

Note that the optimal steady-state asset holdings (11) are linear functions of $\bar{\boldsymbol{p}}$ and, as a consequence, $\bar{\pi}^{\Delta}$ in (13) is a quadratic function of $\bar{\boldsymbol{p}}$, whereas (12) is nonlinear in $\bar{\pi}^{\Delta}$. It follows that general condition (14) characterizing steady-state prices is a nonlinear equation in $\bar{\boldsymbol{p}}$, for which an explicit solution does not appear possible.

However, a substantial simplification arises from assuming that (*i*) all investors have the same risk-aversion coefficient ($\theta_f = \theta_c = \theta$) and (*ii*) the perceived expected price under fundamental belief is consistent with the fundamental steady-state price, namely, $\mathbf{p}^* = \bar{\boldsymbol{p}}$.[9] Consequently, one obtains

$$\bar{z}^f = \bar{z}^c = \mathbf{z}^s = \theta^{-1}\boldsymbol{\Omega}_0^{-1}(\mathbf{d} - r\bar{\boldsymbol{p}}), \quad \bar{m} = \tanh\left(-\frac{\beta}{2}\Delta C\right), \tag{15}$$

which implies that all investors have the same consistent and unbiased belief about the fundamental. In fact, from Eq. (15), the *fundamental* steady-state price turns out to be given by

$$\bar{\boldsymbol{p}} = \boldsymbol{p}^* = \frac{1}{r}\left(\mathbf{d} - \theta\boldsymbol{\Omega}_0\mathbf{z}^s\right). \tag{16}$$

This is consistent with the standard dividend discounting fundamental value. The adjustment for risk appearing in Eq. (16), via the 'risk-adjusted dividend' $\mathbf{d}^{adj} := \mathbf{d} - \theta\boldsymbol{\Omega}_0\mathbf{z}^s$, is consistent with the fact that higher risk (and risk aversion) and larger asset supply reduce the equilibrium price and hence increase the return required by investors in equilibrium.

## 3 Investor behavior and steady-state stability

This section examines the stability of the FSS (15)–(16) of dynamical system (7)–(10) and provides economic intuition on the stability conditions with respect to investor behavior. We provide the analytical derivation of the local stability of FSS in the Appendix.

Note that, as shown in the Appendix, our model can be represented through the iteration of a discrete map of dimension $N = \frac{1}{2}(n^2 + 5n + 2)$, which also represents the dimension of the Jacobian matrix at the FSS, denoted by $\mathbf{J}$. However, a preliminary result established in the Appendix shows that the stability of the steady state actually

[9] Assumption (*i*) is common practice in the heterogeneous-agent literature. Assumption (*ii*)—which is implicit in a large body of literature—avoids that the perceived long-run equilibrium prices of fundamental investors are systematically biased, which seems unreasonable, especially in an asymptotically stable regime of the steady-state solution.

depends on the behavior of $2n$ eigenvalues of $\mathbf{J}$, since the remaining eigenvalues are smaller than one in modulus for any value of the parameters.[10]

More importantly, by assuming that matrix $\mathbf{\Omega}_0$ is diagonal ($\mathbf{\Omega}_0 = diag[\sigma_1^2, \cdots, \sigma_n^2]$), a much simplified set of conditions on the stability can be obtained, which is stated in Proposition 1 below. This assumption corresponds to a situation in which the $n$ assets (or asset classes, or industry segments) are uncorrelated, at least in investors' beliefs about fundamentals and dividends of such assets. Besides highly simplifying our analysis, this benchmark assumption will ensure that asset comovements arising when the market is not at its steady state are purely driven by investor behavioral characteristics. The following Proposition, proven as a special case in the Appendix, states our main result about the stability of the FSS.

**Proposition 1** *Suppose that $\Omega_0 = diag[\sigma_1^2, \ldots, \sigma_n^2]$. Then the fundamental steady state (15)–(16) of dynamical system (7)–(10) is Locally Asymptotically Stable (LAS) in the region of the parameter space defined by the following set of double inequalities:*

$$B_i^F < \gamma_i(1 - \bar{s}) < B_i^{\mathrm{NS}}, \quad i = 1, 2, ..., n, \tag{17}$$

*when*[11]

$$\alpha_i(1 + \bar{s}) + 2r < \frac{8\theta\sigma_i^2}{\mu(1 - \delta)}, \tag{18}$$

*where*

$$B_i^F := \frac{1 + \delta}{2\delta}\left[\alpha_i(1 + \bar{s}) + 2r - \frac{4\theta\sigma_i^2}{\mu}\right], \quad B_i^{\mathrm{NS}} := \alpha_i(1 + \bar{s}) + 2r + \frac{(1 - \delta)}{\delta}\frac{2\theta\sigma_i^2}{\mu} \tag{19}$$

*and $\bar{s} := (1 - n_0)\bar{m} + n_0 m_0$. In addition, a parameter change which determines the violation of any of the right inequalities in (17)—all other inequalities being satisfied—results in a Neimark–Sacker bifurcation. Similarly, violation of any of the left inequalities in (17) determines a Flip bifurcation.*

Note that $\bar{s}$ represents the steady-state level of aggregate variable $s_t = (1 - n_0)m_t + n_0 m_0 = q_t^f - q_t^c$ on the difference of shares between the total fundamental investors, $q_t^f$, and trend followers, $q_t^c$. At the steady state, a higher (lower) $\bar{s}$ indicates the dominance of the fundamental investors (trend followers). Thus, $\gamma_i(1 - \bar{s})$ measures the overall activity of the trend followers in terms of their dominance and extrapolation behavior, while $\alpha_i(1 + \bar{s})$ measures the overall activity of the fundamental investors in terms of their dominance and mean-reverting behavior. Therefore, by examining the trading activity of heterogenous investors, Proposition 1 provides the stability conditions in a neat way to capture the joint impact of extrapolation, mean-reverting,

[10] For example, in the case of two assets ($n = 2$), we can focus on the 4 eigenvalues of the submatrix associated to variables $\mathbf{p}_t$ and $\mathbf{u}_t$, whereas the full Jacobian matrix has dimension 8. Note that, as explained in the Appendix, we regard matrix $\mathbf{V}_t$ as a set of $n(n + 1)/2$ dynamic variables, namely, $n$ variances and $n(n - 1)/2$ different covariances.

[11] Note that condition (18) corresponds to $B_i^F < B_i^{\mathrm{NS}}$, which basically ensures that the region of stability of the FSS in the parameter space is non-empty.

strategy switching, and impatience of investors. More precisely, we provide following implications of Proposition 1.

### 3.1 Extrapolative trading and mean-reverting

The stability conditions (17) and (18) in Proposition 1 characterize a trade-off between the mean-reverting trading of the fundamental investors and extrapolative trading of trend followers. Intuitively, mean-reverting trading of the fundamental investors stabilizes market prices, while extrapolative trading from the trend followers on the price trend results in price momentum in short-run, destabilizing market prices. This trade-off leads to the following three implications.

First, when the mean-reverting trading is sufficiently weak [so that (18) holds], the FSS is stable when the extrapolative trading is moderate [so that (17) holds]. In this case, the mean-reverting of the fundamental investors dominates the trade-off, stabilizing the FSS.

