



The maximin support method: an extension of the D'Hondt method to approval-based multiwinner elections

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Received: 31 January 2021 / Accepted: 16 March 2022 / Published online: 29 April 2022
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Abstract

We propose the maximin support method, a novel extension of the D'Hondt apportionment method to approval-based multiwinner elections. The maximin support method is a sequential procedure that aims to maximize the voter support of the least supported elected candidate. It can be computed efficiently and satisfies (adjusted versions of) the main properties of the original D'Hondt method: house monotonicity, population monotonicity, and proportional representation. We also establish a close relationship between the maximin support method and alternative D'Hondt extensions due to Phragmén.

Mathematics Subject Classification 91B14 · 90C05 · 68Q25

A short version of this article has appeared in the Proceedings of the 35th AAAI Conference on Artificial Intelligence [39].

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1 Introduction

In a multiwinner election, the goal is to select a fixed number of candidates (a so-called committee) based on the preferences of a set of agents [13]. Multiwinner voting rules have a wide variety of applications including political elections [6], medical diagnostic decision-making [16], and the selection of “validators” who participate in a blockchain consensus protocol [11].

Recent years have witnessed an increasing interest in settings where the agents express their preferences via approval ballots: For each candidate, an agent has the choice between approving or disapproving the candidate. A particular focus of the approval-based multiwinner voting literature (see [23] for a recent survey) has been on the proportional representation of agents’ preferences in the committee [3, 7, 9, 27–29, 37].

A simpler setting in which proportional representation has been extensively studied is that of apportionment [5]. Here, both candidates and agents have attributes and the goal is to select a committee such that the distribution over attributes in the committee resembles as closely as possible the distribution over attributes among the agents. In classical applications of apportionment, attributes refer to either geographical location or political party affiliation; proportionality then suggests, for instance, that the number of seats assigned to a state of a union in a representative body should be proportional to the population size of the state, or that the number of seats assigned to a political party in a parliament should be proportional to the number of votes the party received in an election. An important apportionment method is named after Victor D’Hondt.¹

The apportionment problem has an illustrious history and has given rise to an elegant mathematical theory [5, 32], but it is not without limitations. For example, requiring voters in a parliamentary election to choose among political parties is often described as restrictive, as it prevents them from expressing more fine-grained preferences [33]. Furthermore, apportionment methods are not applicable in scenarios where attributes (such as party affiliation) are not available. These limitations have led a number of scholars to explore more general settings [e.g., 17, 20]. One important generalization is the setting of approval-based multiwinner elections mentioned above. The apportionment setting corresponds to the special case in which the set of candidates can be partitioned into parties and the approval set of every voter contains precisely the candidates belonging to a single party. Since apportionment problems can be formulated as multiwinner voting problems, every approval-based multiwinner voting rule induces an apportionment method [8]. Indeed, some of the most studied approval-based multiwinner voting rules, those of Phragmén [30] and Thiele [42], have been devised as extensions of the D’Hondt method of apportionment [18].

In this paper, we introduce a novel approval-based multiwinner voting rule. Like Phragmén and Thiele, we take the D’Hondt apportionment method as our point of departure. In contrast to earlier proposals, we focus on a compelling “maximin” characterization of the method in terms of voter support: The D’Hondt method always selects committees maximizing the voter support for the least supported candidate

¹ In the US, the method is named after Thomas Jefferson, the third president of the United States. In fact, Jefferson introduced the method already in 1792, whereas D’Hondt described it in 1878.

in the committee. We generalize the notion of maximin support to approval-based multiwinner elections. When applying this concept iteratively (adding candidates to the committee one at a time), we obtain the *maximin support method (MMS)*. We establish that MMS is an efficiently computable extension of the D'Hondt method that satisfies committee monotonicity, (weak) support monotonicity, and proportional justified representation (PJR).

We also establish a close relationship between the maximin support method and Phragmén's voting rules. In particular, we show that the (computationally intractable) non-sequential variant of Phragmén's rule always produces committees that optimize the maximin support objective globally. From this perspective, both MMS and Phragmén's sequential rule can be considered (axiomatically desirable and polynomial-time computable) approximation algorithms for the maximin support problem. Interestingly, recent independent² work by Cevallos and Stewart [11] has shown that MMS strictly outperforms Phragmén's sequential rule regarding this perspective.

2 Preliminaries

Let C be a finite set of *candidates* and $N = \{1, \dots, n\}$ be a set of n *voters*. Furthermore, $k \in \mathbb{N}$ denotes the number of winners to be selected. We assume $1 \leq k \leq |C|$ and $n \geq 1$.

For each $i \in N$, we let $A_i \subseteq C$ denote the *approval ballot* of voter i . That is, A_i is the subset of candidates that voter i approves of. An *approval profile* is a list $A = (A_1, \dots, A_n)$ of approval ballots, one for each voter $i \in N$. Given an approval profile A and a candidate c , we let $N_c = \{i \in N : c \in A_i\}$ denote the set of approvers of c and we call $|N_c|$ the *approval score* of c . To avoid trivialities, we assume $N_c \neq \emptyset$ for all $c \in C$.

An (*approval-based multiwinner*) *election* E can be represented by a tuple $E = (N, C, A, k)$. An (*approval-based multiwinner voting*) *rule* R is a function that maps an election $E = (N, C, A, k)$ to a subset $R(E) \subseteq C$ of candidates of size $|R(E)| = k$, referred to as the *committee*. We often refer to committee members as *winners*. During the execution of a voting rule, ties between candidates can occur. We assume that ties are broken using a fixed priority ordering over the candidates. An example of a priority ordering is the alphabetic order over the candidates' names, which we use in our examples.

An important subdomain of approval-based multiwinner elections is defined by *party-list elections*, where the set of candidates is partitioned into parties and voters can vote for exactly one party. Formally, a party-list election satisfies $C = P_1 \dot{\cup} P_2 \dot{\cup} \dots \dot{\cup} P_p$ and every approval ballot A_i coincides with one party list P_j . The ballot profile for a party-list election can be summarized by a *vote vector* $V = (v_1, v_2, \dots, v_p)$, where v_j is the number of votes for party P_j (i.e., $v_j = |\{i \in N : A_i = P_j\}|$).

An *apportionment method* takes as input a vote vector $V = (v_1, \dots, v_p)$ and a natural number k and outputs a *seat distribution* $x = (x_1, \dots, x_p) \in \mathbb{N}_0^p$ with $\sum_{j=1}^p x_j = k$. The interpretation is that party P_j is allocated x_j seats. Apportionment

² The work of Cevallos and Stewart [11] is based on a preprint of this paper [38].

methods have been extensively studied in the literature [5, 32]. Since the party-list setting is a special case of the general approval-based multiwinner setting, every approval-based multiwinner rule induces an apportionment method [8]. An approval-based multiwinner rule is called an *extension* of an apportionment method if it induces it. In this paper, we will introduce a novel extension of the apportionment method due to D'Hondt.

The *D'Hondt method* (aka *Jefferson method*) is a particular example from a family of apportionment methods known as *divisor methods* [5, p. 99]. Given a vote vector $V = (v_1, v_2, \dots, v_p)$ and $k \in \mathbb{N}$, the method finds a divisor $d \in \mathbb{R}_{>0}$ such that $\sum_{j=1}^p \lfloor \frac{v_j}{d} \rfloor = k$ and assigns $x_j := \lfloor \frac{v_j}{d} \rfloor$ seats to party P_j . Equivalently, the D'Hondt method selects seat distributions x satisfying $\sum_j x_j = k$ and the following min-max inequality [5, p. 100]:

$$\min_{i: x_i > 0} \frac{v_i}{x_i} \geq \max_j \frac{v_j}{x_j + 1}.$$

Furthermore, the method can also be described as an iterative procedure that assigns seats sequentially: At each iteration, a seat is assigned to the party maximizing $\frac{v_j}{x_j+1}$, where x_j is the number of seats assigned to party P_j in previous iterations [5, p. 100].

To illustrate the D'Hondt method, consider the vote vector (510, 320, 180) and assume that five seats need to be assigned among the three parties P_1 , P_2 , and P_3 . In this example, the D'Hondt method selects $x = (3, 1, 1)$, i.e., it assigns three seats to party P_1 and one seat each to parties P_2 and P_3 . To verify this, observe that $\min_{i: x_i > 0} \frac{v_i}{x_i} = 170 \geq 160 = \max_j \frac{v_j}{x_j+1}$ and that any divisor $d \in (160, 170]$ satisfies $(\lfloor \frac{v_1}{d} \rfloor, \lfloor \frac{v_2}{d} \rfloor, \lfloor \frac{v_3}{d} \rfloor) = (3, 1, 1)$. Using the iterative description, the five seats are allocated to parties in the order $(P_1, P_2, P_1, P_3, P_1)$.

For a vote vector V and a seat distribution x with $x_j > 0$, the value $\frac{v_j}{x_j}$ corresponds to the number of voters that are represented by each seat assigned to party P_j . Intuitively, the lower this value, the better off a party P_j is, because the party needs fewer votes per assigned seat. The D'Hondt method always selects a seat assignment x maximizing $\min_{j: x_j > 0} \frac{v_j}{x_j}$. In the words of Balinski and Young, the D'Hondt method “makes the most advantaged [party] as little advantaged as possible” [5, p. 104]. In the next section, we will interpret $\frac{v_j}{x_j}$ as the “support” of each seat assigned to party P_j . Using this terminology, the D'Hondt method maximizes the minimum support among all seat distributions. In the example from the previous paragraph, each of the three elected candidates from party P_1 is supported by $510/3 = 170$ voters, the elected candidate from party P_2 is supported by 320 voters, and the elected candidate from party P_3 is supported by 180 voters. Therefore, $\min_{j: x_j > 0} v_j/x_j = \min\{170, 320, 180\} = 170$, and all other seat distributions lead to smaller values.

An important proportionality axiom for apportionment methods is *lower quota*, which requires that each party P_j is allocated *at least* $\lfloor k \frac{v_j}{n} \rfloor$ seats. It is well known that the D'Hondt method is the only divisor method satisfying lower quota [5, p. 117]. Moreover, the D'Hondt method satisfies two prominent monotonicity properties: *house monotonicity*, which states that no party loses a seat when the house size k is increased, and *population monotonicity*, which states that if the ratio $\frac{v_i}{v_j}$ increases,

then it should not be the case that x_i decreases and x_j increases [5, p. 117]. Further properties and axiomatic characterizations of the D'Hondt method are discussed in Sect. 5.6.

