



# Matroid-based TSP rounding for half-integral solutions

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Received: 11 July 2022 / Accepted: 19 January 2024  
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## Abstract

We show how to round any half-integral solution to the subtour-elimination relaxation for the TSP, while losing a less-than-1.5 factor. Such a rounding algorithm was recently given by Karlin, Klein, and Oveis Gharan based on sampling from max-entropy distributions. We build on an approach of Haddadan and Newman to show how sampling from the matroid intersection polytope, combined with a novel use of max-entropy sampling, can give better guarantees.

## 1 Introduction

The (symmetric) traveling salesman problem asks: given an graph  $G = (V, E)$  with edge-lengths  $c_e \geq 0$ , find the shortest tour that visits all vertices at least once. The Christofides-Serdyukov algorithm [4, 13] gives a  $3/2$ -approximation to this APX-hard problem; this was recently improved to a  $(3/2 - \varepsilon)$ -approximation by the breakthrough work of Karlin, Klein, and Oveis Gharan, where  $\varepsilon > 0$  [10]. A related question is: *what is the integrality gap of the subtour-elimination polytope relaxation for the TSP?* Wolsey had adapted the Christofides-Serdyukov analysis to show an upper bound of  $3/2$  [16] (also [14]), and there exists a lower bound of  $4/3$ . Building on their above-mentioned work, Karlin, Klein, and Oveis Gharan gave an integrality gap of  $1.5 - \varepsilon'$  for another small constant  $\varepsilon' > 0$  [11], thereby making the first progress towards the conjectured optimal value of  $4/3$  in nearly half a century.

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Most of this work was done when the author was a postdoc at NYU and supported in part by the Simons Collaboration on Algorithms and Geometry.

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Both these recent results are based on a randomized version of the Christofides-Serdyukov algorithm proposed by Oveis Gharan, Saberi, and Singh [12]. This algorithm first samples a spanning tree (plus perhaps one edge) from the *max-entropy distribution* with marginals matching the LP solution, and adds an  $O$ -join on the odd-degree vertices  $O$  in it, thereby getting an Eulerian spanning subgraph. Since the first step has expected cost equal to that of the LP solution, these works then bound the cost of this  $O$ -join by strictly less than half the optimal value, or the LP value. The proof uses a cactus-like decomposition of the near min-cuts of the graph with respect to the values  $x_e$ , like in [12].

Given the  $3/2$  barrier has been broken, we can ask: what other techniques can be effective here? How can we make further progress? These questions are interesting even for cases where the LP has additional structure. The half-integral cases (i.e., points for which  $x_e \in \{0, 1/2, 1\}$  for all  $e$ ) are particularly interesting due to the Schalekamp, Williamson, and van Zuylen conjecture, which says that the integrality gap is achieved on instances where the LP has optimal half-integral solutions [15]. The team of Karlin, Klein, and Oveis Gharan first used their max-entropy approach to get an integrality gap of 1.49993 for half-integral LP solutions [9], before they moved on to the general case in [10] and obtained an integrality gap of  $1.5 - \varepsilon$ ; the latter improvement is considerably smaller than in the half-integral case. It is natural to ask: can we do better for half-integral instances?

In this paper, we answer this question affirmatively. We show how to get tours of expected cost at most 1.4983 times the linear program value using an algorithm based on matroid intersection. Moreover, some of these ideas can be used to strengthen the max-entropy sampling approach in the half-integral case. The matroid intersection approach and the strengthened max-entropy approach each yield improvements over the bound in [9]. Combining the techniques gives our final quantitative improvement:

**Theorem 1.1** *Let  $x$  be a half-integral solution to the subtour elimination polytope with cost  $c(x)$ . There is a randomized algorithm that rounds  $x$  to an integral solution whose cost is at most  $(1.5 - \varepsilon) \cdot c(x)$ , where  $\varepsilon = 0.001695$ .*

We view our work as showing a proof-of-concept of the efficacy of combinatorial techniques (matroid intersection, and flow-based charging arguments) in getting an improvement for the half-integral case. We hope that these techniques, ideally combined with max-entropy sampling techniques, can give further progress on this central problem. Our randomized algorithm does run in polynomial time, if we allow exponentially small error in the marginal  $x$ .

**Our Techniques** The algorithm is again in the Christofides-Serdyukov framework. It is easiest to explain for the case where the graph (a) has an even number of vertices, and (b) has no (non-trivial) proper min-cuts with respect to the LP solution values  $x_e$ —specifically, the only sets for which  $x(\partial S) = 2$  correspond to the singleton cuts. Here, our goal is that each edge is “even” with some probability: i.e., both of its endpoints have even degree with probability  $p > 0$ . In this case we use an idea due to Haddadan and Newman [7]: we *shift* and get a  $\{1/3, 1\}$ -valued solution  $y$  to the subtour elimination polytope  $K_{TSP}$ . Specifically, we find a random perfect matching  $M$  in the

support of  $x$ , and set  $y_e = 1$  for  $e \in M$ , and  $1/3$  otherwise, thereby ensuring  $\mathbb{E}[y] = x$ . To pick a random tree from this shifted distribution  $y$ , we do one of the following:

1. We pick a random “independent” set  $M'$  of matching edges (so that no edge in  $E$  is incident to two edges of  $M'$ ). For each  $e' \in M'$ , we place partition matroid constraints enforcing that exactly one edge is picked at each endpoint—which, along with  $e'$  itself, gives degree 2 and thereby makes the edge even as desired. Finding spanning trees subject to another matroid constraint can be implemented using matroid intersection. (We refer the interested reader to [3] for a similar application of the matroid intersection technique.)
2. Or, instead we sample a random spanning tree from the max-entropy distribution, with marginals being the shifted value  $y$ . (In contrast, [9] sample trees from  $x$  itself; our shifting allows us to get stronger notions of evenness than they do: e.g., we can show that every edge is “even-at-last” with constant probability, as opposed to having at least one even-at-last edge in each tight cut with some probability.)

(Our algorithm randomizes between the two samplers to achieve the best guarantees.) For the  $O$ -join step, it suffices to give fractional values  $z_e$  to edges so that for every odd cut in  $T$ , the  $z$ -mass leaving the cut is at least 1. In the special case we consider, each edge only participates in two min-cuts—those corresponding to its two endpoints. So set  $z_e = x_e/3$  if  $e$  is even, and  $x_e/2$  if not; the only cuts with  $z(\partial S) < 1$  are minimum cuts, and these cuts will not show up as  $O$ -join constraints, due to evenness. For this setting, if an edge is even with probability  $p$ , we get a  $(3/2 - p/6)$ -approximation!

It remains to get rid of the two simplifying assumptions. To sample trees when  $|V|$  is odd (an open question from [7]), we add a new vertex to fix the parity, and perform local surgery on the solution to get a new TSP solution and reduce to the even case. The challenge here is to show that the losses incurred are small, and hence each edge is still even with constant probability.

Finally, what if there are proper tight sets  $S$ , i.e., where  $x(\partial S) = 2$ ? We use the cactus decomposition of a graph (also used in [9, 12]) to sample spanning trees from pieces of  $G$  with no proper min-cuts, and union these trees together. These pieces are formed by contracting sets of vertices in  $G$ , and have a hierarchical structure. Moreover, each such piece is either of the form above (a graph with no proper min-cuts) for which we have already seen samplers, or else it is a double-edged cycle (which is easily sampled from). Since each edge may now lie in many min-cuts, we no longer just want an edge to have both endpoints be even. Instead, we use an idea from [9] that uses the hierarchical structure on the pieces considered above. Every edge of the graph is “settled” at exactly one of these pieces, and we ask for both of its endpoints to have even degree *in the piece at which it is settled*. The  $z_e$  value of such an edge may be lowered from an initial value of  $x_e/2$  in the  $O$ -join without affecting constraints corresponding to cuts *in the piece at which it is settled*.

Since cuts at other levels of the hierarchy may now be deficient because of the lower values of  $z_e$ , we may need to increase the  $z_f$  values for other “lower” edges  $f$  to satisfy these deficient cuts. This last part requires a charging argument, showing that each edge  $e$  has  $z_e$  that is strictly smaller than  $x_e/2$  in expectation. For our samplers, the naïve approach of distributing charge uniformly as in [9] does not work, so we instead formulate this charging as a flow problem.

## 2 Notation and preliminaries

Given a multigraph  $G = (V, E)$ , and a set  $S \subseteq V$ , let  $\partial S$  denote the cut consisting of the edges connecting  $S$  to  $V \setminus S$ ;  $S$  and  $\bar{S} := V \setminus S$  are called *shores* of the cut. (For a singleton set  $\{v\}$ , we write  $\partial v$  instead of  $\partial\{v\}$ .) A subset  $S \subseteq V$  is *proper* if  $1 < |S| < |V| - 1$ ; a cut  $\partial S$  is called *proper* if the set  $S$  is a proper subset. A set  $S$  is *tight* if  $|\partial S|$  equals the size of the minimum edge-cut in  $G$ . Two sets  $S$  and  $S'$  are *crossing* if  $S \cap S'$ ,  $S \setminus S'$ ,  $S' \setminus S$ , and  $V \setminus (S \cup S')$  are all non-empty.

Define the *subtour elimination polytope*  $K_{TSP}(G) \subseteq \mathbb{R}^{|E|}$ :

$$\begin{aligned} x(\partial v) &= 2 & \forall v \in V & \quad (\text{LP-TSP}) \\ x(\partial S) &\geq 2 & \forall \text{ proper } S \\ x &\geq 0. \end{aligned}$$

Let  $x$  be half-integral and feasible for (LP-TSP). W.l.o.g. we can focus on solutions with  $x_e = 1/2$  for each  $e \in E$ , doubling edges if necessary. The support graph  $G$  is then a 4-regular 4-edge-connected (henceforth 4EC) multigraph.

The *spanning tree polytope*  $K_{spT}(G) \subseteq \mathbb{R}^{|E(G)|}$  for a multigraph  $G$  is:

$$\begin{aligned} x(E(S)) &\leq |S| - 1 & \forall S \subseteq V(G) & \quad (\text{LP-spT}) \\ x(E(V(G))) &= |V(G)| - 1 \\ x &\geq 0. \end{aligned}$$

where  $E(S)$  is the set of edges with both endpoints in  $S$ . This is the graphic matroid polytope, and the convex hull of the spanning trees.

**Definition 2.1** (*r-Trees*) Given a multigraph  $G$  and “root” vertex  $r \in V(G)$ , an *r-tree*  $T$  is a connected subgraph with  $n$  edges: the vertex  $r$  has degree exactly 2, and the subgraph restricted to the other vertices  $V(G) \setminus \{r\}$  is a spanning tree on them. This is a matroid, though we do not use this fact.

The (integral) *perfect matching polytope*  $K_{PM} \subseteq \mathbb{R}^{|E(G)|}$  is defined as follows:

$$\begin{aligned} x(\partial v) &= 1 & \forall v \in V & \quad (\text{LP-PM}) \\ x(\partial S) &\geq 1 & \forall S \text{ with } |S| \text{ odd} \\ x &\geq 0. \end{aligned}$$

The (integral) *O-join dominator polytope*  $K_{join}(G, O)$  is defined as follows. Let  $O \subseteq V$ ,  $|O|$  even.

$$\begin{aligned} z(\partial(S)) &\geq 1 & \forall S \subseteq V, |S \cap O| \text{ odd} \\ z &\geq 0 \end{aligned}$$

**Fact 2.2** For any solution  $x \in K_{TSP}(G)$ , it holds that  $x|_{E(V(G) \setminus \{r\})} \in K_{spT}(G[V(G) \setminus \{r\}])$ ,  $x/2 \in K_{PM}(G)$  (when  $|V(G)|$  is even), and  $x/2 \in K_{join}(G, O)$  for  $O \subseteq V(G)$ ,  $|O|$  even.

**Lemma 2.3** Consider a sub-partition  $\mathcal{P} = \{P_1, P_2, \dots, P_t\}$  of the edge set of  $G$ . Let  $x$  be a fractional solution to (LP-spT) that satisfies  $x(P_i) \leq 1$  for all  $i \in [t]$ . Then we can efficiently sample from a probability distribution  $\mathcal{D}$  over spanning trees which contain at most one edge from each of the parts  $P_i$ , such that  $\Pr_{T \leftarrow \mathcal{D}}[e \in T] = x_e$ .

**Proof** This follows immediately from the integrality of the matroid intersection polytope and Carathéodory's Theorem. (See, for e.g., [5], for how to do this sampling in poly-time.)  $\square$

## 2.1 The max-entropy distribution over spanning trees

**Definition 2.4** (Strong Rayleigh Distributions, [2]) Let  $\mu$  be a probability distribution on spanning trees. The generating polynomial for  $\mu$  is  $p(z) = \sum_{T \in \mathcal{T}} \mathbb{P}(e \in T) \prod_{e \in T} z_e$ . We say  $\mu$  is *strongly Rayleigh* (SR) if  $p$  is a real stable polynomial (i.e.,  $p(z) \neq 0$  if  $\text{Im}(z_e) > 0$  for all  $e \in T$ , i.e., if  $z$  lies in the upper half plane).

A distribution  $\mu$  over spanning trees is called  $\lambda$ -uniform or *weighted uniform* if there exist non-negative weights  $\lambda : E \rightarrow \mathbb{R}$  such that  $\mathbb{P}(T) \propto \prod_{e \in T} \lambda(e)$ . Borcea et al. [2] showed that  $\lambda$ -uniform spanning tree distributions are SR.

**Theorem 2.5** ([1], Theorem 5.1) There exists a  $\lambda$ -uniform distribution  $\mu$  over spanning trees such that given  $z$  in the relative interior of the spanning tree polytope of  $G = (V, E)$ ,

$$\sum_{T \ni e} \mathbb{P}_\mu(T) = z_e.$$

The max-entropy distribution, which is the distribution  $\mu$  on spanning trees maximizing the entropy of  $\mu$  subject to preserving the marginals  $z_e$  (that is,  $\mathbb{P}_\mu(e \in T) = z_e$ ), is one such distribution.

Asadpour et al [1] showed that we can find weights  $\tilde{\lambda} : E \rightarrow \mathbb{R}^+$  which approximately respect the marginals given by a vector  $z$  in the spanning tree polytope in polynomial time.

**Theorem 2.6** ([1]) Given  $z$  in the relative interior of the spanning tree polytope of  $G = (V, E)$  and some  $\varepsilon > 0$ , values  $\tilde{\lambda}_e$  for  $e \in E$  can be found such that, for all  $e \in E$ , the  $\tilde{\lambda}$ -uniform distribution  $\mu$  satisfies

$$\sum_{T \ni e} \mathbb{P}_\mu(T) \leq (1 + \varepsilon)z_e.$$

The running time is polynomial in  $|V|$ ,  $\log 1/\min_e z_e$  and  $\log 1/\varepsilon$ .

