Errata to “On Self-mapping Degrees of S^3 -geometry Manifolds”

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The set of self-mapping degrees of S^3 -geometry 3-manifolds in [2] have some mistakes when the fundamental group of the 3-manifold is D_{4n}^* , O_{48}^* , $D'_{n \cdot 2^q}$ or $T'_{8 \cdot 3^q}$. So we need to make this errata.

The self-mapping degrees of all closed and oriented 3-manifolds are listed in [1] and [5]. The table of spherical case in [5] is mostly quoted from [2]. The results in [5] which do not involve the corrections made here still valid. The results in [5] involving the corrections made here should be also changed.

In [2], the sets of self-mapping degrees in the cases D_{4n}^* , O_{48}^* , $D'_{n \cdot 2^q}$ and $T'_{8 \cdot 3^q}$ were:

$$\begin{aligned} D(S^3/D_{4n}^*) &= \{h^2 | 2 \nmid h \text{ or } h = n \text{ or } h = 0\} + 4n\mathbb{Z}; \\ D(S^3/O_{48}^*) &= \{0, 1, 25\} + 48\mathbb{Z}; \\ D(S^3/D'_{n \cdot 2^q}) &= \{k^2 \cdot [1 - (n)^{2^q-1}]^i \cdot [1 - 2^{(2^p-q)(n-1)}]^{j-1} | q \geq p > 0, i, j, k, p \in \mathbb{Z}\} + n \cdot 2^q\mathbb{Z}; \\ D(S^3/T'_{8 \cdot 3^q}) &= \begin{cases} \{k^2 \cdot (3^{2q-2p} - 3^q) | 3 \nmid k, q \geq p > 0, k, p \in \mathbb{Z}\} + 8 \cdot 3^q\mathbb{Z} & (2|q), \\ \{k^2 \cdot (3^{2q-2p} - 3^{q+1}) | 3 \nmid k, q \geq p > 0, k, p \in \mathbb{Z}\} + 8 \cdot 3^q\mathbb{Z} & (2 \nmid q). \end{cases} \end{aligned}$$

Now they should be changed into:

$$\begin{aligned} D(S^3/D_{4n}^*) &= \begin{cases} \{h^2 | 2 \nmid h\} \cup \{0, 2n\} + 4n\mathbb{Z} & \text{if } 4|n, \\ \{h^2 | 2 \nmid h\} \cup \{0\} + 4n\mathbb{Z} & \text{if } 4 \nmid n; \end{cases} \\ D(S^3/O_{48}^*) &= \{0, 1, 24, 25\} + 48\mathbb{Z}; \\ D(S^3/D'_{n \cdot 2^q}) &= \{k^2 | (k, 2n) = 1 \text{ or } n^{2^{q-2}} | k, k \in \mathbb{Z}\} + n \cdot 2^q\mathbb{Z}; \end{aligned}$$

$$D(S^3/T'_{8 \cdot 3^q}) = \begin{cases} \{k^2 \cdot (3^{2q-2p} - 3^q) | 3 \nmid k, q \geq p > 0, k, p \in \mathbb{Z}\} \\ \cup \{0\} \cup \{h^2 | (h, 6) = 1\} + 8 \cdot 3^q \mathbb{Z} & (2|q), \\ \{k^2 \cdot (3^{2q-2p} - 3^{q+1}) | 3 \nmid k, q \geq p > 0, k, p \in \mathbb{Z}\} \\ \cup \{0\} \cup \{h^2 | (h, 6) = 1\} + 8 \cdot 3^q \mathbb{Z} & (2 \nmid q). \end{cases}$$

The D_{4n}^* and O_{48}^* Cases:

The new sets of D_{4n}^* cases are different from the sets in [2]. For example, when $n = 4$, the set in [2] is $\{0, 1, 9\} + 4n\mathbb{Z}$. The new set is $\{0, 1, 8, 9\} + 4n\mathbb{Z}$.

As in [2], under the presentation $D_{4n}^* = \langle a, b | a^2 = b^n = (ab)^2, a^4 = 1 \rangle$, two of the endomorphisms of D_{4n}^* are:

(I) $\varphi_1 : a \mapsto 1, b \mapsto a^2$ $\ker \varphi_1 = \langle a, b^2 \rangle \cong D_{4 \cdot \frac{n}{2}}^*$, $\text{Im } \varphi_1 \cong \{1, a^2\}$;

(II.A) $\varphi_2 : a \mapsto -1, b \mapsto 1$, $\ker \varphi_2 = \langle b \rangle \cong \mathbb{Z}_{2n}$, $\text{Im } \varphi_2 \cong \{1, a^2\}$.