3 A formal model of voter support

In this section, we formalize the notion of *voter support*,³ on which our extension of the D'Hondt method will be based. In party-list elections, where voters support all candidates of their chosen party, the D'Hondt method maximizes the support of the least supported selected candidate (see Sect. 2). We now generalize the notion of voter support to arbitrary approval profiles, by distributing votes in the form of approval ballots among the candidates.

There are many different ways of distributing the support (i.e., the vote) of a voter $i \in N$ among the candidates in the set $A_i \subseteq C$ of approved candidates, leading to different support values for candidates. For an approval profile $A = (A_1, \dots, A_n)$ and a nonempty subset $D \subseteq C$ of candidates, we define the family $\mathcal{F}_{A,D}$ of (*fractional*) *vote assignments for* (A, D) as the set of all functions that distribute voter support *only* among the candidates in D . Formally, $\mathcal{F}_{A,D}$ consists of all functions $f : (N \times D) \rightarrow [0, 1]$ satisfying $f(i, c) = 0$ for all $i \in N$ and $c \in D \setminus A_i$, and

$$\sum_{c \in A_i \cap D} f(i, c) = 1 \quad \text{for all } i \in N \text{ with } A_i \cap D \neq \emptyset.$$

For each voter $i \in N$, $f(i, c)$ is the fraction of voter i 's vote that is assigned to candidate c . Note that the definition requires that $f(i, c) = 0$ whenever $c \notin A_i$. Thus, the support of a voter is distributed only among those candidates that are approved by the voter. Given a vote assignment $f \in \mathcal{F}_{A,D}$ and a candidate $c \in D$, we let $supp_f(c)$ denote the total support received by c under f , i.e.,

$$supp_f(c) = \sum_{i \in N} f(i, c).$$

Example 3.1 Consider the following approval profile A over the candidate set $C = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$:

$$\begin{array}{lll} 10,000 \times \{c_1, c_2\} & 6000 \times \{c_1, c_3\} & 4000 \times \{c_2\} \\ 5500 \times \{c_3\} & 9500 \times \{c_4\} & 3000 \times \{c_5\} \\ 5000 \times \{c_5, c_6, c_7\} & & \end{array}$$

Consider the candidate subset $D = \{c_1, c_3, c_5\}$ and let f be the (unique) vote assignment in $\mathcal{F}_{A,D}$ with $f(i, c_1) = 0.4$ for each voter i with $A_i = \{c_1, c_3\}$ (thus $f(i, c_3) = 0.6$ for those voters). The function f assigns 2400 out of the 6000 $\{c_1, c_3\}$ -votes to c_1 and the remaining 3600 $\{c_1, c_3\}$ -votes to c_3 , resulting in the following support values:

³ In this paper we use the term "support" to refer to the *backing* or *endorsement* of a candidate by a voter. This should not be confused with the usage of the term in measure theory or probability theory.

$$\text{supp}_f(c_1) = 10,000 + 2400 = 12,400,$$

$$\text{supp}_f(c_3) = 3600 + 5500 = 9100,$$

$$\text{supp}_f(c_5) = 5000 + 3000 = 8000.$$

We will be interested in those vote assignments in $\mathcal{F}_{A,D}$ that maximize the support for the least supported candidate in D . To this end, let $\text{maximin}(A, D)$ denote the maximal support for the least supported candidate in D , where the maximum is taken over all vote assignments in $\mathcal{F}_{A,D}$. Formally,

$$\text{maximin}(A, D) = \max_{f \in \mathcal{F}_{A,D}} \min_{c \in D} \text{supp}_f(c).$$

Furthermore, we let $\mathcal{F}_{A,D}^{\text{opt}}$ denote the nonempty⁴ set of *optimal* vote assignments for (A, D) , i.e.,

$$\mathcal{F}_{A,D}^{\text{opt}} = \{f \in \mathcal{F}_{A,D} : \text{supp}_f(c) \geq \text{maximin}(A, D) \text{ for all } c \in D\}.$$

The vote assignment specified in Example 3.1 is optimal, as $|N_{c_5}| = 8000$ and the approval score of a candidate in D is a natural upper bound for $\text{maximin}(A, D)$.

In the remainder of this paper, we will be interested in finding committees W with a large maximin support value $\text{maximin}(A, W)$ for a given approval profile A . An interesting rationale for maximizing the minimum support was given by Cevallos and Stewart [11]: The maximin support value of a committee puts a limit on the overrepresentation of voter groups. To illustrate this, consider a committee W together with an optimal vote assignment f for (A, W) . Let $D \subseteq W$ be a subset of winning candidates, and consider the set $N_D := \bigcup_{c \in D} N_c$ of voters who approve at least one candidate in D . It follows from the definition of a vote assignment that the total support for candidates in D is upper-bounded by $|N_D|$, i.e., $\sum_{c \in D} \text{supp}_f(c) \leq |N_D|$. On the other hand, $\sum_{c \in D} \text{supp}_f(c) \geq |D| \cdot \text{maximin}(A, W)$. Combining these inequalities yields $|D| \leq |N_D| / \text{maximin}(A, W)$. In other words, the voter group N_D cannot have a number of representatives in the committee that is higher than the size of the voter group divided by the maximin support value.

4 The Maximin Support method

We now propose an extension of the D'Hondt method to approval-based multiwinner elections. It is based on the same principle as the D'Hondt method, in that the voter support for the least supported elected candidate should be as large as possible. We therefore refer to this novel method as *maximin support method (MMS)*. The maximin

⁴ Since $\mathcal{F}_{A,D}$ is an infinite set, we need to make sure that the function $\min_{c \in D} \text{supp}_f(c)$ attains a maximum over this set. We will see in the proof of Theorem 4.2 that the corresponding optimization problem can be formulated as a feasible and bounded linear program. It follows that $\mathcal{F}_{A,D}^{\text{opt}} \neq \emptyset$ and that $\max_{f \in \mathcal{F}_{A,D}} \min_{c \in D} \text{supp}_f(c)$ indeed exists.

support method chooses candidates sequentially⁵ until the desired number of candidates has been selected. In every iteration, a candidate with the greatest support is chosen, under the condition that only vote assignments maximizing the support for the least supported candidate are considered.

Given an approval-based multiwinner election $E = (N, C, A, k)$, the set $W = MMS(E)$ is determined by starting with $W = \emptyset$ and iteratively adding candidates until $|W| = k$. In each iteration, we add to W an unelected candidate receiving the greatest support, under the condition that only optimal vote assignments are considered.⁶ More precisely, for each candidate $c \in C \setminus W$, we compute an optimal vote assignment f_c for the set $W \cup \{c\}$ and determine the total support $s_c = \text{supp}_{f_c}(c)$ that c receives under f_c . The candidate maximizing this value is then added to W . (In the case that two or more candidates have the same s_c value, the priority ordering over candidates is used to break the tie.) See Algorithm 1 for a formal description.

Algorithm 1: Maximin Support Method (*MMS*)

Data: approval-based multiwinner election (N, C, A, k)
Result: subset $W \subseteq C$ of candidates with $|W| = k$

```

1  $W = \emptyset$ 
2 for  $j = 1$  to  $k$  do
3   foreach  $c \in C \setminus W$  do
4     compute  $f_c \in \mathcal{F}_{A, W \cup \{c\}}^{\text{opt}}$ 
5      $s_c = \text{supp}_{f_c}(c)$ 
6    $w \in \arg \max_{c \in C \setminus W} s_c$ 
7    $W = W \cup \{w\}$ 
8 return  $W$ 

```

Since the set $\mathcal{F}_{A, W \cup \{c\}}^{\text{opt}}$ of optimal vote assignments may contain more than one function, the value of $\text{supp}_{f_c}(c)$ could potentially depend on the choice of f_c . The following result implies that this is not the case.

Theorem 4.1 *Let $E = (N, C, A, k)$ be an approval-based multiwinner election. The following holds for each $j \in \{0, \dots, k - 1\}$.*

Let W^j denote the set of the first j candidates chosen by the maximin support method when applied to E . Then, for each candidate $c \in C \setminus W^j$ and for each optimal vote assignment $f_c \in \mathcal{F}_{A, W^j \cup \{c\}}^{\text{opt}}$,

$$\text{supp}_{f_c}(c) = \text{maximin}(A, W^j \cup \{c\}).$$

Theorem 4.1 states that in every iteration the candidate c added to W is among the least supported candidates under every optimal vote assignment. The support of this

⁵ One can also define a non-sequential (optimization) variant of the maximin support method; we discuss this variant in Sect. 6.2.

⁶ Restricting attention to optimal vote assignments ensures that support for previously elected candidates is not ignored when searching for new vote assignments.

candidate thus equals $\text{maximin}(A, W \cup \{c\})$, which (by definition) is independent of the particular vote assignment f_c that was chosen in line 4 of the algorithm. The proof of Theorem 4.1 employs linear programming duality theory and can be found in the appendix.

This result gives rise to an interesting alternative formulation of the maximin support method. In this equivalent formulation, there is no need to choose an optimal vote assignment for $(A, W \cup \{c\})$; rather, s_c is directly defined as $\text{maximin}(A, W \cup \{c\})$. A natural interpretation of this definition is that the value s_c measures the effect that the addition of a potential candidate would have on the maximal support for the least supported candidate.

The next theorem establishes that the maximin support method can be computed efficiently.

Theorem 4.2 *The maximin support method can be computed in polynomial time.*

Proof Since the innermost for-loop is executed at most $|C|$ times for every $j \in \{1, \dots, k\}$ and $k \leq |C|$, lines 4 and 5 are executed at most $k|C| < |C|^2$ times. It is therefore sufficient to show that, for any subset $D \subseteq C$ of candidates, an optimal vote assignment $f \in \mathcal{F}_{A,D}^{\text{opt}}$ can be computed in polynomial time. For a given approval profile A and a subset $D \subseteq C$ of candidates, consider the following linear program, containing a variable $f(i, c)$ for each $i \in N$ and $c \in A_i \cap D$, and an additional variable s .

$$\begin{aligned}
 & \text{maximize} && s \\
 & \text{subject to} && \sum_{i \in N: c \in A_i \cap D} f(i, c) \geq s, && \text{for all } c \in D \\
 & && \sum_{c \in A_i \cap D} f(i, c) = 1, && \text{for all } i \in N \text{ with } A_i \cap D \neq \emptyset \\
 & && f(i, c) \geq 0, && \text{for all } i \in N \text{ and } c \in D
 \end{aligned}$$

The first set of constraints requires that the support for the least supported candidate in D is at least s , while the remaining constraints ensure that the variables $f(i, c)$ encode a valid vote assignment.⁷ Therefore, optimal solutions of this linear program correspond to optimal vote assignments. Since linear programming problems can be solved in polynomial time [19], this concludes the proof. \square

We conclude this section by illustrating the maximin support method with an example.