**Fact 2.7** ([2]) Let  $\mu$  be an SR distribution. Let  $F \subset E$  be a subset of edges. Then the projection of  $\mu$  onto  $F$ , i.e.,  $\mu|_F(A) = \sum_{S: S \cap F = A} \mu(S)$ , is also SR. Moreover, for any edge  $e \in E$ , conditioning on  $e \in T$  or on  $e \notin T$  preserves the SR property.

**Theorem 2.8** (Negative Correlation, [12]) *Let  $\mu$  be an SR distribution on spanning trees.*

1. *Let  $S$  be a set of edges and  $X_S = |S \cap T|$ , where  $T \sim \mu$ . Then,  $X_S \sim \sum_{i=1}^{|S|} Y_i$ , where the  $Y_i$  are independent Bernoulli random variables with success probabilities  $p_i$  and  $\sum_i p_i = \mathbb{E}[X_S]$ .*
2. *For any set of edges  $S$  and  $e \notin S$ ,*
  - (i)  $\mathbb{E}_\mu[X_S] \leq \mathbb{E}_\mu[X_S \mid X_e = 0] \leq \mathbb{E}_\mu[X_S] + \mathbb{P}_\mu(e \in T)$ , and
  - (ii)  $\mathbb{E}_\mu[X_S] - 1 + \mathbb{P}_\mu(e \in T) \leq \mathbb{E}_\mu[X_S \mid X_e = 1] \leq \mathbb{E}_\mu[X_S]$ .

**Theorem 2.9** ([8], Corollary 2.1) *Let  $g : \{1, \dots, m\} \rightarrow \mathbb{R}$  and let  $0 \leq p \leq m$ . Let  $B_1, \dots, B_m$  be Bernoulli random variables with probabilities  $p_1^*, \dots, p_m^*$  that maximize (or minimize)  $\mathbb{E}[g(B_1 + \dots + B_m)]$  over all possible success probabilities  $p_i$  for  $B_i$  for which  $p_1 + \dots + p_m = p$ . Then  $\{p_1^*, \dots, p_m^*\} \in \{0, x, 1\}$  for some  $x \in (0, 1)$ .*

### 3 Samplers

In this section, we describe the MAXENT and MATINT samplers for graphs with an even number of vertices that contain no proper min-cuts. We give bounds on certain correlations between edges that will be used in Sect. 5 to prove that every edge is “even” with constant probability. The samplers for the case where the graph has an odd number of vertices are more technical and are deferred to Appendix B.

Suppose the graph  $H = (V, E)$  is 4-regular and 4-edge-connected (4EC), contains at least four vertices, and has no proper min-cuts.<sup>1</sup> This means all proper cuts have six or more edges. We are given a dedicated *external* vertex  $r \in V(H)$ ; the vertices  $I := V \setminus \{r\}$  that are not external are called *internal*. (In future sections, this vertex  $r$  will be given by a cut hierarchy.) Call the edges in  $\partial r$  *external edges*; all other edges are *internal*. An internal vertex is called a *boundary vertex* if it is adjacent to  $r$ . An edge is said to be *special* if both of its endpoints are non-boundary vertices.

We show two ways to sample a spanning tree on  $H[I]$ , the graph induced on the internal vertices, being faithful to the marginals  $x_e$ , i.e.,  $\mathbb{P}_T(e \in T) = x_e$  for all  $e \in E(H) \setminus \partial r$ . Moreover, we want that for each internal edge, both its endpoints have even degree in  $T$  with constant probability. This property will allow us to lower the cost of the  $O$ -join in Sect. 6. While both samplers will satisfy this property, each will do better in certain cases. The MATINT sampler targets special edges; it allows us to “hand-pick” edges of this form and enforce that both endpoints of such edges have degree 2 in the tree. We enforce this property via partition matroid constraints, so the conclusion that the endpoints have degree 2 will be immediate. The MAXENT sampler, on the other hand, relies on maximizing the randomness of the spanning tree sampled (subject to being faithful to the marginals); negative correlation properties allow us to obtain the evenness property, and in particular, better probabilities than MATINT for non-special edges, and a worse probability for the special edges.

<sup>1</sup> This implies that  $H$  is a simple graph: since parallel edges between  $u, v$  means that  $\partial(\{u, v\})$  is a proper min-cut; we use this simplicity of the graph often in the arguments of this section.

Our samplers will depend on the parity of  $|V|$ : when  $|V|$  is even, the MATINT sampler is the one given by [7, Theorem 13], which we describe in Sect. 3.1. They left the case of odd  $|V|$  as an open problem, and in Appendix B we extend their procedure to the odd case.

### 3.1 Samplers for Even $|V(H)|$

Since  $H$  is 4-regular and 4EC and  $|V(H)|$  is even, setting a value of  $1/4 = x_e/2$  on each edge gives a solution to the perfect matching polytope  $K_{PM}(H)$  by Fact 2.2.

1. Sample a perfect matching  $M$  such that  $\mathbb{P}(e \in M) = 1/4 = x_e/2$  for all  $e \in E(H)$ .
2. (Shift) Define a new fractional solution  $y$  (that depends on  $M$ ): set  $y_e = 1$  for  $e \in M$ , and  $y_e = 1/3$  otherwise. We have  $y \in K_{TSP}(H)$ , and hence  $y|_I \in K_{spT}(H[I])$  by Fact 2.2: indeed, each vertex has  $y(\partial v) = 1 + 3 \cdot 1/3 = 2$  because  $M$  is a perfect matching. Moreover, every proper cut  $U$  in  $H$  has at least six edges, so  $y(\partial U) \geq |\partial U| \cdot 1/3 \geq 2$ . Furthermore,

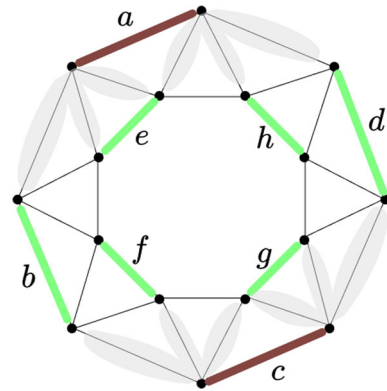
$$\mathbb{E}_M[y_e] = 1/4 \cdot 1 + 3/4 \cdot 1/3 = 1/2 = x_e. \quad (1)$$

3. Sample a spanning tree faithful to the marginals  $y$ , using one of two samplers:

- (a) MAXENT Sampler: Sample from the max-entropy distribution on spanning trees with marginals  $y$ . (Since  $y$  may not be in the relative interior of the spanning tree polytope, contract the 1-valued edges to obtain a 6-regular,  $1/3$ -uniform solution. This may have nontrivial min-cuts, so once again use a cactus hierarchy to decompose the graph into pieces (see Sect. 4.2); the induced solution on each piece is in the relative interior of the spanning tree polytope. For each piece, sample a  $\lambda$ -uniform spanning tree that preserves marginals, and then take the union of these trees.)
- (b) MATINT Sampler:
  - i. Color the edges of  $M$  using 7 colors such that no edge of  $H$  is adjacent to two edges of  $M$  having the same color; e.g., by greedily 7-coloring the 6-regular graph  $H/M$  (Fig. 1). Let  $M'$  be one of these color classes picked uniformly at random. Hence,  $\mathbb{P}(e \in M') = 1/28$ , and  $\mathbb{P}(\partial v \cap M' \neq \emptyset) = 1/7$ .
  - ii. For each edge  $e = uv \in M'$ , let  $L_{uv}$  and  $R_{uv}$  be the sets of edges incident at  $u$  and  $v$  other than  $e$ . Note that  $|L_{uv}| = |R_{uv}| = 3$ . Place partition matroid constraints  $y(L_{uv}) \leq 1$  and  $y(R_{uv}) \leq 1$  on each of these sets. Finally, restrict the partition constraints to the internal edges of  $H$ ; this means some of these constraints are no longer tight for the solution  $y$ .
- (c) Given the sub-matching  $M' \subseteq M$ , and the partition matroid  $\mathcal{M}$  on the internal edges defined using  $M'$ , use Lemma 2.3 to sample a tree on  $H[I]$  (i.e., on the internal vertices and edges of  $H$ ) with marginals  $y_e$ , subject to this partition matroid  $\mathcal{M}$ . (Recall this is valid as  $y|_I \in K_{spT}(H[I])$ .)

Conditioned on the matching  $M$ , we have  $\mathbb{P}(e \in T \mid M) = y_e$ ; now using (1), we have  $\mathbb{P}(e \in T) = x_e$  for all  $e \in (E \setminus \partial r)$ .

**Fig. 1** The matching  $M$  consists of all highlighted edges (both brown and green), one possible choice of  $M'$  has edges  $\{a, c\}$  highlighted in brown, and the constraints are placed on the edges adjacent to those in  $M'$  (marked in gray) (color figure online)



**Remark 3.1** The issue when  $|V(H)|$  is odd is that  $x/2$  is not a point in the perfect matching polytope. To fix this, we split the external vertex  $r$  into two vertices  $r_1, r_2$ , but this action introduces a proper min-cut. So, we can no longer directly sample a spanning tree on the interval vertices  $I$ . To fix this, we make careful local changes to the fractional solution  $y$  to get a new solution  $\hat{y}|_I$  that has the correct marginals and is still in the spanning tree polytope. Finally, we show that, with some small loss in constants, we can reduce the analysis to that of when  $|V(H)|$  is even. See Appendix B.

### 3.2 Correlation properties of samplers

Let  $T$  be a tree sampled using either the MATINT or the MAXENT sampler. The following claims will be used to prove the evenness property in Sect. 5. Each table gives lower bounds on the corresponding probabilities for each sampler. The proofs for  $|V(H)|$  odd are in Appendix B.

**Lemma 3.2** *If  $f, g$  are internal edges incident to a vertex  $v$ , then*

Probability statement	MATINT	MAXENT
$\mathbb{P}( T \cap \{f, g\}  = 2)$	$1/9$	$1/9$
$\mathbb{P}(T \cap \{f, g\} = \{f\})$	$1/9$	$12/72$

**Lemma 3.3** *If edges  $e, f, g, h$  incident to a vertex  $v$  are all internal, then*

**Lemma 3.4** *For an internal edge  $e = uv$ :*

(a) *if both endpoints are non-boundary vertices, then*



Probability statement	MATINT	MAXENT
$\mathbb{P}( T \cap \{e, f, g, h\}  = 2)$	$2/21$	$8/27$
$\mathbb{P}( T \cap \{e, f\}  =  T \cap \{g, h\}  = 1)$	$4/63$	$16/81$

Probability statement	MATINT	MAXENT
$\mathbb{P}( \partial_T(u)  =  \partial_T(v)  = 2)$	$1/36$	$128/6561$

(b) if both  $u, v$  are boundary vertices, then

Probability statement	MATINT	MAXENT
$\mathbb{P}(\text{exactly one of } u, v \text{ has odd degree in } T)$	$1/9$	$5/18$

### 3.2.1 Correlation properties: $|V(H)|$ even

We now prove the correlation properties for the even case: the numerical bounds for the even case are better than those claimed above (which will be dictated by the proofs of the odd case; see Appendix B).

**Proof of Lemma 3.2, Even Case** To prove  $\mathbb{P}(|T \cap \{f, g\}| = 2) \geq 1/9$ , we need only knowledge of the marginals and not the specific sampler. If one of  $f, g$  lies in  $M$  (which happens w.p.  $1/2$ ), then its  $y$ -value equals 1 and it belongs to  $T$  w.p. 1, and the other edge is chosen w.p.  $1/3$ , making the unconditional probability  $1/2 \cdot 1/3 = 1/6 \geq 1/9$ . Similarly, conditioned on  $f$  lying in  $M$  and hence belonging to  $T$ , edge  $g$  is not chosen w.p.  $1 - y_e = 2/3$ , so  $\mathbb{P}(T \cap \{f, g\} = \{f\}) \geq 1/4 \cdot 2/3 = 1/6 \geq 1/9$ .

*The MAXENT claim:* It remains to show that  $\mathbb{P}(T \cap \{f, g\} = \{f\}) \geq 12/72$ . Conditioned on  $f \in M$ , we have  $g \notin T$  w.p.  $2/3$ . Now condition on neither  $f$  nor  $g$  in  $M$  (happens w.p.  $1/2$ ). By Theorem 2.8,

$$1/3 \leq \mathbb{E}[f \in T \mid g \notin T]$$

By Theorem 2.9,  $\mathbb{P}(f \in T \mid g \notin T) \geq 1/3$ , so  $\mathbb{P}(f \in T \wedge g \notin T) \geq 1/3 \cdot 2/3 = 2/9$ . Putting all of this together, we get

$$\mathbb{P}(f \in T \wedge g \notin T) \geq 1/4 \cdot 2/3 + 1/2 \cdot 2/9 = 5/18 \geq 12/72.$$

□

**Proof of Lemma 3.3, Even Case** *The MATINT claims:* Each perfect matching  $M$  contains one of these four edges in  $\partial v$ . Say that edge is  $e$ . If  $e$  also belongs to  $M'$  (w.p.  $1/7$ ),

then  $T$  contains exactly one of  $\{f, g, h\}$ . Since  $1/7 \geq 2/21$ , this gives us the first bound in Lemma 3.3. Moreover, the probability of this edge in  $T$  belonging to the other pair (in this case,  $\{g, h\}$ ) is  $2/3$ . Hence  $\mathbb{P}(|T \cap \{e, f\}| = 1 \wedge |T \cap \{g, h\}| = 1) \geq 2/21 > 4/63$ , giving us the other bound in Lemma 3.3 for the MATINT sampler.

*The MAXENT claims:* For the first bound in Lemma 3.3, we have that one of the four edges must be in  $M$  and have  $y$ -value 1. W.l.o.g., call that edge  $e$ . The other three,  $f, g, h$ , will be  $1/3$ -valued edges. Since  $\mathbb{E}[X_{f,g,h}] = 1$ , we may apply Theorem 2.9 and obtain a lower bound of

$$\mathbb{P}(X_{f,g,h} = 1) \geq 3 \cdot 1/3 \cdot (2/3)^2 = 4/9 \geq 8/27.$$

For the second bound in Lemma 3.3, call  $S_1 = \{e, f\}$  and  $S_2 = \{g, h\}$ . W.l.o.g., we once again label  $e \in M$ . Then,  $\mathbb{P}(f \notin T) = 2/3$ . So for  $S_2$ , we condition on  $f \notin T$ . Then by Theorem 2.8,

$$2/3 \leq \mathbb{E}[X_{S_2} \mid f \notin T] \leq 2/3 + 1/3 = 1.$$

Hence, we use Theorem 2.9 again and obtain  $\mathbb{P}(X_{S_2} = 1 \mid f \notin T) \geq 2 \cdot 1/3 \cdot 2/3 = 4/9$ . In total, we obtain

$$\mathbb{P}(|T \cap \{e, f\}| = |T \cap \{g, h\}| = 1) \geq 2/3 \cdot 4/9 = 8/27 \geq 16/81.$$

□

**Proof of Lemma 3.4a, Even Case** *The MATINT claims:* The event happens when  $e \in M'$ , which happens w.p.  $1/28$ , which is at least  $1/36$ .

