Correction to: Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs

Samuel Burer¹ · Yinyu Ye²

Published online: 8 July 2021

© Springer-Verlag GmbH Germany, part of Springer Nature and Mathematical Optimization Society 2021

Correction to: Mathematical Programming (2020) 181:1–17 <https://doi.org/10.1007/s10107-019-01367-2>

This note reports an error in Theorem 3 of [1], which was pointed about by Prof. Fatma Kılınç-Karzan and Mr. Alex Wang of Carnegie Mellon University, whom we sincerely thank for bringing this to our attention.

The error can be seen in the proof of Theorem 3, where system (12) is analyzed for given diagonal matrices $\{C, B_i, D_i\}$ and variable diagonal matrices $\{X, Y_i\}$. In this setting, $C, B_i \succ 0$, while each D_i is indefinite. In fact, multiple systems of the form (12) are analyzed. In each, one scalar variable in $\{X, Y_i\}$ is allowed to be free, while all others are nonnegative. Then, within the proof, the goal is to show that, with high probability, all such systems are simultaneously feasible. However, in truth the proof fails because, with this setup, the systems cannot be simultaneously feasible. Said differently, the probability that all systems are feasible is zero.

To see this, we assume all such systems are feasible and derive a contradiction. Fix i and let $\omega > 0$ be minimal such that $D_i + \omega B_i \succeq 0$. In particular, let j be such that $[D_i + \omega B_i]_{jj} = 0$. Then consider the specific system (12) with $[Y_i]_{jj}$ free. Because this system is feasible, the inequality

$$\omega C \bullet X + \epsilon(D_i + \omega B_i) \bullet Y_i + \epsilon \omega \sum_{k \neq i} B_k \bullet Y_k \leq -\omega,$$

The original article can be found online at <https://doi.org/10.1007/s10107-019-01367-2>.

✉ Samuel Burer
samuel-burer@uiowa.edu

Yinyu Ye
yinyu-ye@stanford.edu

¹ Department of Business Analytics, University of Iowa, Iowa City, IA 52242-1994, USA

² Department of Management Science and Engineering, Institute of Computational and Mathematical Engineering, Stanford University, Stanford, CA 94305-4121, USA

is also feasible. Note that $[Y_i]_{jj}$ does not actually appear in this system because its coefficient $\epsilon[D_i + \omega B_i]_{jj}$ equals 0, and so we see that a positive combination of nonnegative variables is negative, thus providing the desired contradiction.

Instead of the purported Theorem 3, we now prove an analogous theorem, except that it only applies to diagonal instances of (1). We first describe a geometric condition on the data $\{C, A_i\}$ and $\{c, a_i\}$ ensuring $r^* = 1$. Define, for each $j = 1, \dots, n$,

$$f_j := (C_{jj}, [A_1]_{jj}, \dots, [A_m]_{jj})^T \in \mathbb{R}^{1+m} \quad \text{and} \quad d_j := (c_j, a_{1j}, \dots, a_{mj})^T \in \mathbb{R}^{1+m}. \quad (\dagger)$$

We remark that, using an epigraph variable t , we may rewrite (1) equivalently as follows:

$$\min \left\{ t : \sum_{j=1}^n (x_j^2 f_j + 2 x_j d_j) \leq \begin{pmatrix} t \\ b \end{pmatrix} \right\} \quad (\ddagger)$$

where $(t, b) \in \mathbb{R}^{1+m}$. This formulation $(\ddagger)$ emphasizes the separable nature of each quadratic function in (1). In addition to $(\dagger)$, define $K_j := \text{cone}\{f_k - f_j : k \neq j\} \subseteq \mathbb{R}^{1+m}$ and $L_j := \{\lambda d_j : \lambda \in \mathbb{R}\} \subseteq \mathbb{R}^{1+m}$ for each j . K_j is the cone of feasible directions at f_j in the polytope $Q := \text{conv}\{f_1, \dots, f_n\}$, and L_j is the line spanned by the directions $\pm d_j$.

Lemma *Given an instance of the diagonal QCQP (1), if $\text{int}(K_j) \cap L_j \neq \emptyset$ for all $j = 1, \dots, n$, then $r^* = 1$.*

Proof We use Assumptions 1–2, which guarantee that the feasible set of (1) is bounded, to add the redundant constraint $x^T x \leq \rho^2$ for some radius ρ . Note that we do not need to know ρ explicitly.