Example 4.1 Consider the election $E = (N, C, A, k)$, where A is the approval profile from Example 3.1 and $k = 3$.

In the first iteration, the value $s_c = \text{maximin}(A, \{c\})$ equals the approval score of candidate c , i.e., $s_c = |N_c|$ for all c . Therefore, the approval winner c_1 (with

⁷ Note that constraints of the form $f(i, c) \leq 1$ are not necessary because each variable $f(i, c)$ is non-negative and appears in a constraint of the form $\sum_{c \in A_i \cap D} f(i, c) = 1$.

$s_{c_1} = 16,000$) is chosen. The corresponding vote assignment f satisfies $f(i, c_1) = 1$ for all $i \in N_{c_1}$.

In the second iteration, we have $W = \{c_1\}$ and we need to compute the value $s_x = \text{maximin}(A, \{c_1, x\})$ for all $x \in C \setminus \{c_1\}$. For example, for candidate c_2 we get $s_{c_2} = \text{maximin}(A, \{c_1, c_2\}) = 10,000$; the corresponding vote assignment assigns 4000 out of the 10,000 $\{c_1, c_2\}$ -votes to c_1 and the remaining 6000 to c_2 . A better value is achieved by candidate c_3 . The vote assignment realizing $s_{c_3} = \text{maximin}(A, \{c_1, c_3\}) = 10,750$ assigns all 10,000 $\{c_1, c_2\}$ -votes to c_1 , all 5500 $\{c_3\}$ -votes to c_3 , and divides the 6000 $\{c_1, c_3\}$ -votes between c_1 and c_3 such that both candidates have a total support of 10,750 each. Computing the other values, we get $s_{c_4} = 9500$, $s_{c_5} = 8000$, and $s_{c_6} = s_{c_7} = 5000$. Therefore, c_3 is selected.

In the third iteration, we have $W = \{c_1, c_3\}$ and we need to compute the value $s_x = \text{maximin}(A, \{c_1, c_3, x\})$ for all $x \in C \setminus \{c_1, c_3\}$. It can be checked that $s_{c_2} = 8500$, $s_{c_4} = 9500$, $s_{c_5} = 8000$, and $s_{c_6} = s_{c_7} = 5000$. Thus, candidate c_4 is chosen. There are several optimal vote assignments realizing $\text{maximin}(A, \{c_1, c_3, c_4\}) = 9500$; each of them assigns all 9500 $\{c_4\}$ -votes to c_4 and distributes the 6000 votes that approve of c_1 and c_3 in such a way that c_1 and c_3 have a total support of at least 9500 each.

In summary, we have $MMS(E) = \{c_1, c_3, c_4\}$.

5 Axiomatic properties of MMS

In this section, we show that the maximin support method is indeed an extension of the D'Hondt method and that it satisfies (adjusted versions of) several important properties that the latter satisfies. In particular, we show that the maximin support method satisfies *committee monotonicity* (aka house monotonicity), *weak support monotonicity* (a variant of population monotonicity), and *proportional justified representation*. We also discuss properties that are not satisfied by MMS (Sect. 5.5) and the possibility to generalize axiomatic characterizations of the D'Hondt method (Sect. 5.6).

5.1 D'Hondt extension

We first show that the maximin support method coincides with the D'Hondt method in the party-list domain.

Theorem 5.1 *The maximin support method is an extension of the D'Hondt method.*

Proof Consider a party-list election $E = (N, C, A, k)$ with $C = P_1 \cup \dots \cup P_p$ and vote vector $V = (v_1, \dots, v_p)$, where $v_j = |\{i \in N : A_i = P_j\}|$. We show that the maximin support method chooses a committee $W = MMS(E)$ that corresponds to the seat distribution $x = (x_1, \dots, x_p)$ selected by the D'Hondt method, in the sense that $|W \cap P_j| = x_j$ for every $j \in \{1, \dots, p\}$.

We consider the iterative formulation of the D'Hondt method and prove the statement by induction on the committee size k . For $k = 1$, the statement holds because the maximin support method as well as D'Hondt give the only available seat to (a candidate from) a party P_j maximizing v_j .

Now let $k > 1$ and assume that the maximin support method and D'Hondt agree on the first $k - 1$ rounds. Let W denote the set of $k - 1$ candidates selected by MMS in the first $k - 1$ rounds and let $y = (y_1, \dots, y_p)$ denote the seat assignment constructed by the D'Hondt method in the first $k - 1$ rounds. By our assumption, we have $|W \cap P_j| = y_j$ for all j . Recall that the D'Hondt method assigns the last seat to the party P_{j^*} maximizing $\frac{v_j}{y_j+1}$. We need to show that the maximin support method selects a candidate from that party.

In the party-list domain, the maximin support value $\text{maximin}(A, D)$ of any subset $D \subseteq C$ is given by $\min_{r:|D \cap P_r|>0} \frac{v_r}{|D \cap P_r|}$, because distributing voter support uniformly among the candidates of a party is clearly optimal. Therefore, when we add a candidate c from a party P_j to W , thereby obtaining a seat distribution x satisfying $x_j = y_j + 1$ and $x_\ell = y_\ell$ for all $\ell \neq j$, the new maximin value is $\text{maximin}(A, W \cup \{c\}) = \min_{r:x_r>0} \frac{v_r}{x_r}$. Applying the min-max inequality for D'Hondt seat distributions (see Sect. 2) to y , we get $\frac{v_j}{x_j} = \frac{v_j}{y_j+1} \leq \frac{v_\ell}{y_\ell} = \frac{v_\ell}{x_\ell}$ for all $\ell \neq j$. Thus, the minimum of $\frac{v_r}{x_r}$ is attained at $r = j$ and $\text{maximin}(A, W \cup \{c\}) = \frac{v_j}{x_j} = \frac{v_j}{y_j+1}$. Since the maximin support method selects a candidate maximizing $\text{maximin}(A, W \cup \{c\})$, it selects a candidate from the party P_{j^*} maximizing $\frac{v_j}{y_j+1}$, as desired. \square

5.2 Committee monotonicity

Committee monotonicity requires that all selected candidates are still selected when the committee size k is increased. Since the maximin support method selects winners iteratively, committee monotonicity is trivially satisfied.

Observation 5.2 *The maximin support method satisfies committee monotonicity.*

5.3 Support monotonicity

In its most basic version, support monotonicity requires that additional support for a selected candidate does not lead to that candidate dropping out of the winning committee. A natural extension of this idea has been proposed by Sánchez-Fernández and Fisteus [35] and considers what happens when the support of several of the winners is increased. One version of the axiom, referred to as *weak support monotonicity*, requires that, when the support of a subset of the winners is increased, at least one of those candidates must remain in the winning set.

Definition 5.1 An approval-based multiwinner voting rule R satisfies *weak support monotonicity* if the following statements hold for all approval-based multiwinner elections $E = (N, C, A, k)$ and for all nonempty subsets $G \subseteq R(E)$ of winning candidates:

1. (weak support monotonicity without population increase) Let $i \in N$ be a voter with $A_i \cap G = \emptyset$ and consider the election $E' = (N, C, A', k)$, where $A'_j = A_j$ for all $j \in N \setminus \{i\}$ and $A'_i = A_i \cup G$. Then, $R(E') \cap G \neq \emptyset$.

- (weak support monotonicity with population increase) Let $n + 1$ be a new voter and consider the election $E'' = (N \cup \{n + 1\}, C, A'', k)$, where $A''_j = A_j$ for all $j \in N$ and $A''_{n+1} = G$. Then, $R(E'') \cap G \neq \emptyset$.

We note that this definition reduces to the basic version of support monotonicity when $|G| = 1$, and thus, despite its name, weak support monotonicity is slightly stronger than the basic version.

Theorem 5.3 *The maximin support method satisfies weak support monotonicity.*

Proof First of all, we observe that for any nonempty set $X \subseteq C$ that is disjoint from G , the maximin support value of X is the same for $E, E',$ and E'' . This is because the changes made in E' and E'' do not affect how the votes can be distributed among the candidates in X .

Let r be the MMS iteration in which the first candidate from G is elected in election E and let $c^* \in G$ be this candidate. For $0 \leq j \leq k$, let $W^j, W^j_{A'}$, and $W^j_{A''}$ be the first j candidates chosen by MMS for elections $E, E',$ and E'' . (For $j = 0$, we have $W^0 = W^0_{A'} = W^0_{A''} = \emptyset$.)

If at least one candidate from G is selected within the first $r - 1$ iterations of the execution of MMS for election E' (respectively, for election E''), the statement of the theorem holds. Therefore, we assume that in the first $r - 1$ iterations no candidate from G is selected for election E' (respectively, for election E''). In this case, in each iteration the candidate added to the committee will be the same for E and E' (respectively, E and E'') because the computation of *maximin* is done over sets of candidates disjoint from G . Consequently, $W^{r-1} = W^{r-1}_{A'}$ (respectively, $W^{r-1} = W^{r-1}_{A''}$).

We are going to prove that the candidate chosen in iteration r for election E' (respectively, for election E'') belongs to G . First, we observe that since $W^{r-1} \cap G = \emptyset$, for each candidate $c \in C \setminus (W^{r-1} \cup G)$ the maximin support value of $W^{r-1} \cup \{c\}$ is the same for elections $E, E',$ and E'' . It is therefore sufficient to prove that the maximum support value of $W^{r-1} \cup \{c^*\} = W^r$ for election E' (respectively, for election E'') is greater than or equal to the maximin support value of W^r for election E . Further, it is enough to find a vote assignment $g \in \mathcal{F}_{A', W^r}$ (respectively, a vote assignment $h \in \mathcal{F}_{A'', W^r}$) such that for each candidate c in W^r the support of c under g (respectively, the support of c under h) is greater than or equal to the maximin support value of W^r for election E .