Second, again when (18) holds, the FSS becomes unstable when the extrapolative trading is strong [so that the right part of (17) is violated, i.e., $\gamma_i(1-\bar{s}) > B_i^{\mathrm{NS}}$]. In this case, the extrapolative trading of trend followers dominates the trade-off, destabilizing the FSS. Furthermore, due to the Neimark–Sacker bifurcation, market price fluctuates around the steady-state fundamental price. This destabilizing mechanism through Neimark–Sacker bifurcation is well documented in agent-based literature for single risky asset. We show that the same mechanism also carries to the market with many risky assets. Intuitively, when the extrapolative trading becomes more significant, the FSS becomes unstable. Combining with mean-reverting trading from the fundamental investors, this then generates price momentum in short-run and mean-reverting in long-run. Numerically, we show in Sect. 4 that such instability mechanism for individual asset can spill over to other assets. This nonlinear mechanism via Neimark–Sacker bifurcation can potentially explain time-series momentum in short-run and mean-reverting in long-run, along with cross-section momentum, that have been studied extensively in empirical finance literature.[12]

Third, when the mean-reverting trading becomes more significant [so that (18) is violated], it dominates the trade-off, by making the stability region empty. Interestingly, a similar mechanism is also at work when the extrapolative trading is weak while mean-reverting trading is moderate such that

$$\gamma_i(1-\bar{s}) < \frac{2\theta\sigma_i^2}{\mu}\frac{(1-\delta)}{\delta} \tag{20}$$

and

$$\frac{2\delta}{1+\delta}\gamma_i(1-\bar{s}) + \frac{4\theta\sigma_i^2}{\mu} < \alpha_i(1+\bar{s}) + 2r < \frac{8\theta\sigma_i^2}{\mu(1-\delta)}. \tag{21}$$

[12] See He et al. (2018) for an account of such empirical literature and a dynamic heterogeneous-agent model of cross-section momentum trading.

In this case, Proposition 1 implies that a flip bifurcation occurs for $\alpha_i$ at $\alpha_i = \alpha_i^*$ satisfying $\frac{2\delta}{1+\delta}\gamma_i(1-\bar{s}) + \frac{4\theta\sigma_i^2}{\mu} = \alpha_i^*(1+\bar{s}) + 2r$. Therefore, when extrapolative trading is less active [such that (20) holds], a moderate mean-reverting trading [so that (21) holds] can also lead to instability of the fundamental price via a flip bifurcation.[13]

### 3.2 Impatience and investor sentiment

We now explore the effect of patience, measured by the population of long-term investors $n_o$, and investor sentiment, measured by the difference of population shares between the fundamental investors and trend followers $s_t = (1-n_0)m_t + n_0 m_0 = q_t^f - q_t^c$.

The effect of investors' patience or confidence depends on the sign of quantity $m_0 - \bar{m}$. Note that $m_0$ measures the population difference between the fundamental investors and trend followers among long-term patient investors, while $\bar{m}$ is the analogous difference among the short-term impatient investors, at the steady state. We may interpret condition $m_0 > \bar{m}$ as the 'relative dominance' of fundamental investors over the trend followers, namely[14]

$$n^f > \frac{1+\bar{m}}{1-\bar{m}} n^c. \tag{22}$$

In this case, the steady-state value $\bar{s}$ of investor sentiment $s_t$ increases in $n_0$, which increases the impact of the fundamental investors but reduces the impact of the trend followers. Generally speaking, based on (17), this tends to reduce the chance of a Neimark–Sacker bifurcation. Thus, whenever investor sentiment increases in the share of patient long-term investors, this tends to stabilize the market, while the opposite is true when investor sentiment increases in the share of impatient short-term investors. This however goes opposite with the relative dominance of the trend followers [so that condition (22) is violated]. In this case, even an increase in the patient long-term investors can destabilize the market. Therefore, more patient investors do not necessarily stabilize market, particularly when there is more extrapolative trading among them. Intuitively, the market becomes more stable when there are more patient and fundamental traders and less impatient and trend followers.

### 3.3 Speculative switching and information cost

In the literature with single risky asset, it is well documented that increasing speculative switching, measured by parameter $\beta$, can destabilize the market. Proposition 1 shows that this also holds with many risky assets, provided that extrapolation is large enough for at least one asset. Since $\bar{m} = \tanh\left(-\frac{\beta}{2}\Delta C\right) < 0$, $\bar{s}$ decreases with $\beta$. As a

13 Note that the generic $i$th left-hand inequality in (17) is automatically satisfied whenever $B_i^F < 0$ for asset $i$. This typically happens if $\alpha_i$ is sufficiently small. On the other hand, violation of the condition $B_i^F < \gamma_i(1-\bar{s})$ typically requires sufficiently large mean-reverting trading $\alpha_i$, sufficiently weak extrapolation $\gamma_i$ and large $\mu$ (i.e., strong price reaction to the excess demand).

14 Note that since $\bar{m} < 0$, the dominance of $n^f$ over $n^c$ in (22) can hold even with a lower share of patient fundamental traders.

consequence, starting from a situation of stability, increasing values of $\beta$ shift quantity $\gamma_i(1-\bar{s})$ upwards and boundaries $B_i^{\text{NS}}$ and $B_i^F$ downwards, until possibly $\gamma_i(1-\bar{s}) \geq B_i^{\text{NS}}$ for some $i$, at which value a Neimark–Sacker bifurcation occurs. Intuitively, increasing speculative switching increases the speculative trading among short-term investors and hence destabilizes the market. Note also that, as long as information cost differential $\Delta C = C_f - C_c > 0$, its impact on the local stability of the steady state is similar to that of parameter $\beta$, i.e., an increasing cost of fundamental analysis is destabilizing, *ceteris paribus*.[15] This not only shows that the information costs parameter is quite relevant (including its sign) but may also affect the stability of the market. This observation may have interesting policy implications for our model. For instance, policy makers may promote better information with respect to the market's fundamental values, thereby decreasing information costs. Alternatively, some papers also study how policy makers may use transaction taxes or profit and/or wealth taxes to reduce the fitness disadvantage of the stabilizing fundamental expectation rule (see, e.g., Westerhoff and Dieci 2006; Martin et al. 2021).

Finally, perhaps more surprisingly, an increase in speculative switching can increase the dominance of fundamental investors among patient investors and stabilize the market to a certain extent. From (22), an increase in $\beta$ makes the relative dominance hold even with a smaller population of patient fundamental investors $n^f$, which reduces the flip bifurcation boundary and hence the chances of price overshooting and regime switching.

### 3.4 Cross-section spill-over effect

As remarked above, the reduction of the stability conditions to a set of *independent* inequalities associated to each asset is a consequence of the simplified structure of $\mathbf{J}$ which arises from assuming that $\mathbf{\Omega_0}$ is diagonal. However, this does not mean that fluctuations arising due to changes of parameters 'associated' to one asset (e.g., when $\gamma_i$ increases and/or $\alpha_i$ decreases until the boundary $B_i^{\text{NS}}$ is crossed), remain confined to that asset. In fact, the crossing of boundary $B_i^{\text{NS}}$ leads to a *local* bifurcation of the steady state. However, due to the endogenous covariance estimation of trend followers (which is not captured in the local stability condition), this potentially affects the behavior of all dynamic variables, including the prices of all assets. This benchmark scenario of independence provides two further insights to rich cross-section phenomena in financial markets. (i) With many risky assets, the spill-over of short-run momentum and long-run reversal of one asset (explored above) to others can generate cross-section momentum, which is well documented in empirical finance literature. Our analytical results and the following numerical illustration - carried out in the benchmark scenario—highlight the fact that cross-section momentum may be actually driven by behavioral factors. Of course, in general, such forces can amplify the impact of other existing interlinkages due to correlated fundamentals. (ii) Focusing on the independent stability condition for each asset provides flexibility to accommodate

[15] However, under the more general interpretation of $\Delta C$ in terms of predisposition effects, discussed in Sect. 2.4, the sign of both $\Delta C$ and $\bar{m}$ may turn out to be reversed and such parameters may have a completely different impact on the dynamics and stability.

many different price behaviors among different asset, particularly among different asset classes, including the coexistence of stability of some assets and instability of the others, cross-section momentum among some asset classes and regime switching price dynamics among the others, depending on asset characteristics.