*The MAXENT claims:* Condition on  $e \in M$ . Let  $S_1 = \partial(u) \setminus e$  and  $S_2 = \partial(v) \setminus e$ . Denote  $S_1 = \{a, b, c\}$ . Lower bound  $\mathbb{P}(|S_1 \cap T| = 1)$  using Theorem 2.9:  $\mathbb{E}[|S_1 \cap T|] = 3 \cdot 1/3 = 1$ , so  $\mathbb{P}(|S_1 \cap T| = 1) \geq 3 \cdot 1/3 \cdot (2/3)^{2/3} = 4/9$ . Consider the distribution over the edges in  $S_2$  conditioned on  $a \in T$ ; this distribution is also SR. By Theorem 2.8,  $1/3 \leq \mathbb{E}[X_{S_2} \mid X_a = 1] \leq 1$ . Applying Theorem 2.8 twice more,

$$1/3 \leq \mathbb{E}[X_{S_2} \mid X_a = 1, X_{b,c} = 0] \leq 1 + 1/3 + 1/3 = 5/3.$$

By Theorem 2.9,  $\mathbb{P}(X_{S_2} = 1 \mid X_a = 1, X_{b,c} = 0) \geq 3 \cdot 1/9 \cdot (8/9)^2 = 64/243$ . Using symmetry, we obtain  $\mathbb{P}(X_{S_2} = 1 \wedge X_{S_1} = 1 \wedge e \in M) \geq 64/243 \cdot 4/9 \cdot 1/4 \geq 128/6561$ . □

**Proof of Lemma 3.4b, Even Case** *The MATINT claims:* For part (b), suppose  $e \in M$  (w.p.  $1/4$ ), then each of  $u, v$  have two other internal edges, each with  $y$ -value  $1/3$ . Let us say the good cases are when exactly one of these four is chosen; exactly one of  $u, v$  has degree 2 and the other has degree 1 in these cases. We cannot choose zero of these four edges, because of the connectivity of  $T$ , so all bad cases choose at least two of these four. Given the  $y$ -values of  $1/3$  on all four edges, the expected number of these edges chosen are  $4/3$ , so the the probability of a bad case at most  $1/3$ . This means that with probability at least  $1/4 \cdot (1 - 1/3) = 1/6$ , exactly one of  $u, v$  has odd degree in  $T$ . Since  $1/6 \geq 1/9$ , we have proven (b).

*The MAXENT claims:* Observe that the edge  $e$  will always contribute either: 0 to both the degree of  $u$  in  $T$  and the degree of  $v$  in  $T$  OR 1 to both of these degrees. Let  $a, b$  be the two internal edges incident to  $u$  and  $c, d$  be the two edges incident to  $v$ .

1. *Case 1:  $e \in M$ .* This implies  $a, b, c, d$  are all  $1/3$ -valued edges, so  $\mathbb{E}[X_{a,b,c,d}] = 4/3$ . Note that we must have  $X_{a,b,c,d} \geq 1$ . Therefore,  $\mathbb{P}(X_{a,b,c,d} = 1) \geq 2/3$ .
2. *Case 2:  $e \notin M$ , exactly two of  $a, b, c, d$  are in  $M$ .* W.l.o.g., say  $a, c \in M$ , so  $a$  and  $c$  are in  $T$ .  $\mathbb{E}[X_{b,d}] = 2/3$ , which implies (by Theorem 2.9) that  $\mathbb{P}(X_{b,d} = 1) \geq 4/9$ .
3. *Case 3:  $e \notin M$ , exactly one of  $a, b, c, d$  is in  $M$ .* W.l.o.g., say  $a \in M$ , so we have  $\mathbb{E}[X_{b,c,d}] = 1$ . We condition on  $e \notin T$ , which happens w.p.  $2/3$ . Since we sample a spanning tree on the internal vertices, this means that at least one of  $c, d$  is in  $T$ . W.l.o.g., say  $c \in T$ . So in order to bound the probability that the internal parities of  $u, v$  are different, equivalently we would like to bound  $\mathbb{P}(X_{b,d} = 1 \mid e \notin T)$ . Since  $2/3 \leq \mathbb{E}[X_{b,d} \mid e \notin T] \leq 2/3 + 1/3$ , Theorem 2.9 implies that  $\mathbb{P}(X_{b,d} = 1 \mid e \notin T) \geq 4/9$ . So removing the conditioning on  $e \notin T$  gives a lower bound of  $2/3 \cdot 4/9 = 8/27$ .

Taking the minimum of the bounds  $2/3$ ,  $4/9$ , and  $8/27$  in the three cases gives

$$\mathbb{P}(\text{exactly one of } u, v \text{ has odd degree in } T) \geq 8/27 \geq 5/18.$$

□

## 4 Sampling algorithm for general solutions; and cut Hierarchy

Now that we can sample a spanning tree from a graph with no proper min-cuts, we introduce the algorithm to sample an  $r_0$ -tree from a 4-regular, 4EC graph, perhaps with proper min-cuts.

### 4.1 The algorithm

Assume that the graph  $G = (V, E)$  has a set of three distinguished vertices  $\{r_0, u_0, v_0\}$ , with each pair  $r_0, u_0$  and  $r_0, v_0$  having a pair of edges between them. This is without loss of generality (used in line 18). (We can introduce dummy nodes to ensure this property, which is for simplicity—it guarantees that the top set in the cut hierarchy is a cycle set.) Define a *double cycle* to be a cycle graph in which each edge is replaced by a pair of parallel edges, and call each such pair *partner edges*. For  $A \subseteq V$ , we use  $G/A$  to denote the graph that results from contracting the vertex set  $A$ .

As in [9], we refer to the sets in line 4 as *critical sets*. The algorithm samples from the same pieces as in [9], with the key differences being randomizing between the MATINT and MAXENT samplers; the MAXENT sampler defined with respect to marginals  $y$ ; and the critical optimization for  $K_5$ 's. The  $r_0$ -tree sampled is the union of the spanning trees, edges, and paths sampled during the algorithm.

Since  $G$  is a 4-regular, 4EC graph at every stage of the algorithm, if  $|S| = 2$  or  $3$ , then  $S$  must be a cycle set, whereas if  $|S| \geq 4$ , then  $S$  may be a degree or cycle set.