Consider any optimal vote assignment $f \in \mathcal{F}_{A, W^r}^{\text{opt}}$. For election E' we can define g as follows. If $A_i \cap W^r \neq \emptyset$, then let $g(j, c) = f(j, c)$ for each voter $j \in N$ and each candidate $c \in W^r$. If voter i does not approve any of the candidates in W^r in election E (that is, if $A_i \cap W^r = \emptyset$), then for each candidate $c \in W^r$ we have $f(i, c) = 0$. In that case, we define $g(j, c) = f(j, c)$ for each voter $j \in N \setminus \{i\}$ and each candidate $c \in W^r$, $g(i, c^*) = 1$, and $g(i, c) = 0$ for each candidate $c \in W^{r-1}$.

For election E'' , let $h(j, c) = f(j, c)$ for each voter $j \in N$ and each candidate $c \in W^r$, $h(n + 1, c^*) = 1$, and $h(n + 1, c) = 0$ for each candidate $c \in W^{r-1}$.

Clearly, each candidate in W^r receives a support under g and h that is greater than or equal to the support that the same candidate receives under f . Moreover, since $f \in \mathcal{F}_{A, W^r}^{\text{opt}}$, all candidates in W^r receive a support under f that is greater than or equal to the maximin support value of W^r for election E .

In summary, we have:

- The maximin support value of $W^{r-1} \cup \{c^*\} = W^r$ for elections E' and E'' is greater than or equal to that for election E .
- For each candidate $c \in C \setminus (W^{r-1} \cup G)$ the maximin support value of $W^{r-1} \cup \{c\}$ is the same for elections $E, E',$ and E'' .
- A similar argument to that used for c^* shows that for each candidate $c \in G$ the maximin support value of $W^{r-1} \cup \{c\}$ for elections E' and E'' is greater than or equal to that for election E .

It follows that, for elections E' and E'' , the maximin support value of W^r is greater than or equal to the maximin support value of $W^{r-1} \cup \{c\}$ for each candidate $c \in C \setminus (W^{r-1} \cup G)$. Equality can hold for elections E' and E'' only if it also holds for election E . In such case, we assume that a priority order of the candidates is used to break ties, and therefore, no candidate in $C \setminus (W^{r-1} \cup G)$ can be selected at iteration r for elections E' and E'' . \square

5.4 Proportional representation

Axioms capturing the proportional representation of voter preferences in approval-based multiwinner elections have been studied extensively in recent years [3, 7, 27–29]. In this paper, we focus on an axiom known as *proportional justified representation (PJR)* [37]. PJR is a generalization of the lower quota axiom to the general approval-based multiwinner setting: If a voting rule satisfies PJR, then its induced apportionment method satisfies lower quota [8].

In order to define PJR, we need some terminology. Consider an election (N, C, A, k) . Given a positive integer $\ell \in \{1, \dots, k\}$, we say that a subset $N^* \subseteq N$ of voters is ℓ -cohesive if $|N^*| \geq \ell \frac{n}{k}$ and $|\bigcap_{i \in N^*} A_i| \geq \ell$. The first condition requires that the group is large enough to “claim” ℓ representatives (in a perfectly proportional committee, every candidate represents $\frac{n}{k}$ voters) and the second condition requires that there are at least ℓ candidates unanimously approved by the group. Intuitively, an ℓ -cohesive group deserves to have ℓ of “their” candidates in the committee. A subset $D \subseteq C$ of candidates of size k provides *proportional justified representation (PJR)* if for all $\ell \in \{1, \dots, k\}$ and all ℓ -cohesive subsets $N^* \subseteq N$, it holds that $|D \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$.

Definition 5.2 An approval-based multiwinner voting rule R satisfies *proportional justified representation (PJR)* if $R(E)$ provides PJR for every election E .

We now show that MMS satisfies this property.

Theorem 5.4 *The maximin support method satisfies proportional justified representation.*

Proof For the sake of contradiction, suppose that there exists an election $E = (N, C, A, k)$ and an ℓ -cohesive group $N^* \subseteq N$ such that for the set $W = \text{MMS}(E)$ of winners output by the maximin support method we have $|W \cap (\bigcup_{i \in N^*} A_i)| < \ell$. Thus, there are $x > k - \ell$ candidates in W that are not approved of by any voter in N^* ,

and therefore, the support of some of these x candidates (and the maximum support of the least supported candidate in W) has to be strictly less than $\frac{n}{k}$, because

$$\frac{|N| - |N^*|}{x} \leq \frac{n - \ell \frac{n}{k}}{x} = n \frac{k - \ell}{kx} < n \frac{k - \ell}{k(k - \ell)} = \frac{n}{k}.$$

By Theorem 4.1, at each iteration the candidate that is added to the set of winners is one of the least supported when we maximize the support of the least supported candidate in the current set of winners. Therefore, at some iteration during the execution of *MMS* for election E , the support of the candidate that we add to the set of winners is strictly less than $\frac{n}{k}$ (this happens for sure in the very last iteration).

Let j be the first iteration of *MMS* for election E such that the maximum support of the least supported candidate in W is less than $\frac{n}{k}$ and let c be the candidate elected in this iteration. Let c^* be a candidate that is approved by all the voters in N^* and that does not belong to W (such a candidate exists because $|\bigcap_{i \in N^*} A_i| \geq \ell$ but $|W \cap (\bigcup_{i \in N^*} A_i)| < \ell$). Since all the voters in N^* approve c^* and there are at most $\ell - 1$ candidates in W that are approved by some voters in N^* , if we add candidate c^* to the set of winners instead of candidate c at iteration j , the support of c^* when we maximize the support of the least supported candidate would be at least $\frac{|N^*|}{\ell}$ (observe that if the support of c^* were less than $\frac{|N^*|}{\ell}$, then there must exist some candidates in W that are approved by some voters in N^* that receive a support from the voters in N^* greater than $\frac{|N^*|}{\ell}$; we could then iteratively pick each of such candidates and give the surplus coming from voters in N^* to c^*). Observe now that $\frac{|N^*|}{\ell} \geq \frac{\ell \frac{n}{k}}{\ell} = \frac{n}{k}$, and therefore candidate c^* would be elected ahead of candidate c at iteration j , a contradiction. \square

Remark 5.1 The maximin support method also satisfies the recently introduced notion of *priceability* [27], which implies PJR. Since priceability characterizes D'Hondt seat distributions in the party-list domain [27], this constitutes an alternative proof of Theorem 5.1.

In the same spirit as lower quota, PJR ensures that each cohesive group of voters is represented in the committee by *at least* the number of candidates that is proportional to the group size. In light of the discussion at the end of Sect. 3, it can be argued that *MMS* strikes an attractive compromise between two competing representation goals.

5.5 Properties not satisfied by *MMS*

In this section, we discuss axiomatic properties from the literature that are violated by the maximin support method. We start with two proportionality properties: extended justified representation and perfect representation.

Extended Justified Representation (EJR) [3]. A subset $D \subseteq C$ of candidates of size k provides *extended justified representation (EJR)* if for all $\ell \in \{1, \dots, k\}$ and all ℓ -cohesive subsets $N^* \subseteq N$, there exists a voter $i \in N^*$ with $|A_i \cap D| \geq \ell$.

An approval-based multiwinner voting rule R satisfies *EJR* if $R(E)$ provides *EJR* for every election $E = (N, C, A, k)$.

EJR is a strictly stronger requirement than *PJR*. The following example shows that *MMS* does not satisfy *EJR*.

Example 5.1 Consider the election $E = (N, C, A, k)$ with $k = 4$ and $C = \{a_1, a_2, a_3, c_1, c_2, c_3, c_4\}$. There are 16 voters casting the following ballots:

$$\begin{array}{lll} 5 \times \{a_1, c_1, c_2, c_3, c_4\} & 4 \times \{a_2, c_1, c_2, c_3, c_4\} & 3 \times \{a_3, c_1, c_2, c_3, c_4\} \\ 2 \times \{a_1\} & 1 \times \{a_2\} & 1 \times \{a_3\} \end{array}$$

The maximin support method selects $MMS(E) = \{c_1, a_1, a_2, a_3\}$ (selected in this order). The 12 voters whose approval set contains $\{c_1, c_2, c_3, c_4\}$ form a 3-cohesive group, but none of these voters approves at least 3 candidates in the committee. Therefore, $MMS(E)$ fails to provide *EJR* for this election.

Perfect representation (PR) [37]. Consider an election $E = (N, C, A, k)$ such that k divides $n = |N|$. A candidate subset W of size k provides *perfect representation* for E if it is possible to partition N into k pairwise disjoint subsets of size $\frac{n}{k}$ and assign a distinct candidate from W to each of these subsets so that all the voters in a subset approve their assigned candidate. A rule satisfies *perfect representation (PR)* if it outputs a committee that provides *PR* whenever such a committee exists.

An example of a committee that provides *PR* is the output of the maximin support method in Example 5.1.⁸ We can assign a_1 to the 2 voters who approve only candidate a_1 and to 2 of the 5 voters who approve $\{a_1, c_1, c_2, c_3, c_4\}$, a_2 to the voter who approves only candidate a_2 and to 3 of the 4 voters who approve $\{a_2, c_1, c_2, c_3, c_4\}$, a_3 to all the voters who approve a_3 , and c_1 to the four remaining voters. Sánchez-Fernández et al. [37] proved that no polynomial-time computable rule can satisfy *PR*, from which it directly follows that *MMS* fails *PR*. It is also not difficult to find examples in which *MMS* fails *PR*.

Example 5.2 Consider the election $E = (N, C, A, k)$ with $k = 2$ and $C = \{a, b, c\}$. There are 8 voters casting the following ballots:

$$3 \times \{a, b\} \quad 2 \times \{a, c\} \quad 1 \times \{b\} \quad 2 \times \{c\}$$

The maximin support method selects $MMS(E) = \{a, c\}$. It is evident that this committee does not provide *PR* since the voter who approves only candidate b is unrepresented. However, the committee $\{b, c\}$ provides *PR* for this election.

Strong support monotonicity. With respect to support monotonicity, Sánchez-Fernández and Fisteus [35] also consider a stronger axiom called *strong support monotonicity* (with and without population increase) that requires that, if the support of a subset G of the winners is increased, all candidates in G must remain in the committee. The following example shows that *MMS* does not satisfy this stronger requirement.

