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Erratum to "On a Classical Theorem on the Diameter and Minimum Degree of a Graph"

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Abstract The original version of the article was published in [1]. Unfortunately, the original version of this article contains a mistake: in Theorem 6.2 appears that $\beta(n,\Delta) = (n-\Delta+5)/4$ but the correct statement is $\beta(n,\Delta) = (n-\Delta+4)/4$. In this erratum we correct the theorem and give the correct proof.

Keywords Extremal problems on graphs, diameter, minimum degree, maximum degree, Gromov hyperbolicity, hyperbolicity constant, finite graphs

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6 Computation of $\beta(n,\Delta)$

Lemma 6.1 *For all* $\Delta \ge 6$ *and* $n \ge \Delta+1$, *we have* $\beta(n,\Delta) \le (n-\Delta+4)/4$.

Proof Seeking for a contradiction assume that $\beta(n,\Delta) > (n-\Delta+4)/4$. Thus, there exists $G \in \mathcal{J}(n,\Delta)$ such that $\delta(G) > (n-\Delta+4)/4$. By Theorem 3.5, $\delta(G) \ge (n-\Delta+5)/4$. By Theorem 3.6 there exist a geodesic triangle $T = \{x,y,z\}$ with $x,y,z \in J(G)$ and $p \in [xy]$ such that $\delta(G) = d(p,[xz]\cup[yz])$. Since $L(T) \ge 4\delta(G)$, we have $L(T) = n-\Delta+t$ with $t \ge 5$, and $|V(G)\setminus T| = \Delta - t$.

Fix $v \in V(G)$ with $\deg(v) = \Delta$. We consider now several cases:

Case (A) Assume first that $v \notin T$. Since $|V(G)\setminus T| = \Delta - t$ and $v \in V(G)\setminus T$, we have $|N(v)\cap T| \ge t+1$. Define $t_1 = |N(v)\cap[xy]|$ and $t_2 = |N(v)\cap(T\setminus[xy])|$. Thus, $t_1+t_2 \ge t+1$. Since $[xy]$ is a geodesic, we have $0 \le t_1 \le 3$ and, therefore, $t_2 \ge t-2 \ge 3$ and $N(v)\cap(T\setminus[xy]) \ne \emptyset$.

Case (A.1) If $t_1 \ge 2$, then let us choose $\alpha_x,\alpha_y \in V(G)\cap[xy]$ and $\beta_x,\beta_y \in V(G)\cap(T\setminus[xy])$, with

$$\begin{aligned}
d_G(x,\alpha_x) &= \min\{d_G(x,w) \mid w \in N(v)\cap[xy]\},\\
d_G(y,\alpha_y) &= \min\{d_G(y,w) \mid w \in N(v)\cap[xy]\},\\
d_{[xz]\cup[zy]}(x,\beta_x) &= \min\{d_{[xz]\cup[zy]}(x,w) \mid w \in N(v)\cap(T\setminus[xy])\},\\
d_{[xz]\cup[zy]}(y,\beta_y) &= \min\{d_{[xz]\cup[zy]}(y,w) \mid w \in N(v)\cap(T\setminus[xy])\}.
\end{aligned}$$

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We have $L([xy]) \ge d_G(x,\alpha_x)+t_1-1+d_G(\alpha_y,y)$ and $L([xz]\cup[zy]) \ge d_{[xz]\cup[zy]}(x,\beta_x)+t_2-1+d_{[xz]\cup[zy]}(\beta_y,y)$. We can assume that $p\in[\alpha_y y]$, since otherwise the argument is similar (if $p\in[x\alpha_x]$ the argument is symmetric, if $p\in[\alpha_x\alpha_y]$ the argument is similar and simpler). Since there exists $v'\in N(v)\cap(T\setminus[xy])$, we have $L([\alpha_y y]\cup[\alpha_y,v]\cup[v,v'])\ge 2\delta(G)$ and, consequently, $d_G(\alpha_y,y)\ge 2\delta(G)-2$. Thus, $L([xy])\ge d_G(x,\alpha_x)+t_1-1+2\delta(G)-2$. We also have

$$2\delta(G)\le d_G(x,y)\le d_G(x,\beta_x)+d_G(\beta_x,v)+d_G(v,\beta_y)+d_G(\beta_y,y)=d_G(x,\beta_x)+d_G(\beta_y,y)+2,$$
$$L([xz]\cup[zy])\ge 2\delta(G)-2+t_2-1,$$
$$n-\Delta+t=L(T)\ge d_G(x,\alpha_x)+t_1+t_2-6+4\delta(G)\ge d_G(x,\alpha_x)+t-5+n-\Delta+5,$$

which is a contradiction if $d_G(x,\alpha_x)>0$, i.e., $x\notin N(v)$.

If $x\in N(v)$, then $L([xz]\cup[zy])\ge t_2+d_{[xz]\cup[zy]}(\beta_y,y)$, and the previous argument gives

$$n-\Delta+t=L(T)\ge t_1+t_2-5+4\delta(G)\ge t-4+n-\Delta+5,$$

which is a contradiction.