**Algorithm 1** Sampling an  $r_0$ -tree for a half-integral solution

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```

1: let  $G$  be the support graph of a half-integral solution  $x$ .
2: let  $T = \emptyset$ .
3: while there exists a proper tight set of  $G$  that is not crossed by another proper tight set do
4:   let  $S$  be a minimal such set (and choose  $S$  such that  $r_0 \notin S$ ).
5:   Define  $G' = G/(V \setminus S)$ .
6:   if  $G'$  is a double cycle then
7:     Label  $S$  a cycle set.
8:     sample a random edge from each set of partner edges in  $G[S]$ ; add these edges to  $T$ .
9:   else //  $G'$  has no proper min-cuts (Claim 4.2).
10:    Label  $S$  a degree set.
11:    if  $G' = K_5$  then
12:      sample a random path on  $G[S]$ 
13:    else
14:      W.p.  $\lambda$ , let  $\mu$  be the MAXENT distribution over  $E(S)$ 
15:      W.p.  $1 - \lambda$ , let  $\mu$  be the MATINT distribution.
16:      sample a spanning tree on  $G[S]$  from  $\mu$  and add its edges to  $T$ .
17:   let  $G = G/S$ 
18: Due to  $r_0, u_0, v_0$ , at this point  $G$  is a double cycle (Claim 4.1). Sample one edge between each pair of
    adjacent vertices in  $G$ .

```

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The following two claims are proved in Appendix A.1, and show that Algorithm 1 is well-defined.

**Claim 4.1** *The graph remaining at the end of the algorithm (line 18) is a double cycle.*

**Claim 4.2** *In every iteration in Algorithm 1,  $G'$  is either a double cycle or a graph with no proper min-cuts.*

We will prove the following theorem in Sect. 6. This in turn gives Theorem 1.1.

**Theorem 4.3** *Let  $T$  be the  $r_0$ -tree chosen from Algorithm 1, and  $O$  be the set of odd degree vertices in  $T$ . The expected cost of the minimum cost  $O$ -join for  $T$  is at most  $(1/2 - \varepsilon) \cdot c(x)$ .*

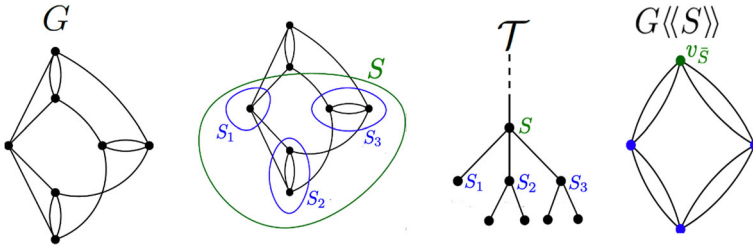
## 4.2 The cut Hierarchy

Recall our ultimate goal is to create a low cost, feasible solution in the  $O$ -join polytope, where  $O$  is the set of odd degree vertices in our sampled tree  $T$ . We start with the fractional solution  $z = x/2$  and then reduce the  $z_e$  value of some edges. In the process, we may violate some constraints corresponding to min-cuts. To fix these cuts, we need a complete description of the min-cuts of a graph. This is achieved by the implicit hierarchy of critical sets that Algorithm 1 induces.

The hierarchy is given by a rooted tree  $\mathcal{T} = (V_{\mathcal{T}}, E_{\mathcal{T}})$ .<sup>2</sup> The vertex set  $V_{\mathcal{T}}$  corresponds to all critical sets found by the algorithm, along with a root node and leaf nodes representing the vertices in  $V_G \setminus \{r_0\}$ . If  $S$  is a critical set, we label the node in  $V_{\mathcal{T}}$  with  $S$ , where we view  $S \subseteq V_G$  and not  $V_{G'}$ . The root node is labelled  $V_G \setminus \{r_0\}$

---

<sup>2</sup> Since there are several graphs under consideration, the vertex set of  $G$  is called  $V_G$ . Moreover, for clarity, we refer to elements of  $V_G$  as vertices, and elements of  $V_{\mathcal{T}}$  as nodes.



**Fig. 2** A portion of the cut hierarchy  $\mathcal{T}$  and the local multigraph  $G\langle\langle S \rangle\rangle$

and each leaf nodes is labelled by the vertex in  $V_G \setminus \{r_0\}$  corresponding to it. Now we define the edge set  $E_{\mathcal{T}}$ . A node  $S$  is a child of  $S'$  if  $S \subset S'$  and  $S'$  is the first superset of  $S$  contracted after  $S$  in the algorithm. In addition, the root node is a parent of all nodes corresponding to critical sets that are not strictly contained in any other critical set (i.e., the critical sets corresponding to the vertices in the graph  $G$  from line 17 when the **while** loop terminates). Finally, each leaf node is a child of the smallest critical set that contains it (or if no critical set contains it, is a child of the root node). Thus by construction, vertex sets labelling the children of a node are a partition of the vertex set labelling that node. A node in  $V_{\mathcal{T}}$  is a *cycle* or *degree* node if the corresponding critical set labelling it is a cycle or degree set, respectively. We say the root node is a cycle node (since the graph  $G$  in line 18 is a double cycle), and accordingly call  $V_G \setminus \{r_0\}$  a cycle set. (The leaf nodes are not labelled as degree or cycle nodes.)

**Definition 4.4** (Local multigraph) Let  $S \subseteq V_G$  be a set labelling a node in  $\mathcal{T}$ . Define the *local multigraph*  $G\langle\langle S \rangle\rangle$  to be the following graph: take  $G$  and contract the subsets of  $V_G$  labelling the children of  $S$  in  $\mathcal{T}$  down to single vertices and contract  $\bar{S}$  to a single vertex  $v_{\bar{S}}$ . Remove any self-loops. The vertex  $v_{\bar{S}}$  is called the *external vertex*; all other vertices are called *internal vertices*. An internal vertex is called a *boundary vertex* if it is adjacent to the external vertex. The edges in  $G\langle\langle S \rangle\rangle \setminus v_{\bar{S}}$  are called *internal edges*. Observe that  $G\langle\langle S \rangle\rangle$  is precisely the graph  $G'$  in line 5 of Algorithm 1 when  $S$  is a critical set, and is a double cycle when  $S = V_G \setminus \{r_0\}$ .

### Properties of $\mathcal{T}$ :

1. Let  $G$  be a 4-regular, 4EC graph with associated hierarchy  $\mathcal{T}$ . Let  $S \subseteq V_G$  be a set labelling a node in  $\mathcal{T}$ . If  $S$  is a degree node in  $\mathcal{T}$ , then  $G\langle\langle S \rangle\rangle$  has at least five vertices and no proper min-cuts, and hence every proper cut in  $G\langle\langle S \rangle\rangle$  has at least 6 edges. If  $S$  is a cycle node in  $\mathcal{T}$ , then  $G\langle\langle S \rangle\rangle$  is exactly a double cycle. These follow from Claim 4.2 and the equivalence between  $G'$  and  $G\langle\langle S \rangle\rangle$ .
2. Algorithm 1 can be restated as follows: For each non-leaf and non-root node  $S$  in  $\mathcal{T}$ , sample a random path on  $G\langle\langle S \rangle\rangle \setminus v_{\bar{S}}$  if it is a double cycle or  $K_5$ , and otherwise use the MAXENT or MATINT samplers w.p.  $\lambda$  and  $1 - \lambda$ , respectively, on  $G\langle\langle S \rangle\rangle \setminus v_{\bar{S}}$ . Sample a uniformly random cycle on the double cycle in line 18.
3. For a degree set  $S$ , the graph  $G\langle\langle S \rangle\rangle$  having no proper min-cuts implies that it has no parallel edges. In particular, no vertex has parallel edges to the external vertex in  $G\langle\langle S \rangle\rangle$ . Hence we get the following:

**Corollary 4.5** For a set  $S$  labeling a non-leaf node in  $\mathcal{T}$  and any internal vertex  $v \in G\langle\langle S \rangle\rangle$ : if  $S$  is a cycle set then  $|\partial v \cap \partial S| \in \{0, 2\}$ , and if  $S$  is a degree set then  $|\partial v \cap \partial S| \in \{0, 1\}$ .

Finally, we show how the hierarchy  $\mathcal{T}$  allows us to characterize the min-cut structure of  $G$ . The cactus representation of min-cuts ([6]) is a compact representation of the min-cuts of a graph, and it can be constructed from the cut hierarchy; we defer the details to Appendix A.2. In turn we obtain the following complete characterization of the min-cuts of  $G$  in terms of local multigraphs.

**Claim 4.6** Any min-cut in  $G$  is either (a)  $\partial S$  for some node  $S$  in  $\mathcal{T}$ , or (b)  $\partial X$  where  $X$  is obtained as follows: for some cycle set  $S$  in  $\mathcal{T}$ ,  $X$  is the union of vertices corresponding to some contiguous segment of the cycle  $G\langle\langle S \rangle\rangle$ .

## 5 Analysis part I: the even-at-last property

The proof of Theorem 4.3 proceeds in two parts:

- (1) In this section, we show that each edge  $e$  is “even-at-last” with constant probability. (This is an extension of the property that both of its endpoints have even degree.)
- (2) Then we construct the fractional  $O$ -join. As always,  $z = x/2$  is a feasible join, but we show how to save a constant fraction of the LP value for an edge when it is even-at-last. This savings causes other cuts to be deficient, so other edges *raise* their  $z$  values in response. However, a charging argument shows that the  $z$ -value for an edge does decrease by a constant factor, in expectation. This argument appears in Sect. 6.

To address part (1), let us define a notion of evenness for every edge in  $G$ . In the case where  $G$  has no proper min-cuts, we called an edge *even* if both of its endpoints were even in  $T$ . Now, the general definition of evenness will depend on where an edge belongs in the hierarchy  $\mathcal{T}$ .

**Definition 5.1** We say an edge  $e \in E(G)$  is *settled* at  $S$  if  $S$  is the (unique) set such that  $e$  is an internal edge of  $G\langle\langle S \rangle\rangle$ ; call  $S$  the *last set* of  $e$ . If  $S$  is a degree or cycle set, we call  $e$  a *degree edge* or *cycle edge*, respectively.

**Definition 5.2** (Even-at-Last) Let  $S$  be the last set of  $e$ , and  $T\langle\langle S \rangle\rangle$  be the restriction of  $T$  to  $G\langle\langle S \rangle\rangle$ .

1. A degree edge  $e$  is called *even-at-last* (EAL) if both of its endpoints have even degree in  $T\langle\langle S \rangle\rangle$ .
2. For a cycle edge  $e = uv$ , the graph  $G\langle\langle S \rangle\rangle \setminus \{v_{\bar{S}}\}$  is a chain of vertices  $v_{\ell}, \dots, u, v, \dots, v_r$ , with consecutive vertices connected by two parallel edges. Let  $C := \{v_{\ell}, \dots, u\}$ , and  $C' := \{v, \dots, v_r\}$  be the partition of this chain. The cuts  $\partial C$  and  $\partial C'$  are called the *canonical cuts* for  $e$ . Cycle edge  $e$  is called *even-at-last* (EAL) if both canonical cuts are crossed an even number of times by  $T\langle\langle S \rangle\rangle$ ; in other words, if there is exactly one edge in  $T\langle\langle S \rangle\rangle$  from each of the two pairs of external partner edges leaving  $v_{\ell}$  and  $v_r$ .

Informally, a degree edge is EAL in the general case if it is even in the tree at the level at which it is settled. Also note that *cycle* edges settled at the same (cycle) set are either all EAL or none are EAL.

**Definition 5.3** (Special and Half-Special Edges) Let  $e$  be settled at a *degree* set  $S$ . We say that  $e$  is *special* if both of its endpoints are non-boundary vertices in  $G\langle\langle S \rangle\rangle$  and *half-special* if exactly one of its endpoints is a boundary vertex in  $G\langle\langle S \rangle\rangle$ .

We now prove a key property used in Sect. 6 to reduce the  $z$ -values of edges in the fractional  $O$ -join.

**Theorem 5.4** (The Even-at-Last Property) *The table below gives lower bounds on the probability that special, half-special, and all other types of degree edges are EAL in each of the two samplers.*

	Special	Half-special	Other degree edges
MATINT	$1/36$	$1/21$	$1/18$
MAXENT	$128/6561$	$4/27$	$1/12$

Moreover, a cycle edge is EAL w.p. at least  $\lambda \cdot 1/12 + (1 - \lambda) \cdot 1/18$ .

**Proof** Let  $e$  be settled at  $S$ . Let  $T_S$  be the spanning tree sampled on the internal vertices of  $G\langle\langle S \rangle\rangle$  (in the notation of Algorithm 1, the spanning tree sampled on  $G[S]$ ).

First, assume that  $S$  is a degree set:

1. If none of the endpoints of  $e$  are boundary vertices in  $G\langle\langle S \rangle\rangle$  (i.e.,  $e$  is special), then it is EAL exactly when both its endpoints have even degree in  $T_S$ . By Lemma 3.4(a), this happens w.p.  $1/36$  for the MATINT sampler and w.p.  $128/6561$  for the MAXENT sampler.
2. Now suppose that  $e$  is half-special, so that exactly one of the endpoints of  $e = uv$  (say  $u$ ) is a boundary vertex in  $G\langle\langle S \rangle\rangle$ , with edge  $f$  incident to  $u$  leaving  $S$ . By Lemma 3.3, the other endpoint  $v$  is even in  $T_S$  w.p.  $2/21$  for the MATINT sampler and  $8/27$  for the MAXENT sampler. Moreover, the edge  $f$  is chosen at a higher level than  $S$  and is therefore independent of  $T_S$ , and hence can make the degree of  $u$  even w.p.  $1/2$ . Thus  $e$  is EAL w.p.  $1/21$  for the MATINT sampler and  $4/27$  for the MAXENT sampler.
3. Suppose both endpoints of  $e$  are boundary vertices of  $S$ , with edges  $f, g$  leaving  $S$ . Let  $q_{=}$  be the probability that the degrees of vertices  $u, v$  in the tree  $T_S$  chosen within  $S$  have the same parity, and  $q_{\neq} = 1 - q_{=}$ . Now, when  $S$  is contracted and we choose a  $r_0$ -tree  $T'$  on the graph  $G/S$  consistent with the marginals, let  $p_{=}$  be the probability that either both or neither of  $f, g$  are chosen in  $T'$ , and  $p_{\neq} = 1 - p_{=}$ . Hence

$$\Pr[e \text{ EAL}] = q_{oo}p_{11} + q_{oe}p_{10} + q_{eo}p_{01} + q_{ee}p_{00} = 1/2(p_{=q_{=}} + p_{\neq q_{\neq}}), \quad (2)$$

where  $q_{oo}, q_{oe}, q_{eo}$  and  $q_{ee}$  correspond to different parity combinations of  $u$  and  $v$  in  $T_S$  and  $p_{00}, p_{01}, p_{10}, p_{11}$  correspond to whether  $f$  and  $g$  are chosen in  $T'$ . The second inequality follows from Claim C.1 applied to the random  $r_0$ -tree  $T'$ .

- (a) If  $f, g$  are settled at different levels, then they are independent. This gives  $p_{=} = p_{\neq} = 1/2$ , and hence  $\Pr[e \text{ EAL}] = 1/4$  regardless of the sampler.
