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REGULAR PAPER

New cosine similarity and distance measures for Fermatean fuzzy sets and TOPSIS approach

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Abstract

The most straightforward approaches to checking the degrees of similarity and differentiation between two sets are to use distance and cosine similarity metrics. The cosine of the angle between two n-dimensional vectors in n-dimensional space is called cosine similarity. Even though the two sides are dissimilar in size, cosine similarity may readily find commonalities since it deals with the angle in between. Cosine similarity is widely used because it is simple, ideal for usage with sparse data, and deals with the angle between two vectors rather than their magnitude. The distance function is an elegant and canonical quantitative tool to measure the similarity or difference between two sets. This work presents new metrics of distance and cosine similarity amongst Fermatean fuzzy sets. Initially, the definitions of the new measures based on Fermatean fuzzy sets were presented, and their properties were explored. Considering that the cosine measure does not satisfy the axiom of similarity measure, then we propose a method to construct other similarity measures between Fermatean fuzzy sets based on the proposed cosine similarity and Euclidean distance measures and it satisfies the axiom of the similarity measure. Furthermore, we obtain a cosine distance measure between Fermatean fuzzy sets by using the relationship between the similarity and distance measures, then we extend the technique for order of preference by similarity to the ideal solution method to the proposed cosine distance measure, which can deal with the related decision-making problems not only from the point of view of geometry but also from the point of view of algebra. Finally, we give a practical example to illustrate the reasonableness and effectiveness of the proposed method, which is also compared with other existing methods.

Keywords Similarity measure · Euclidean distance · Multi-criteria decision-making · Fermatean fuzzy set · Similarity measure · Euclidean distance · Cosine distance · TOPSIS

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1 Introduction

1.1 Similarity and distance measures

Almost every scientific field relies heavily on the concept of similarity. Geometric methods for assessing similarity, for example, are used in studies of congruence and homothety, as well as in allied fields such as trigonometry. Semantics, for example, makes use of topological methods. Graph theory is widely used in taxonomy to assess cladistic similarities. Fuzzy set theory has also developed similarity measures, which are used in fields such as management, medicine, and meteorology. Measuring the sequence similarity of two proteins is an important problem in molecular biology.

Not surprisingly, similarity has played a critical role in psychological experiments and theories. Many experiments, for example, ask participants to make direct or indirect judgments about the similarity of two objects. In these studies, a variety of experimental techniques are used. Still, the most common is to ask subjects whether the objects are the same or different, or to ask them to produce a number, say between 1 and 7, that corresponds to their feelings about how similar the objects appear (e.g., with 1 meaning very dissimilar and 7 meaning very similar). Similarity also plays an important but less direct role in modeling many other psychological tasks. This is especially true in theories of object recognition, identification, and categorization, where it is commonly assumed that the greater the similarity between two things, the more likely one will be confused with the other.

In data science, the similarity measure is a way of determining how closely related or similar data samples are. A dissimilarity measure is used to determine how distinct the data objects are. When similar data samples are grouped into one cluster, these terms are commonly used in clustering. The similarity is somewhat subjective, and it is heavily dependent on context and application. For instance, the similarity of vegetables can be determined based on their taste, size, color, and so on.

Statistical measures of similarity allow scholars to think computationally about how similar or dissimilar their objects of study might be, and these measures also serve as the foundation for many other clustering and classification techniques. The similarity of two texts in the context of text analysis can be assessed in its most basic form by representing each text as a series of word counts and calculating distance using those word counts as features. The similarity measure is typically expressed as a numerical value, which rises as the data samples become more similar. It is commonly expressed as a number between zero and one by conversion: zero indicates a low level of similarity (the data objects are dissimilar) and one indicates a high level of similarity (the data objects are very similar). The similarity is essentially a broad umbrella term that encompasses a diverse set of scores and measures for evaluating the differences between various types of data.

As previously stated, similarity can refer to a broader category of similarity measures, whereas distance is a more narrow category that measures the difference in Cartesian space. In the context of text analysis, these terms are often used interchangeably—distance is simply the inverse of similarity and vice versa. When measuring distance, the closest points have the shortest distance, but when measuring similarity, the closest points have the highest degree of similarity.

The dissimilarity metrics indicate how distinct and distinct two things (observations) are from one another. In contrast, distance describes the distance between two objects in a certain mathematical space. Distance is defined as the quantifiable degree to which two objects differ from one another in scientific and mathematical terms. Differences are measured by distance.

The difference is a measurement of the contrast or incompatibility of two items based on their numerous properties. Dissimilarity measurements calculate the distance between two things. The degree of dissimilarity between two items is determined using distance metrics. The larger the difference between two things, the higher the distance value.

The concepts of distance and difference are frequently employed interchangeably, even though distance refers to a subset of differences. The distance is a measure of disorder or disorder between two items, whereas the difference is a measure of contrast or incongruity between two objects based on numerous qualities. In brief, distance measures are used to determine the degree of separation between two things. As a result, the differences between different pairs of items will be great, and the value of the distance measure will be small, whereas the differences between more similar pairs of objects will be lower, and the value of the distance measure will be less.

1.2 Uncertainty

Many fields, including psychology, philosophy, cognitive science, and artificial intelligence, study people's thinking and decision-making processes in the face of everyday situations. These processes are typically described using various mathematical and statistical models. The issue of decision-making emerges during this procedure. Decision-making (DM) is described as the process of selecting one or more of the different forms of behavior presented to a person or institution to attain a certain objective. According to research, while making many daily judgements instinctively is fine, this method alone is insufficient for complicated and critical decisions. Multi-Criteria Decision Making (MCDM) is a set of analytical methodologies that analyze the benefits and drawbacks of alternatives based on a variety of criteria. MCDM approaches are used to aid the DM process by selecting or ranking one or more alternatives from a group of alternatives with different attributes based on competing criteria. In other words, using MCDM methodologies, decision-makers evaluate options with various features by analyzing them against a variety of criteria. MCDM is a collection of strategies that are widely employed in all aspects of life and at all levels.