Under the presentation $O_{48}^* = \langle a, b | a^4 = b^3 = (ab)^2, a^8 = 1 \rangle$, an endomorphisms of O_{48}^* is:

(III) $\phi : a \mapsto a^4, b \mapsto 1$, $\ker \phi \cong T_{24}^*$, $\text{Im } \phi \cong \{1, a^4\}$.

For Case (I) and Case (II.A), the construction of self-mappings from S^3 to S^3 in [2] began with self-mappings on $\mathbb{C}^2 - \{0\}$. The induced self-mapping on S^3 turn out NOT to be covariant with the group action of D_{4n}^* , and therefore can not produce self-mappings on S^3/D_{4n}^* . The calculation of Case (III) of the O_{48}^* case in [2] relies on that of the D_{4n}^* case. Hence it is also not correct.

Now, we use a new method to calculate the mapping degrees in the above three cases. Consider the covering $p : \mathbb{R}P^3 \rightarrow M$, where M is S^3/D_{4n}^* or S^3/O_{48}^* . In the Case (I) and (II.A), the images of the endomorphisms on the fundamental groups are $\{1, a^2\}$. In the Case (III), the image of the endomorphism on the fundamental group is $\{1, a^4\}$. Hence in all cases there are liftings $\tilde{f} : M \rightarrow \mathbb{R}P^3$ such that $p \circ \tilde{f} = f$.

It is obvious that $\deg(p) = |\pi_1(M)|/2$, and $\deg(f) = \deg(\tilde{f}) \cdot \deg(p)$. In order to calculate the mod $|\pi_1(M)|$ congruence of $\deg(f)$, since $|\pi_1(\mathbb{R}P^3)| = 2$, we only need to calculate the mod 2 congruence of $\deg(\tilde{f})$. When $\deg(\tilde{f}) \equiv 0 \pmod{2}$, we have $\deg(f) \equiv 0 \pmod{|\pi_1(M)|}$. When $\deg(\tilde{f}) \equiv 1 \pmod{2}$, we have $\deg(f) \equiv |\pi_1(M)|/2 \pmod{|\pi_1(M)|}$.

[4, Theorem 4.1] gives a criterion: the existence of a degree 1 map from a 3-manifold M to $\mathbb{R}P^3$ is equivalent to the existence of an element c in $H^1(M; \mathbb{Z}_2)$ with $c \cup c \cup c \neq 0$. The cohomology ring structures of all spherical 3-manifolds have been calculated in [6]. There exists such an element c for S^3/O_{48}^* and for S^3/D_{4n}^* when $4|n$, but not for S^3/D_{4n}^* when $4 \nmid n$ ([6, Theorem 2.2]). Hence $|\pi_1(M)|/2$ lies in $D(M)$ when $M = S^3/D_{4n}^*$ or $M = S^3/O_{48}^*$.

The $T'_{8 \cdot 3^q}$ Case:

The new sets of $T'_{8 \cdot 3^q}$ cases are different from the sets in [2]. For example, when $q = 2$, the set in [2] is $\{16, 40, 64\} + 72\mathbb{Z}$. The new set is $\{0, 1, 16, 25, 40, 49, 64\} + 72\mathbb{Z}$.

The solution of congruence equation for endomorphism with non-trivial kernel is correct in [2]:

$$\deg(f) \equiv \begin{cases} k^2 \cdot (3^{2q-2p} - 3^q) & (2|q) \\ k^2 \cdot (3^{2q-2p} - 3^{q+1}) & (2 \nmid q) \end{cases} \pmod{8 \cdot 3^q} \quad (3 \nmid k, q \geq p > 0).$$

All possible endomorphisms include also the trivial endomorphism and automorphisms. The

degrees of mappings inducing automorphisms on the fundamental groups are squares of integers which are co-prime to the order of the fundamental group (see [3]). So the set of all possible mapping degrees should be

$$D(S^3/T'_{8 \cdot 3^q}) = \begin{cases} \{k^2 \cdot (3^{2q-2p} - 3^q) | 3 \nmid k, q \geq p > 0, k, p \in \mathbb{Z}\} \\ \cup \{0\} \cup \{h^2 | (h, 6) = 1\} + 8 \cdot 3^q \mathbb{Z} & (2|q), \\ \{k^2 \cdot (3^{2q-2p} - 3^{q+1}) | 3 \nmid k, q \geq p > 0, k, p \in \mathbb{Z}\} \\ \cup \{0\} \cup \{h^2 | (h, 6) = 1\} + 8 \cdot 3^q \mathbb{Z} & (2 \nmid q). \end{cases}$$

The $D'_{n \cdot 2^q}$ Case:

The new sets of $D'_{n \cdot 2^q}$ cases are different from the sets in [2]. For example, when $n = 3$, $q = 3$, the new set is $\{0, 1, 9, 12\} + 24\mathbb{Z}$.

