



# Nonnegative forms with sublevel sets of minimal volume

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## Abstract

We show that the Euclidean ball has the smallest volume among sublevel sets of nonnegative forms of bounded Bombieri norm as well as among sublevel sets of sum of squares forms whose Gram matrix has bounded Frobenius or nuclear (or, more generally,  $p$ -Schatten) norm. These volume-minimizing properties of the Euclidean ball with respect to its representation (as a sublevel set of a form of fixed even degree) complement its numerous intrinsic geometric properties. We also provide a probabilistic interpretation of the results.

**Keywords** Nonnegative homogeneous polynomials · Sublevel sets · Lebesgue volume · Orthogonally invariant norms · Extremal properties

**Mathematics Subject Classification** 49Q10 · 65K10 · 90C25 · 26B15 · 28A75 · 97G30

## Introduction

It is well-known that the unit Euclidean ball  $B_n = \{x \in \mathbf{R}^n : \sum_{i=1}^n x_i^2 \leq 1\}$  has numerous (intrinsic) geometric properties. For example,  $B_n$  has the smallest surface area among all domains in  $\mathbf{R}^n$  of a given volume or, equivalently, it has the largest volume among all domains of a given surface area. Hilbert and Cohn-Vossen [8] describe ten more geometric properties of  $B_n$  or of its boundary  $\partial B_n = \{x \in \mathbf{R}^n :$

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$\sum_{i=1}^n x_i^2 = 1$ }, the Euclidean sphere. In [11] it was shown that  $B_n$  exhibits some interesting extremal properties relative to its representation as a sublevel set of a nonnegative form.

More generally, in [11] the author was interested in properties of  $n$ -variate forms  $f$  of a given degree whose sublevel set  $\{f \leq 1\} = \{x \in \mathbf{R}^n : f(x) \leq 1\}$  has fixed Lebesgue volume. For instance, it was proved that the form  $x \in \mathbf{R}^n \mapsto f^*(x) = \sum_{i=1}^n x_i^{2d}$  minimizes the sparsity-inducing  $\ell_1$ -norm of coefficients among all  $n$ -variate forms of degree  $2d$  whose sublevel set has the same Lebesgue volume as  $\{f^* \leq 1\}$ , the unit  $L^{2d}$ -ball in  $\mathbf{R}^n$ . Equivalently, by homogeneity,  $f^*$  minimizes  $\text{vol}\{f \leq 1\}$ , the Lebesgue volume of the sublevel set, among all  $n$ -variate forms  $f$  of even degree  $2d$  with bounded  $\ell_1$ -norm.

Similarly, it was proved that the form  $x \in \mathbf{R}^n \mapsto b_{2d,n}(x) = (\sum_{i=1}^n x_i^2)^d$ , whose sublevel set  $\{b_{2d,n} \leq 1\} = B_n$  is the unit Euclidean ball, minimizes  $\text{vol}\{f \leq 1\}$  among all  $n$ -variate forms  $f$  of degree  $2d$  with bounded Bombieri norm when  $d=1, 2, 3, 4$ . In addition, for some values of  $d$ , the form  $b_{2d,n}$  also minimizes  $\text{vol}\{f \leq 1\}$  among all  $n$ -variate sum of squares forms  $f$  of degree  $2d$  whose Gram matrix has bounded trace.

Hence, the abovementioned results from [11] suggest that the Euclidean ball has volume-minimizing properties with regard to its representation as the sublevel set of a form of fixed even degree  $d$ , when considering nonnegative forms of degree  $d$  with bounded Bombieri norm or sum of squares forms of degree  $d$  with Gram matrix of bounded trace.

## Contribution

This paper shows that indeed these results for the unit Euclidean ball  $B_n$  are true for all even degrees  $d$  and not only for the special cases considered in [11]. In fact we prove a more general result. The unit Euclidean ball  $B_n$  minimizes  $\text{vol}\{f \leq 1\}$ :

- Over all nonnegative  $n$ -variate forms  $f$  of fixed (arbitrary) even degree  $d$  with bounded norm, when the norm is invariant under orthogonal changes of variables, which includes *Bombieri norm* as important special case;
- Over all sum of squares  $n$ -variate forms  $f$  of fixed (arbitrary) even degree  $d$ , whose Gram matrix has bounded norm, when the norm is invariant under conjugation by orthogonal matrices. This includes *Schatten  $p$ -norms* and, in particular, *nuclear* and *Frobenius* norms.