The rich dynamics provided by our model can be further illustrated numerically. Within certain parameter ranges outside the stability region, the prices of some assets may still converge to their steady-state values in the long run, as shown in the numerical examples in Sect. 4.

## 4 Cross-section price dynamics

This section presents the results of some numerical simulations about the impact of extrapolation parameters, intensity of choice and investor impatience. The main focus is to confirm and illustrate the different ways and mechanisms a Neimark–Sacker bifurcation may occur (as proven analytically in the previous section) and to discuss some typical *qualitative* scenarios emerging as a consequence of this phenomenon, in particular, the co-existence of assets with stable and unstable long-run price behavior. The baseline parameter setting, which includes three risky assets, is one for which the FSS is locally asymptotically stable, yet not far from the Neimark–Sacker bifurcation boundary defined by the right-hand sides of (17), as shown in Table 1.

As a consequence, from Eq. (16), fundamental asset prices become $\mathbf{p}^* = [100, 100, 100]^\top$, while the steady-state mix of switching investors is determined as $\bar{m} = -0.1974$ (which implies $\bar{s} = -0.0987$), that is, there are roughly 40% fundamental traders and 60% trend followers in the population of *impatient*, *short-term* investors, at the steady state. On the other hand, by assuming $m_0 = 0$, we keep a fixed $(50\%, 50\%)$ mix of fundamental investors and trend followers among the population of *patient*, *long-term traders*. We would like to stress that our parameter selection in Table 1 has the main purpose of *qualitatively* producing boom-bust cycles as a consequence of the stability loss of the steady state, without worrying about calibration issues, e.g., regarding the duration of the cycles. However, just for the sake of concreteness, based on the assumed values of the parameters (in particular $\mathbf{d}$, $r$ and $\sigma_i^2$, $i = 1, 2, 3$), we may roughly regard the time unit as being one quarter or one month.[16]

### 4.1 The effect of extrapolation

The 'critical thresholds' $\gamma_i^{\text{NS}} := B_i^{\text{NS}}/(1 - \bar{s})$ for the extrapolation parameters are given by 0.1350, 0.2050, 0.1714 for assets 1, 2 and 3, respectively. Under the baseline selection, with an initial condition characterized by $p_i$ slightly above or $u_i$ slightly below their steady-state levels, $i = 1, 2, 3$, and by an initial value of $m$ below $\bar{m}$ (implying more initial trend followers than at the steady state), prices slowly converge to their steady-state levels after an initial, moderate price spike and, accordingly,

[16] For instance, by regarding the time unit as one quarter, the assumed variance parameter $\sigma_i^2$, $i = 1, 2, 3$, imply volatilities $\sigma_i$ of 5%, 10%, 8% of the fundamental price levels, respectively, in annual terms.

**Table 1** Baseline parameter setting

| Parameter | Description | Baseline value |
|---|---|---|
| $\mu$ | Price adjustment coefficient | 0.125 |
| $n_0 := n^f + n^c$ | Share of non-switching agents | 0.5 |
| $m_0 := (n^f - n^c)/n_0$ | Mix of non-switching agents | 0 |
| $r$ | Risk-free rate | 0.005 |
| $\mathbf{d} := [d_1, d_2, d_3]^\top$ | Asset dividends | $[1.125, 1, 1.3]^\top$ |
| $\mathbf{z}^s := [z_1^s, z_2^s, z_3^s]^\top$ | Asset supply stocks | $[10, 2, 5]^\top$ |
| $\theta$ | Risk aversion coefficient | 0.01 |
| $\beta$ | Intensity of choice | 0.8 |
| $\Delta C$ | Cost differential | 0.5 |
| $[\alpha_1, \alpha_2, \alpha_3]$ | Mean-reversion coefficients | [0.125, 0.125, 0.125] |
| $[\gamma_1, \gamma_2, \gamma_3]$ | Extrapolation coefficients | [0.1325, 0.2025, 0.1625] |
| $\delta$ | Memory parameter | 0.975 |
| $\lambda$ | Sensitivity to past comovements | 0.5 |
| $\mathbf{\Omega_0} = \text{diag}[\sigma_1^2, \sigma_2^2, \sigma_3^2]$ | 2nd moment beliefs (exogenous) | diag[6.25, 25, 16] |

impatient (variable) investor shares approach their long-run levels, as captured by the slow convergence of variable $m$ (Fig. 1, left panels).

By increasing (only) extrapolation parameter $\gamma_3$ beyond its threshold $\gamma_3^{\text{NS}}$ to the new level $\gamma_3 = 0.1875$ (Fig. 1, right panels), endogenous and wider price oscillations emerge, as the result of the loss of stability. Accordingly, large fluctuations also affect the dynamics of variable $m$. It is remarkable that, although an increase of parameter $\gamma_3$ reflects a change in investors' beliefs about the trend of asset 3 only, the prices of other assets can also be indirectly destabilized, even in the long run. Note that, with this particular parameter selection, asset 2 still tends to approach its steady-state level in the long run, while assets 1 and asset 3 display persistent and somehow irregular fluctuations, as one can check numerically over (much) longer time horizons. In fact, due to the assumed diagonal structure of matrix $\mathbf{\Omega_0}$ (implying no exogenous correlations between assets), a Neimark–Sacker bifurcation does not necessarily lead to persistent fluctuations of *all* assets, as already reported in multi-asset models related to ours (e.g., Chiarella et al. 2013).

Relatedly, as one can easily prove, the state space of dynamical system (7)–(10) has further *invariant sets* $S_i$, $i = 1, 2, ..., n$, that are characterized by $p_i = u_i = p_i^*$ as well as by the $i$th row (and column) of matrix $\mathbf{V}$ being equal to zero, i.e., by the price of asset $i$ being at rest, regardless of the behavior of other assets. Thus, the dynamics depicted in Fig. 1, right panels, will end up taking place on an attractor belonging to $S_2$ in the long run, featuring fluctuations of assets 1 and 3 only.[17] Finally, the emerging

[17] The dynamics along similar attractors—lying in invariant submanifolds of the phase space—is driven by suitable restrictions of the dynamical system. Our numerical examples suggest that such objects may in fact be attractors for the complete dynamical system, as well (thanks to some *transverse stability* properties), which is not easy to prove in general. See Bischi and Cerboni Baiardi (2015, 2017) for a theoretical discussion and insightful applications to economic dynamics. The outstanding work of Gian Italo and col-