The endomorphisms of $D'_{n \cdot 2^q}$ with non-trivial kernels should be changed to

$$\begin{aligned} (A) \quad & \varphi_1(u) = 1, \quad \varphi_1(w) = w^{h \cdot 2^r} \quad (2 \nmid h), \\ (B) \quad & \varphi_2(u) = u^m, \quad \varphi_2(w) = w \quad ((m, n) = 1). \end{aligned}$$

The corresponding congruence equations are

$$\begin{aligned} (A) \quad & \begin{cases} \deg(f) \equiv 0 & (\text{mod } n) \\ \deg(f) \equiv (h \cdot 2^r)^2 & (\text{mod } 2^q) \end{cases} \quad (2 \nmid h), \\ (B) \quad & \begin{cases} \deg(f) \equiv m^2 & (\text{mod } n) \\ \deg(f) \equiv 1 & (\text{mod } 2^q) \end{cases} \quad ((m, n) = 1). \end{aligned}$$

These lead to the result that

$$\begin{aligned} (A) \quad & \deg(f) \equiv (h \cdot 2^r)^2 \cdot n^{2^{q-1}} \equiv (h \cdot 2^r \cdot n^{2^{q-2}})^2 (\text{mod } n \cdot 2^q) \quad (2 \nmid h), \\ (B) \quad & \deg(f) \equiv (1 - n^{2^{q-1}}) \cdot m^2 + n^{2^{q-1}} \equiv [(1 - n^{2^{q-1}}) \cdot m + n^{2^{q-1}}]^2 (\text{mod } n \cdot 2^q) \quad ((m, n) = 1). \end{aligned}$$

The degrees of mappings inducing automorphisms on the fundamental groups are squares of integers which are co-prime to the order of the fundamental group (see [3]). Considering the possibility of the composition of the mappings inducing trivial endomorphisms and the mappings inducing nontrivial endomorphisms of the fundamental group, we have

$$\begin{aligned} D(S^3/D'_{n \cdot 2^q}) &= \{0\} \cup \{k^2 | (k, 2n) = 1\} \cup \{k^2 \cdot (h \cdot 2^r \cdot n^{2^{q-2}})^2 | (k, 2n) = 1, 2 \nmid h\} \\ &\cup \{k^2 \cdot [(1 - n^{2^{q-1}}) \cdot m + n^{2^{q-1}}]^2 | (k, 2n) = 1, (m, n) = 1\} \\ &= \{k^2 | (k, 2n) = 1 \text{ or } n^{2^{q-2}} | k\} + n \cdot 2^q \mathbb{Z}. \end{aligned}$$

References

- [1] Derbez, P., Sun, H. B., Wang, S. C.: Finiteness of Mapping Degree Sets for 3-Manifolds. *Acta Mathematica Sinica, English Series*, **27**(5), 807–812 (2011)
- [2] Du, X. M.: On self-mapping degrees of S^3 -geometry manifolds. *Acta Mathematica Sinica, English Series*, **25**(8), 1243–1252 (2009)
- [3] Hayat-Legrand, C., Kudryavtseva, E., Wang, S., et al.: Degrees of Self-Mappings of Seifert Manifolds with Finite Fundamental Groups, *Rend. Istit. Mat. Univ. Trieste Suppl.* 1 Vol. XXXII, 131–147 (2001)
- [4] Hayat-Legrand, C., Wang, S. C.: Heiner Zieschang. Degree-one maps onto Lens spaces. *Pacific Journal of Mathematics*, **176**(1), 19–32 (1996)

- [5] Sun, H. B., Wang, S. C., Wu, J. C., et al.: Self-mapping degrees of 3-manifolds. *Osaka Journal of Mathematics*, **49**(1), 247–269 (2012)
- [6] Tomoda, S., Zvengrowski, P.: Remarks on the cohomology of finite fundamental groups of 3-manifolds. *Geometry and Topology Monographs*, **14**, 519–556 (2008)