For DM difficulties, uncertainty is a critical notion. Uncertainty implies unpredictability. In unpredictable scenarios, routine decisions cannot be mentioned. The benefits and drawbacks of potential repercussions in unknown situations must be carefully considered. At this time, it is critical to thoroughly examine the environmental elements. In the face of ambiguity, drawing on previous experiences and judgments is not always useful, and ultimate decisions are not in doubt. Linguistic terms that we use unknowingly in our everyday lives have become "computable" because of Zadeh's notion of "fuzzy sets" (FS) [41]. With the grading system, fuzzy logic enlarged the universe of classical mathematics from certainty to uncertainty, and this notion achieved a paradigm shift that spread all over the world as a result of its successful applications in real life. An element with a characteristic function is either an element or not an element of a set, according to the traditional set idea. However, whether an apple belongs to a set or not in the FS idea is specified by a membership function (MF) that assigns the item a degree of membership in the range [0, 1]. The degree of belonging of an element to the set in the FS A is $\rho(A)$, whereas the degree of not belonging is $1 - \rho(A)$. As a result, the sum of the degrees of belonging and non-belonging equals one. This circumstance, however, is insufficient to explain the ambiguity in some cases. As a result, Atanassov [1] introduced the intuitionistic fuzzy set (IFS) theory, a generalization of the FS theory. In IFS theory, the non-membership degree (ND) is defined in addition to the membership degree (MD), whereas FS theory is designed to only reveal the membership degree (MD) defined in the range [0, 1]. According to IFS theory, MD and ND both fall between [0, 1]. Yager [37] proposed Pythagorean fuzzy sets (PFS) and in certain circumstances developed them as an extension of IFSs because IFSs cannot adequately convey uncertainty. PFSs employ the notion that the sum of the squares of MD and ND is less than or equal to 1 for circumstances when decision-making is impossible when MD and ND are added together. In the literature, there is a ton of study on FS and its many expansions [5–10, 12–15, 26, 27, 31, 37, 39].

Senapati and Yager [29] is credited with creating the Fermatean fuzzy set (FFS). The MD and ND in the FFS achieve the property $0 \leq m_A^3 + n_A^3 \leq 1$. When identifying uncertainties, the FFS, a novel idea in the literature, performs better than the IFS and PFS. As an illustration, consider $0.9+0.6 > 1$, $0.9^2+0.6^2 > 1$, and $0.9^3+0.6^3 < 1$. Numerous investigations have originated with FFSs [2, 3, 11, 16, 18–21, 25]. Senapati and Yager [29] offers some of the FFS characteristics, score, and accuracy attributes. The FFS problem has been addressed using the TOPSIS approach, which is widely applied to MCDM issues. Additionally, Senapati and Yager claim that FFS was carried out utilizing the TOPSIS method, which is frequently applied to MCDM issues. Senapati and Yager [29] further mention that the TOPSIS method, which is frequently used in MCDM issues, has been used for FFS. Senapati and Yager [30] continued this work by investigating several novel operations, including Fermatean arithmetic mean, division, and subtraction operations on FFSs, and they used the Fermatean fuzzy weighted product model to tackle MCDM issues. Novel aggregation operators that are part of the FFS have been defined and their associated attributes have been studied in Senapati and Yager [30].

The new measurement combines the Euclidean distance measure with the cosine similarity measure. Fermatean fuzzy soft sets (FFSS) were defined by Kirisci [16], who also provided a measure of entropy based on them. A brand-new hesitant fuzzy set (HFS) known as the Fermatean hesitant fuzzy set (FHFS) has been presented and some of its attributes have been examined in Kirisci [19]. The ELECTRE I technique is described in Kirisci et al. [17] with FFSs by the group DM operate in which several people engage simultaneously. To increase the efficacy of anti-virus masks, Shahzadi and Akram [28] presents a decision support method based on the FFSS notion.

1.3 Motivation

In the field of fuzzy set theory, the similarity metric is a crucial concept. In pattern recognition, medical diagnostics, and other fields, it is commonly employed. On FSs, IFSs, and PFSs, several similarity metrics have been investigated [23, 32, 33, 36, 40, 42, 43].