These new volume-minimizing properties of the Euclidean ball are attached to its representation as a sublevel set of a form and complement its intrinsic geometric properties.

Our results admit a probabilistic interpretation. The Gaussian-like probability measure with density  $x \mapsto \exp(-\kappa |x|^d)$  minimizes an  $O(n)$ -invariant norm  $\|f\|$  over all probability measures with density  $x \mapsto \exp(-f(x))$ , where  $f$  is a nonnegative form of degree  $d$ .

### 1 Main results

In the following we denote by  $\mathcal{F}_{d,n}$  the space of  $n$ -ary real forms (real homogeneous polynomials) of degree  $d$ . For any form  $f \in \mathcal{F}_{d,n}$  let  $\{f \leq 1\} = \{x \in \mathbf{R}^n : f(x) \leq 1\}$  be its sublevel set at level one and let  $v(f)$  denote the Lebesgue volume of  $\{f \leq 1\}$ ,

$$v(f) = \text{vol}\{f \leq 1\}. \tag{1.1}$$

If for  $f \in \mathcal{F}_{d,n}$  the volume  $v(f)$  of the sublevel set is finite, then  $f$  is necessarily *nonnegative*, that is,  $f(x) \geq 0$  for all  $x \in \mathbf{R}^n$ . In particular, the degree  $d$  must be even which we implicitly assume in the sequel.

The *volume function*  $v : \mathcal{F}_{d,n} \rightarrow \mathbf{R}_{\geq 0} \cup \{+\infty\}$  is lower-semicontinuous and homogeneous of degree  $-n/d$ . Moreover, forms  $f \in \mathcal{F}_{d,n}$  with finite  $v(f)$  constitute a convex subcone  $\mathcal{V}_{d,n}$  of the cone of nonnegative forms in  $\mathcal{F}_{d,n}$ <sup>1</sup> and the restriction  $v|_{\mathcal{V}_{d,n}} : \mathcal{V}_{d,n} \rightarrow \mathbf{R}_{\geq 0}$  is strictly convex. We refer to [11, Thm. 2.2] for these results.

Let  $\|\cdot\| : \mathcal{F}_{d,n} \rightarrow \mathbf{R}$  be any norm and consider the following convex optimization problem

$$\mathbf{P}_{\|\cdot\|} : \text{opt}_{\|\cdot\|} = \inf\{v(f) : \|f\| \leq 1, f \in \mathcal{F}_{d,n}\}. \tag{1.2}$$

**Remark 1.1** Note that  $\mathbf{P}_{\|\cdot\|}$  is the problem of minimization of the volume of the sublevel set  $\{f \leq 1\}$  of a form  $f \in \mathcal{F}_{d,n}$  over the unit ball in  $\mathcal{F}_{d,n}$  defined by the norm  $\|\cdot\|$ .

Consider the following standard action of the group  $O(n) = \{\rho \in \mathbf{R}^{n \times n} : \rho\rho^t = \text{id}\}$  of orthogonal transformations on forms  $\mathcal{F}_{d,n}$ :

$$\rho \in O(n), f \in \mathcal{F}_{d,n} \mapsto \rho^* f \in \mathcal{F}_{d,n}, \rho^* f(x) = f(\rho^{-1}x). \tag{1.3}$$

A norm  $\|\cdot\| : \mathcal{F}_{d,n} \rightarrow \mathbf{R}$  is  $O(n)$ -invariant if  $\|\rho^* f\| = \|f\|$  for all  $\rho \in O(n)$  and  $f \in \mathcal{F}_{d,n}$ .

In the following theorem we show that  $\mathbf{P}_{\|\cdot\|}$  has a unique optimal solution and we find it explicitly in the case of an  $O(n)$ -invariant norm.