For fixed j , system (5) is certainly feasible if the following more restrictive system is feasible:

$$\begin{aligned} C \bullet X + c_j x_j &< 0 \\ A_i \bullet X + a_{ij} x_j &< 0 \quad \forall i = 1, \dots, m \\ I \bullet X &= 0 \\ X &\text{ diagonal, } X_{kk} \geq 0 \quad \forall k \neq j \\ X_{jj} &\text{ free, } x_j \text{ free.} \end{aligned}$$

Eliminating the free variable X_{jj} via the equation $I \bullet X = 0$ and using the definitions of f_j and d_j in $(\dagger)$, we have the equivalent vector-inequality system

$$\begin{aligned} \sum_{k \neq j} X_{kk} (f_k - f_j) + x_j d_j &< 0 \\ X_{kk} &\geq 0 \quad \forall k \neq j, \quad x_j \text{ free,} \end{aligned}$$

which is feasible if and only if there exists a direction in K_j , which equals the sum of a vector in L_j and a strictly negative vector, i.e., if and only if the intersection $K_j \cap (L_j + \text{int}(\mathbb{R}_-^{1+m}))$ is nonempty. Since L_j is a line, we have $L_j + \text{int}(\mathbb{R}_-^{1+m}) = \text{int}(L_j + \mathbb{R}_-^{1+m})$.

So, to complete the proof, it suffices to show $K_j \cap \text{int}(L_j + \mathbb{R}_-^{1+m}) \neq \emptyset$, which is true by the following argument. Since $\text{int}(K_j) \cap L_j \neq \emptyset$ by assumption, K_j is full-dimensional. In addition, the polyhedron $L_j + \mathbb{R}_-^{1+m}$ is full-dimensional with $\emptyset \neq \text{int}(K_j) \cap L_j \subseteq \text{int}(K_j) \cap (L_j + \mathbb{R}_-^{1+m})$, i.e., the interior of the full-dimensional polyhedron K_j intersects the full-dimensional polyhedron $L_j + \mathbb{R}_-^{1+m}$. It follows that K_j intersects the interior of $L_j + \mathbb{R}_-^{1+m}$, as desired. $\square$

Theorem Given fixed $m > 0$ and $\theta \in (0, \frac{\pi}{2}]$, consider the class of random diagonal QCQPs (1) generated as follow for $n > 0$:

- first, the diagonal entries of C and $A_1, \dots, A_m$ are generated randomly such that, for each $j = 1, \dots, n$ independently, $f_j/\|f_j\|$ is uniformly distributed on the unit sphere in $\mathbb{R}^{1+m}$, where f_j is defined in $(\dagger)$;
- next, the entries of c and $a_1, \dots, a_m$ are generated randomly such that, for each $j = 1, \dots, n$ independently, the angle between d_j and f_j is outside the interval $\pi/2 \pm \theta$, where d_j is defined by $(\ddagger)$;
- finally, the entries of b are chosen such that Assumptions 1 and 3 are satisfied, and for some finite $\rho > 0$, the constraint $x^T x \leq \rho^2$ is added to ensure that Assumption 2 is satisfied, while not violating Assumptions 1 and 3.

Then $\text{Prob}(r^* = 1) \rightarrow 1$ as $n \rightarrow \infty$.

Proof We apply the lemma as $n \rightarrow \infty$. Specifically, we wish to show the probability that $\text{int}(K_j) \cap L_j \neq \emptyset$ for all $j = 1, \dots, n$ goes to 1 as $n \rightarrow \infty$, while m and θ stay fixed. Also define Q as before the lemma.

We first note that, by using the formulation $(\ddagger)$ along with independent variable substitutions $x_j = \hat{x}_j/\sqrt{\|f_j\|}$ for all $j = 1, \dots, n$, (1) is equivalent to

$$\min \left\{ t: \sum_{j=1}^n \left(\hat{x}_j^2 \cdot \frac{f_j}{\|f_j\|} + 2 \hat{x}_j \cdot \frac{d_j}{\sqrt{\|f_j\|}} \right) \leq \binom{t}{b} \right\}.$$

Note that this corresponds to a new instance of (1) with data $\hat{f}_j := f_j/\|f_j\|$ and $\hat{d}_j := d_j/\sqrt{\|f_j\|}$ such that each $\hat{f}_j$ has norm 1 and the angle between $\hat{d}_j$ and $\hat{f}_j$ equals the angle between d_j and f_j . Since the tightness of the Shor SDP relaxation is not affected by simple positive variable scalings, we thus may assume without loss of generality that $\|f_j\| = 1$.

Since $\{f_j\}$ follows a uniform distribution on the unit sphere, as $n \rightarrow \infty$, Q approaches the unit ball and the cones K_j approach the halfspaces $\{v: f_j^T v \leq 0\}$. Since L_j makes an angle with f_j outside of $\pi/2 \pm \theta$, L_j intersects $\text{int}(K_j)$ for all j with probability going to 1 as $n \rightarrow \infty$. $\square$

Remark It is well known that, if all of the diagonal entries of $\{C, A_i\}$ are generated *i.i.d.* standard Gaussian, then $f_j/\|f_j\|$ is uniform on the unit sphere. In addition, d_j can be generated with entries *i.i.d.* standard Gaussian along with a simple accept-reject procedure to satisfy the angle condition of the theorem.

Reference

1. Burer, S., Ye, Y.: Exact semidefinite formulations for a class of (random and non-random) nonconvex quadratic programs. *Math. Program.* **181**(1, Ser. A), 1–17 (2020)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.