- (b) If  $f, g$  have the same last set which is a degree set, then by Lemma 3.2 each of the quantities  $p_{11}, p_{01}, p_{10} \geq 1/9$ . By Claim C.1,  $p_{00} \geq 1/9$  as well. Hence, (2) gives  $\Pr[e \text{ EAL}] \geq 1/9$ .
- (c) If  $f, g$  have the same last set which is a cycle set, then first consider the case where  $f, g$  are partners, in which case  $p_{\neq} = 1$ . Now (2) implies that  $\Pr[e \text{ EAL}] = q_{\neq}/2$ , which by Lemma 3.4(b) is  $\geq 1/2 \cdot 1/9 = 1/18$  in the MATINT sampler, and  $1/2 \cdot 5/18 = 5/36$  in the MAXENT sampler.  
If  $f, g$  are not partners, then they are chosen independently, in which case again  $p_{=} = p_{\neq} = 1/2$ , and hence  $\Pr[e \text{ EAL}] = 1/4$ .

Next, let  $S$  be a cycle set. Let  $e = uv$  be an edge inside the cycle, and let  $\{a, b\}, \{c, d\}$  be the four edges crossing  $\partial S$ . Let  $v_S$  be the vertex obtained by contracting down  $S$  (whose incident edges are then  $\{a, b, c, d\}$ ). Now in order for  $e$  to be EAL, one each of  $\{a, b\}$  and  $\{c, d\}$  must belong to  $T$ ; call this event  $\mathcal{E}$ . We again consider cases based on where these edges are settled. Let node  $P$  be the parent of node  $S$ , and let  $v_S$  be the vertex in  $G\langle\langle P \rangle\rangle$  obtained from contracting  $S$ .

1. If all four edges are settled at  $P$ , and  $P$  is a degree set, then Lemma 3.3 says that for the MATINT sampler, we have  $\Pr[\mathcal{E}] \geq 4/63$ . In contrast, the MAXENT sampler gives us  $\Pr[\mathcal{E}] \geq 16/81$ .
2. If all four edges are settled at  $P$ , and  $P$  is a cycle set, then no matter how these four edges are distributed,  $\Pr[\mathcal{E}] \geq \frac{1}{2}$ .
3. If three of them  $\{a, b, c\}$  are settled at  $P$ , and the fourth (say  $d$ ) at a higher level, then since  $P$  has a vertex  $v_S$  with a single edge leaving it,  $P$  must be a degree set. Now exactly one of  $\{a, b\}$  is chosen in  $T$  w.p. at least  $1/9$  in the MATINT sampler and  $12/72$  in the MAXENT sampler, by Lemma 3.2. And since  $d$  is independently picked at a different level, exactly one of  $\{c, d\}$  is chosen in  $T$  w.p.  $1/2$ , giving an overall probability of  $1/18$  in the MATINT sampler and  $1/12$  in the MAXENT sampler.
4. Finally, if only two edges are settled at  $P$ , and two others go to higher levels, then  $P$  is a cycle set by Corollary 4.5. In this case, exactly one of the two edges that are settled in  $P$  is chosen. Now we want a specific one of the edges going to a higher level to be chosen into  $T$  (and the other to not be chosen), which in the worst case happens w.p. at least  $1/9$ , due to Lemma 3.2.

□

## 6 Analysis part II: the O-join and charging

To prove Theorem 4.3 and thereby finish the proof, we construct an  $O$ -join for the random tree  $T$ , and bound its expected cost via a charging argument. The structure here is similar to [9]; however, we use a flow-based argument to perform the charging



instead of the naive one, and also use our stronger property that *every* edge is EAL with constant probability (versus the weaker property obtained in [9] that every tight cut contains an EAL edge with constant probability).

Let  $O$  denote the (random set of) odd-degree vertices in  $T$ . The dominant of the  $O$ -join polytope  $K_{\text{join}}(G, O)$  is given by

$$x(\partial S) \geq 1 \quad \forall S \subseteq V, |S \cap O| \text{ odd}.$$

This polytope is integral, so it suffices to exhibit a fractional  $O$ -join solution  $z \in K_{\text{join}}(G, O)$ , with low expected cost. (The expectation is taken over  $O$ .) Note that  $|S \cap O|$  odd if and only if  $|\partial S \cap T|$  odd.

Now Theorem 4.3 follows from the claim below, which we will prove in this section.

**Lemma 6.1** *There is an  $\varepsilon > 0$  such that if the  $r_0$ -tree  $T$  is sampled using the procedure described in Algorithm 1, and  $O$  is the set of odd-degree vertices in  $T$ , then there is a fractional solution  $z \in K_{\text{join}}(G, O)$  where  $\mathbb{E}[z_e] \leq (1/2 - \varepsilon)x_e$  for all edges  $e \in E(G)$ .*

## 6.1 Construction of the fractional $O$ -join

The construction of the fractional  $O$ -join  $z$  goes as follows: We start with the solution  $z = x/2$ . Notice that  $z(\partial S) \geq 1$  is a tight constraint in this initial solution when  $S$  is a min-cut. Now we describe how to reduce the  $z_e$  values.

Define

$$\begin{aligned} p_{sp}^{MI} &:= 1/36, p_{hs}^{MI} := 1/21, p^{MI} := 1/18, p_{sp}^{ME} := 128/6561, p^{ME} := 1/12 \\ p_{sp} &:= \lambda p_{sp}^{ME} + (1 - \lambda)p_{sp}^{MI}, p_{hs} := \lambda p^{ME} + (1 - \lambda)p_{hs}^{MI}, p := \lambda p^{ME} + (1 - \lambda)p^{MI}. \end{aligned}$$

Let  $p_{sp}(e)$ ,  $p_{hs}(e)$ , and  $p_d(e)$  denote the probabilities that  $e$  is EAL if  $e$  is a special degree edge, half-special degree edge, or other degree edge, respectively. Let  $p_c(e)$  denote the probability that a cycle edge  $e$  is EAL. These constants are just lower bounds on the probability an edge is EAL. Indeed, by Theorem 5.4,

$$p_{sp}(e) \geq p_{sp}, p_{hs}(e) \geq p_{hs}, p_d(e) \geq p, p_c(e) \geq p.$$

(We do not distinguish the half-special case in the MAXENT sampler, as the half-special bound of  $4/27$  in Theorem 5.4 is greater than  $p^{ME}$ .) Call an edge  $e$  a  $K_5$  degree edge if  $e$  is settled at a degree set  $S$  where  $G \langle\langle S \rangle\rangle$  is a  $K_5$ . By the inequalities above, the random variables below are well-defined.

1. Define a Bernoulli random variable for each edge  $e$ :

- (a) If  $e$  is a special degree edge, set  $B_e \sim \text{Ber}(p_{sp}/p_{sp}(e))$ .
- (b) If  $e$  is a half-special degree edge, set  $B_e \sim \text{Ber}(p_{hs}/p_{hs}(e))$ .
- (c) If  $e$  is any other degree edge, set  $B_e \sim \text{Ber}(p/p_d(e))$ .
- (d) If  $e$  is a cycle edge, set  $B_e \sim \text{Ber}(p/p_c(e))$ . Further, if  $e$  and  $f$  are partners, perfectly correlate their coin flips, i.e., set  $B_e = B_f$ .

2. If  $e$  is EAL and  $B_e = 1$ , reduce  $z_e$  by
  - (a)  $\tau$  if  $e$  is a non- $K_5$  degree edge.
  - (b)  $\gamma$  if  $e$  is a  $K_5$  degree edge.
  - (c)  $\beta$  if  $e$  is a cycle edge.
3. We enforce that  $\tau \leq \gamma \leq \beta \leq 1/12$ ,  $\beta \geq 2\tau$ , and  $\beta \geq 2\gamma$ . We will optimize  $\tau$ ,  $\gamma$ , and  $\beta$  via a linear program later, in Sect. 6.7.

This reduction scheme may make  $z$  infeasible for the  $O$ -join polytope. We now discuss how to maintain feasibility.

The purpose of the Bernoulli coin flips is to flatten the probability that an edge is reduced down to the lower bound on the probability that it is EAL from Theorem 5.4:

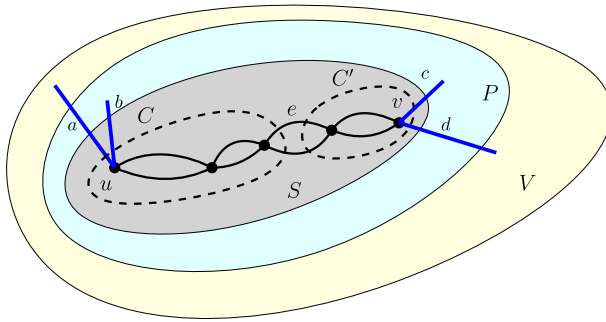
**Observation 6.2** If  $e$  is a special, half-special, other degree, or cycle edge, then  $z_e$  is reduced with probability exactly  $p_{sp}$ ,  $p_{hs}$ ,  $p$ , or  $p$ , respectively.

## 6.2 Maintaining feasibility of the fractional $O$ -join via charging

Suppose  $f$  is EAL and that we reduce edge  $z_f$  (per its coin flip  $B_f$ ). Say  $f$  is settled at  $S$ . If  $S$  is a degree set, then the only min-cuts of  $G\langle S \rangle$  are the degree cuts. So the only min-cuts that the edge  $f$  is part of in  $G\langle S \rangle$  are the degree cuts of its endpoints, call them  $U, V$ , in  $G\langle S \rangle$  ( $U$  and  $V$  are vertices in  $G\langle S \rangle$  representing sets  $U$  and  $V$  in  $G$ ). But since  $|\partial U \cap T|$  and  $|\partial V \cap T|$  are both even by definition of EAL, we need not worry that reducing  $z_f$  causes  $z(\partial U) \geq 1$  and  $z(\partial V) \geq 1$  to be violated. Likewise, if  $S$  is a cycle set, then by definition of EAL all min-cuts  $S'$  in  $G\langle S \rangle$  containing  $e$  have  $|\partial S' \cap T|$  even, so again we need not worry.

Since  $f$  is only an internal edge for its last set  $S$ , only certain  $O$ -join constraints (recall  $O$ -join constraints are of the form  $z(\partial S') \geq 1$ ,  $|\partial S' \cap T|$  odd) may be violated by reducing  $z_f$ . In particular, only the cuts  $S'$  represented in lower levels of the hierarchy (that is, cuts  $S'$  in multigraphs  $G\langle X \rangle$  for  $X \subset S$ ) may be violated, since these are the only multigraphs in which  $f$  can be an external edge. Specifically, let  $f$  be an external edge for some  $G\langle X \rangle$  and  $S'$  be a min-cut of  $G\langle X \rangle$  (either a degree cut or a canonical cut). (For instance, in Fig. 2, let  $f$  be an edge between  $S_1$  and  $S_2$ . We have that  $f$  is settled at  $S$ . Take  $X = S_2$ . Now  $f$  is an external edge in  $G\langle X \rangle$ . Reducing  $z_f$  may now cause a min-cut  $S'$  in  $G\langle X \rangle$  to become violated.) By Claim 4.6, cuts of the form  $S'$  (i.e., min-cuts in  $G\langle X \rangle$  for  $X \subset S$ ) are the only cuts that may be violated in the  $O$ -join polytope as a result of reducing  $z_f$ . Call the internal edges (in  $G\langle X \rangle$ ) of  $\partial S'$  *lower edges*. When  $z_f$  is reduced and  $|\partial S' \cap T|$  is odd, we must distribute an increase (charge) over the lower edges totalling the amount by which  $z_f$  is reduced, so that  $z(\partial S') = 1$ . We say these lower edges receive a *charge* from  $f$ . How the charge is distributed will depend on whether the lower edges are degree edges or cycle edges (see Sect. 6.3 and Sect. 6.4 below). (For instance, in the example from Fig. 2, letting  $f = (u, v)$  where  $v \in S_2$ , if  $z_f$  is reduced by  $\beta$ , we could increase each internal edge in  $G\langle X \rangle$  incident to  $v$  by  $\beta/2$ , i.e., *charge* these edges.)

We claim that this procedure maintains feasibility of the  $O$ -join solution  $z$ . Indeed, by Claim 4.6, no constraint corresponding to a *min-cut* in the  $O$ -join polytope is violated. Further, by capping the amount an edge can be reduced at  $1/4 - 1/6 = 1/12$ ,



**Fig. 3** Case (i)

we will ensure that *none* of the constraints are violated, since every other cut has at least 6 edges.

To prove Lemma 6.1, we will lower bound the expected decrease to  $z_e$ , using different strategies for charging  $e$  when  $e$  is a cycle edge versus a degree edge. In particular, when  $e$  is a cycle edge, we distribute charge from an external edge evenly between  $e$  and its partner. When  $e$  is a degree edge, charge will be distributed according to a maximum-flow solution.

### 6.3 Charging of cycle edges

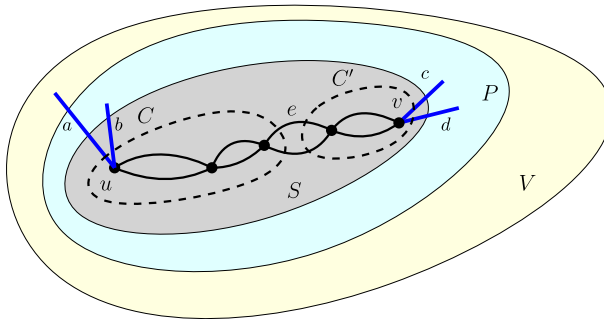
We now analyze the expected net decrease for each edge, starting with cycle edges. The expected decrease for a cycle edge starts off as exactly  $p \cdot \beta$  by Observation 6.2. However, we need to control the amount of charge the edge receives from edges settled at higher levels. Note that in the calculations, we assume the worst-case scenario that such edges settled at higher levels are reduced w.p. exactly  $p$ . The special/half-special edges are reduced with lower probability, which is only better for us because the expected amount of charge will be lower in these cases. Note also that throughout the calculations we may assume that an edge settled at a higher level is a cycle edge. This will follow from the assumption that  $\beta \geq 2\tau, 2\gamma$ .

Consider a cycle edge  $e$  that is settled at set  $S$ . Let its partner on the cycle be  $e'$ , and let the four external edges for  $S$  be  $\{a, b\}$  (leaving the vertex  $u$ ) and  $\{c, d\}$  (leaving vertex  $v$ ). When  $S$  is contracted, vertex  $v_S$  has these four edges  $\{a, b, c, d\}$  incident to it. Let  $P$  be  $S$ 's parent node in  $\mathcal{T}$ . Moreover, let  $C$  and  $C'$  be the canonical cuts, as in the figure below (Figs. 3 and 4).

We start with a small lemma which will be invoked in our casework.

**Lemma 6.3** *If  $P$  has two edges leaving it and  $P$  is a cycle set, one must be from  $\{a, b\}$  and one from  $\{c, d\}$ .*

**Proof** To see this, say both  $c$  and  $d$  (leaving node  $v$ ) both did not leave  $P$ . Since  $P$  is a cycle set,  $c$  and  $d$  are partner edges. Consider the min cut-set  $S'$  which  $c$  and  $d$  are adjacent to in  $G \langle P \rangle$ .  $S' \cup v$  is a proper min cut which crosses  $S$ ; however, this contradicts the fact that  $S$  is a critical set (namely  $S$  is not supposed to be crossed by another proper min cut).  $\square$



**Fig. 4** Case (ii)

We consider the following cases as in [9].

- (i) Two of the four edges incident to  $v_S$  also leave  $P$ —and therefore  $P$  is a cycle set by Corollary 4.5. By Lemma 6.3, of the two edges leaving  $P$ , one must be from  $\{a, b\}$  and one from  $\{c, d\}$ .

Without loss of generality, let  $\{a, d\}$  are the external edges for  $P$ , and  $\{b, c\}$  are internal (and hence partners in this cycle set).

Now, consider the event that  $b$  is reduced; the important observation is that  $c$  is also reduced, because both are EAL at the same time and their coins are perfectly correlated. Not only that, the event that  $b$  is reduced implies that  $|\delta(S) \cap T|$  is even. This means that both cuts  $C, C'$  are deficient by the same amount and at the same time, due to this event of reducing  $b, c$ , and raising  $e$  helps both of them. This means the net expected charge to  $e$  is at most

$$p \frac{\beta}{2} (\Pr[C \text{ odd} \mid b \text{ reduced}] + \Pr[C \text{ odd} \mid a \text{ reduced}] + \Pr[C' \text{ odd} \mid d \text{ reduced}]).$$