A series of distance and similarity measurements between two hesitant fuzzy linguistic word sets is provided in Liao et al. [23]. Second, different weighted or ordinal weighted distance and similarity measurements are provided between two collections of hesitant fuzzy linguistic word sets. Following that, these metrics were examined in both discrete and continuous scenarios. In [40], a cosine similarity measure and a weighted cosine similarity measure between IFSs are proposed based on the concept of the cosine similarity measure for fuzzy sets, taking into account the information carried by the membership degree and the non-membership degree in IFSs as a vector representation with the two elements. Zhou et al. [43] developed the heuristic fuzzy ordered weighted cosine similarity measure by combining the heuristic fuzzy ordered weighted cosine similarity measure and the extended ordinal weighted average operator. The intuitionistic fuzzy ordered weighted cosine similarity measure distinguishes itself by not only being an extension of several frequently used similarity measures, but also by dealing with the correlation of distinct decision matrices or multidimensional arrays for intuitionistic fuzzy values. The entropy of interval-valued fuzzy sets and similarity measures of interval-valued fuzzy sets were presented by Zeng and Li [42]. Based on their axiomatic definitions, Zeng and Li [42] established three theorems that similarity measure and entropy of interval-valued fuzzy sets may be modified by each other and proposed some formulae to compute entropy and similarity measure of interval-valued fuzzy sets. Wei [35] introduced several unique approaches for determining the similarity of picture fuzzy sets. Some similarity metrics across image fuzzy sets are defined in [35], including cosine similarity, weighted cosine similarity, set-theoretic similarity, weighted set-theoretic cosine similarity, grey similarity, and weighted grey similarity. Wei and Wei [36] proposed 10 similarity metrics between PFSs based on the cosine function, taking into account the degree of membership, nonmembership, and reluctance in PFSs. These similarity and weighted similarity scores between PFSs were applied to pattern recognition and medical diagnostics. The axiom definitions of entropy, distance measure, and similarity measure of fuzzy sets are systematically presented in [33], and essential relationships between these measures are examined. Sridevi and Nadarajan [33] presented a new fuzzy similarity measure to determine the degree of similarity of generalized fuzzy numbers (GFNs). The fuzzy similarity measure is created by combining the notion of center of gravity (COG) points with the fuzzy difference of distance between fuzzy number points. Aydin [4] introduced a new MCDM technique using FFSs that employs entropy theory to compute criterion weights and cosine similarity measurements to select the optimal option. Xu and Shen [34] investigated Fermatean fuzzy set similarity measures. The definitions of the Fermatean fuzzy sets similarity measures and its weighted similarity measures on discrete and continuous universes are provided in turn in this work. The fundamental features of the proposed similarity metrics are then addressed. Following that, a decision-making process based on the TOPSIS approach is constructed in the Fermatean fuzzy environment, and a novel method based on the provided Fermatean fuzzy sets similarity measures is designed to tackle medical diagnosis issues.

The major reason we used FFSs in designing the current study's strategy is because of its exibility in dealing with unclear information. The supreme tendency of FFSs to address the inexact human decision makes it more feasible and accurate to model two-dimensional (i.e., membership and non-membership) information in a wider space as compared to IFSs and PFSs. The inner product of two vectors divided by the product of their lengths gives the measure of cosine similarity. The aim of this study is to define cosine similarity and weighted cosine similarity measures based on FFSs. The characteristics of the new cosine similarity measures will be examined and a new decision-making algorithm based on these measures will be given. The algorithm is obtained by combining the new cosine similarity measures with the TOPSIS method.

The originality: There have been various extensions of the classical cosine similarities such as fuzzy, IF, and PF cosine similarities. These extensions have improved the performance of the cosine similarities. FFSs can handle problems with ambiguity and incomplete information more efficiently than that of IFSs and PFSs. In this study, the Fermatean fuzzy cosine and weighted cosine similarity measures were developed considering the intuitionistic fuzzy and Pythagorean fuzzy cosine similarity measures studies. Since the $MD_3 + ND_3 \leq 1$ requirement is satisfied for an object in the use of FFSs, there will be the possibility to cover more elements than IFSs and PFSs. A medical application regarding the new similarities is shown.

The remainder of this article is structured as follows. In Sect. 2, we will give the fundamental information that will be used in the study. In Sect. 3, we will present new cosine similarity and weighted cosine similarity measures and show the properties of these measures. Sect.

4 is devoted to the MCDM algorithm with respect to cosine similarities and the TOPSIS technique. In the fifth chapter, an application to infectious diseases is presented. The medical decision-making model is shown that the cosine similarities given in the study are easy to use and optimum results can be obtained. From the illustrative example study, it has been accomplished that the offered cosine similarities in the FFS framework can conveniently operate the real-life DM problem with their objectives.

2 Preliminaries

Now, some fundamental information that will be used in the study will be given.

Definition 2.1 [29] For $\mathcal{X} = \{x_1, x_2, \dots, x_n\}$, if

$$S = \{(x, \rho_S(x), \tau_S(x)) : x \in \mathcal{X}\}$$

satisfies the following conditions, then the set S is called FFS:

$$\rho_S, \tau_S \in [0, 1], \quad 0 \le \rho_S^3 + \tau_S^3 \le 1.$$

$\theta_S = (1 - \rho_S^3 + \tau_S^3)^{1/3}$ shows the hesitation degree.

The pair $(\rho_S(x), \tau_S(x))$ in the FFS S is defined as a Fermatean Fuzzy Number (FFN). Choose the FFNs $\mathcal{F} = (\rho_{\mathcal{F}}, \tau_{\mathcal{F}})$ and $\mathcal{G} = (\rho_{\mathcal{G}}, \tau_{\mathcal{G}})$.

a. $\overline{\mathcal{F}} = (\tau_{\mathcal{F}}, \rho_{\mathcal{F}})$,
b. $\mathcal{F} \boxplus G = ((\rho_F^3 + \rho_{\mathcal{G}}^3 - \rho_{\mathcal{F}}^3 \rho_{\mathcal{G}\mathcal{F}}^3)^{1/3}, \tau_{\mathcal{F}} \tau_{\mathcal{G}})$,
c. $\mathcal{F} \boxtimes \mathcal{G} = (\rho_{\mathcal{F}} \rho_{\mathcal{G}}, (\tau_{\mathcal{F}}^3 + \tau_{\mathcal{G}}^3 - \tau_{\mathcal{F}}^3 \tau_{\mathcal{G}}^3)^{1/3})$,
d. $z.\mathcal{F} = ((1 - (1 - \rho_{\mathcal{F}}^3)^z)^{1/3}, \tau_{\mathcal{F}}^z)$,
e. $\mathcal{F}^z = (\rho_{\mathcal{F}}^z, (1 - (1 - \tau_{\mathcal{F}F}^3)^z)^{1/3})$.