**Theorem 1.2** *Let  $d$  be even and  $\|\cdot\| : \mathcal{F}_{d,n} \rightarrow \mathbf{R}$  be a norm. Then*

- *The convex optimization problem  $\mathbf{P}_{\|\cdot\|}$  has a unique optimal solution  $f^* \in \mathcal{V}_{d,n}$ .*
- *If the norm  $\|\cdot\|$  is  $O(n)$ -invariant, then  $f^* = b_{d,n}/\|b_{d,n}\|$ , where  $b_{d,n}(x) = |x|^d = (x_1^2 + \dots + x_n^2)^{d/2}$ , and  $\text{opt}_{\|\cdot\|} = \|b_{d,n}\|^{n/d} v(b_{d,n})$ .*

The first claim follows from the fact that the volume function is lower-semicontinuous and strictly convex, see [11, Sect. 7.2]. The  $O(n)$ -invariance of the norm and of the volume function combined with the uniqueness of the optimal solution imply the second claim. We refer to Sect. 3, where Theorem 1.2 is proved in detail.

<sup>1</sup> Note that  $\mathcal{V}_{d,n}$  does not contain the origin.

The sublevel set of  $b_{d,n}$  is the unit Euclidean ball  $B_n = \{|x| \leq 1\}$ , it does not depend on  $d$  and its volume equals

$$v(b_{d,n}) = \text{vol}(B_n) = \frac{\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2} + 1\right)}. \tag{1.4}$$

**Remark 1.3** Theorem 1.2 implies that the Euclidean ball in  $\mathbf{R}^n$  of radius  $\|b_{d,n}\|^{1/d}$  has smallest volume among sublevel sets of forms in the unit ball  $\{f \in \mathcal{F}_{d,n} : \|f\| \leq 1\}$  in  $\mathcal{F}_{d,n}$  defined by an  $O(n)$ -invariant norm.

Observe that if the norm  $\|\cdot\|$  is not  $O(n)$ -invariant, then  $f^* \neq b_{d,n}/\|b_{d,n}\|$  in general. For example, when  $\|\cdot\|$  is the  $\ell_1$ -norm of coefficients of a form written in the basis  $\{x^\alpha\}_{|\alpha|=d}$  of monomials, then [11, Thm. 3.2] implies that  $f^* = \frac{1}{\sqrt{n}}(x_1^d + \dots + x_n^d)$ , a form not proportional to  $b_{d,n}$  for  $d > 2$ .

**Remark 1.4** An interesting extension of Theorem 1.2 pointed out by a referee is to consider the general setting of continuous  $\lambda$ -homogeneous functions on  $\mathbf{R}^n$ , where  $\lambda$  is a positive real number. To establish such a generalization one need to investigate (i) continuity properties of the volume function on an appropriate infinite-dimensional (reflexive) Banach space that contains such functions, and (ii) whether homogeneity is preserved when passing to weak-limits of sequences. We leave this question for future research.

Now we compute the optimal value of  $\mathbf{P}_{\|\cdot\|}$  for some relevant  $O(n)$ -invariant norms, in view of Theorem 1.2 and (1.4) this task reduces to computing  $\|b_{d,n}\|$ .

### 1.1 Bombieri norm

Recall first that any  $f \in \mathcal{F}_{d,n}$  can be written in the basis of rescaled monomials,

$$f(x) = \sum_{|\alpha|=d} f_\alpha \sqrt{\binom{d}{\alpha}} x^\alpha, \quad x \in \mathbf{R}^n, \tag{1.5}$$

where  $\binom{d}{\alpha} = \frac{d!}{\alpha_1! \dots \alpha_n!}$  is the multinomial coefficient. The Bombieri norm of a form  $f$

$$\|f\|_B^2 = \sum_{|\alpha|=d} f_\alpha^2. \tag{1.6}$$

Under different names this norm appears in real algebraic geometry [15], in perturbation theory of roots of univariate polynomials [18], in the truncated moment problem [16], in the study of random polynomials [7,17], in the theory of symmetric tensor decompositions [3] and in many others branches of mathematics. It is well-known that Bombieri norm is  $O(n)$ -invariant (see, eg., [2, Sect. 2.1]).