⁸ This example also shows that *PR* and *EJR* are incompatible requirements.

Example 5.3 Consider the election $E = (N, C, A, k)$ with $k = 6$ and $C = \{a, b, c_1, \dots, c_5\}$. There are 18 voters casting the following ballots:

$$13 \times \{c_1, \dots, c_5\} \quad 2 \times \{a, b\} \quad 2 \times \{a\} \quad 1 \times \{b\}$$

For this election, we have $MMS(E) = \{a, c_1, \dots, c_5\}$ (candidate a is elected in the fourth iteration). If a new voter enters the election and approves of precisely $\{a, c_1, \dots, c_5\}$, then the committee selected by MMS is $\{a, b, c_1, c_2, c_3, c_4\}$ (candidate a is now elected in the third iteration, while candidate b is elected in the last one). This proves that MMS violates strong support monotonicity with population increase.

To prove that MMS violates strong support monotonicity *without* population increase, we modify E by adding a new candidate d and a new voter approving of $\{d\}$. Let E' denote this modified election. It is easy to check that $MMS(E') = MMS(E) = \{a, c_1, \dots, c_5\}$. If the new voter changes her approval set to $\{a, c_1, \dots, c_5, d\}$, then the set of MMS winners is again given by $\{a, b, c_1, c_2, c_3, c_4\}$.

Remark 5.2 Whether strong support monotonicity without population increase and PJR are compatible axioms is an open issue.

Strategyproofness. The maximin support method is not strategyproof. This follows from a known incompatibility between strategyproofness and proportional representation in approval-based multiwinner elections due to Peters [26] and Kluiving et al. [22]. In particular, every approval-based multiwinner rule that satisfies a modest degree of proportional representation is vulnerable to manipulations in which voters misrepresent their approval set by omitting some of their approved candidates. As a consequence, none of the rules considered in Sect. 6 is strategyproof.

5.6 MMS and axiomatic characterizations of the D'Hondt method

Despite the positive axiomatic results in Sects. 5.1–5.4, an axiomatic characterization of the maximin support method remains elusive.⁹ In the following, we review three axiomatic characterizations of the D'Hondt method given by Balinski and Young [5] and we discuss the potential of generalizing the respective properties to approval-based multiwinner elections.

First characterization The first characterization exclusively uses axioms that have already been introduced in Sect. 2.

Theorem 5.5 (Balinski and Young [5, p. 130], Proposition 6.4) *The D'Hondt method is the unique apportionment method satisfying population monotonicity and lower quota.*

We have already seen that the maximin support method satisfies adjusted versions of lower quota and population monotonicity for approval-based multiwinner elections

⁹ The same holds for other sequential rules such as those proposed by Phragmén (see Sect. 6.3) and Thiele (see Sect. 6.4). With respect to the axioms discussed in Sects. 5.1–5.5, Phragmén's sequential rule performs identically to the maximin support method.

(Sects. 5.3 and 5.4). However, there are other well-known approval-based multiwinner rules that satisfy both axioms (see Sect. 6).

Second Characterization. The second axiomatic characterization of the D'Hondt method requires another axiom called *uniformity* [5, p. 141–142]. Informally speaking, uniformity requires “that an apportionment that is acceptable to all [parties] must be acceptable if restricted to any subset of [parties] considered alone” [5, p. 141].

Theorem 5.6 (Balinski and Young [5, p. 149], Proposition 8.2) *The D'Hondt method is the unique apportionment method satisfying uniformity and lower quota.*

It can be shown that the maximin support method satisfies an appropriate generalization of uniformity to approval-based multiwinner elections. The fact that MMS satisfies (generalized) uniformity and PJR is an alternative proof that MMS is a D'Hondt extension. However, this combination of axioms does not characterize MMS because other rules satisfying both axioms can be defined.

Third Characterization. The final axiomatic characterization of the D'Hondt method given by Balinski and Young [5] is concerned with coalition incentives. An apportionment method *encourages coalitions* if, whenever two parties merge (i.e., form a coalition), then the coalition gets at least as many seats as the sum of the seats of the two individual parties [5, p. 150].

Theorem 5.7 (Balinski and Young [5, p. 150], Proposition 9.1) *The D'Hondt method is the unique apportionment method that satisfies population monotonicity and encourages coalitions.*

Whether similar coalition incentives hold for the maximin support method depends on how exactly this property is generalized to approval-based multiwinner elections. For example, it can be shown that MMS encourages the coalition of two groups of voters provided that (1) the members of each group have identical approval sets and (2) the approval set of each group is disjoint from the approval set of the other group and from the approval set of each voter outside the two groups. If those conditions do not hold, groups of voters may suffer from joining forces, as the following example shows.

Example 5.4 Consider the election $E = (N, C, A, k)$ with $k = 3$ and $C = \{a, b, c, d, e, f\}$. There are 47 voters casting the following ballots:

$$\begin{array}{lll} 6 \times \{a, c\} & 6 \times \{a, b, c\} & 4 \times \{c\} \\ 8 \times \{a, e\} & 8 \times \{b, f\} & 15 \times \{d\} \end{array}$$

The maximin support method selects $MMS(E) = \{a, d, b\}$ (in this order). Now, if the 8 voters who approve $\{a, e\}$ and the 8 voters who approve $\{b, f\}$ decide to join forces and all approve $\{a, b, e, f\}$, then the maximin support method selects $\{a, c, d\}$ (in this order).

Remark 5.3 In the context of approval-based single-winner elections, *approval voting* (AV) is the rule that selects the candidate with highest approval score. By definition,

MMS coincides with AV when $k = 1$. Therefore, MMS can be seen as an extension of AV to the multiwinner setting. Axiomatic characterizations of AV have been proposed by Fishburn [14, 15] and Alós-Ferrer [1]. All these characterizations make use of the well-known *consistency* axiom [41, 43]. Consistency has been generalized to approval-based multiwinner elections by Lackner and Skowron [24], who used it to characterize the class of approval-based committee scoring rules. The maximin support method is not a member of this class and violates consistency.

6 Comparison to other D'Hondt extensions

The maximin support method is by no means the only way to extend the D'Hondt method to approval-based multiwinner elections. As mentioned in the introduction, the rules proposed by the Scandinavian mathematicians L. Edvard Phragmén [30] and Thorvald N. Thiele [42] also generalize the D'Hondt method. In this section, we compare the maximin support method to those rules. We start with Phragmén's rules (Sects. 6.1–6.3) and consider Thiele's rules in Sect. 6.4. For an extensive treatment of Phragmén's and Thiele's rules and their properties, we refer to the survey by Janson [18].

6.1 Load distributions

Phragmén's rules can be described as *load distribution* methods [7]. Every selected candidate induces one unit of *load*, and this load needs to be distributed among the approvers of that candidate. For example, if there are 6 voters approving candidate c and we decide to select c for the committee, then one possible way of distributing the load would be to give a load of $\frac{1}{6}$ to each of those voters. However, it is not required that the load is distributed evenly among the approvers: different approvers of c could be assigned different (non-negative) loads, as long as the loads associated with each selected candidate sum up to 1. The goal is to choose a committee such that the load distribution is as *balanced* as possible. Different interpretations of balancedness lead to different optimization goals; the most relevant variant *minimizes the maximal load* of a voter.

In particular, *max-Phragmén* is the rule that returns committees corresponding to load distributions minimizing the maximal voter load. And *seq-Phragmén* is a sequential (greedy) version of max-Phragmén; it selects candidates iteratively, in each round adding a candidate to the committee such that the new maximal voter load is as small as possible.

Definition 6.1 Given an approval profile A and a subset $D \subseteq C$ of candidates, a *load distribution for D given A* is a two-dimensional array $\ell = (\ell_{i,c})_{i \in N, c \in D}$ satisfying

$$\begin{aligned} 0 \leq \ell_{i,c} \leq 1 & \quad \text{for all } i \in N \text{ and } c \in D, \\ \ell_{i,c} = 0 & \quad \text{for all } i \in N \text{ and } c \in D \setminus A_i, \text{ and} \\ \sum_{i \in N} \ell_{i,c} = 1 & \quad \text{for all } c \in D. \end{aligned}$$

We let $\mathcal{L}_{A,D}$ denote the set of all load distributions for (A, D) . For a load distribution $\ell \in \mathcal{L}_{A,D}$, the total load of voter i under ℓ , denoted ℓ_i , is given by $\ell_i = \sum_{c \in D} \ell_{i,c}$. Note that $\sum_{i \in N} \ell_i = |D|$ for all $\ell \in \mathcal{L}_{A,D}$. Finally, a load distribution is called optimal for (A, D) if the maximal total voter load $\max_{i \in N} \ell_i$ is as small as possible. $\mathcal{L}_{A,D}^{\text{opt}}$ denotes the set of all optimal load distributions for (A, D) .

We are now going to establish a close connection between load distributions and vote assignments.

Lemma 6.1 *Let A be an approval profile and $D \subseteq C$ a subset of candidates. Then, the following statements hold.*

1. *For every vote assignment $f \in \mathcal{F}_{A,D}$, there is a load distribution $\ell^f \in \mathcal{L}_{A,D}$ such that*

$$\max_{i \in N} \ell_i^f \leq \frac{1}{\min_{c \in D} \text{supp}_f(c)}.$$

2. *For every load distribution $\ell \in \mathcal{L}_{A,D}$, there is a vote assignment $f^\ell \in \mathcal{F}_{A,D}$ such that*

$$\min_{c \in D} \text{supp}_{f^\ell}(c) \geq \frac{1}{\max_{i \in N} \ell_i}.$$

Proof For a given vote assignment $f \in \mathcal{F}_{A,D}$, define the load distribution $\ell^f \in \mathcal{L}_{A,D}$ by setting $\ell_{i,c}^f = \frac{f(i,c)}{\text{supp}_f(c)}$ for each $i \in N$ and $c \in D$.¹⁰ It follows that the total load of a voter is upper bounded by $\frac{1}{\text{supp}_f(c^*)}$, where c^* is a candidate with minimal support (recall that $\sum_c f(i, c) = 1$ for each voter i such that $A_i \cap D \neq \emptyset$).