Definition 2.2 Consider the two FFNs $\mathcal{F} = (\rho_{\mathcal{F}}, \tau_{\mathcal{F}})$ and $\mathcal{G} = (\rho_{\mathcal{G}}, \tau_{\mathcal{G}})$. For $\mathcal{F}$ and $\mathcal{G}$. The operation laws between them are as follows:

i. $\mathcal{F} \cup \mathcal{G} = (\max\{\rho_{\mathcal{F}}, \rho_{\mathcal{G}}\}, \min\{\tau_{\mathcal{F}}, \tau_{\mathcal{G}}\})$
ii. $\mathcal{F} \cap \mathcal{G} = (\min\{\rho_{\mathcal{F}}, \rho_{\mathcal{G}}\}, \max\{\tau_{\mathcal{F}}, \tau_{\mathcal{G}}\})$
iii. $\mathcal{F}^C = (\tau_{\mathcal{F}}, \rho_{\mathcal{F}})$
iv. $\mathcal{F} \preceq \mathcal{G}$ if and only if $\rho_{\mathcal{F}} \le \rho_{\mathcal{G}}, \tau_{\mathcal{F}} \le \tau_{\mathcal{G}}$.

Definition 2.3 [29] Consider the two FFNs $\mathcal{F} = (\rho_{\mathcal{F}}, \tau_{\mathcal{F}})$ and $\mathcal{G} = (\rho_{\mathcal{G}}, \tau_{\mathcal{G}})$. For $\mathcal{F}$ and $\mathcal{G}$, the score functions $\mathrm{SC}(\mathcal{F}) = \rho_{\mathcal{F}}^3 - \tau_{\mathcal{F}}^3$ and $\mathrm{SC}(\mathcal{G}) = \rho_{\mathcal{G}}^3 - \tau_{\mathcal{G}}^3$ and the accuracy functions $\mathrm{AC}(\mathcal{F}) = \rho_{\mathcal{F}}^3 + \tau_{\mathcal{F}}^3$ and $\mathrm{AC}(\mathcal{G}) = \rho_{\mathcal{G}}^3 + \tau_{\mathcal{G}}^3$.

In this definition, the following situations are held:

Lemma 2.1 *For the two FFNs $\mathcal{F} = (\rho_{\mathcal{F}}, \tau_{\mathcal{F}})$ and $\mathcal{G} = (\rho_{\mathcal{G}}, \tau_{\mathcal{G}})$,*

– *If $SC(\mathcal{F}) < \mathrm{SC}(\mathcal{G})$, then $\mathcal{F} < \mathcal{G}$,*
– *If $SC(\mathcal{F}) = \mathrm{SC}(\mathcal{G})$, $AC(\mathcal{F}) < AC(\mathcal{G})$, then $\mathcal{F} < \mathcal{G}$,*
– *If $SC(\mathcal{F}) = \mathrm{SC}(\mathcal{G})$, $AC(\mathcal{F}) = AC(\mathcal{G})$, then $\mathcal{F} = \mathcal{G}$.*

Lemma 2.2 *Choose any two FSs $\mathcal{F}, \mathcal{G}$. If the conditions [i.]–[iv.] are held, then $S : \mathrm{FS} \times \mathrm{FS} \to [0, 1]$ is said to be an SM between $\mathcal{F}, \mathcal{G}$.*

i. $0 \le S(\mathcal{F}, \mathcal{G}) \le 1$,
ii. $S(\mathcal{F}, \mathcal{G}) = 1 \Leftrightarrow \mathcal{F} = \mathcal{G}$,
iii. $S(\mathcal{F}, \mathcal{G}) = S(\mathcal{G}, \mathcal{F})$,
iv. $S(\mathcal{F}, \mathcal{H}) \le S(\mathcal{F}, \mathcal{G})$ and $S(\mathcal{F}, \mathcal{H}) \le S(\mathcal{G}, \mathcal{H})$ if $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$.

3 New measures

It is known that the inner product of two vectors divided by the product of their lengths gives the cosine similarity(CS) measure.

Definition 3.1 Take a fixed set $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$. Choose any two FFSs $\mathcal{F} = \{(x, [\rho_{\mathcal{F}}(x_i), \tau_{\mathcal{F}}(x_i)]) : x_i \in \mathcal{X}\}$ and $\mathcal{G} = \{(x, [\rho_{\mathcal{G}}(x_i), \tau_{\mathcal{G}}(x_i)]) : x_i \in \mathcal{X}\}$. Therefore, a CS measure $C_{\text{FFS}}(\mathcal{F}, \mathcal{G})$ between $\mathcal{F}$ and $\mathcal{G}$ can be defined as

$$C_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = \frac{1}{n} \sum_{i=1}^{n} \frac{\rho_{\mathcal{F}}^3(x_i)\rho_{\mathcal{G}}^3(x_i) + \tau_{\mathcal{F}}^3(x_i)\tau_{\mathcal{G}}^3(x_i) + \theta_{\mathcal{F}}^3(x_i)\theta_{\mathcal{G}}^3(x_i)}{\sqrt[3]{\rho_{\mathcal{F}}^6(x_i) + \tau_{\mathcal{F}}^6(x_i) + \theta_{\mathcal{F}}^6(x_i)}\sqrt[3]{\rho_{\mathcal{G}}^6(x_i) + \tau_{\mathcal{G}}^6(x_i) + \theta_{\mathcal{G}}^6(x_i)}}.$$