(Note that there is no term for  $c$  here because of the discussion above; moreover, the  $\beta/2$  term reflects that the charge of  $\beta$  is split between  $e, e'$ .)

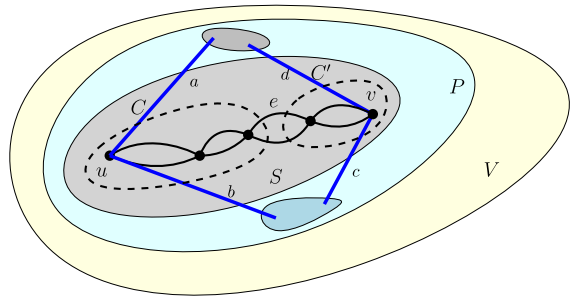
Since  $a$  and  $b$  are settled at different levels, the parity of cut  $C$  is unbiased, even either conditioned on choices within  $P$ , or else conditioned on choices outside  $P$ . The same holds for the parity of  $C'$ , so each of the probabilities above is at most  $1/2$ , making the net expected decrease

$$p(\beta - \beta/2 \cdot 3 \cdot 1/2) = \boxed{p\beta/4}. \quad (3)$$

- (ii) Only one of the edges incident to  $v_S$ , say edge  $a$ , is external for  $P$ . By Corollary 4.5,  $P$  is a degree set, and in the worst case, a  $K_5$  degree set since  $\gamma \geq \tau$ . The charge is maximized when  $a$  is a cycle edge, in which case the expected charge to  $e$  is at most

$$p(\beta/2 \Pr[C \text{ odd} \mid a \text{ reduced}] + \gamma/2 \Pr[C \text{ odd} \mid b \text{ reduced}] + 2 \cdot \gamma/2).$$

Fig. 5 Case (iii)



Since  $a$  and  $b$  are settled at different levels, the parity of the cut  $C$  is unbiased, either conditioning on the process inside  $P$  or else outside  $P$ , which makes the above expression  $p(\beta/4 + 5\gamma/4)$ , and the net expected decrease to be

$$p(\beta - \beta/4 - 5\gamma/4) = \boxed{p(3\beta/4 - 5\gamma/4)}. \quad (4)$$

- (iii) All four of the edges are internal to  $P$ , which is a cycle set. The partners must now be one edge from each of the two pairs  $\{a, b\}$  and  $\{c, d\}$ —let's say  $\{a, d\}$  and  $\{b, c\}$  are partners (due to the proof of Lemma 6.3). The cut  $C$  is odd in  $S$  if and only if  $C'$  is odd; indeed,  $C$  is odd means we choose both or neither of  $\{a, b\}$ , and since  $\{c, d\}$  are the partners of  $\{a, b\}$ , we choose both or neither of them. Hence, increasing  $e$  fixes both cuts at the same time, similar to the argument in (i), so we focus only on the increase due to cut  $C$ . Since  $a$  and  $b$  belong to different pairs and are chosen independently into the tree  $T$ ,  $\Pr[C \text{ odd}] = 1/2$ . This means both  $a$  and  $b$  can give a expected charge of  $p(\beta/2) \cdot 1/2$  to  $e$ . The net expected decrease is  $p(\beta - 2 \cdot \beta/2 \cdot 1/2) = p\beta/2$ , which is no worse than (3).
- (iv) All four of the edges are internal to  $P$ , and  $P$  is a degree set (Figs. 5 and 6. Note that  $G\langle\langle P \rangle\rangle$  cannot be a  $K_5$  since the vertex  $v_S$  has four internal edges incident to it. We simply bound the charge due to each edge by  $\tau/2$ , and hence get net expected decrease of

$$p(\beta - 4 \cdot \tau/2) = \boxed{p(\beta - 2\tau)}. \quad (5)$$

## 6.4 Charging degree edges: max-flow formulation

So far we have shown that no cycle edge will receive too much charge. We did this by considering four different configurations for the external edges of  $S$ , and distributing charge evenly between a cycle edge and its partner. Before showing that no degree edge receives too much charge, we will define a charging scheme for degree edges that achieves better bounds than distributing charge uniformly. A similar flow-based charging argument was independently used in [10].

Let  $G\langle\langle S \rangle\rangle$  be the local multigraph for some degree set  $S$ , so  $G\langle\langle S \rangle\rangle$  is 4-regular and has no proper min-cuts. For an external edge  $e$ , let  $u_e$  denote the internal boundary

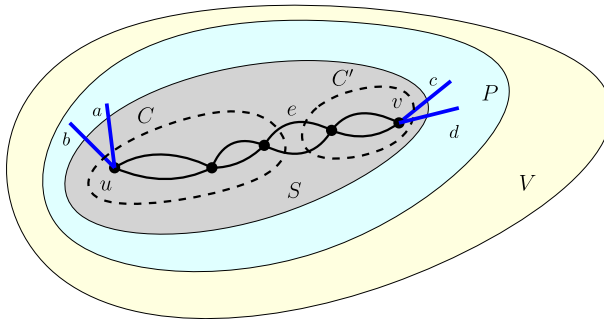


Fig. 6 Case (iv)

vertex to which  $e$  is incident. We create a bipartite graph  $H = (B, F, E)$  from  $G\langle\langle S \rangle\rangle$  as follows. The vertices of  $B$  are labelled with the external edges of  $G\langle\langle S \rangle\rangle$ . The vertices of  $F$  are labelled with the internal edges of  $G\langle\langle S \rangle\rangle$ . So  $|B| = 4$ . Place an edge between  $e \in B$  and  $f \in F$  in the edge set  $E$  if  $e$  and  $f$  share  $u_e$  as an endpoint. We call  $H$  the *bipartization* of  $G\langle\langle S \rangle\rangle$ .

The following lemma follows from the Max-Flow Min-Cut theorem:

**Lemma 6.4** *Given a bipartite graph  $G = (B, F, E)$  and  $c \geq 0$ , there exists  $m : B \times F \rightarrow \mathbb{R}_+$  satisfying*

$$\sum_{f \in \partial u} m(u, f) = 1 \text{ for all } u \in B, \quad (6)$$

$$\sum_{u \in \partial f} m(u, f) \leq c \text{ for all } f \in F \quad (7)$$

*if and only if*

$$|N(R)| \geq \frac{|R|}{c} \text{ for all } R \subseteq B \quad (8)$$

Lemma 6.4 now gives the following result, which follows from straightforward casework.

**Lemma 6.5** *Let  $H = (B, F, E)$  be the bipartization of  $G\langle\langle S \rangle\rangle$ , as described above. If  $G\langle\langle S \rangle\rangle$  is not a  $K_5$ , then the smallest  $c$  satisfying (8) is given by  $c = 1/2$ . If  $G\langle\langle S \rangle\rangle$  is a  $K_5$ , then the smallest  $c$  satisfying (8) is given by  $c = 2/3$ .*

The fact that we may achieve a value of  $c = 1/2$  instead of  $c = 2/3$  as long as  $G\langle\langle S \rangle\rangle$  is not a  $K_5$  motivates why we utilize a different sampler for  $K_5$ . In fact,  $c = 2/3$  on  $K_5$  is achieved by distributing uniform charge over internal edges.

## 6.5 Charging of non- $K_5$ degree edges

We now analyze the expected net decrease for degree edges. By Observation 6.2, the expected decrease for a non-special degree edge starts off as at least  $p_{hs} \cdot \tau$  or  $p_{hs} \cdot \gamma$ , since  $p_{hs} \leq p$ . We will analyze the charge on non- $K_5$  degree and  $K_5$  degree edges separately, as a different sampler is used when  $G\langle\langle S \rangle\rangle = K_5$ . We begin with the former.

### 6.5.1 Charging of non-special edges

1. *Case 1: All external edges are non- $K_5$  degree edges.* Let  $m : B \times F \rightarrow \mathbb{R}_+$  be any function satisfying the constraints in Lemma 6.4 for  $c = 1/2$  (by Lemma 6.5 such an  $m$  exists, and can be found efficiently using max-flow min-cut). The charging scheme is as follows: for each external edge  $e$  and boundary vertex  $u_e$ , if the cut  $\delta(u)$  is odd and the edge  $e$  is reduced, charge each internal edge  $f$  incident to  $u_e$  by  $\tau \cdot m(e, f)$ . The constraint (6) ensures that the charge neutralizes the reduction of  $e$  by  $\tau$ . For any internal edge  $f$ , let  $\delta(f)$  denote the external edges with which  $f$  shares an endpoint, i.e., the neighbors of  $f$  in the bipartization  $H = (B, F, E)$ . We have

$$\begin{aligned} \mathbb{E}[\text{charge to } f] &= \sum_{e \in \delta(f)} \tau \cdot m(e, f) \cdot \mathbb{P}(u_e \text{ odd} \wedge e \text{ reduced}) \\ &\leq \tau \cdot c \cdot \mathbb{P}(u_e \text{ odd} \mid e \text{ reduced}) \cdot \mathbb{P}(e \text{ reduced}) \end{aligned} \quad (9)$$

Using the naive bound that  $\mathbb{P}(u_e \text{ odd} \mid e \text{ reduced}) \leq 1$ , we obtain that the expected decrease is at least  $p_{hs}\tau - \frac{\tau}{2} \cdot p$ .

2. *Case 2: All external edges are  $K_5$  degree edges.* An analogous argument to (9) (with  $\gamma$  instead of  $\tau$ ) shows that the expected decrease is at least  $p_{hs}\tau - \frac{\gamma}{2} \cdot p$ .
3. *Case 3: All external edges are cycle edges.* We have that  $\mathbb{P}(u_e \text{ odd} \mid e \text{ reduced}) = \frac{1}{2}$  in this case, since  $f \in T$  is independent of whether  $f$  is reduced since  $f$  has a partner edge. So an analogous argument to (9) (with  $\beta/2$  instead of  $\tau$ ) shows that the expected decrease is at least  $p_{hs}\tau - \frac{\beta}{2} \cdot \frac{1}{2} \cdot p$ .
4. *Case 4: The external edges are a combination of non- $K_5$  degree edges,  $K_5$  degree edges, and cycle edges.* In this case, An analogous argument to (9) (with  $\max\{\frac{\tau}{2}, \frac{\gamma}{2}, \frac{\beta}{4}\}$  instead of  $\tau$ ) shows that the expected decrease is at least  $p_{hs}\tau - p \cdot \max\{\frac{\tau}{2}, \frac{\gamma}{2}, \frac{\beta}{4}\}$ .

### 6.5.2 Charging of special edges

These edges are never charged, as their endpoints are both non-boundary. Also, these cannot be  $K_5$  degree edges by definition. So the expected decrease is equal to the expected reduction, which is at least  $p_{sp}\tau = \frac{\tau}{36}$ .

### 6.6 Charging of $K_5$ degree edges

$K_5$  degree edges have an initial decrease of exactly  $p\gamma$ , as they are neither special nor half-special. We will use the following lemma, whose proof is straightforward.

**Lemma 6.6** *Let  $G\langle\langle S \rangle\rangle = K_5$ . Let  $f$  be an external edge and  $u$  be the boundary vertex that is one of its endpoints. Then*

$$Pr(u \text{ odd} \mid f \text{ reduced}) = \frac{1}{2}.$$

We will factor in this probability into the charging of  $K_5$  degree edges below.

1. *All external edges are non- $K_5$  degree edges.* Since the edge being charged is a  $K_5$  edge, Lemma 6.5 gives us a worst-case expected charge of  $p \cdot \frac{2\tau}{3}$ . However, factoring in  $Pr(u \text{ odd} \mid f \text{ reduced}) = \frac{1}{2}$  for every external edge  $f$ , we have a worst-case expected charge of  $p \cdot \frac{2\tau}{3} \cdot \frac{1}{2}$ , which by (9) gives us an expected net decrease of at least

$$p\gamma - p\frac{\tau}{3}.$$

2. *All external edges are  $K_5$  degree edges.* Using similar reasoning as above, we have a worst-case expected charge of  $p \cdot \frac{2\gamma}{3} \cdot \frac{1}{2}$ , which gives us an expected net decrease of at least

$$p\gamma - p\frac{\gamma}{3}.$$

3. *All external edges are cycle edges.* Using the same reasoning as above, we get an expected net decrease of at least

$$p\gamma - p\frac{\beta}{3}.$$

4. *The external edges are a combination of non- $K_5$  degree edges,  $K_5$  degree edges, and cycle edges.* An argument as in the previous section shows that the three cases above are the only possible worst cases.

## 6.7 Setting the parameters

For fixed  $\lambda$ , finding the optimal values of  $\beta, \gamma, \tau$  can be done with a linear program. In particular, we maximize the minimum expected decrease  $\delta$ , which is the minimum of the boxed expressions in §6.3 and Sect. 6.4. The additional constraints are that  $\tau \leq \gamma \leq \beta$ ,  $\beta \leq 1/12$ ,  $\beta \geq 2\tau$ , and  $\beta \geq 2\gamma$ . We then optimize over  $\lambda$ . This gives  $\lambda = 0.4715$ ,  $\beta = 1/12$ ,  $\gamma = 0.0401$ ,  $\tau = 0.0355$ , and  $\delta = 0.0008475$ . We have  $\varepsilon = 2\delta = 0.001695$ , proving Theorem 4.3.

**Funding** Open Access funding provided by Carnegie Mellon University



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## A details of the cut Hierarchy construction

### A.1 The algorithm

**Fact A.1** (Lemma 2 in [6]) *Let  $k \geq 1$  be an integer,  $H = (W, F)$  be a 4-regular, 4EC graph in which there is a proper min-cut, and every proper min-cut is crossed by some proper min-cut. Then  $H$  is a double cycle.*

**Proof of Claim 4.1** Let  $G_f$  refer to the graph in the claim, and  $G$  refer to the input to the algorithm.

*Case 1: There is no proper tight set in  $G_f$ .* Let  $u$  be the vertex in  $G_f$  such that the two edges that between  $r_0$  and  $u_0$  in  $G$  are between  $r_0$  and  $u$  in  $G_f$ . Then either  $G_f$  is a graph on 2 vertices with four parallel edges (i.e., a double cycle on 2 vertices), or  $\{r_0, u\}$  is a tight set in  $G_f$ . In the latter case, since  $\{r_0, u\}$  is not proper,  $G_f$  is a double cycle with 3 vertices. *Case 2: There is a proper tight set in  $G_f$ , and every proper tight set is crossed by another proper tight set.* By Fact A.1,  $G_f$  is a double cycle.  $\square$

**Proof of Claim 4.2** Let  $v_{\bar{S}}$  denote the vertex corresponding to the contraction of  $V \setminus S$  in  $G'$ . Suppose  $G'$  has a proper min-cut. Let  $S'$  be the shore of this min-cut not containing  $v_{\bar{S}}$ . Then  $S' \subset S$ . But by minimality of  $S$ ,  $S'$  is crossed by another proper tight set. Since this must be true for every proper min-cut  $S'$  in  $G'$ , by Fact A.1,  $G'$  must be a double cycle.  $\square$

### A.2 Cactus representation of min-cuts

**Claim 4.6** *Any min-cut in  $G$  is either (a)  $\partial S$  for some node  $S$  in  $\mathcal{T}$ , or (b)  $\partial X$  where  $X$  is obtained as follows: for some cycle set  $S$  in  $\mathcal{T}$ ,  $X$  is the union of vertices corresponding to some contiguous segment of the cycle  $G \langle\langle S \rangle\rangle$ .*

To prove the above claim, we recall the cactus representation of min-cuts (see, for e.g., [6], for a full exposition). The key idea to prove Claim 4.6 is that Algorithm 1 implicitly constructs the cactus representation of min-cuts.

**Definition A.2** *A cactus is a loopless, 2EC graph in which each edge belongs to exactly one cycle.*

Note that this includes cycles on two vertices, containing exactly two parallel edges. The min-cuts of a cactus are obtained by cutting any two edges belonging to the same cycle.

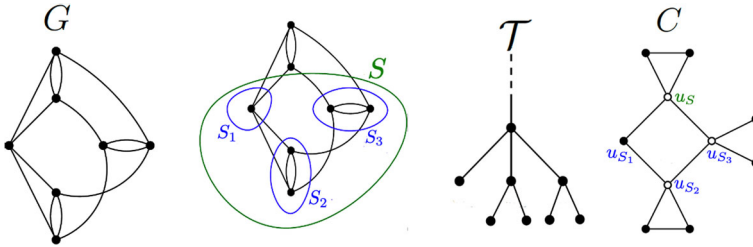


Fig. 7 Construction of the cactus  $C$  from the hierarchy  $\mathcal{T}$  in the proof of Claim 4.6