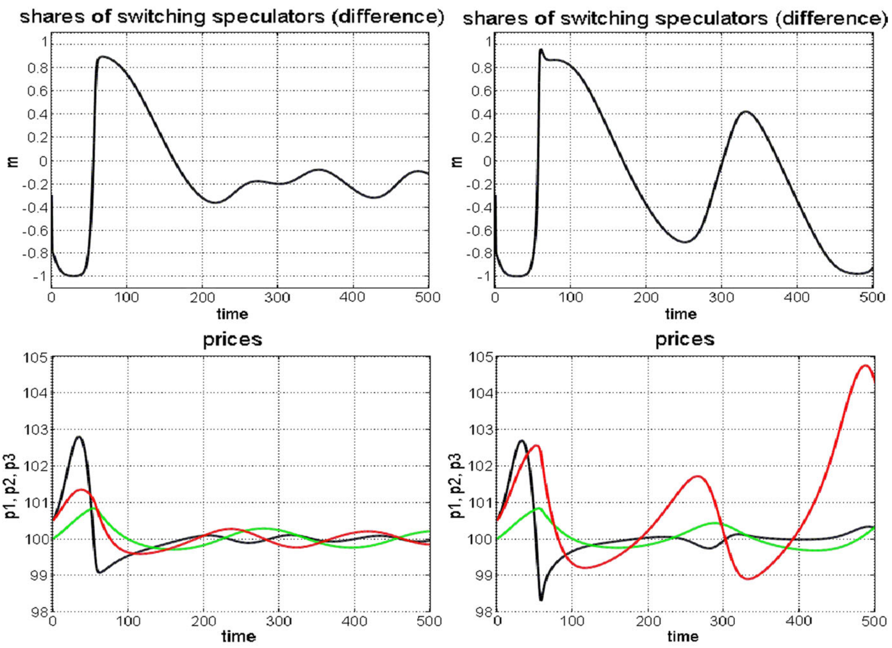


**Fig. 1** Neimark–Sacker bifurcation as the effect of stronger extrapolative behavior on asset 3 (due to parameter $\gamma_3$). Parameters according to the baseline setting in Table 1, except $\gamma_3 = 0.1875$ in the right panels. Left panels: difference $m$ between fundamental investors and trend followers shares of switching agents (top) and asset prices $p_i$, $i = 1, 2, 3$ (bottom) under the baseline setting, where $\gamma_3 = 0.1625$ and the FSS is stable. Right panels: variables $m$ (top) and $p_i$, $i = 1, 2, 3$ (bottom) beyond the Neimark–Sacker boundary for parameter $\gamma_3$, namely, with $\gamma_3 = 0.1875$. Asset prices 1, 2 and 3 are in black, green and red, respectively. Initial condition with all variables at their steady-state level, except $p_{1,0} = p_1^* + 0.5 = 100.5$, $p_{3,0} = p_3^* + 0.5 = 100.5$, $u_{2,0} = p_2^* - 0.2 = 99.8$ and initial value of $m$ equal to $\bar{m} - 0.1 = -0.2974$ (Color figure online)

patterns of co-movement between different assets seem nontrivial and varying over time. Similar phenomena can also be observed by assuming increased extrapolation on other assets, starting from the baseline case. Such spill-over is most likely driven by the optimal portfolio effect through the endogenous covariance updating from the trend followers.

## 4.2 The effect of speculative switching

Increasing values of the intensity of choice $\beta$ have destabilizing effects, as well, as commonly reported by the heterogeneous-agent literature.[18] Note that, by solving for

laborators in the field of *global properties* of discrete-time dynamical systems has long been, and continues to be, a source of inspiration for us.

[18] Parameter $\beta$ is inversely related to the variance of the noise term in random utility models (see, e.g., Anderson et al. 1993). The larger $\beta$ is, the stronger investors perceive the signal coming from observed performance differentials among strategies/forecasting rules, which generally has a destabilizing impact

$\beta$ the set of (right-hand sides of) conditions (17)–(19), one obtains:

$$\beta < \min_i \beta_i^{\mathrm{NS}}, \quad \beta_i^{\mathrm{NS}} := \frac{1}{\Delta C} \ln\left(\frac{1+\Phi_i}{1-\Phi_i}\right) \tag{23}$$

where

$$\Phi_i := \frac{1}{(1-n_0)(\alpha_i+\gamma_i)}\left[n_0 m_0(\alpha_i+\gamma_i)+\alpha_i-\gamma_i+2r+\frac{1-\delta}{\delta}\frac{2\theta\sigma_i^2}{\mu}\right],$$

provided that $-1 < \Phi_i < 1$ for any $i$.[19] Figure 2 shows the dynamics generated under a value of parameter $\beta$ beyond the Neimark–Sacker bifurcation boundary defined by conditions (23).

More precisely, against the baseline parameters of Table 1 corresponding to the left panels of Fig. 1 (where $\beta = 0.8$), Fig. 2 corresponds to $\beta = 1.40$, a value which exceeds each of the analytical thresholds $(\beta_1^{\mathrm{NS}}, \beta_2^{\mathrm{NS}}, \beta_3^{\mathrm{NS}}) = (0.889, 0.870, 1.087)$. Simulations are run over a very long time window, in order to both show the market's reaction to its initial displacement from steady state and provide the intuition of its asymptotic dynamics. In this particular case, the fluctuations emerging from the Neimark–Sacker bifurcation will remain alive only for assets 1 and 2, whereas the price of asset 3 will approach its steady-state level $p_3^*$ in the long run (second panel). However, the final outcome appears to be sensitive to the particular value of $\beta$ and may deviate from that displayed in Fig. 2, under slightly different parameter selections. For instance, one can check that while $\beta = 1.15$ generates again an asymptotic situation similar to Fig. 2, for $\beta = 1.30$ the price of asset 2 will converge to $p_2^*$, with persistent oscillatory behavior affecting only assets 1 and 3. Therefore, based on the invariant sets introduced above, the asymptotic dynamics for $\beta = 1.15$, $\beta = 1.30$, $\beta = 1.40$, take place on $S_3$, $S_2$ and again $S_3$, respectively. Together with risk-adjusted optimal portfolios based on the endogenous covariance estimation of trend followers, this illustrates another possible way speculative switching may turn out to be 'stabilizing'. That is, even in a situation when the FSS is not locally stable, investors may cyclically adjust their portfolio in a way that involves only a limited number of assets, by keeping their demand of the remaining assets constant, as shown by the demand patterns of fundamental traders and trend followers in the bottom panels of Fig. 2. Such demand patterns, along with their varying impact captured by variable $m$ in the top panel, provide some intuition about the mechanisms behind the price movements represented in the second panel.

on the dynamics. An even more general interpretation and micro-foundation is provided by Mattsson and Weibull (2002).

[19] More generally, and more precisely, no Neimark–Sacker bifurcation can take place on increasing parameter $\beta$ if $\Phi_i \geq 1$ for any $i$, which is typically the case when extrapolation parameters $\gamma_i$ are low enough for all assets. If $\Phi_i < 1$ only for some $i$, condition (23) must be formally restricted to a suitable subset of assets.