Theorem 3.1 *Take any two FFSs $\mathcal{F}$ and $\mathcal{G}$. Therefore the CS measure $C_{\text{FFS}}(\mathcal{F}, \mathcal{G})$ satisfies the following conditions:*

i. $0 \le C_{\text{FFS}}(\mathcal{F}, \mathcal{G}) \le 1$
ii. $C_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = C_{\text{FFS}}(\mathcal{G}, \mathcal{F})$
iii. $C_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 1$, *if* $\mathcal{F} = \mathcal{G}$, $(\rho_{\mathcal{F}}(x_i) = \rho_{\mathcal{G}}(x_i), \tau_{\mathcal{F}}(x_i) = \tau_{\mathcal{G}}(x_i))$.

Definition 3.2 Choose any two FFSs $\mathcal{F} = \{(x, [\rho_{\mathcal{F}}(x_i), \tau_{\mathcal{F}}(x_i)]) : x_i \in \mathcal{X}\}$, $\mathcal{G} = \{(x, [\rho_{\mathcal{F}}(x_i), \tau_{\mathcal{F}}(x_i)]) : x_i \in \mathcal{X}\}$. For $x_i \in \mathcal{X}$, take the weight ω_i. The weighted cosine similarity(WCS) measure $C_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G})$ is given as

$$C_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = \frac{1}{n} \sum_{i=1}^{n} \omega_i \frac{\rho_{\mathcal{F}}^3(x_i)\rho_{\mathcal{G}}^3(x_i) + \tau_{\mathcal{F}}^3(x_i)\tau_{\mathcal{G}}^3(x_i) + \theta_{\mathcal{F}}^3(x_i)\theta_{\mathcal{G}}^3(x_i)}{\sqrt[3]{\rho_{\mathcal{F}}^6(x_i) + \tau_{\mathcal{F}}^6(x_i) + \theta_{\mathcal{F}}^6(x_i)}\sqrt[3]{\rho_{\mathcal{G}}^6(x_i) + \tau_{\mathcal{G}}^6(x_i) + \theta_{\mathcal{G}}^6(x_i)}}.$$

When take $\omega = \{\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\}$, the WCS $C_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G})$ is reduced to the CS measure $C_{\text{FFS}}(\mathcal{F}, \mathcal{G})$.

Theorem 3.2 *Take any two FFSs $\mathcal{F}$ and $\mathcal{G}$. Therefore, the WCS measure $C_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G})$ satisfies the [i.]–[iii.] conditions:*

i. $0 \le C_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) \le 1$
ii. $C_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = C_{\text{FFS}}^{\omega}(\mathcal{G}, \mathcal{F})$
iii. $C_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = 1$, *if* $\mathcal{F} = \mathcal{G}$, *that is,* $\rho_{\mathcal{F}}(x_i) = \rho_{\mathcal{G}}(x_i)$, $\tau_{\mathcal{F}}(x_i) = \tau_{\mathcal{G}}(x_i)$.

Example 3.1 For two FFSs $\mathcal{F} = \{(x_1, [0.4, 0.7]), (x_2, [0.5, 0.6]), (x_3, [0.3, 0.9]), (x_4, [0.5, 0.5]))\}$ and $\mathcal{G} = \{(x_1, [0.4, 0.5]), (x_2, [0.8, 0.3]), (x_3, [0.6, 0.3]), (x_4, [0.6, 0.6]))\}$. If $\omega = \{0.32, 0.23, 0.18, 0.27\}$, then $C_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 0.375$ and $C_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = 0.156$.

If a SM $S(\mathcal{F}, \mathcal{G})$ satisfies the conditions [i]–[iii] of Lemma 2.2, then $S(\mathcal{F}, \mathcal{G})$ is called genuine SM. It is known that an SM satisfies the conditions of Lemma 2.2, then we can give the following statement:

Let $D(\mathcal{F}, \mathcal{G})$ show the distance measure(DiMe) between $\mathcal{F}, \mathcal{G}$. Then $D = 1 - S$ is the SM between $\mathcal{F}, \mathcal{G}$.

Since Definitions 3.1 and 3.2 do not the hold the condition [ii] of Lemma 2.2 in several situation, these CS measures are not the genuine SMs.

Example 3.2 For $X = \{x_1, x_2\}$, take two FFSs $\mathcal{F} = \{(x_1, [0.3, 0.3]), (x_2, [0.2, 0.2])\}$ and $\mathcal{G} = \{(x_1, [0.4, 0.4]), (x_2, [0.3, 0.3])\}$ and $\omega = (0.4, 0.6)$. The WCS measure $C_{\text{FFS}}^{\omega}(\mathcal{F}F, \mathcal{G}) = 1$. However, the given numbers are not equal to each other.

In this example, there is a problem in obtaining the desired result. To solve this problem, the following definition is given.