**Theorem A.3** (Cactus Representation of Min-Cuts, [6], Theorem 7) *Let  $G$  be a 4-regular, 4EC graph. There is a cactus  $C = (U, F)$  and a mapping  $\phi : V \rightarrow U$  so that if  $U_1$  and  $\bar{U}_1$  are two shores of a min-cut in  $C$ , then  $\phi^{-1}(U_1)$  and  $\phi^{-1}(\bar{U}_1)$  are two shores of a min-cut in  $G$ . Further, every min-cut in  $G$  can be obtained this way for some  $U_1$ .*

**Proof sketch of Claim 4.6** Construct a cactus  $C = (U, F)$  as follows. Create a vertex  $u_S$  in  $U$  for every node  $S$  in the hierarchy. If  $S$  is a cycle node, make a cycle between  $u_S$  and the vertices  $u_T$  for all nodes  $T$  which are children of  $S$ . If  $S$  is a degree node, include two parallel edges from  $u_S$  to each  $u_T$  for which  $T$  is a child of  $S$ . See Fig. 7. All of the terminal vertices in  $U$ , except for  $u_{V_G \setminus r_0}$ , correspond to all of the leaf nodes in  $\mathcal{T}$ , which in turn correspond to all of the vertices in  $V_G \setminus \{r_0\}$ . We identify the vertex  $u_{V_G \setminus r_0}$  in  $U$  with the vertex  $r_0$  in  $V_G$ . In other words, the mapping  $\phi$  in Theorem A.3 can be defined as  $\phi(v) = u_v$  for all  $v \in V_G \setminus r_0$ , and  $\phi(r_0) = u_{V_G \setminus r_0}$ .

Since the construction of the cactus in the proof of Theorem A.3 given in [6] and the construction of  $C$  above are equivalent,  $\phi$  satisfies the conclusions of Theorem A.3. Since the min-cuts of a cactus are obtained by cutting any two edges belonging to the same cycle, Claim 4.6 follows.  $\square$

## B Samplers for odd $|V(H)|$

The main change from the case where  $V(H)$  is even is that  $x/2 = 1/4 \cdot \mathbf{1}$  is not a point in the perfect matching polytope: in fact,  $H$  has no perfect matchings because it has an odd number of vertices.

To fix this, split the external vertex  $r$  into two external vertices  $r_1, r_2$ , each incident to some two of the edges in  $\partial r$ , and then add two parallel edges between  $r_1, r_2$ . Call this multigraph  $\hat{H}$ . Again, all vertices and edges between vertices in  $V \setminus \{r_1, r_2\}$  are called internal, vertices in  $\{r_1, r_2\}$  and edges in  $\partial r_1 \cup \partial r_2$  are external, and internal vertices adjacent to external vertices are boundary vertices.

**Fact B.1** *The graph  $\hat{H}$  is also 4-regular and 4EC; it has an even number of vertices. However, it contains a single proper min-cut separating the internal vertices  $I := V \setminus \{r_1, r_2\}$  from the two external vertices. Any perfect matching  $M$  of  $\hat{H}$  contains either zero or two edges in  $\partial I$ .*

The sampling procedure for trees on  $H[I] = \widehat{H}[I]$  is conceptually similar to the one above, but with some crucial changes because of the proper min-cut.

1. Since  $\widehat{H}$  is 4-regular and 4EC, and  $|V(\widehat{H})|$  is even, setting  $1/4 = x_e/2$  on each edge gives a solution  $K_{PM}(\widehat{H})$  in the perfect matching polytope. Sample a perfect matching  $M$  such that for each edge  $e$ , we have  $\Pr[e \in M] = 1/4$ . Again define the fractional solution

$$y_e = 1/3 + \mathbf{1}_{(e \in M)} \cdot 2/3.$$

This ensures  $\mathbb{E}[y] = x$ , but  $y|_I$  may no longer be in the spanning tree polytope  $K_{spT}(\widehat{H}[I])$ . Indeed,  $y(\partial I) \in \{4/3, 8/3\}$  depending on whether  $M$  has zero or two edges crossing the cut, so  $y(E(I)) \in \{|I| - 2/3, |I| - 4/3\}$  and not  $|I| - 1$  as required. Hence we cannot directly sample a spanning tree on  $\widehat{H}[I]$ , as we did above. We fix this in step 3 below.

2. Pick a random induced sub-matching  $M'$  of  $M$ , and define partition matroid constraints on the sets  $L_{uv}, R_{uv}$  incident on the endpoints of edges  $uv$  in this sub-matching  $M'$  exactly as in steps i and ii of the sampling procedure for the even-size case above.
3. We now address that  $y|_I$  is not in the spanning tree polytope by locally changing the fractional solution as follows:
  - (a) (Local Decrease) If  $M \cap \partial I = \emptyset$ , then  $y(E(I)) = |I| - 2/3$  and  $y(\partial I) = 4/3$ . Hence  $y|_I$  is not a solution to  $K_{spT}(\widehat{H}[I])$ . To fix this, pick a random edge  $e \in \partial I$ , pick a random edge  $f \in E(I)$  adjacent to  $e$ , and reduce the  $y$ -value of  $f$  by  $1/3$  to get solution  $\widehat{y}$ . (Note this may reduce either a 1-edge to  $2/3$ , or a  $1/3$ -edge to zero.)
  - (b) (Local Increase) If  $|M \cap \partial I| = 2$ , then  $y(E(I)) = |I| - 4/3$ , and hence  $y|_I \notin K_{spT}(\widehat{H}[I])$ . To fix this, pick a random edge  $e$  from the two matching edges in  $M \cap \partial I$ , pick a random edge  $f \in E(I)$  adjacent to  $e$ , and increase the  $y$ -value of  $f$  by  $1/3$  to get solution  $\widehat{y}$ . (Note this increases one  $1/3$  edge to  $2/3$ .) Suppose this edge  $f$  is contained in some matroid constraint, say on set  $P$  of edges:
    - i. If all three edges  $f, g, h$  of this constraint are in  $E(I)$ , pick a random one of  $\{g, h\}$ , say  $h$ , and redefine  $P \leftarrow \{f, g\}$ . Note that  $\widehat{y}_f + \widehat{y}_g = 1$  and the constraint is tight.
    - ii. If  $f$  belongs to a constraint  $P = \{f, g\}$ , that used to be  $\{f, g, h\}$  but the third edge  $h$  was in  $\partial I$  and hence was dropped, retain the constraint  $P = \{f, g\}$ . Again  $\widehat{y}_f + \widehat{y}_g = 1$ .

The following claims show that  $\widehat{y}|_I$  has the right marginals, and that it belongs to the spanning tree polytope. So now we can sample a spanning tree  $T$  from it (subject to the partition matroid  $\mathcal{M}$ ). Then we show, in §B.2, that this tree also satisfies negative correlation properties.

## B.1 Feasibility of the solution $\hat{y}$

**Claim B.2** (*Marginal Preserving*)  $\mathbb{E}[\hat{y}_e] = x_e = 1/2$  for all  $e \in E(I)$ .

**Proof** Since  $\mathbb{E}[y_e] = 1/2$  for all  $e$ , we focus on the expected difference  $\mathbb{E}[\hat{y}_e - y_e]$ . We only need to worry about internal edges incident to boundary vertices, since those are the only edges whose  $y$ -values are changed.

- Let  $e$  be an edge incident only on one boundary vertex  $v$ , that has a boundary edge  $f$ . Now consider choosing  $M$  randomly. When  $M \cap \partial I = \emptyset$ , which happens w.p.  $1/2$ ,  $e$  is decreased w.p.  $1/12$ , since the total internal degree of boundary vertices is 12. On the other hand, when  $f \in M$  (w.p.  $1/4$ , and this implies the event  $|M \cap \partial I| = 2$ ),  $e$  is increased with probability  $1/6$ , since the total internal degree of the boundary vertices incident to internal endpoints of  $M$  is 6.
- Let  $e = uv$  such that both  $u, v$  are boundary vertices with edges  $f, g \in \partial I$  respectively. Now, when  $M \cap \partial I = \emptyset$ , which happens w.p.  $1/2$ ,  $e$  is decreased w.p.  $1/6$  (since  $e$  is counted from both ends). When  $f \in M$  (w.p.  $1/4$ , and this implies the event  $|M \cap \partial I| = 2$ ),  $e$  is increased with probability  $1/6$  “through  $f$ ”. And same happens for  $g$ . Therefore, regardless of how  $f$  and  $g$  are correlated,  $e$  is increased with probability  $2 \cdot 1/4 \cdot 1/6 = 1/12$ .

In both cases, the expected difference is zero, giving the proof.  $\square$

**Claim B.3** (*Feasible*) The solutions  $\hat{y}|_I$  belong to the spanning tree polytope.

**Proof** The equality constraint  $\sum_{e \in I} \hat{y}_e = |I| - 1$  is satisfied by design, so we need to check that  $\sum_{e \in E[S']} \hat{y}_e \leq |S'| - 1$  for all proper subsets  $S' \subsetneq I$ . Recall that  $\partial_H S'$  has at least six edges, including the edges leaving  $I$ . Let  $a := |E_H(S', I \setminus S')|$  and  $b := |E_H(S', V \setminus I)|$  so that  $a + b \geq 6$ .

Observe that

$$\sum_{e \in E[S']} y_e = \frac{1}{2} \left( \sum_{v \in S'} \sum_{e \in \partial_H v} y_e - \sum_{e \in \partial_H S'} y_e \right) \leq \frac{1}{2} (2|S'| - (a + b)/3) \leq |S'| - 1. \quad (10)$$

- In a local decrease, since we decrease  $y$  to get  $\hat{y}$ , the constraint is still satisfied.
- In a local increase, since we increase only one edge by  $1/3$ , we only need to consider the case when (10) is tight, which implies that every edge going out of  $S'$  has value exactly  $1/3$ . In particular, every vertex in  $S'$  is matched inside  $S'$ . By construction, the edge whose  $y$ -value we increase must have a neighboring edge matched by  $M$  outside  $I$ . This  $M$ -edge cannot be incident to  $S'$ , so we are safe.

$\square$

## B.2 Correlation properties: $|V(H)|$ odd

To prove the correlation properties in the odd case, we observe that each of the proofs from §3.2.1 focuses on a small set of edges. Hence, if we can show that with constant

probability the fractional solution for these edges is unchanged, the proofs go through with small changes (and weaker constants). A more careful case-analysis is likely to give better constants.

We will often make arguments that condition on certain sets of edges not being perturbed by a local increase or decrease. The following lemma gives lower bounds on the probabilities that these desired events occur.

**Lemma B.4** *Let  $f, g$  be internal edges incident to vertex  $v$ . Then, if  $f$  or  $g$  lies in  $M$ ,  $y_f$  and  $y_g$  remain untouched by a local increase or decrease w.p. at least  $2/3$ . Similarly, if  $v$  is an internal vertex with internal edges  $e, f, g, h$ , the four edges remain untouched with probability at least  $2/3$*

**Proof** If one of  $f, g$  lies in  $M$  (which happens w.p.  $1/2$ ), then its  $y$  value equals 1 and it belongs to  $T$  w.p. 1. If there is a local increase, then the increase can only affect the edge not in  $M$  (through its other endpoint), so the  $y$ -value remains unchanged w.p.  $2/3$ . If there is a local decrease, both edges  $f, g$  may have both endpoints that are boundary vertices in worst case, giving a  $4/12$  chance of being decreased, and hence a  $8/12 = 2/3$  chance of  $y$ -values of these two edges remaining unchanged.

The second statement follows from the fact that there are three internal edges adjacent to an external edge.  $\square$

We move on to the proofs of the correlation properties. We restate the lemmas here for convenience.

**Lemma 3.2** *If  $f, g$  are internal edges incident to a vertex  $v$ , then*

Probability statement	MATINT	MAXENT
$\mathbb{P}( T \cap \{f, g\}  = 2)$	$1/9$	$1/9$
$\mathbb{P}(T \cap \{f, g\} = \{f\})$	$1/9$	$12/72$

**Proof of Lemma 3.2, Odd Case** *The MATINT claims:* As in the even case, to prove  $\mathbb{P}(|T \cap \{f, g\}| = 2) \geq 1/9$ , we need only knowledge of the marginals and not the specific sampler. If one of  $f, g$  lies in  $M$  (which happens w.p.  $1/2$ )  $y$ -values of these two edges remaining unchanged with probability at least  $2/3$ , due to Lemma B.4. Now, conditional on these  $y$ -values remaining unchanged, the other edge (not in  $M$ ) is chosen with probability  $1/3$ , making the unconditional probability  $1/2 \cdot 2/3 \cdot 1/3 = 1/9$ , as desired. Similarly, conditioned on  $f$  lying in  $M$  and hence belonging to  $T$ , edge  $g$  is not chosen w.p.  $1 - y_e = 2/3$ , giving a probability at least  $1/4 \cdot 2/3 \cdot 2/3 = 1/9$ , proving the second part of Lemma 3.2 for the MATINT sampler.

*The MAXENT claim:* It remains to show that  $\mathbb{P}(T \cap \{f, g\} = \{f\}) \geq 12/72$ . In the case that  $f \in M$ , we once again have a probability of at least  $2/3$  that  $f$  and  $g$  are untouched, due to Lemma B.4. Since  $\mathbb{P}(g \notin T) = 2/3$  in this case, we have

$$\mathbb{P}(T \cap \{f, g\} = \{f\} \wedge f \in M) \geq 1/4 \cdot 2/3 \cdot 2/3 = 1/9.$$

In the case that neither  $f$  nor  $g$  is in  $M$ , the probability  $f$  or  $g$  is chosen in a local decrease is at most  $1/3$  and the probability  $f$  or  $g$  is chosen in a local increase is at most  $1/2$ . (For the first bound, the worst case is when each of the three endpoints of  $f$ ,  $g$  is incident to an external edge. Then the probability that neither  $f$  nor  $g$  is chosen for a local decrease is at least  $1/2 \cdot 2/3 + 1/4 \cdot 1/3 + 1/4 = 2/3$ . For the second bound, the worst case is when each edge in  $M \cap \partial I$  is incident to an endpoint of  $f$ , w.l.o.g. Then the probability that neither  $f$  nor  $g$  is chosen for a local increase is at least  $1/2 \cdot 2/3 + 1/2 \cdot 1/3 = 1/2$ .) Hence, the edges remain untouched with probability at least  $1/2$ . In total, this gives

$$\mathbb{P}(T \cap \{f, g\} = \{f\} \wedge f, g \notin M) \geq 1/2 \cdot 1/2 \cdot 2/9 = 4/72$$

where the bound  $\mathbb{P}(T \cap \{f, g\} = \{f\} \mid f, g \notin M) \geq 1/9$  was computed in the even case (3.2.1). Hence,

$$\mathbb{P}(T \cap \{f, g\} = \{f\}) \geq 1/9 + 4/72 = 12/72.$$

□

**Lemma 3.3** *If edges  $e, f, g, h$  incident to a vertex  $v$  are all internal, then*

Probability statement	MATINT	MAXENT
$\mathbb{P}( T \cap \{e, f, g, h\}  = 2)$	$2/21$	$8/27$
$\mathbb{P}( T \cap \{e, f\}  =  T \cap \{g, h\}  = 1)$	$4/63$	$16/81$

**Proof of Lemma 3.3, Odd Case** *The MATINT claims:* Each perfect matching  $M$  contains one of these four edges in  $\partial v$ . Say that edge is  $e$ . The vertex  $v$  has no external edges, so regardless of whether we do a local increase or decrease, these four edges remain untouched w.p.  $2/3$ , due to Lemma B.4. If this happens, and if  $e$  also belongs to  $M'$  (w.p.  $1/7$ ), tree  $T$  contains exactly one of  $\{f, g, h\}$ , which gives us the bound  $2/21$ . Moreover, the probability of this edge in  $T$  belonging to the other pair (in this case,  $\{g, h\}$ ) is  $2/3$ , giving the bound  $4/63$ .

*The MAXENT claims:* Since  $v$  is not a boundary vertex, none of  $e, f, g, h$  are picked for a local increase or decrease w.p. at least  $2/3$ , due to Lemma B.4. Conditioning on  $e, f, g, h$  remaining untouched, we have the same analysis as that of the even case (3.2.1). Hence, we obtain