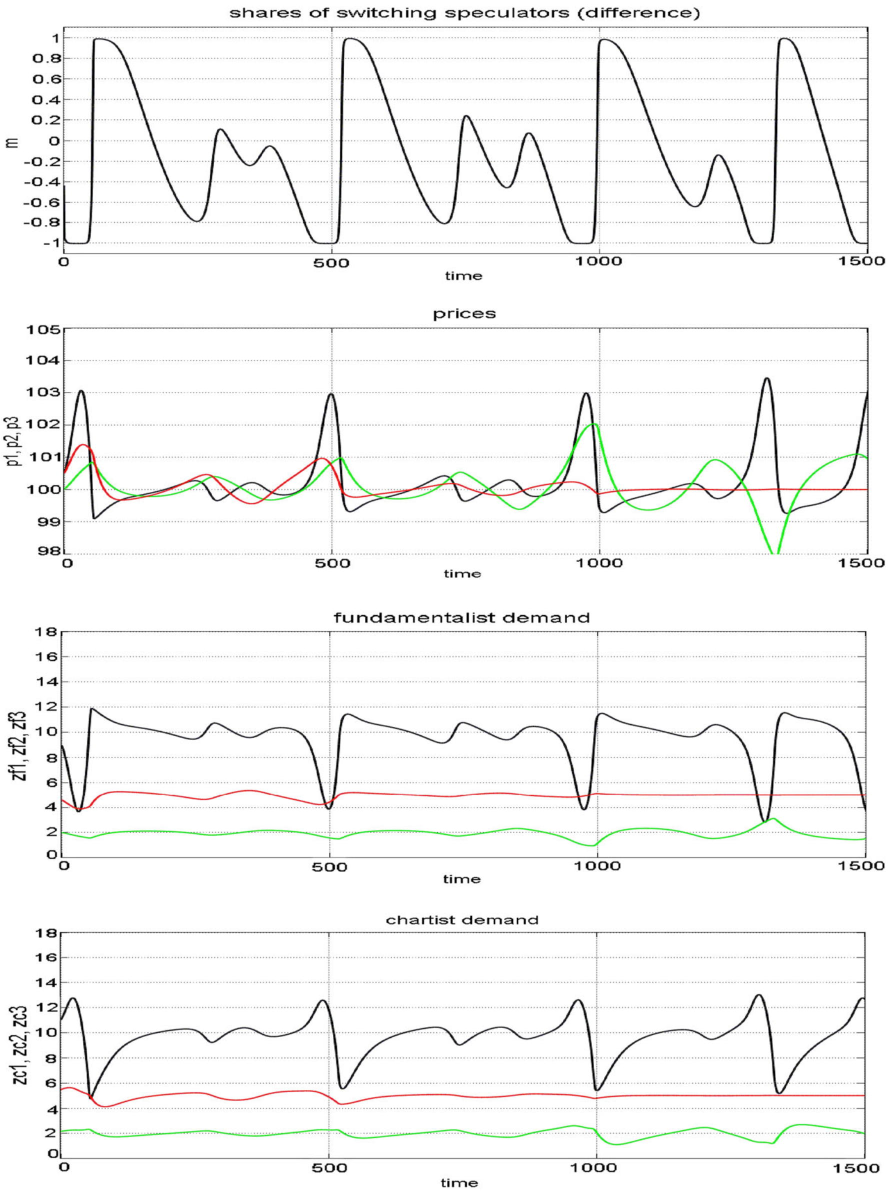


**Fig. 2** Dynamics beyond the Neimark–Sacker bifurcation boundary, as the effect of a higher value of the intensity of choice $\beta$. Top panel: difference $m$ between fundamental investors and trend followers shares of switching agents. Second panel: asset prices $p_i$, $i = 1, 2, 3$. Third panel and bottom panel: fundamental investors and trend followers demand (asset holdings) $z_i^f$ and $z_i^c$, respectively, $i = 1, 2, 3$. Assets 1, 2 and 3 are in black, green and red, respectively. Parameters according to the baseline setting in Table 1, except $\beta = 1.4$. Initial condition as in Figure 1 (though the initial value of $m$ is equal to $\bar{m} - 0.1 = -0.4364$ in this case). The dynamics of $m$ and $p_i$, $i = 1, 2, 3$, can be contrasted with the left panels of Fig. 1, where $\beta = 0.8$, other things being equal (Color figure online)

### 4.3 The effect of impatience

Finally, by assuming $m_0 - \bar{m} > 0$, an increase in the share of potentially switching agents (i.e., a reduction of parameter $n_0$) has an effect similar to increasing the intensity of choice. As discussed above, this may be interpreted as increasing investors' degree of impatience (see also Palczewski et al. 2016). For fixed extrapolation parameters $\gamma_i$, $i = 1, 2, 3$, and intensity of choice $\beta$, a larger population of impatient investors who are ready to change their strategy/forecasting rule can destabilize the dynamics and prevent long-run convergence to steady state. The stability threshold for parameter $n_0$ can be made explicit by rearranging (the right-hand sides of) stability conditions (17)–(19), as follows:

$$n_0 > \max_i n_{0,i}^{\text{NS}} \quad n_{0,i}^{\text{NS}} := \frac{1}{m_0 - \bar{m}} \left[ \frac{1}{\alpha_i + \gamma_i} \left( \gamma_i - \alpha_i - 2r - \frac{1-\delta}{\delta} \frac{2\theta\sigma_i^2}{\mu} \right) - \bar{m} \right], \tag{24}$$

provided that $0 < n_{0,i}^{\text{NS}} < 1$.[20]

In Figure 3 we slightly deviate from our baseline parameter setting by assuming $\gamma_3 = 0.17$, $\lambda = 1$. Under this setting, the above thresholds $n_{0,i}^{\text{NS}}$ are given as $(n_{0,1}^{\text{NS}}, n_{0,2}^{\text{NS}}, n_{0,3}^{\text{NS}}) = (0.446, 0.458, 0.474)$ and therefore, by decreasing parameter $n_0$, a Neimark–Sacker bifurcation takes place at $n_0 = \max_i n_{0,i}^{\text{NS}} \cong 0.474$. While for, e.g., $n_0 = 0.6$ we observe dynamics very similar to that in the left panels of Fig. 1, Fig. 3 shows the evolution of the market corresponding to $n_0 = 0.35$ (left panels) and $n_0 = 0.25$ (right panels), under our baseline initial condition. In the top and middle panels, in spite of quite similar dynamics of variable $m$ and prices $p_i$ in the two cases during a long initial time interval, the two situations become substantially different at the end of the selected time interval (from about $t = 400$ to $t = 500$), due to a drastic increase of the *variable* share of trend followers (which adds to the assumed reduction of the exogenous share of patient investors from 0.35 to 0.25). In addition, the bottom line of panels shows the price developments after a very long transient, when it is clear that the existing attractors have qualitative different properties in the two cases, with the price of asset 3 being at rest for $n_0 = 0.25$, whereas all three asset are subject to endogenous long-run fluctuations for $n_0 = 0.35$.

As already pointed out in Sect. 3, both an increase of parameter $\beta$ and a decrease of $n_0$ result— via different mechanisms—in an increase of the total share $q_t^c$ of trend followers (i.e., in a reduction of the investor sentiment variable $s_t$) in the steady-state solution, which is what really affects the local asymptotic stability of the FSS. However, the endogenous covariance estimation and risk-adjusted optimal portfolio choice of trend followers may sometimes have stabilizing effects.

[20] Similar to the previous analysis of parameter $\beta$, a Neimark–Sacker bifurcation cannot occur by varying parameter $n_0$, if $n_{0,i}^{\text{NS}} \le 0$ for any $i$, which is again typically the case when extrapolation $\gamma_i$ is weak for all assets. If $n_{0,i}^{\text{NS}} > 0$ only for some $i$, Eq. (24) must be formally restricted to a suitable subset of assets.