For a given load distribution $\ell \in \mathcal{L}_{A,D}$, define the vote assignment $f^\ell \in \mathcal{F}_{A,D}$ by setting $f^\ell(i, c) = \frac{\ell_{i,c}}{\ell_i}$ for each voter $i \in N$ such that $\ell_i > 0$. That is, the support for a candidate is proportional to the load received from that candidate, scaled such that the total support by the voter is 1. It follows that the minimal support of a candidate is lower bounded by $\frac{1}{\ell_{i^*}}$, where i^* is a voter with maximal load. To see this, let i^* denote a voter with maximal load. For $c \in D$, we get

$$\text{supp}_{f^\ell}(c) = \sum_{i \in N} f^\ell(i, c) \geq \sum_{i \in N: \ell_i > 0} \frac{\ell_{i,c}}{\ell_i} \geq \frac{1}{\ell_{i^*}} \sum_{i \in N} \ell_{i,c} = \frac{1}{\ell_{i^*}}. \quad \square$$

Lemma 6.1 has particularly interesting implications for load distributions and vote assignments that are optimal.

6.2 Phragmén’s optimal rule

The construction used in the proof of Lemma 6.1 establishes a one-to-one correspondence between elements of $\mathcal{L}_{A,D}^{\text{opt}}$ and elements of $\mathcal{F}_{A,D}^{\text{opt}}$. Therefore, the objective of

¹⁰ If $\text{supp}_f(c) = 0$ for some candidate c , the first part of the lemma trivially holds.

minimizing the maximal voter load is equivalent to the objective of maximizing the minimal support. As a consequence, max-Phragmén (the method that globally minimizes the maximal voter load) is identical to the rule that globally maximizes the minimal support.

Theorem 6.1 *Let $E = (N, C, A, k)$ be an approval-based multiwinner election. Then, $\text{max-Phragmén}(E) = \arg \max_{W \subseteq C, |W|=k} \text{maximin}(A, W)$.*

Since it is NP-hard to compute winners under max-Phragmén [7], the same is true for finding a set of candidates maximizing the minimum support. Brill et al. [7] proved that max-Phragmén satisfies PJR (when combined with an appropriate tie-breaking rule) and PR but not EJR. With respect to monotonicity axioms, Mora and Oliver [25] proved that max-Phragmén fails committee monotonicity and Sánchez-Fernández and Fisteus [35] have extended previous results by Phragmén [31], showing that max-Phragmén satisfies weak support monotonicity but fails strong support monotonicity.

6.3 Phragmén's sequential rule

The rule seq-Phragmén can be viewed as a greedy algorithm for max-Phragmén. Candidates are added to the committee iteratively. When a candidate is added, the load corresponding to the candidate is distributed among the voters approving it. For $j = 0, \dots, k$, let $x_i^{(j)}$ be the load of voter i after j iterations of seq-Phragmén. The initial load $x_i^{(0)}$ of each voter i is set to 0.

In iteration $j + 1$, the potential maximum load $s_c^{(j+1)}$ associated to a candidate c that has not yet been added to the committee is computed as

$$s_c^{(j+1)} = \frac{1 + \sum_{i \in N_c} x_i^{(j)}}{|N_c|}.$$

The underlying idea of this expression is to distribute the unit of load corresponding to candidate c in such a way that the resulting maximum voter load is as small as possible. When doing so, the load that voters accumulated in earlier iterations are taken into account as well. Then, in each iteration the candidate w with the lowest potential maximal load is added to the committee and the loads of the voters are updated as follows: for each voter i with $w \in A_i$, we have $x_i^{(j+1)} = s_w^{(j+1)}$; the load of other voters does not change.

There is a close relationship between the maximin support method and Phragmén's sequential rule. Both MMS and seq-Phragmén construct the committee by iteratively adding candidates: MMS chooses candidates such that the minimal support of the new set is maximized; seq-Phragmén chooses candidates such that the maximal voter load incurred by the new set is minimized. However, there is a subtle difference between the two methods concerning the *redistribution* of support/load. Under MMS, support assigned to candidates in earlier rounds can be freely redistributed when looking for maximin vote assignments for the new set of candidates. This is not the case for the loads under seq-Phragmén, however: once a voter is assigned some load from some

candidate, this load is “frozen” and will always stay with the voter.¹¹ As a consequence, the two methods might give different results, as Example 5.1 illustrates: The committee according to the maximin support method is given by $MMS(E) = \{c_1, a_1, a_2, a_3\}$ (selected in this order), while seq-Phragmén selects (in this order) $\{c_1, c_2, c_3, a_1\}$.

It is straightforward to check that seq-Phragmén can be computed in polynomial time [7]. With respect to the axiomatic properties considered in this paper, seq-Phragmén is indistinguishable from the maximin support method: seq-Phragmén satisfies committee monotonicity by definition; it satisfies PJR but fails EJR and PR [7]; and it satisfies weak support monotonicity [18, 25, 31] but fails strong support monotonicity [34].

An interesting distinction between seq-Phragmén and the maximin support method concerns their ability to approximate the optimal solution of the maximin support problem. Building on a preliminary version of this article [39], Cevallos and Stewart [11] have recently shown that MMS provides a 2-approximation for this problem, whereas seq-Phragmén does not offer a constant-factor approximation.¹² To state these results formally, we let $OPT(A, k)$ denote the optimal maximin support value $\max_{W \subseteq C, |W|=k} \text{maximin}(A, W)$ for election (N, C, A, k) , and H_k the k -th harmonic number $H_k = \sum_{i=1}^k 1/i$.

Proposition 6.1 (Cevallos and Stewart [11])

1. For each approval-based multiwinner election $E = (N, C, A, k)$,

$$\text{maximin}(A, MMS(E)) \geq \frac{1}{2} OPT(A, k).$$

2. For each committee size $k \in \mathbb{N}$ and each $\epsilon > 0$, there is an approval-based multiwinner election $E^{(k)} = (N, C, A^{(k)}, k)$ such that

$$OPT(A^{(k)}, k) \geq (H_k - \epsilon) \cdot \text{maximin}(A^{(k)}, W_k),$$

where W_k is the committee selected by seq-Phragmén in election $E^{(k)}$.

Proposition 6.1 implies that seq-Phragmén can behave arbitrarily worse than max-Phragmén (and also than the maximin support method) in terms of maximizing the minimum support.

We note that the maximin support value of a committee can be seen as a measure of its representativeness: the optimal value of $|N|/k$ can only be achieved when all voters are represented in the committee (in the sense that each voter approves at least one winning candidate) and, furthermore, the support can be evenly distributed

¹¹ Since redistributions are explicitly allowed in the rule *Generalized Phragmén’s sequential rule* (GPseq) proposed by Aziz et al. [4] for the participatory budgeting setting, this rule is actually a generalization of MMS, rather than of Phragmén’s sequential rule. In particular, GPseq is identical to MMS when the costs of all projects are equal to 1.

¹² Due to Lemma 6.1, the same bounds hold for the problem of minimizing the maximal voter load. The first part of Proposition 6.1 is stated as Theorem 11 in the paper by Cevallos and Stewart [11]. The second part of Proposition 6.1 follows from the proof of Lemma 8 of the same paper and was explicitly stated in an earlier version of the paper [10, Lemma 10].

among the committee members. Committees with smaller maximin support values can thus be interpreted as providing a lesser degree of representation. From this perspective, Proposition 6.1 shows an important advantage of the maximin support method compared to seq-Phragmén. Note, however, that this advantage comes at the price of increased computational complexity (see also [11]): as we have seen in Sect. 4, in each iteration of MMS, we need to solve one linear program for every remaining candidate.

6.4 Thiele's rules

The D'Hondt extensions due to Thiele [42] are based on a score optimization problem: For a given approval-based multiwinner election $E = (N, C, A, k)$, the goal is to find a size- k committee W maximizing $s(W) = \sum_{i \in N} s(i, W)$, where $s(i, W)$ is defined by $s(i, W) = \sum_{j=1}^{|A_i \cap W|} \frac{1}{j}$. Like Phragmén's rules, Thiele's rules come in two variants.

Thiele's non-sequential rule is known as *Proportional Approval Voting (PAV)* [3, 21] and chooses committees W maximizing $s(W)$. PAV satisfies EJR [3] (and thus PJR) but not PR [37] and is NP-hard to compute [2, 40]. It was already known by Thiele [42] that PAV fails committee monotonicity. PAV satisfies strong support monotonicity with population increase (in fact, PAV is the only rule that is known to satisfy this axiom together with PJR) and weak support monotonicity without population increase.

Thiele's sequential rule, often referred to as *sequential PAV*, is a greedy heuristic for the score optimization problem defined above. The rule starts with $W = \emptyset$ and iteratively adds candidates c maximizing the score $s(W \cup \{c\})$. Sequential PAV is polynomial-time computable and trivially satisfies committee monotonicity; weak support monotonicity (with *and* without population increase) holds as well [34, 35]. Sequential PAV fails PR, PJR, and even a weaker property known as *justified representation* [3, 37].

Thiele's rules show a poor behaviour in terms of worst-case maximin support: Cevallos and Stewart [11] observed that both PAV and sequential PAV can select committees with a maximin support value that is an $O(\sqrt{k})$ factor away from the optimal value. We complement their observation by showing that the factor can even be linear in k .

Proposition 6.2 *For each committee size $k \in \mathbb{N}$, there is an approval-based multiwinner election $E^{(k)} = (N, C, A^{(k)}, k)$ such that $OPT(A^{(k)}, k) \geq (k - 1) \cdot \text{maximin}(A^{(k)}, W_k)$, where W_k is the committee selected by PAV and sequential PAV in election $E^{(k)}$.*

Proof For a natural number $k \geq 3$, consider the election $E^{(k)} = (N, C, A^{(k)}, k)$ with $C = \{c_1, \dots, c_{k+1}\}$. There are $k(k - 1) + 1$ voters casting the following ballots: For each $i \in \{1, \dots, k - 1\}$, there are $k - 1$ voters who approve $\{c_i\}$; there are $k - 1$ voters who approve $\{c_1, \dots, c_k\}$; and there is one voter who approves $\{c_{k+1}\}$. For this election, the optimal maximin support value $OPT(A^{(k)}, k)$ is equal to $k - 1$ and can be achieved with the committee $W = \{c_1, \dots, c_k\}$. However, both PAV and sequential PAV select $W_k = \{c_1, \dots, c_{k-1}, c_{k+1}\}$, and the maximin support value for this committee is $\text{maximin}(A^{(k)}, W_k) = 1$, because candidate c_{k+1} is only approved by a single voter. \square

7 Conclusion

We have proposed the maximin support method (MMS) as a novel extension of the D'Hondt method to approval-based multiwinner elections. Like the method of D'Hondt, MMS aims to maximize the support of the least supported winning candidate. We have shown that MMS can be computed efficiently and satisfies an attractive combination of axiomatic properties. In particular, we have argued that MMS strikes a balance between sufficiently representing the interests of cohesive voter groups, while at the same time trying not to overrepresent groups. We have also established a close relationship between MMS and Phragmén's rules. This novel connection allows us to formulate Phragmén's rules as support maximization (rather than load minimization) problems, and to view MMS as a tractable approximation of Phragmén's (intractable) optimal rule.