Definition 3.3 For any two FFSs $\mathcal{F} = \{(x, [\rho_{\mathcal{F}}(x_i), \tau_{\mathcal{F}}(x_i)]) : x_i \in \mathcal{X}\}$ and $\mathcal{G} = \{(x, [\rho_{\mathcal{G}}(x_i), \tau_{\mathcal{G}}(x_i)]) : x_i \in \mathcal{X}\}$. The Euclidean DiMe $D_{\text{FFS}}(\rho, \tau)$ is defined as

$$D_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = \left(\frac{1}{2n} \sum_{x_i \in X} \left(|\rho_{\mathcal{F}}^3 - \rho_{\mathcal{G}}^3|^2 + |\tau_{\mathcal{F}}^3 - \tau_{\mathcal{G}}^3|^2 + |\theta_{\mathcal{F}}^3 - \theta_{\mathcal{G}}^3|^2 \right) \right)^{1/2}.$$

For $x_i \in X$, take the weight ω_i. The weighted Euclidean DiMe $D_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G})$ is described as

$$D_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = \left(\frac{1}{2} \sum_{x_i \in X} \omega_i \left(|\rho_{\mathcal{F}}^3 - \rho_{\mathcal{G}}^3|^2 + |\tau_{\mathcal{F}}^3 - \tau_{\mathcal{G}}^3|^2 + |\theta_{\mathcal{F}}^3 - \theta_{\mathcal{G}}^3|^2 \right) \right)^{1/2}.$$

Theorem 3.3 *For any two FFSs $\mathcal{F}, \mathcal{G}$, the weighted Euclidean DiMe $D_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G})$ satisfies the [i.]–[iii.] conditions:*

i. $0 \leq D_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) \leq 1$
ii. $D_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = D_{\text{FFS}}^{\omega}(\mathcal{G}, \mathcal{F})$
iii. $D_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = 1$, *if* $\mathcal{F} = \mathcal{G}$, *that is,* $\rho_{\mathcal{F}}(x_i) = \rho_{\mathcal{G}}(x_i)$, $\tau_{\mathcal{F}}(x_i) = \tau_{\mathcal{G}}(x_i)$.

Proof i. Since $0 \leq \rho_{\mathcal{F}}, \rho_{\mathcal{G}}, \tau_{\mathcal{F}}, \tau_{\mathcal{G}}, \theta_{\mathcal{F}}, \theta_{\mathcal{G}} \leq 1$, $0 \leq |\rho_F^3 - \rho_G^3|^2 \leq 1$, $0 \leq |\tau_F^3 - \tau_G^3|^2 \leq 1$ and, $|\theta_F^3 - \theta_G^3|^2 \leq 1$. Hence, $0 \leq D_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) \leq (1/2)^{1/2}(2\sum_{i=1}^{n} \omega_i)^{1/2} = 1$.
ii. From Definition 3.3, it can easily show.
iii. $D_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = 0 \Leftrightarrow |\rho_{\mathcal{F}}^3 - \rho_{\mathcal{G}}^3| = 0$, $|\tau_{\mathcal{F}}^3 - \tau_{\mathcal{G}}^3| = 0$, and, $|\theta_{\mathcal{F}}^3 - \theta_{\mathcal{G}}^3| = 0$ if and only if $\mathcal{F} = \mathcal{G}$. □

Definition 3.4 For any two FFSs $\mathcal{F} = \{(x, [\rho_{\mathcal{F}}(x_i), \tau_{\mathcal{F}}(x_i)]) : x_i \in \mathcal{X}\}$ and $\mathcal{F} = \{(x, [\rho_{\mathcal{G}}(x_i), \tau_{\mathcal{G}}(x_i)]) : x_i \in \mathcal{X}\}$. The new SM $S_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G})$ can be given as

$$S_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = \frac{C_{\text{FFS}}(\mathcal{F}, \mathcal{G}) + 1 - D_{\text{FFS}}(\mathcal{F}, \mathcal{G})}{2}$$

Definition 3.5 For any two FFSs $\mathcal{F} = \{(x, [\rho_{\mathcal{F}}(x_i), \tau_{\mathcal{F}}(x_i)]) : x_i \in \mathcal{X}\}$ and $\mathcal{G} = \{(x, [\rho_{\mathcal{G}}(x_i), \tau_{\mathcal{G}}(x_i)]) : x_i \in X\}$, the WCS measure $S_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G})$ between $\mathcal{F}$ and $\mathcal{G}$ can be defined as

$$S_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = \frac{C_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) + 1 - D_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G})}{2}$$

where ω_i denote the weight of $x_i \in \mathcal{X}$ ($\sum_{i=1}^{n} \omega_i = 1, 0 \leq \omega_i \leq 1$).

Example 3.3 Consider the information in Example 3.2. The weighted similarity measure $S_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = 0.98$. That is when $F \neq G$, $S_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G})$ does not equal 1.

Theorem 3.4 *For the two FFSs F, G, the new WCS measure $S_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G})$ satisfies the [i.]–[iii.] conditions:*

i. $0 \leq S_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) \leq 1$
ii. $S_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = S_{\text{FFS}}^{\omega}(\mathcal{G}, \mathcal{F})$
iii. $S_{\text{FFS}}^{\omega}(\mathcal{F}, \mathcal{G}) = 1$, *if* $\mathcal{F} = \mathcal{G}$, *(*$\rho_{\mathcal{F}}(x_i) = \rho_{\mathcal{G}}(x_i)$, $\tau_{\mathcal{F}}(x_i) = \tau_{\mathcal{G}}(x_i)$*).*

Proof (i) Form Theorem 3.2, $0 \leq C^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) \leq 1$, we know the DiMe $0 \leq D^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) \leq 1$ according to Lemma 2.1. Hence, $0 \leq \frac{C^{\omega}_{\text{FFS}}(\mathcal{F},\mathcal{G})+1-D^{\omega}_{\text{FFS}}(\mathcal{F},\mathcal{G})}{2} \leq 1$, that is, $0 \leq S^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) \leq 1$ obtained.