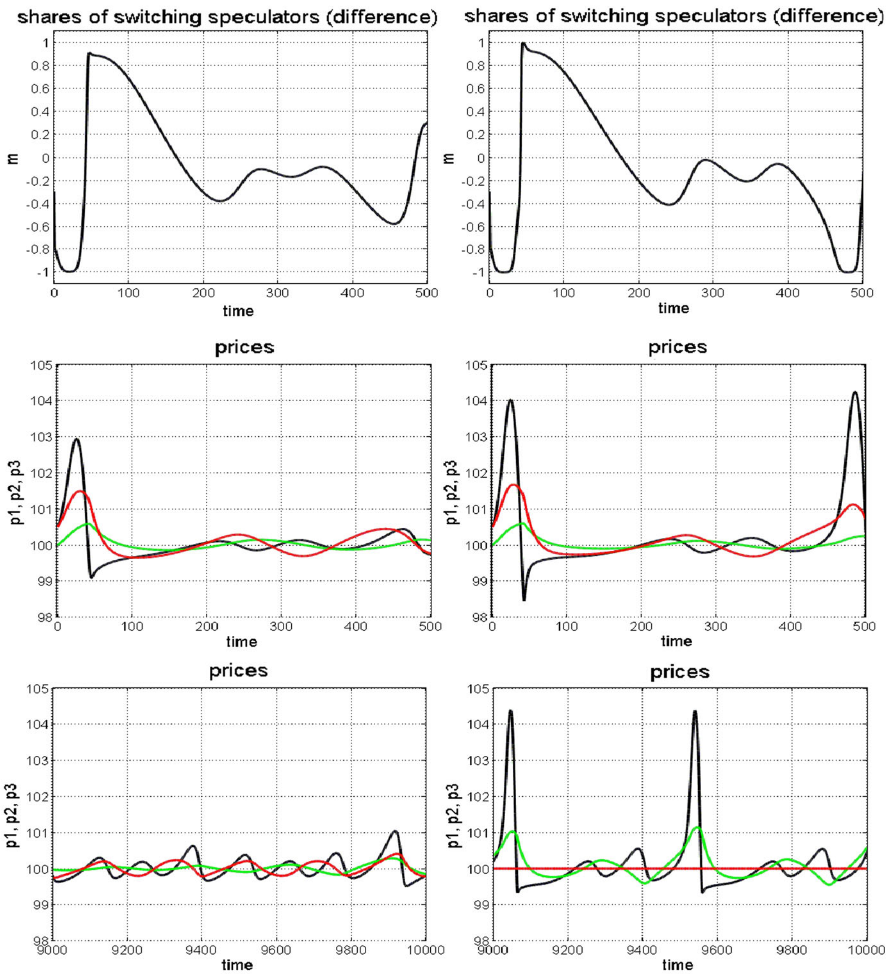


**Fig. 3** Dynamics of $m$ and $p_i$, $i = 1, 2, 3$ (black, green and red, respectively), with different values of parameter $n_0$, selected outside the region of stability in the parameter space: $n_0 = 0.35$ in the left panels, $n_0 = 0.25$ in the right panels. Top panels: difference $m$ between fundamental trader and trend follower shares of switching agents. Middle panels: asset prices. Bottom panels: asymptotic behavior of asset prices, after removing a very long transient. Other parameters as in our baseline selection, except $\gamma_3 = 0.17$, $\lambda = 1$. Initial condition as in Fig. 1 (Color figure online)

## 5 Conclusion

In this paper, we introduce a stylized dynamic model of a multi-asset financial market in the presence of investors with heterogeneous expectations and having some degree of 'impatience' (i.e., being ready to change their strategy over time up to a certain extent) to provide insight into the potentially destabilizing effect of various behavioral forces in financial markets and the complex way it can bring about cross-section instabilities. The dynamics of asset prices is driven by disequilibrium adjustments as a consequence of excess demand/supply of myopic mean-variance investors with different types of beliefs about future price developments. In addition to price expectations, investors possibly update their second moment beliefs, as well. A steady-state solution to this model (Fundamental Steady State—FSS) reflects well-established equilibrium conditions between dividends, the risk-free rate and investors' risk perceptions, resulting in standard 'fundamental prices' of the risky assets. Such equilibrium prices represent the long-term outcome of the model under 'normal' ranges of the main behavioral parameters, even if the market is initially displaced from such fundamental equilibrium. However, a loss of asymptotic stability of the FSS may take place under other ranges of the behavioral parameters. Besides the impact of sufficiently large extrapolation, characterizing trend-followers' beliefs, further behavioral elements play a major role in the dynamics and can act as destabilizing forces, leading to a regime of persistent joint price fluctuations and cross-section instabilities. These are the intensity of choice, characterizing investors' propensity to switch their strategy based on observed performance differentials, and a parameter capturing the population share of potentially switching investors (or, symmetrically, the complementary share of fixed-strategy investors), which is related to the degree of investors' 'impatience' in this setting. While the destabilizing impact of the intensity of choice is well documented in the literature, the latter is still largely unexplored. Building on, and generalizing the elementary one-risky-asset framework of Dieci et al. (2006), this paper provides a complete analytical treatment of the local asymptotic stability conditions of the FSS in the parameter space, in the benchmark case when investors' perceived correlations among assets are fully determined endogenously (i.e., in the absence of exogenous correlations between dividends/fundamentals of different assets). Among other things, the paper shows that an increase of the proportion of impatient, potentially switching investors can have dramatic destabilizing effects. This may also be interpreted as the effect of an increase of the share of short-term speculators, in contrast to long-term investors. Besides generalizing the preliminary insights offered by Dieci et al. (2006), our results are in agreement with Palczewski et al. (2016), who report similar effects of an increase of the amount of capital which flows freely among competing fund managers with different investment styles, in an evolutionary finance framework with multiple risky assets. Further extensions of our model that incorporate exogenous covariances, noisy dividends and a richer supply structure, can potentially generate more realistic cross-section price dynamics and market anomalies.

A second contribution of our paper consists in emphasizing the substantial (and somehow surprising) differences between the way instabilities arise and show up in a multiple-risky-asset setting, in contrast to a baseline one-risky-asset setting. Intuitively, in a single-risky-asset setting, the expectations feedback process in disequilibrium conditions (by which observed price movements determine expectations updating and subsequent changes to asset demand and portfolio compositions, which again contributes to keep price fluctuations alive) remains confined to a unique asset (as the price and return of the riskless asset are fixed, by assumption). Generally speaking, and different from the one-asset case, in a multiple asset framework expectations updating for one asset has immediate effect on the quantity demanded of other assets, as well [see Eq. (3)], particularly if investors also update their second-moment beliefs based on observed asset comovements [Eq. (4)]. Even a few examples provided in our paper indicate how wide a range of possible dynamic patterns and joint price movements a loss of stability of the FSS can bring about in a multi-asset setting. Therefore, once a regime of persistent price fluctuation arises due to, e.g., a Neimark–Sacker bifurcation, this also implies a continuing evolution of investors first- and second-moment beliefs and portfolio compositions, which may result in cross-section instabilities involving multiple assets. However, this scenario is compatible with long run-patterns in which only a subset of the existing assets display price fluctuations. That is, optimal diversification in an unstable regime may require changes in amounts held for only a subset of available assets, with fixed optimal amounts for the remaining assets. Moreover, additional simulations not reported here[21], confirm that such long-run patterns are very sensitive to small parameter changes.

Finally, our analytical results in the general case of $n$ assets correspond to a simplified situation in which investors regard dividends (and fundamental prices) as uncorrelated across assets, so that the only possible price correlation patterns emerge from observed comovements in disequilibrium conditions. Although analytical results seem impossible in the case of general correlations (resulting in a non-diagonal matrix $\mathbf{\Omega}_0$), preliminary simulations show that the effect of the main parameters on steady-state stability is still qualitatively similar to that illustrated in this paper.[22] Anyway, further analysis should focus more systematically on the case when an 'exogenous' asset correlation component is also at work, as the effect of common shocks or of broader, economy-wide interdependencies, while an 'endogenous' correlation similar to the one illustrated here may still come into play and add to the exogenous component due to behavioral forces. Chiarella et al. (2005, 2007) offer preliminary insights in a deterministic setting and Schmitt and Westerhoff (2014) provide interesting developments and empirical calibrations in a more general setting with different sources of noise. Furthermore, suitable stochastic versions of our baseline model could be useful to improve our knowledge on how changing risk-return patterns in asset markets can be affected by investors' behavioral characteristics, along the lines indicated in Chiarella et al. (2013), Chiarella et al. (2013).

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[21] See, however, Fig. 3 (bottom panels) for an example of two very different qualitative long-run scenarios.

[22] Moreover, in the general case when $\Omega_0$ is non diagonal, the fluctuations generated by the Neimark–Sacker bifurcation of the FSS affect the long-run stability properties of all assets (since invariant sets like those outlined in Sect. 4 do not exist in this case).