Case (A.2) If $t_1<2$, then $t_2\ge t$. The argument in (A.1) also gives $L([xz]\cup[zy])\ge 2\delta(G)+t_2-3$ in this case. Since $L([xy])\ge 2\delta(G)$, we have

$$n-\Delta+t=L(T)\ge 4\delta(G)-3+t\ge n-\Delta+5-3+t,$$

a contradiction.

Case (B) Now, assume that $v\in T$. Since $|V(G)\setminus T|=\Delta-t$, we have $|N(v)\cap T|\ge t$.

Case (B.1) Assume that $v\in[xy]$. Since $[xy]$ is a geodesic and $v\in[xy]$, we have $|N(v)\cap[xy]|\le 2$ and, therefore, $|N(v)\cap(T\setminus[xy])|\ge t-2$ and $N(v)\cap(T\setminus[xy])\ne\emptyset$. Let us choose $\beta_x,\beta_y\in V(G)\cap(T\setminus[xy])$ as in case (A.1).

We can assume that $p\in[vy]$, since otherwise the argument is similar. We have

$$\begin{aligned}
&2\delta(G)\le d_G(x,y)\le d_G(x,v)+d_G(v,\beta_y)+d_G(\beta_y,y)=d_G(x,v)+d_G(\beta_y,y)+1,\\
&L([xz]\cup[zy])\ge d_G(x,\beta_x)+d_G(\beta_x,\beta_y)+d_G(\beta_y,y)\ge 1/2+d_G(\beta_y,y)+t-3,\\
&L([xy])\ge d_G(x,v)+d_G(v,y)\ge d_G(x,v)+2\delta(G)-1,\\
&n-\Delta+t=L(T)\ge d_G(x,v)+2\delta(G)-1+1/2+d_G(\beta_y,y)+t-3\\
&\qquad\qquad\ge 4\delta(G)+1/2+t-5\ge n-\Delta+t+1/2,
\end{aligned}$$

which is a contradiction.

Case (B.2) Assume that $v\in T\setminus[xy]$. Define $t_1=|N(v)\cap[xy]|$ and $t_2=|N(v)\cap(T\setminus[xy])|$. Thus, $t_1+t_2\ge t$. Since $[xy]$ is a geodesic, we have $0\le t_1\le 3$ and, therefore, $t_2\ge t-3\ge 2$ and $N(v)\cap(T\setminus[xy])\ne\emptyset$.

Case (B.2.1) If $t_1\ge 2$, then let us choose $\alpha_x,\alpha_y\in V(G)\cap[xy]$ and $\beta_x,\beta_y\in V(G)\cap(T\setminus[xy])$ as in case (A.1).

We have $L([xy])\ge d_G(x,\alpha_x)+t_1-1+d_G(\alpha_y,y)$ and $L([xz]\cup[zy])\ge d_{[xz]\cup[zy]}(x,\beta_x)+t_2-1+d_{[xz]\cup[zy]}(\beta_y,y)$. We can assume that $p\in[\alpha_y y]$, since otherwise the argument is similar. Since $L([\alpha_y y]\cup[\alpha_y,v])\ge 2\delta(G)$, we have $d_G(\alpha_y,y)\ge 2\delta(G)-1$. Thus, $L([xy])\ge d_G(x,\alpha_x)+t_1-1+2\delta(G)-1$. We also have

$$2\delta(G)\le d_G(x,y)\le d_G(x,\beta_x)+d_G(\beta_x,v)+d_G(v,\beta_y)+d_G(\beta_y,y)=d_G(x,\beta_x)+d_G(\beta_y,y)+2,$$

$L([xz] \cup [zy]) \geq 2\delta(G) - 2 + t_2 - 1,$

$n - \Delta + t = L(T) \geq d_G(x, \alpha_x) + t_1 + t_2 - 5 + 4\delta(G) \geq d_G(x, \alpha_x) + t - 5 + n - \Delta + 5,$

which is a contradiction if $d_G(x, \alpha_x) > 0$, i.e., $x \notin N(v)$.

If $x \in N(v)$, then $L([xz] \cup [zy]) \geq t_2 + d_{[xz]\cup[zy]}(\beta_y, y)$, and the previous argument gives

$$n - \Delta + t = L(T) \geq t_1 + t_2 - 4 + 4\delta(G) \geq t - 4 + n - \Delta + 5,$$

a contradiction.

Case (B.2.2) If $t_1 < 2$, then $t_2 \geq t - 1$. The argument in (B.2.1) also gives $L([xz] \cup [zy]) \geq 2\delta(G) + t_2 - 3$ in this case. Since $L([xy]) \geq 2\delta(G)$, we have

$$n - \Delta + t = L(T) \geq 4\delta(G) + t - 4 \geq n - \Delta + 5 + t - 4,$$

which is a contradiction.

Hence, $\beta(n, \Delta) \leq (n - \Delta + 4)/4$. □

Theorem 6.2 *Consider any* $1 \leq \Delta \leq n - 1$.