There are several intriguing questions for future work, including the following: Is the approximation factor of MMS for the optimal maximin support problem tight? Do there exist polynomial-time computable (and axiomatically desirable) voting rules providing a better approximation factor? Do there exist committee-monotonic rules satisfying stronger proportionality guarantees such as extended justified representation? Finally, another direction for future work is to find axiomatic characterizations of MMS and other sequential rules.

Acknowledgements A previous version of this paper has circulated under the title "Fully Open Extensions to the D'Hondt Method" [36] and was presented at the 13th Meeting of the Society for Social Choice and Welfare (Lund, June 2016). In this earlier version, the maximin support method was referred to as the *Open D'Hondt (ODH)* method. We would like to thank the anonymous reviewers and the participants of the COMSOC Video Seminar (December 2020) for valuable feedback and discussions. This work was supported in part by the project Real time social sensor AnaLysis and deep learning based resource ESTimations for multimodal transport (MaGIST-RALES), funded by MCIN/AEI/10.13039/501100011033 under Grant PID2019-105221RB-C44, by the project FLATCITY-APP: Mobile application for FlatCity, funded by MCIN/AEI/10.13039/501100011033 under Grant PDC2021-121239-C33, by the Madrid Government (Comunidad de Madrid-Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M21), and in the context of the V PRICIT (Regional Programme of Research and Technological Innovation), by a Feodor Lynen return fellowship of the Alexander von Humboldt Foundation, and by the Deutsche Forschungsgemeinschaft (DFG) under Grant BR 4744/2-1.

Funding Open Access funding provided by Universidad Carlos III de Madrid (Read & Publish Agreement CRUE-CSIC 2022 with Springer Nature).

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Appendix

A Proof of Theorem 4.1

We employ linear programming duality theory (see, e.g., Chvátal [12, Chapter 5]).

Let A be an approval profile and $D \subseteq C$ a nonempty subset of candidates. We have seen in Sect. 4 that $\text{maximin}(A, D)$ can be computed with the following linear program.

$$\begin{aligned} &\text{maximize} && s \\ &\text{subject to} && \sum_{i \in N: c \in A_i \cap D} f(i, c) \geq s, && \text{for all } c \in D \quad (\text{A.1}) \\ & && \sum_{c \in A_i \cap D} f(i, c) = 1, && \text{for all } i \in N \text{ with } A_i \cap D \neq \emptyset \quad (\text{A.2}) \\ & && f(i, c) \geq 0, && \text{for all } i \in N \text{ and } c \in D \end{aligned}$$

We now consider the *dual* of this linear program. For every inequality constraint of the form (A.1), there is an associated non-negative dual variable y_c ($c \in D$), and for every equality constraint of the form (A.2), there is an unrestricted dual variable z_i ($i \in N$ and $A_i \cap D \neq \emptyset$). The dual looks as follows.

$$\begin{aligned} &\text{minimize} && \sum_{i \in N: A_i \cap D \neq \emptyset} z_i \\ &\text{subject to} && \sum_{c \in D} y_c = 1 \\ & && z_i \geq y_c, && \text{for all } i \in N \text{ and } c \in A_i \cap D \\ & && y_c \geq 0, && \text{for all } c \in D \end{aligned}$$

Let (s^*, f^*) be an optimal solution for the primal linear program and (y^*, z^*) an optimal solution for the dual linear program. Then,

$$s^* = \sum_{i \in N: A_i \cap D \neq \emptyset} z_i^* = \text{maximin}(A, D).$$

Moreover, from *complementary slackness* we get that the implication

$$\text{supp}_{f^*}(c) = \sum_{i \in N_c} f^*(i, c) > s^* \Rightarrow y_c^* = 0 \quad (\text{A.3})$$

holds for all candidates $c \in D$. That is, if the support of a candidate c under an optimal vote assignment f^* exceeds $\text{maximin}(A, D)$, then the dual variable y_c (associated with the primal constraint $\sum_i f(i, c) \geq s$) equals zero in the optimal solution (y^*, z^*) for the dual.

We first prove a lemma that relates maximin support values of different sets. In particular, it states that candidates receiving more than the minimal support (under an optimal vote assignment) can be removed without affecting the maximin value of the set.

Lemma A.1 *Let A be an approval profile, $D \subseteq C$ a nonempty subset of candidates, and $f \in \mathcal{F}_{A,D}^{opt}$ an optimal vote assignment for (A, D) . If there exists a candidate $\hat{c} \in D$ such that $supp_f(\hat{c}) > maximin(A, D)$, then $maximin(A, D) = maximin(A, D \setminus \{\hat{c}\})$.*

Proof It is easy to verify that $maximin(A, D) \leq maximin(A, D \setminus \{\hat{c}\})$. We will now show that $maximin(A, D) \geq maximin(A, D \setminus \{\hat{c}\})$ also holds.

Consider the primal and dual linear programs corresponding to the computation of $maximin(A, D)$ together with optimal solutions (s^*, f^*) and (y^*, z^*) , where $f^* = f$ and $s^* = maximin(A, D)$. Since $supp_f(\hat{c}) > maximin(A, D)$ holds by assumption, (A.3) implies that $y_{\hat{c}}^* = 0$. Furthermore, if there exist voters $i \in N$ with $A_i \cap D = \{\hat{c}\}$, then $z_i^* = 0$ for all such voters (because variable z_i only appears in the single constraint $z_i \geq y_{\hat{c}}$).

Now consider the dual linear program corresponding to the computation of $maximin(A, D \setminus \{\hat{c}\})$. This linear program has a variable y_c for each $c \in D \setminus \{\hat{c}\}$ and a variable z_i for each $i \in N$ with $A_i \cap (D \setminus \{\hat{c}\}) \neq \emptyset$. We are going to construct a feasible solution (\hat{y}, \hat{z}) for this linear program by restricting (y^*, z^*) to the smaller domain.

For each $c \in D \setminus \{\hat{c}\}$, let $\hat{y}_c = y_c^*$. Moreover, for each $i \in N$ with $A_i \cap (D \setminus \{\hat{c}\}) \neq \emptyset$, let $\hat{z}_i = z_i^*$. The solution (\hat{y}, \hat{z}) is feasible for the linear program in question because $y_{\hat{c}}^* = 0$ and thus

$$\sum_{c \in D \setminus \{\hat{c}\}} \hat{y}_c = \sum_{c \in D} y_c^* = 1.$$

And the objective function value of the solution (\hat{y}, \hat{z}) is equal to that of the solution (y^*, z^*) in the original dual because

$$\begin{aligned} \sum_{\substack{i \in N: \\ A_i \cap (D \setminus \{\hat{c}\}) \neq \emptyset}} \hat{z}_i &= \sum_{\substack{i \in N: \\ A_i \cap (D \setminus \{\hat{c}\}) \neq \emptyset}} z_i^* = \sum_{\substack{i \in N: \\ A_i \cap (D \setminus \{\hat{c}\}) \neq \emptyset}} z_i^* \\ &+ \sum_{\substack{i \in N: \\ A_i \cap D = \{\hat{c}\}}} z_i^* = \sum_{\substack{i \in N: \\ A_i \cap D \neq \emptyset}} z_i^* = s^*. \end{aligned}$$

It follows that the objective function value of the dual linear program for $D \setminus \{\hat{c}\}$ is less than or equal to the objective function value s^* of the dual linear program for D .¹³ In other words, $maximin(A, D \setminus \{\hat{c}\}) \leq s^* = maximin(A, D)$. \square

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1 The proof is by induction on j . For $j = 0$, the statement clearly holds because $W^0 \cup \{c\} = \{c\}$ and there is a unique optimal vote assignment $f_c \in \mathcal{F}_{A,\{c\}}^{opt}$ that furthermore satisfies $supp_{f_c}(c) = maximin(A, \{c\})$.

¹³ Recall that the duals are minimization problems.

For the inductive step, let us assume that the statement holds for $j = m$. We show that it also holds for $j = m + 1$.

Suppose for contradiction that this is not the case. Then, there exists a candidate $c \in C \setminus W^{m+1}$ and an optimal vote assignment $f_c \in \mathcal{F}_{A, W^{m+1} \cup \{c\}}^{\text{opt}}$ such that

$$\text{supp}_{f_c}(c) > \text{maximin}(A, W^{m+1} \cup \{c\}). \tag{A.4}$$

By Lemma A.1, this implies that

$$\text{maximin}(A, W^{m+1} \cup \{c\}) = \text{maximin}(A, W^{m+1}). \tag{A.5}$$

Let c_{m+1} be the $(m + 1)$ th candidate chosen by MMS for election (N, C, A, k) . Thus, $W^{m+1} = W^m \cup \{c_{m+1}\}$. Let g be a vote assignment on $W^m \cup \{c\}$ such that

$$\text{supp}_g(c') \geq \text{supp}_{f_c}(c') \quad \text{for all } c' \text{ in } W^m \cup \{c\}. \tag{A.6}$$

Such a function can easily be constructed by considering f_c and redistributing support that is assigned to candidate c_{m+1} .

We now distinguish two cases: either the function g maximizes the support for the least supported candidate in $W^m \cup \{c\}$, or it does not.