## Appendix: Local asymptotic stability of the steady state

By omitting time subscripts and using the symbol $'$ to denote a one-step advancement in time, we rewrite the map driving the time evolution of the model into four groups of equations, corresponding to variables $\mathbf{p}$, $\mathbf{u}$, $\mathbf{V}$, $m$, as follows:

$$\mathbf{p}' = \mathbf{p} + \frac{\mu}{2}\left\{\left(\mathbf{z}^f + \mathbf{z}^c\right) + [(1-n_0)m + n_0 m_0] \left(\mathbf{z}^f - \mathbf{z}^c\right) - 2\mathbf{z}^s\right\} := \mathbf{F}(\mathbf{p},\mathbf{u},\mathbf{V},m) \tag{25}$$

$$\mathbf{u}' = \delta\mathbf{u} + (1-\delta)\mathbf{p}' = \delta\mathbf{u} + (1-\delta)\mathbf{F}(\mathbf{p},\mathbf{u},\mathbf{V},m) \tag{26}$$

$$\mathbf{V}' = \delta\mathbf{V} + \delta(1-\delta)(\mathbf{p}'-\mathbf{u})(\mathbf{p}'-\mathbf{u})^\top = \delta\mathbf{V} + \delta(1-\delta)\left(\mathbf{F}(\mathbf{p},\mathbf{u},\mathbf{V},m)-\mathbf{u}\right)\left(\mathbf{F}(\mathbf{p},\mathbf{u},\mathbf{V},m)-\mathbf{u}\right)^\top \tag{27}$$

$$\begin{aligned} m' &= \tanh\left\{\frac{\beta}{2}\left[\left(\mathbf{z}^f - \mathbf{z}^c\right)^\top\left(\mathbf{p}' + \mathbf{d} - (1+r)\mathbf{p}\right) - \Delta C\right]\right\} \\ &= \tanh\left\{\frac{\beta}{2}\left[\left(\mathbf{z}^f - \mathbf{z}^c\right)^\top\left(\mathbf{F}(\mathbf{p},\mathbf{u},\mathbf{V},m) + \mathbf{d} - (1+r)\mathbf{p}\right) - \Delta C\right]\right\} \\ &:= G(\mathbf{p},\mathbf{u},\mathbf{V},m) \end{aligned} \tag{28}$$

where

$$\begin{aligned} \mathbf{z}^f &= \mathbf{z}^f(\mathbf{p}) = \theta^{-1}\boldsymbol{\Omega}_0^{-1}\left[\mathbf{A}(\mathbf{p}^* - \mathbf{p}) + \mathbf{d} - r\mathbf{p}\right], \\ \mathbf{z}^c(\mathbf{p},\mathbf{u},\mathbf{V}) &= \theta^{-1}(\boldsymbol{\Omega}_0 + \lambda\mathbf{V})^{-1}\left[\boldsymbol{\Gamma}(\mathbf{p}-\mathbf{u}) + \mathbf{d} - r\mathbf{p}\right] \end{aligned}$$

As will become clear from the structure of the Jacobian matrix of the map (25)–(28), computed at the FSS, local asymptotic stability is affected only by certain blocks of derivatives, for which we provide the necessary details below. In doing so, it is useful to regard (with some abuse of notation) the block of variables corresponding to symmetric matrix $\mathbf{V}$ as rearranged into a vector with $q = n + (n^2 - n)/2 = n(n+1)/2$ components ($n$ variances plus $(n^2-n)/2$ different covariances).

Denote, in general, by $D_{\mathbf{x}}\boldsymbol{\Phi}$ the (partial) Jacobian matrix of function $\boldsymbol{\Phi}$ with respect to the block of variables $\mathbf{x}$, at the FSS. For the demand functions, one obtains:

$$D_{\mathbf{p}}\mathbf{z}^f = -\theta^{-1}\boldsymbol{\Omega}_0^{-1}(\mathbf{A} + r\mathbf{I}), \quad D_{\mathbf{p}}\mathbf{z}^c = \theta^{-1}\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\Gamma} - r\mathbf{I}), \quad D_{\mathbf{u}}\mathbf{z}^c = -\theta^{-1}\boldsymbol{\Omega}_0^{-1}\boldsymbol{\Gamma}$$

and, in particular:

$$D_{\mathbf{p}}\left(\mathbf{z}^f + \mathbf{z}^c\right) = \theta^{-1}\boldsymbol{\Omega}_0^{-1}(\boldsymbol{\Gamma} - \mathbf{A} - 2r\mathbf{I}), \quad D_{\mathbf{p}}\left(\mathbf{z}^f - \mathbf{z}^c\right) = -\theta^{-1}\boldsymbol{\Omega}_0^{-1}(\mathbf{A} + \boldsymbol{\Gamma}).$$

Therefore, since recursive equation for prices (25) is linear in $\mathbf{p}$ and $\mathbf{u}$, one obtains:

$$D_{\mathbf{p}}\mathbf{F} = \mathbf{I} + \frac{\mu}{2}\theta^{-1}\boldsymbol{\Omega}_0^{-1}\left[(\boldsymbol{\Gamma} - \mathbf{A} - 2r\mathbf{I}) - \bar{s}(\mathbf{A} + \boldsymbol{\Gamma})\right],$$

$$D_{\mathbf{u}}\mathbf{F} = \frac{\mu}{2}\theta^{-1}\boldsymbol{\Omega}_0^{-1}\left(\bar{s} - 1\right)\boldsymbol{\Gamma},$$

where $\bar{s} := n_0 m_0 + (1 - n_0)\bar{m}$. Also, since (25) is linear in $m$, demand functions $\mathbf{z}^f$ and $\mathbf{z}^c$ do not depend on $m$, and $\bar{\mathbf{z}}^f - \bar{\mathbf{z}}^c = \mathbf{0}$ at the steady state, it turns out that:

$$D_m \mathbf{F} = \frac{\mu}{2}(1 - n_0)\left(\bar{\mathbf{z}}^f - \bar{\mathbf{z}}^c\right) = \mathbf{0}.$$

As regards the recursive definitions for $\mathbf{u}$ and $\mathbf{V}$, denoting by $\mathbf{Q}(\mathbf{p}, \mathbf{u}, \mathbf{V}, m)$ and $\mathbf{Z}(\mathbf{p}, \mathbf{u}, \mathbf{V}, m)$ the right-hand sides of (26) and (27), respectively, one immediately obtains:

$$\begin{aligned} D_{\mathbf{p}}\mathbf{Q} =&(1-\delta)D_{\mathbf{p}}\mathbf{F}, \quad D_{\mathbf{u}}\mathbf{Q} =\delta\mathbf{I} + (1-\delta)D_{\mathbf{u}}\mathbf{F}, \quad D_m\mathbf{Q} =(1-\delta)D_m\mathbf{F} = \mathbf{0}, \\ & D_{\mathbf{V}}\mathbf{Z} =\delta\mathbf{I}, \quad D_{\mathbf{p}}\mathbf{Z} =D_{\mathbf{u}}\mathbf{Z} = \mathbf{0}, \quad D_m\mathbf{Z} = \mathbf{0}. \end{aligned}$$

In particular, the great simplifications arising in the last set of derivatives depend on the fact that all the partial derivatives of the term $\left(\mathbf{p}' - \mathbf{u}\right)\left(\mathbf{p}' - \mathbf{u}\right)^\top$ include some multiplicative factor of the type $(p_i' - u_i)$, which vanishes at the FSS.