- *If* $\Delta = 1$, *then* $n = 2$ *and* $\beta(2, 1) = 0$.
- *If* $2 \leq \Delta \leq 4$, *then* $\beta(n, \Delta) = n/4$.
- *If* $\Delta \geq 5$, *then* $\beta(n, \Delta) = (n - \Delta + 4)/4$.

Proof For every n and Δ, Theorem 4.7 gives $\beta(n, \Delta) \leq n/4$.

If $\Delta = 1$ and $G \in \mathcal{J}(n, 1)$, then G is isomorphic to the path graph P_2. Thus, $n = 2$ and $\beta(2, 1) = 0$.

If $\Delta = 2$, then every graph $G \in \mathcal{J}(n, 2)$ is isomorphic to either the path graph P_n (if $\delta = 1$) or the cycle graph C_n (if $\delta = 2$). Since $\delta(C_n) = n/4$, we conclude $\beta(n, \Delta) = n/4$.

If $\Delta = 3$ or $\Delta = 4$, then [43, Proposition 29 and Theorem 30] provide graphs $G_{n,\Delta} \in \mathcal{J}(n, \Delta)$ with $\delta(G_{n,\Delta}) = n/4$, which implies $\beta(n, \Delta) = n/4$.

Assume $\Delta = 5$ (thus $n \geq 6$). Note that [43, Proposition 29 and Theorem 30] give $\beta(n, 5) < n/4$, and Theorem 3.5 gives $\beta(n, 5) \leq (n - 1)/4$. Since $\beta(n, 4) = n/4$ for every $n \geq 5$, there exists a graph $F_n \in \mathcal{J}(n - 1, 4)$ with $\delta(F_n) = (n - 1)/4$ and $w \in V(F_n)$ such that $\deg w = 4$ for each $n \geq 6$. Consider a graph Γ isomorphic to P_2 and fix a vertex $v \in V(\Gamma)$. Identify v and w in a single vertex v^*. We obtain in this way a graph $G_n \in \mathcal{J}(n, 5)$ from F_n and Γ, since $\Delta = \deg v^* = 4 + 1$. Furthermore, $\{F_n, \Gamma\}$ is the biconnected decomposition of G_n and Theorem 3.1 gives $\delta(G_n) = \delta(F_n) = (n - 1)/4$. Therefore, $\beta(n, 5) \geq \delta(G_n) = (n - 1)/4$, and we conclude $\beta(n, 5) = (n - 1)/4$.

Assume now $\Delta \geq 6$. Since $n - \Delta \geq 1$ we can consider a graph G_1 isomorphic to the cycle graph $C_{n-\Delta+5}$. Consider two points $x, y \in G_1$, with $x \in V(G_1)$ and $d_{G_1}(x, y) = (n - \Delta + 4)/2$. Denote by Γ_1, Γ_2 the geodesics in G_1 joining x and y with $G_1 = \Gamma_1 \cup \Gamma_2$. Denote by v_i^j the vertex in Γ_i with $d_{G_1}(v_i^j, x) = j$, for $i = 1, 2$ and $1 \leq j \leq (n - \Delta + 5)/2$. Note that $(n - \Delta + 5)/2 \geq 3$.

Consider a graph G_2 isomorphic to the star graph with $\Delta + 1$ vertices $S_\Delta \in \mathcal{J}(\Delta + 1, \Delta)$. Denote by $v^* \in V(G_2)$ the vertex of maximum degree in G_2, that is, $\deg v^* = \Delta$. Since $\Delta \geq 6$, we can choose vertices $w_j \in V(G_2) \setminus \{v^*\}$ $(j = 1, \dots, 6)$.

Identify x and w_6 in a single vertex w^*. For each $1 \leq j \leq 2$, identify $v_1^j \in V(\Gamma_1)$ and $w_j \in V(G_2)$ in a single vertex v_j^*, and for each $1 \leq j \leq 3$, identify $v_2^j \in V(\Gamma_2)$ and $w_{j+2} \in V(G_2)$

in a single vertex v^*_{j+2}. We obtain in this way a graph $G \in \mathcal{J}(n, \Delta)$ from G_1 and G_2, since $|V(G)| = n - \Delta + 5 + \Delta + 1 - 6 = n$ and $\Delta = \deg v^*$.

Consider the geodesic triangle $T = \{x, y, z\}$ in G with $z = v^*_4$, $\Gamma_1 = [xy]$ and $\Gamma_2 = [xz] \cup [zy]$. If we consider the midpoint p of Γ_1, then

$$\beta(n, \Delta) \geq \delta(G) \geq d_G(p, \Gamma_2) = d_G(p, \{x, y\}) = \frac{1}{2} L(\Gamma_1) = \frac{n - \Delta + 4}{4}.$$

Since Lemma 6.1 implies $\beta(n, \Delta) \leq (n - \Delta + 4)/4$, we conclude $\beta(n, \Delta) = (n - \Delta + 4)/4$. □

References

[1] Hernández, V., Pestana, D., Rodríguez, J.: On a classical theorem on the diameter and minimum degree of a graph. *Acta Mathematica Sinica, English Series*, **33**(11), 1477–1503 (2017)