Case 1 g is an optimal vote assignment for $(A, W^m \cup \{c\})$. In this case,

$$\text{maximin}(A, W^m \cup \{c\}) = \min_{c' \in W^m \cup \{c\}} \text{supp}_g(c'),$$

and thus, by the induction hypothesis,

$$\text{maximin}(A, W^m \cup \{c\}) = \text{supp}_g(c). \tag{A.7}$$

By combining (A.4), (A.5), (A.6), and (A.7), we have

$$\begin{aligned} \text{maximin}(A, W^m \cup \{c\}) &= \text{supp}_g(c) \geq \text{supp}_{f_c}(c) \\ &> \text{maximin}(A, W^{m+1} \cup \{c\}) \\ &= \text{maximin}(A, W^m \cup \{c_{m+1}\}). \end{aligned}$$

However, this is a contradiction because it implies that candidate c should have been selected instead of c_{m+1} in iteration $m + 1$.

Case 2 g is not an optimal vote assignment for $(A, W^m \cup \{c\})$. In this case,

$$\text{maximin}(A, W^m \cup \{c\}) > \min_{c' \in (W^m \cup \{c\})} \text{supp}_g(c'). \tag{A.8}$$

Furthermore, since $f_c \in \mathcal{F}_{A, W^{m+1} \cup \{c\}}^{\text{opt}}$, we have

$$\min_{c' \in (W^{m+1} \cup \{c\})} \text{supp}_{f_c}(c') = \text{maximin}(A, W^{m+1} \cup \{c\}). \quad (\text{A.9})$$

By combining (A.5), (A.6), (A.8) and (A.9), we have

$$\begin{aligned} \text{maximin}(A, W^m \cup \{c\}) &> \min_{c' \in (W^m \cup \{c\})} \text{supp}_g(c') \\ &\geq \min_{c' \in (W^{m+1} \cup \{c\})} \text{supp}_{f_c}(c') \\ &= \text{maximin}(A, W^{m+1} \cup \{c\}) \\ &= \text{maximin}(A, W^m \cup \{c_{m+1}\}). \end{aligned}$$

Again, this is a contradiction because it implies that candidate c should have been selected instead of c_{m+1} in iteration $m + 1$. \square

References

1. Alós-Ferrer, C.: A simple characterization of approval voting. *Soc. Choice Welf.* **27**(3), 621–625 (2006)
2. Aziz, H., Gaspers, S., Gudmundsson, J., Mackenzie, S., Mattei, N., Walsh, T.: Computational aspects of multi-winner approval voting. In: Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems, pp. 107–115. IFAAMAS, (2015)
3. Aziz, H., Brill, M., Conitzer, V., Elkind, E., Freeman, R., Walsh, T.: Justified representation in approval-based committee voting. *Soc. Choice Welf.* **48**(2), 461–485 (2017)
4. Aziz, H., Lee, B.E., Talmon, N.: Proportionally representative participatory budgeting: axioms and algorithms. In: Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS), pp. 23–31. IFAAMAS, (2018)
5. Balinski, M., Young, H.P.: Fair Representation: Meeting the Ideal of One Man. Yale University Press, One Vote (1982)
6. Brams, S.J., Kilgour, D.M., Potthoff, R.F.: Multiwinner approval voting: an apportionment approach. *Pub. Choice* **178**(1–2), 67–93 (2019)
7. Brill, M., Freeman, R., Janson, S., Lackner, M.: Phragmén’s voting methods and justified representation. In: Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI), pp. 406–413. AAAI Press, (2017a)
8. Brill, M., Laslier, J.-F., Skowron, P.: Multiwinner approval rules as apportionment methods. *J. Theor. Polit.* **30**(3), 358–382 (2018)
9. Brill, M., Gözl, P., Peters, D., Schmidt-Kraepelin, U., Wilker, K.: Approval-based apportionment. In: Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI), pp. 1854–1861. AAAI Press, (2020)
10. Cevallos, A., Stewart, A.: A verifiably Secure and Proportional Committee Election Rule. Technical report, [arXiv:2004.12990v2](https://arxiv.org/abs/2004.12990v2) [cs.DS], (2020)
11. Cevallos, A., Stewart, A.: A verifiably secure and proportional committee election rule. In: Proceedings of the 3rd ACM Conference on Advances in Financial Technologies (AFT), pp. 29–42. ACM (2021)
12. Chvátal, V.: Linear Programming. Series of Books in the Mathematical Sciences. W. H. Freeman (1983)
13. Faliszewski, P., Skowron, P., Slinko, A., Talmon, N.: Multiwinner voting: A new challenge for social choice theory. In: Endriss, U. (ed) Trends in Computational Social Choice, chapter 2. (2017)
14. Fishburn, P.C.: Axioms for approval voting: direct proof. *J. Econ. Theor.* **19**(1), 180–185 (1978)
15. Fishburn, P.C.: Symmetric and consistent aggregation with dichotomous voting. In: Aggregation and Revelation of Preferences. North Holland, Amsterdam, pp. 201–218. North-Holland, (1979)

16. Gangl, C., Lackner, M., Maly, J., Woltran, S.: Aggregating expert opinions in support of medical diagnostic decision-making. In: Knowledge Representation for Health Care/ProHealth (KR4HC), pp. 56–62, (2019)
17. Hylland, A.: Proportional representation without party lists. In: Malnes, R., Underda, A. (eds.) *Rationality and Institutions: Essays in Honour of Knut Midgaard on the Occasion of his 60th Birthday*, February 11, 1991, pp. 126–153. Universitetsforlaget Oslo, (1992)
18. Janson, S.: Phragmén's and Thiele's Election Methods. Technical report, [arXiv:1611.08826](https://arxiv.org/abs/1611.08826) [math.HO], (2016)
19. Khachian, L.: A polynomial algorithm for linear programming. *Doklady Akademiai Nauk USSR* **244**, 1093–1096 (1979)
20. Kilgour, D.C., Brams, S.J., Sanver, M.R.: How to elect a representative committee using approval balloting. In: Simeone B., Pukelsheim, F. (eds.) *Mathematics and Democracy*. Springer, (2006)
21. Kilgour, D.M.: Approval balloting for multi-winner elections. In: Laslier, J.-F., Sanver, M.R. (eds.) *Handbook on Approval Voting*, pp. 105–124. Springer, (2010)
22. Kluyving, B., de Vries, A., Vrijbergen, P., Boixel, A., Endriss, U.: Analysing irresolute multiwinner voting rules with approval ballots via SAT solving. In: Proceedings of the 24th European Conference on Artificial Intelligence (ECAI), pp. 131–138. IOS Press, (2020)
23. Lackner, M., Skowron, P.: Multi-winner voting with approval preferences. Technical report, [arXiv:2007.01795](https://arxiv.org/abs/2007.01795) [cs.GT] (2022)
24. Lackner, M., Skowron, P.: Consistent approval-based multi-winner rules. *J. Econ. Theor.* **192**, 105173 (2021)
25. Mora, X., Oliver, M.: Eleccions mitjançant el vot d'aprovació. el mètode de phragmén i algunes variants. *Butlletí de la Societat Catalana de Matemàtiques* **30**(1), 57–101 (2015)
26. Peters, D.: Proportionality and strategyproofness in multiwinner elections. In: Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS), Vol. 1549–1557, (2018)
27. Peters, D., Skowron, P.: Proportionality and the limits of welfarism. In: Proceedings of the 21st ACM Conference on Economics and Computation (ACM-EC), pp. 793–794, (2020)
28. Peters, D., Pierczyński, G., Shah, N., Skowron, P.: Market-based explanations of collective decisions. In: Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI), pp. 5656–5663. AAAI Press, (2021a)
29. Peters, D., Pierczyński, G., Skowron, P.: Proportional participatory budgeting with additive utilities. In: Ranzato, M., Beygelzimer, A., Dauphin, Y., Liang, P., Vaughan, J.W. (eds.) *Advances in Neural Information Processing Systems (NeurIPS)*, vol. 34, pp. 12726–12737. Curran Associates, Inc. (2021)
30. Phragmén, E.: Sur une méthode nouvelle pour réaliser, dans les élections, la représentation proportionnelle des partis. *Öfversigt af Kongliga Vetenskaps-Akademiens Förhandlingar* **51**(3), 133–137 (1894)
31. Phragmén, E.: Sur la théorie des élections multiples. *Öfversigt af Kongliga Vetenskaps-Akademiens Förhandlingar* **53**, 181–191 (1896)
32. Pukelsheim, F.: *Proportional Representation: Apportionment Methods and Their Applications*. Springer, (2014)
33. Renwick, A., Pilet, J.-B.: *Faces on the Ballot: The Personalization of Electoral Systems in Europe*. Oxford University Press, (2016)
34. Sánchez-Fernández, L., Fisteus, J.A.: Monotonicity Axioms in Approval-Based Multi-Winner Voting Rules. Technical report, [arXiv:1710.04246](https://arxiv.org/abs/1710.04246) [cs.GT], (2017)
35. Sánchez-Fernández, L., Fisteus, J.A.: Monotonicity axioms in approval-based multi-winner voting rules. In: Proceedings of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS), pp. 485–493, (2019)
36. Sánchez-Fernández, L., Fernández, N., Fisteus, J.A.: Fully Open Extensions to the D'Hondt Method. Technical report, [arXiv:1609.05370v1](https://arxiv.org/abs/1609.05370v1) [cs.GT], (2016)
37. Sánchez-Fernández, L., Elkind, E., Lackner, M., Fernández, N., Fisteus, J.A., Basanta Val, P., Skowron, P.: Proportional justified representation. In: Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI), pp. 670–676. AAAI Press, (2017)
38. Sánchez-Fernández, L., Fernández, N., Fisteus, J.A., Brill, M.: The Maximin Support Method: An Extension of the D'Hondt Method to Approval-Based Multiwinner Elections. Technical report, [arXiv:1609.05370v2](https://arxiv.org/abs/1609.05370v2) [cs.GT] (2018)

39. Sánchez-Fernández, L., Fernández, N., Fisteus, J.A., Brill, M.: The maximin support method: An extension of the D'Hondt method to approval-based multiwinner elections. In: Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI), pp. 5690–5697. AAAI Press (2021)
40. Skowron, P., Faliszewski, P., Lang, J.: Finding a collective set of items: from proportional multirepresentation to group recommendation. *Artif. Intell.* **241**, 191–216 (2016)
41. Smith, J.H.: Aggregation of preferences with variable electorate. *Econometrica* **41**(6), 1027–1041 (1973)
42. Thiele, T.N.: Om flerfoldsvvalg. Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandling, pp. 415–441, (1895)
43. Young, H.P.: A note on preference aggregation. *Econometrica* **42**(6), 1129–1131 (1974)

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