Finally, as regards the recursive Eq. (28) for variable $m$, one obtains, at the steady state:

$$\frac{\partial G}{\partial m} = \frac{\beta}{2}\left[1 - \tanh^2\left(\frac{\beta}{2}(\bar{\pi}^\Delta - \Delta C)\right)\right]\frac{\partial}{\partial m}\left[\left(\bar{z}^f - \bar{z}^c\right)^\top (\mathbf{d} - r\bar{\boldsymbol{p}})\right] = 0,$$

again because $\mathbf{z}^f$ and $\mathbf{z}^c$ do not depend on $m$ and $\bar{z}^f = \bar{z}^c$.

The Jacobian matrix $\mathbf{J}$ of map (25)–(28) has dimension $N = (n + n + q + 1) = \frac{5n+n^2}{2} + 1$. As regards the study of the roots of the characteristic polynomial of $\mathbf{J}$, $\mathcal{S}(\zeta) := \det(\mathbf{J}{-}\zeta\mathbf{I})$, our previous results show that both matrix $\mathbf{J}{-}\zeta\mathbf{I}$ and its square submatrix associated with variables $(\mathbf{p}, \mathbf{u}, \mathbf{V})$, have block triangular structures, which greatly simplifies this task. More precisely, matrix $\mathbf{J}{-}\zeta\mathbf{I}$ can be partitioned as follows (here we indicate, for clarity, the dimension of each identity matrix involved):

$$\mathbf{J}{-}\zeta\mathbf{I}_N = \begin{pmatrix} D_{\mathbf{p}}\mathbf{F} - \zeta\mathbf{I}_n & D_{\mathbf{u}}\mathbf{F} & D_{\mathbf{V}}\mathbf{F} & \mathbf{0} \\ (1-\delta)D_{\mathbf{p}}\mathbf{F} & (\delta-\zeta)\mathbf{I}_n + (1-\delta)D_{\mathbf{u}}\mathbf{F} & (1-\delta)D_{\mathbf{V}}\mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\delta-\zeta)\mathbf{I}_q & \mathbf{0} \\ D_{\mathbf{p}}G & D_{\mathbf{u}}G & D_{\mathbf{V}}G & -\zeta \end{pmatrix}$$

and therefore:

$$\mathcal{S}(\zeta) := \det(\mathbf{J}{-}\zeta\mathbf{I}_N) = -\zeta(\delta-\zeta)^q\mathcal{P}(\zeta),$$

where $\mathcal{P}(\zeta)$ is the determinant of the upper-left block of $\mathbf{J}{-}\zeta\mathbf{I}_N$, corresponding to variables $(\mathbf{p}, \mathbf{u})$, namely:

$$\begin{aligned} \mathcal{P}(\zeta) &= \det(\mathbf{M}(\zeta)), \\ \mathbf{M}(\zeta) &:= \begin{pmatrix} D_{\mathbf{p}}\mathbf{F} - \zeta\mathbf{I}_n & D_{\mathbf{u}}\mathbf{F} \\ (1-\delta)D_{\mathbf{p}}\mathbf{F} & (\delta-\zeta)\mathbf{I}_n + (1-\delta)D_{\mathbf{u}}\mathbf{F} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_1(\zeta) & \mathbf{M}_2 \\ \mathbf{M}_3 & \mathbf{M}_4(\zeta) \end{pmatrix}. \end{aligned} \quad (29)$$

It follows that one root of $\mathcal{S}(\zeta)$ is equal to 0 and $q$ additional roots are equal to $\delta$, $0 < \delta < 1$. Stability thus depends only on the $2n$ roots of $\mathcal{P}(\zeta)$. For instance, in the case of $n = 2$ risky assets, stability depends on the joint behavior of four eigenvalues and a loss of stability occurs when one of them becomes larger than 1 in modulus, as a consequence of parameter changes. Although the above result represents a substantial simplification to the original problem, the analytical study of (29) still appears prohibitive, in general. However, a complete analytical treatment is still possible in a benchmark case, in which matrix $\mathbf{\Omega}_0$ is diagonal (the 'exogenous' component of asset covariances is zero). Note that in this case, all four square blocks $\mathbf{M}_1(\zeta)$, $\mathbf{M}_2$, $\mathbf{M}_3$, $\mathbf{M}_4(\zeta)$ are themselves diagonal matrices. Based on Schur's formula for the determinant of block matrices (see, e.g., Gantmacher 1959, p. 46), one has:

$$
\begin{aligned}
\det(\mathbf{M}(\zeta)) &= \det(\mathbf{M}_1(\zeta)\mathbf{M}_4(\zeta) - \mathbf{M}_2\mathbf{M}_3) \\
&= \det\left[(\delta - \zeta)\left(D_{\mathbf{p}}\mathbf{F} - \zeta \mathbf{I}_n\right) - (1-\delta)\zeta D_{\mathbf{u}}\mathbf{F}\right] \\
&= \prod_{i=1}^{n} [(\delta - \zeta)(a_i - \zeta) - (1-\delta)\zeta b_i], \qquad (30)
\end{aligned}
$$

where, for $i = 1, 2, ..., n$:

$$
\begin{aligned}
a_i &= 1 + \frac{\mu}{2\theta\sigma_i^2}\left[(\gamma_i - \alpha_i - 2r) - \bar{s}(\alpha_i + \gamma_i)\right], \\
b_i &= \frac{\mu}{2\theta\sigma_i^2}\gamma_i(\bar{s} - 1),
\end{aligned}
$$

are the diagonal elements of $D_{\mathbf{p}}\mathbf{F}$ and $D_{\mathbf{u}}\mathbf{F}$, respectively, in the case when $\mathbf{\Omega}_0 = \mathrm{diag}(\sigma_1^2, \sigma_2^2, ...\sigma_n^2)$. Each factor of (30), namely

$$
\mathcal{Q}_i(\zeta) := \zeta^2 - (a_i + \delta + (1-\delta)b_i)\zeta + \delta a_i \quad i = 1, 2, ..., n
$$

is a second-degree polynomial 'associated' to a particular asset. The roots of $\mathcal{Q}_i(\zeta)$ are smaller than one in modulus if and only if the following conditions jointly hold:

$$
\mathcal{Q}_i(1) > 0, \quad \mathcal{Q}_i(-1) > 0, \quad \mathcal{Q}_i(0) < 1,
$$

which results, respectively, in the following system of inequalities:

$$
\begin{cases}
\dfrac{\mu}{2\theta\sigma_i^2}[\alpha_i(1+\bar{s}) + 2r] > 0 \\
\dfrac{\mu}{2\theta\sigma_i^2}(1+\delta)[\alpha_i(1+\bar{s}) + 2r] < 2(1+\delta) + \dfrac{\mu}{2\theta\sigma_i^2}2\delta\gamma_i(1-\bar{s}) \\
\dfrac{\mu}{2\theta\sigma_i^2}\left[\gamma_i(1-\bar{s}) - \alpha_i(1+\bar{s}) - 2r\right] < \dfrac{1-\delta}{\delta}
\end{cases} \qquad (31)
$$

In particular, whenever the third (*resp.* second) condition of (31) is violated—all other conditions being satisfied—the modulus of the two complex conjugate roots of $\mathcal{Q}_i(\zeta)$

becomes larger than one (*resp.* one real root of $\mathcal{Q}_i(\zeta)$ becomes smaller than $-1$). Since the first inequality of (31) is always satisfied in the parameter space, by making quantity $\gamma_i(1-\bar{s})$ explicit in the second and third inequalities, one finally obtains the more synthetic and intuitive form of the stability conditions stated in Proposition 1, Eqs. (17)–(19), expressed in terms of $n$ stability intervals for quantities $\gamma_i(1-\bar{s})$, $i = 1, 2, ..., n$, provided that condition (18) holds.

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