**Corollary 1.5** (Bombieri norm). *For any  $f \in \mathcal{F}_{d,n}$  with  $\|f\|_B \leq 1$*

$$v\left(\frac{b_{d,n}}{\|b_{d,n}\|_B}\right) = \left(\prod_{i=0}^{d/2-1} \frac{2i+n}{2i+1}\right)^{n/2d} \frac{\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2}+1\right)} \leq v(f) \tag{1.7}$$

and equality holds if and only if  $f = b_{d,n}/\|b_{d,n}\|_B$ .

The second author of the present paper conjectured in [11, p. 249] the result of Corollary 1.5 and proved it for any  $n$  and  $d = 2, 4, 6$  and  $8$ .

**1.2  $L^p$ -norms on  $\mathbb{S}^{n-1}$**

The following class of norms plays a fundamental role in the study of boundary value problems for partial differential equations (see, for example, [1]). Let  $p \geq 1$  and define  $L^p$ -norm on the unit sphere  $\mathbb{S}^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$  as

$$\|f\|_{L^p(\mathbb{S}^{n-1})} = \left(\int_{\mathbb{S}^{n-1}} |f(x)|^p d\mathbb{S}^{n-1}\right)^{1/p}, \quad f \in \mathcal{F}_{d,n}, \tag{1.8}$$

where  $d\mathbb{S}^{n-1}$  is the Riemannian volume density on  $\mathbb{S}^{n-1}$ . The integral in (1.8) is convergent for any  $f \in \mathcal{F}_{d,n}$  and the norms  $\|\cdot\|_{L^p(\mathbb{S}^{n-1})}$ ,  $p \geq 1$ , are obviously  $O(n)$ -invariant.

**Corollary 1.6** ( $L^p$ -norm on  $\mathbb{S}^{n-1}$ ). *For any  $f \in \mathcal{F}_{d,n}$  with  $\|f\|_{L^p(\mathbb{S}^{n-1})} \leq 1$*

$$v\left(\frac{b_{d,n}}{\|b_{d,n}\|_{L^p(\mathbb{S}^{n-1})}}\right) = \left(\frac{2\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2}\right)}\right)^{n/dp} \frac{\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2}+1\right)} \leq v(f) \tag{1.9}$$

and equality holds if and only if  $f = b_{d,n}/\|b_{d,n}\|_{L^p(\mathbb{S}^{n-1})}$ .

**1.3 Uniform norm on  $\mathbb{S}^{n-1}$**

As the limiting case of  $L^p(\mathbb{S}^{n-1})$ -norms when  $p \rightarrow +\infty$  one obtains the uniform norm on the unit sphere  $\mathbb{S}^{n-1}$ ,

$$\|f\|_{L^\infty(\mathbb{S}^{n-1})} = \max_{x \in \mathbb{S}^{n-1}} |f(x)|. \tag{1.10}$$

**Corollary 1.7** (Uniform norm on  $\mathbb{S}^{n-1}$ ). *For any  $f \in \mathcal{F}_{d,n}$  with  $\|f\|_{L^\infty(\mathbb{S}^{n-1})} \leq 1$*

$$v\left(\frac{b_{d,n}}{\|b_{d,n}\|_{L^\infty(\mathbb{S}^{n-1})}}\right) = \frac{\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2}+1\right)} \leq v(f) \tag{1.11}$$

and equality holds if and only if  $f = b_{d,n}$ .

Note that (1.11) can be considered as the limiting case of (1.9) when  $p \rightarrow +\infty$ .

### 1.4 Nuclear norm

*Nuclear norm* appears in the study of tensor decompositions [6] and in the theory of rank-one approximations of tensors [2,12]. For  $f \in \mathcal{F}_{d,n}$  it is defined as

$$\|f\|_* = \inf \left\{ \sum_{k=1}^r |\lambda_k| : f(x) = \sum_{k=1}^r \lambda_k (y^k \cdot x)^d, \lambda_k \in \mathbf{R}, y^k \in \mathbb{S}^{n-1} \right\}, \quad (1.12)$$

where  $(y \cdot x) = y_1x_1 + \dots + y_nx_n$  denotes the dot product of two vectors in  $\mathbf{R}^n$ .