$$\mathbb{P}(|T \cap \{e, f, g, h\}| = 2) \geq 2/3 \cdot 4/9 = 8/27$$

and

$$\mathbb{P}(|T \cap \{e, f\}| = |T \cap \{g, h\}| = 1) \geq 2/3 \cdot 8/27 = 16/81.$$

□

**Lemma 3.4** For an internal edge  $e = uv$ :

(a) if both endpoints are non-boundary vertices, then

Probability statement	MATINT	MAXENT
$\mathbb{P}( \partial_T(u)  =  \partial_T(v)  = 2)$	$1/36$	$128/6561$

(b) if both  $u, v$  are boundary vertices, then

Probability statement	MATINT	MAXENT
$\mathbb{P}(\text{exactly one of } u, v \text{ has odd degree in } T)$	$1/9$	$5/18$

**Proof of Lemma 3.4a, Odd Case** The MATINT claims: In this proof, we will often use that  $G$  is a simple graph. Suppose  $e \in M'$ , which happens w.p.  $1/28$ . Call the edges  $(\partial u \cup \partial v) \setminus \{e\}$  interesting edges. Condition on the matching  $M$  (which decides whether the change is a local increase or a decrease), and the edge  $f$  in  $\partial I$  whose incident internal edge is increased/decreased. At most two of the three internal edges incident to  $f$  can be interesting. Since  $u, v$  are non-boundary vertices, the value of  $e$  is never changed by the alteration.

First suppose it is a local decrease, then with probability at least  $1/3$  a non-interesting edges is reduced, and hence the matroid constraint ensures we always succeed. With probability at most  $2/3$  an interesting edge is reduced, in which case we still have a  $2/3$  chance of getting degree 2 at both  $u$  and  $v$ . So we have  $1/3 + 2/3 \cdot 2/3 = 7/9$  of success, conditioned on the choice of the edge.

Else, suppose it is a local increase; the argument is similar. With probability at least  $1/3$  a non-interesting edges is increased, and hence the matroid constraint ensures we always succeed. With probability at most  $2/3$  an interesting edge is increased, in which case we change the matroid constraint to drop one of the other interesting edges. This dropped edge can be added in with its  $\hat{y}$ -value  $1/3$ , and hence we have a  $2/3$  probability of getting degree 2 at both  $u$  and  $v$ . Again, the probability of success is  $1/3 + 2/3 \cdot 2/3 = 7/9$  of success. Removing the conditioning on  $e \in M'$ , the net probability of success is  $1/28 \cdot 7/9 = 1/36$ .

The MAXENT claims: Let  $U = \delta(u) \setminus \{e\}$  and  $V = \delta(v) \setminus \{e\}$ . Condition on  $e \in M$ . Since  $u, v$  are both non-boundary vertices, the value of  $e$  will remain unchanged regardless of whether there is a local increase or decrease, so we will have  $y_e = 1$  always. Recall that  $y$  is changed at a single edge when the local increase or decrease is performed. We take cases on whether a local increase or local decrease occurs.

*A local decrease occurs.* If none of the edges in  $U \cup V$  are decreased, then (conditioned on  $e \in M$ ) the analysis from the even case (3.2.1) gives

$$\mathbb{P}(X_U = 1 \wedge X_V = 1) \geq 256/2187.$$

Now consider the case that an edge in  $U \cup V$  is decreased. Let  $S_1 = U$  and let  $S_2 = \{a, b\}$  denote the set of two nonzero edges in  $V \setminus \{e\}$  (see Fig. 8). Applying Theorem 2.9 and the fact that  $\mathbb{E}(|S_2 \cap T|) = 2/3$ , we have  $\mathbb{P}(|S_2 \cap T| = 1) \geq 2 \cdot 2/3 \cdot 1/3 = 4/9$ . By Theorem 2.8,

$$1/3 = 1 - 2/3 \leq \mathbb{E}(|S_1 \cap T| \mid a \in T, b \notin T) \leq 1 + 1/3 = 4/3.$$

We apply Theorem 2.9 again to lower bound  $\mathbb{P}(|S_1 \cap T| = 1 \mid a \in T, b \notin T)$  by

$$\mathbb{P}(|S_1 \cap T| = 1 \mid a \in T, b \notin T) \geq 3 \cdot 1/9 \cdot (8/9)^2 = 64/243.$$

Since the same calculation may be repeated with  $b \in T$  and  $a \notin T$  (as  $y_a = y_b$ ), we have

$$\mathbb{P}(X_U = 1 \wedge X_V = 1) \geq 4/9 \cdot 64/243 = 256/2187.$$

So comparing both cases, we have that conditioned on  $e \in M$ , if there is a local decrease then

$$\mathbb{P}(X_U = 1 \wedge X_V = 1) \geq \min\{256/2187, 256/2187\} = 256/2187.$$

*A local increase occurs.* We first note that the probability of no edge in  $U \cup V$  being increased is at least  $1/3$ , in which case the analysis from the even case (3.2.1) gives

$$\mathbb{P}(X_U = 1 \wedge X_V = 1) \geq 256/2187.$$

Now consider the case that an edge  $c$ , say in  $U$ , is increased from  $1/3$  to  $2/3$ . Let  $U \setminus \{c\} = \{a, b\}$ . Since  $\mathbb{E}[X_{a,b}] = 2/3$ , by Markov's inequality we have  $\mathbb{P}(X_{a,b} = 0) \geq 1/3$ . Furthermore,  $\mathbb{E}[X_c \mid X_{a,b} = 0] \geq \mathbb{E}[X_c] = 2/3$  by Theorem 2.8. In total,  $\mathbb{P}(c \in T, a, b \notin T) \geq 1/3 \cdot 2/3 = 2/9$  (Fig. 9). Applying Theorem 2.8 thrice more, we obtain

$$1/3 \leq \mathbb{E}(|S_2 \cap T| \mid c \in T, a, b \notin T) \leq 5/3$$

which by Theorem 2.9 gives us

$$\mathbb{P}(|S_2 \cap T| = 1 \mid c \in T, a, b \notin T) \geq 64/243.$$

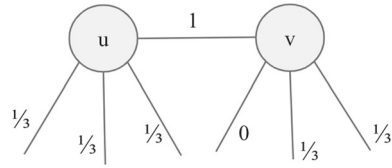
This yields, in the case that an edge in  $U \cup V$  is increased,

$$\mathbb{P}(X_U = 1 \wedge X_V = 1) \geq 64/243 \cdot 2/9 = 128/2187.$$

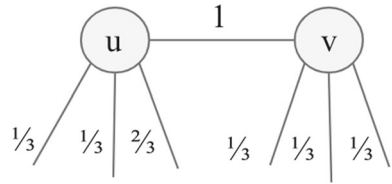
Since the probability that no edge in  $U \cup V$  is increased is at least  $1/3$ , and  $256/2187 > 128/2187$ , in the worst case we have that, conditioned on  $e \in M$ , if there is a local



**Fig. 8** The decrease case in Lemma 3.4a, labelled with  $y$ -values after the decrease is performed



**Fig. 9** The local increase case



increase then

$$\mathbb{P}(\hat{X}_U = 1 \wedge X_V = 1) \geq 1/3 \cdot 256/2187 + 2/3 \cdot 128/2187 = 512/6561.$$

Now comparing the cases of local increase and decrease and removing the conditioning of  $e \in M$ , we obtain in the worst case

$$\mathbb{P}(X_U = 1 \wedge X_V = 1) \geq 1/4 \cdot \min\{256/2187, 512/6561\} = 128/6561$$

which is the bound we sought.

**Proof of Lemma 3.4b, Odd Case** *The MATINT claims.* Fix a matching  $M$  such that  $e \in M$  (w.p.  $1/4$ ); each of  $u, v$  have two other internal edges, each with  $y$ -value  $1/3$ . Call these four *interesting* edges. Suppose  $M \cap \partial I$  has two edges, then we perform a local increase step. Condition on the edge in  $M \cap \partial I$  chosen as part of the local increase: in the worst case it is incident to the endpoint of two of the interesting edges.

- Then w.p.  $2/3$  it increases one of the interesting edges from  $y_f = 1/3$  to  $\hat{y}_f = 2/3$ . The total  $\hat{y}$ -value of interesting edges becomes  $5/3$ . By the same argument as for the even case, the tree  $T$  contains exactly one interesting edge with probability at least  $1/3$ .
- With the remaining probability  $1/3$ , the increase is not to an interesting edge, and hence we succeed with the original probability  $2/3$ .

The total probability of success is therefore  $2/3 \cdot 1/3 + 1/3 \cdot 2/3 = 4/9$ .

On the other hand, if  $M \cap \partial I$  is empty, then we perform a local decrease step. Condition on the boundary edge chosen in the first part of the local decrease. In the worst case it is one of the boundary edges incident to either  $u$  or  $v$ . One of the interesting edges is reduced w.p.  $2/3$ , from  $y_f = 1/3$  to  $\hat{y}_f = 0$ . If that happens, the total  $\hat{y}$ -value of interesting edges sums to 1. The connectivity of the spanning tree means that every tree we sample with these marginals will have exactly one interesting edge.

Hence, in either case, we have exactly one of these four internal edges and satisfy the condition of the claim with probability  $4/9$ , conditioned on having  $e \in M$ . The overall probability is therefore at least  $1/4 \cdot 4/9 = 1/9$ .

*The MAXENT claims.* Observe that the edge  $e$  will always contribute either: 0 to both the degree of  $u$  in  $T$  and the degree of  $v$  in  $T$  OR 1 to both the degree of  $u$  in  $T$  and the degree of  $v$  in  $T$ . Thus, to lower bound the probability that  $u$  and  $v$  have different parities in  $T$ , we will not need to consider whether or not  $e$  is in  $T$ .

Let  $a, b$  be the two *internal* edges in  $\delta(u) \setminus \{e\}$  and  $c, d$  be the two *internal* edges in  $\delta(v) \setminus \{e\}$ . (We know each set has exactly two edges because  $u, v$  are each boundary vertices.) Let  $f, g$  be the two external edges incident to  $u, v$ , respectively. We take cases based on (a) whether a local increase or decrease is performed, (b) whether or not any of  $a, b, c, d$  are changed by the local increase or decrease, and (c) whether or not  $e \in M$ . Note that for (b), we do not take subcases on whether or not  $e$  is changed, by the above paragraph. Also note that for (c), we consider whether  $e \in M$  in order to determine the marginals of  $\{a, b, c, d\}$  prior to a local increase or decrease.

*Case I.N.1 : local increase occurs,  $\{a, b, c, d\}$  not touched,  $e \in M$ .* In this case, we obtain the same bound from Case 1 of the even case in the proof of Lemma 3.4b (see 3.2.1), that is,

$$\mathbb{P}(\text{exactly one of } u, v \text{ has odd degree in } T) \geq \mathbb{P}(X_{a,b,c,d} = 1) \geq \boxed{\frac{2}{3}}.$$

*Case I.N.2 : local increase occurs,  $\{a, b, c, d\}$  not touched,  $e \notin M$ .* This case includes the two subcases  $a, c \in M$  and  $a \in M, c \notin M$ , which are cases 2 and 3 in the proof of Lemma 3.4b (see 3.2.1) and yield lower bounds on the desired probability of  $\boxed{\frac{4}{9}}$  and  $\boxed{\frac{8}{27}}$ , respectively. Finally, there is one more subcase for the case where  $|V(H)|$  is odd, namely,  $a, c \notin T$ , since we may have  $f, g \in M$ . Since in this subcase  $a, b, c, d$  are all  $\frac{1}{3}$ -valued edges, we have the same bound from Case 1 of the even case in the proof of Lemma 3.4, namely,  $\boxed{\frac{2}{3}}$ .

*Case I.T.1 : local increase occurs,  $\{a, b, c, d\}$  touched,  $e \in M$ .* One of the edges  $a, b, c, d$  is increased from  $\frac{1}{3}$  to  $\frac{2}{3}$ , while the rest are still  $\frac{1}{3}$ -valued, since  $e \in M$ . Hence,  $\mathbb{E}[X_{a,b,c,d}] = \frac{5}{3}$ . Furthermore, note that we must have  $X_{a,b,c,d} \geq 1$ . This gives

$$\mathbb{P}(\text{exactly one of } u, v \text{ has odd degree in } T) \geq \mathbb{P}(X_{a,b,c,d} = 1) \geq \boxed{\frac{1}{3}}.$$

*Case I.T.2 : local increase occurs,  $\{a, b, c, d\}$  touched,  $e \notin M$ .* Just as in Case I.N.2., there are three subcases. First consider the subcase of  $a, c \in M$ . In this subcase,  $a, c$  cannot be increased, so we know  $\mathbb{E}[X_{b,d}] = \frac{1}{3}$ . By Theorem 2.9,  $\mathbb{P}(X_{b,d} = 1) \geq \boxed{\frac{1}{2}}$ .

Next, consider the subcase that only  $a \in M$ . Then, we have  $\mathbb{E}(X_{b,c,d}) = \frac{4}{3}$  and hence by Theorem 2.9,  $\mathbb{P}(X_{b,c,d} = 2) \geq 2 \cdot \frac{1}{3} \cdot \frac{1}{3} \geq \boxed{\frac{5}{18}}$ .

For the final subcase that  $f, g \in M$ , we have  $\mathbb{E}[X_{a,b,c,d}] = \frac{5}{3}$  and since  $X_{a,b,c,d} \geq 1$ , we have  $\mathbb{P}(X_{a,b,c,d} = 1) \geq \boxed{\frac{1}{3}}$ .

This concludes the local increase cases. We move on to the local decrease cases.

*Case D.N.1 : local decrease occurs,  $\{a, b, c, d\}$  not touched,  $e \in M$ .* This is the same case as I.N.1., giving a bound of  $\boxed{\frac{2}{3}}$ .

*Case D.N.2 : local decrease occurs,  $\{a, b, c, d\}$  not touched,  $e \notin M$ .* This is the same case as I.N.2., taking only the first subcase ( $a, c \in M$ ), since  $M \cap \partial I = \emptyset$  in a local decrease. Thus the lower bound here is  $\boxed{4/9}$ .

*Case D.T.1 : local decrease occurs,  $\{a, b, c, d\}$  touched,  $e \in M$ .* Since  $e \in M$ , we know  $a, b, c, d$  start off as  $1/3$ -valued. Then, w.l.o.g., say  $a$  is decreased to 0. Then,  $\mathbb{E}[X_{b,c,d}] = 1$ . But since  $X_{b,c,d} \geq 1$ , this gives  $\mathbb{P}(X_{b,c,d} = 1) = \boxed{1}$ .

*Case D.T.2 : local decrease occurs,  $\{a, b, c, d\}$  touched,  $e \notin M$ .* Since in a local decrease  $M \cap \partial I = \emptyset$ , we have w.l.o.g. that  $a, c \in M$  and that  $a$  is not decreased. So  $\mathbb{E}(X_{b,c,d}) = 1 + 2 \cdot 1/3 - 1/3 = 4/3$ . Since we know  $a \in T$ , we instead study  $\mathbb{P}(X_{b,c,d} = 2)$ ; by Theorem 2.9, we have  $\mathbb{P}(X_{b,c,d} = 2) \geq 2 \cdot 1/6 \cdot 5/6 = \boxed{5/18}$ .

The minimum of all the boxed quantities is  $5/18$ . Hence, for the odd case of Lemma 3.4b, we obtain the desired bound

$$\mathbb{P}(\text{exactly one of } u, v \text{ has odd degree in } T) \geq 5/18.$$

□

## C Symmetry lemma

**Calim C.1** (Symmetry Lemma) *For any graph  $H$  with each edge  $e$  having  $x_e = 1/2$ , let  $T$  be a random set of edges faithful to the marginals, i.e.,  $\Pr[e \in T] = x_e$ . For any two edges  $f, g$ , let  $p_{00}, p_{10}, p_{01}, p_{11}$  denote the probabilities of  $T \cap \{f, g\}$  being  $\emptyset, \{f\}, \{g\}, \{f, g\}$  respectively. Then  $p_{00} = p_{11}$  and  $p_{01} = p_{10}$ .*

**Proof** Observe that  $p_{00} + p_{01} + p_{10} + p_{11} = 1$ . Moreover,  $p_{01} + p_{10} + 2p_{11} = \mathbb{E}[|T \cap \{f, g\}|] = x_f + x_g = 1$ . Hence,  $p_{00} = p_{11}$ , proving the first statement. Now,  $p_{10} + p_{11} = x_f = x_g = p_{01} + p_{11}$ , giving  $p_{10} = p_{01}$ , proving the second. □

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