(ii) Form Theorems 3.2 and 3.3, $C^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = C^{\omega}_{\text{FFS}}(\mathcal{G}\mathcal{F}, \mathcal{F})$ and $D^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = D^{\omega}_{\text{FFS}}(\mathcal{G}, \mathcal{F})$. Hence, $S^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = S^{\omega}_{\text{FFS}}(\mathcal{G}, \mathcal{F})$.

(iii) Take $\mathcal{F} = \mathcal{G}$. Therefore $C^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 1$ and $D^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 0$, then $S^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 1$. Conversely, take $S^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 1$. Then, $C^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) + 1 - D^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 2$, that is, $C^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 1 + D^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G})$. For all $\mathcal{F}, \mathcal{G}$, $0 \leq C^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) \leq 1$ and $0 \leq D^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) \leq 1$ should exist simultaneously, therefore $C^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 1$ and $D^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 0$. From Theorem 3.3, when $D^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 0$, it is already known $\mathcal{F} = \mathcal{G}$. Hence, $S^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{F}) = 1$ iff $\mathcal{F}F = \mathcal{G}$.

□

When the SM satisfies the condition of DiMe, then a corresponding DiMe can be obtained concerning the relationship between the DM and SM. Since the suggested SM $S^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G})$ is a genuine SM, the corresponding DiMe $D^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G})$ between any two FFSs $\mathcal{F}, \mathcal{G}$ is obtained as follows:

Definition 3.6 For the two FFSs $\mathcal{F}, \mathcal{G}$. The weighted distance measure(WDM)

$$\text{DM}^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 1 - S^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = \frac{1 - C^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) + D^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G})}{2}.$$

where ω_i denote the weight of $x_i \in \mathcal{X}$ $(\sum_{i=1}^{n} \omega_i = 1)$.

If take $\omega = (1/n, \ldots, 1/n)$, the DiMe $D_{\text{FFS}}(\mathcal{F}, \mathcal{G})$ is obtained.

Example 3.4 Consider the information in Example 3.1. Hence $\text{DM}^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 0.482$.

Theorem 3.5 *For the two FFSs $\mathcal{F}, \mathcal{G}$,* $\text{DM}^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G})$ *satisfies the following conditions:*

i. $0 \leq \text{DM}^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) \leq 1$
ii. $\text{DM}^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = \text{DM}^{\omega}_{\text{FFS}}(\mathcal{G}, \mathcal{F})$
iii. $\text{DM}^{\omega}_{\text{FFS}}(\mathcal{F}, \mathcal{G}) = 1$, *if* $\mathcal{F} = \mathcal{G}$, *that is,* $\rho_{\mathcal{F}}(x_i) = \rho_{\mathcal{G}}(x_i)$, $\tau_{\mathcal{F}}(x_i) = \tau_{\mathcal{G}}(x_i)$.

The distance measure $D_{\text{FFS}}(\mathcal{F}, \mathcal{G})$ also satisfies the properties of Theorem 3.5.

4 TOPSIS approach

This section is dedicated to developing a TOPSIS technique for MCDM with FFS.

Consider that the experts evaluate the alternatives $U = \{U_1, U_2, \ldots, U_m\}$ according to the criteria $K = \{K_1, K_2, \ldots, K_n\}$, which are represented by FFSs $U_{ij} = (\rho_{ij}, \tau_{ij})$ such that $\rho_{ij}, \tau_{ij} \in [0, 1]$ and $\rho^3_{ij} + \tau^3_{ij} \leq 1$.

Let ω be weight vector of criteria satisfying with $\sum_{j=1}^{n} \omega_j = 1$ and $\omega_j \geq 0$. Then the FF decision matrix(FFDMT) $E = (U_{ij})_{n\times n} = ((\rho_{ij}, \tau_{ij}))_{M\times n}$ is shown as: For $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$,

$$E = \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ U_{21} & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ U_{m1} & U_{m2} & \cdots & U_{mn} \end{pmatrix}$$

where U_{ij} are FFSs.

The algorithm based on the suggested CM is developed as follows:

1: Firstly, we will normalize the decision matrix $E = (U_{ij})_{n\times n} = ((\rho_{ij}, \tau_{ij}))_{m\times n}$. For normalization we will use the following negation operator:

$$\widehat{E} = ((\widehat{\rho}_{ij}, \widehat{\tau}_{ij}))\begin{cases}(\rho_{ij}, \tau_{ij}) & \text{for benefit type} \quad \mathrm{K_j}, \\ (\tau_{ij}, \rho_{ij}) & \text{for cost type} \quad \mathrm{K_j}.\end{cases} \tag{1}$$

This operator is comprehended as follows: If the criterion we are considering is benefit-type, no action is taken. If our criterion is cost-type, we will convert this criterion to benefit-type.

2: We will obtain positive and negative ideal solutions determined with the help of the score and accuracy functions and denoted by $U^+ = \{U_1^+, U_2^+, \ldots, U_n^+\}$, $U^- = \{U_1^-, U_2^-, \ldots, U_n^-\}$: For $j = 1, 2, \ldots, n$,

$$U_j^+ = \max\{\mathrm{SC}(U_{1j}), \mathrm{SC}(U_{2j}), \ldots, \mathrm{SC}(U_{nj})\},$$
$$U_j^- = \min\{\mathrm{SC}(U_{1j}), \mathrm{SC}(U_{2j}), \ldots, \mathrm{SC}(U_{nj})\}.$$

If all score values are equal, we need to use accuracy values. That is, we use accuracy values for comparison.

3: We will compute the separations for each alternative between the obtained U^+ and U^- with the suggested DiMe $\mathrm{DM}^{\omega}_{\mathrm{FFS}}$. The separation measures as follows: For $i = 1, 2, \ldots, m$,

$$\mathrm{DM}^{\omega}_{\mathrm{FFS}}(U_i, U^+) = \sum_{j=1}^{n} \omega_j \mathrm{DM}^{\omega}_{\mathrm{FFS}}(U_{ij}, U^+),$$
$$\mathrm{DM}^{\omega}_{\mathrm{FFS}}(U_i, U^-) = \sum_{j=1}^{n} \omega_j \mathrm{DM}^{\omega}_{\mathrm{FFS}}(U_{ij}, U^-).$$

Based on these measures, the closeness index γ_i connected to the U_i will be as follows:

$$\gamma_i = \frac{\mathrm{DM}^{\omega}_{\mathrm{FFS}}(U_i, U^+)}{\mathrm{DM}^{\omega}_{\mathrm{FFS}}(U_i, U^+) + \mathrm{DM}^{\omega}_{\mathrm{FFS}}(U_i, U^-)}.$$

4: We will rank the alternatives according to their γ_i values. As the γ_i value gets smaller, we will take the alternative U_i with the smallest value of γ_i to choose the best alternative.

Application

The infectious diseases example from Kirisci and Simsek [22] was adapted for this study to represent the application of the suggested method in MCDM.

Let s

$$D = \{\text{Hepatitis}\, C, \text{Crimean} - \text{Congo Hemorrhagic Fever(CCHF)}, \text{influenza} A(H1N1)\}$$
$$= \{U_1, U_2, U_3\}$$

be the set of three alternatives. Alternatives in this cluster were selected as infectious diseases, which are common in Turkey, before COVID-19. The set of criteria $S = \{s_1, s_2, s_3, s_4, s_5\}$. The criteria s_1 is cost type and the other criteria s_2, s_3, s_4, s_5 are benefit type. The corresponding weight vector of the attribute is $\omega = (0.25, 0.20, 0.15, 0.18, 0.22)^T$. The evaluation values are represented by FFNs (Table 1).

Table 1 The FFDMT

	s_1	s_2	s_3	s_4	s_5
U_1	(0.7, 0.4)	(0.8, 0.5)	(0.8, 0.7)	(0.7, 0.5)	(0.9, 0.1)
U_2	(0.7, 0.3)	(0.6, 0.5)	(0.8, 0.4)	(0.5, 0.5)	(0.7, 0.2)
U_3	(0.8, 0.4)	(0.8, 0.6)	(0.9, 0.3)	(0.6, 0.4)	(0.7, 0.4

Table 2 The normalized FFDMT

	s_1	s_2	s_3	s_4	s_5
U_1	(0.4, 0.7)	(0.8, 0.5)	(0.8, 0.7)	(0.7, 0.5)	(0.9, 0.1)
U_2	(0.3, 0.7)	(0.6, 0.5)	(0.8, 0.4)	(0.5, 0.5)	(0.7, 0.2)
U_3	(0.4, 0.8)	(0.8, 0.6)	(0.9, 0.3)	(0.6, 0.4)	(0.7, 0.4)

1: We will normalize the decision matrix $E = (U_{ij})_{n\times n} = ((\rho_{ij}, \tau_{ij}))_{n\times n}$. Transform the FFDMT E into the normalized FFDMT by (1)(Table 2).
2: Now we will find the ideal solutions. These solutions:

$$U^+ = \{(0.4, 0.7), (0.8, 0.5), (0.9, 0.3), (0.7, 0.5), (0.9, 0.1)\},$$
$$U^- = \{(0.3, 0.7), (0.6, 0.5), (0.8, 0.7), (0.5, 0.5), (0.7, 0.4)\}.$$

3: We use the suggested FFDMT $\mathrm{DM}^{\omega}_{\mathrm{FFS}}$ to compute the separation of each alternative between positive ideal and negative ideal solutions. The closeness index γ_i (for all U_i) is computed: $\gamma_1 = 0.687$, $\gamma_2 = 0.631$, $\gamma_3 = 0.704$.
4: For $j = 1, 2, 3$, the γ_j values will help rank the alternatives. Hence, the best alternative is U_2.

The advantages of the suggested method:

(1) Since the main characteristic of FFSs is that the sum of cubes of membership and non-membership value of any object can be less than or equal to 1, then using FFSs, we can cover more elements than that of PFSs and IFSs. In other words, the FFS model is a valuable, practical, and impressive extended form of IFSs and PFSs. In this instance, experts become more autonomous in expressing their views on the level of membership.
(2) The choice of the best alternative from a set of alternatives in an MCDM problem is handicapped when uncertain data are strained to adopt the limited form of IFNs and PFNs. The aforenamed cases would cause the mutilation of data. A more generalized model is required to ensure telling solutions in such crucial cases. FFSs give more correct and exact outcomes when used to cope with practical MCDM problems including FF information as they are an effective extension of IFSs and PFSs.
(3) The measures considered in this study are not limited to CS. It has also been studied with Euclidean DiMes. Working with both measures provides a geometric as well as algebraic point of view in the MCDM problem.

5 Conclusion

This study focuses on solving an MCDM problem in which measures of CS and cosine distance between FFSs are considered. Based on FFS values, CS measure and Euclidean DiMe were defined and their basic properties were examined. Therefore, we established new

SMs between FFSs according to the suggested cosine SM and the Euclidean DiMe, which not only satisfy the condition of SM but also deal with the related decision-making problems from both points of view of geometry and algebra. The usefulness, influence, and versatility of the developed method have been demonstrated in a medical case study.

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Declarations

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