**Corollary 1.8** (Nuclear norm). *For any  $f \in \mathcal{F}_{d,n}$  with  $\|f\|_* \leq 1$*

$$v \left( \frac{b_{d,n}}{\|b_{d,n}\|_*} \right) = \left( \prod_{i=0}^{d/2-1} \frac{2i+n}{2i+1} \right)^{n/d} \frac{\sqrt{\pi}^n}{\Gamma(\frac{n}{2}+1)} \leq v(f) \quad (1.13)$$

and equality holds if and only if  $f = b_{d,n}/\|b_{d,n}\|_*$ .

A form  $f \in \mathcal{F}_{d,n}$  of even degree  $d$  is a *sum of squares* if  $f = s_1^2 + \dots + s_r^2$  for some forms  $s_1, \dots, s_r \in \mathcal{F}_{d/2,n}$  of degree  $d/2$ . Any sum of squares form is non-negative. Fix a total order  $\leq$  on the set  $\left\{ \sqrt{\binom{d/2}{\alpha}} x^\alpha : |\alpha| = d/2 \right\}$  of rescaled monomials of degree  $d/2$  (e.g., the lexicographic order) and denote by  $N = \binom{d/2+n-1}{n-1}$  the dimension of  $\mathcal{F}_{d/2,n}$ . Then,  $f \in \mathcal{F}_{d,n}$  is a sum of squares if and only if there exists a positive semidefinite real symmetric matrix  $G \in \mathcal{S}_N$ , called *Gram matrix*, satisfying

$$f(x) = m_{d/2}(x)^t G m_{d/2}(x), \quad x \in \mathbf{R}^n, \quad (1.14)$$

where  $m_{d/2}(x)$  denotes the  $N$ -dimensional column-vector of rescaled monomials  $\sqrt{\binom{d/2}{\alpha}} x^\alpha$ ,  $|\alpha| = d/2$ , ordered with respect to  $\leq$  (see [4, §2] and Lemma 2.2). Note that the cone of sums of squares in  $\mathcal{F}_{d,n}$  is the image of the closed convex cone  $\mathcal{PSD}_N \subset \mathcal{S}_N$  of positive semidefinite matrices under linear map (1.14).

Fix a norm  $\|\cdot\|$  on the space  $\mathcal{S}_N$  of real symmetric  $N \times N$  matrices and consider the following optimization problem:

$$\mathbf{P}_{\|\cdot\|}^{\text{SOS}} : \quad \text{opt}_{\|\cdot\|}^{\text{SOS}} = \inf \{v(f) : f = m_{d/2}(x)^t G m_{d/2}(x), G \in \mathcal{PSD}_N, \|G\| \leq 1\}. \quad (1.15)$$

**Remark 1.9** Note that  $\mathbf{P}_{\|\cdot\|}^{\text{SOS}}$  is the problem of minimization of the volume of the sublevel set  $\{f \leq 1\}$  of a sum of squares  $f = m_{d/2}(x)^t G m_{d/2}(x)$  with Gram matrix  $G$  from the unit ball in  $\mathcal{S}_N$  defined by the norm  $\|\cdot\|$ .

A norm  $\| \cdot \| : \mathcal{S}_N \rightarrow \mathbf{R}$  is said to be  $O(N)$ -invariant if  $\|R^t G R\| = \|G\|$  for all  $R \in O(N)$  and  $G \in \mathcal{S}_N$ . We prove that problem  $\mathbf{P}_{\|\cdot\|}^{\text{sos}}$  has a unique optimal solution, which, in the case of an  $O(N)$ -invariant norm, is proportional to  $b_{d,n}$ .

**Theorem 1.10** *Let  $d$  be even and  $\| \cdot \| : \mathcal{S}_N \rightarrow \mathbf{R}$  be a norm. Then*

- $\mathbf{P}_{\|\cdot\|}^{\text{sos}}$  is a convex optimization problem with a unique optimal solution  $f_{\text{sos}}^*$ .
- If norm  $\| \cdot \|$  is  $O(N)$ -invariant, then  $f_{\text{sos}}^* = b_{d,n}/\|\text{id}_N\|$ , where  $\text{id}_N \in \mathcal{S}_N$  is the identity matrix, and  $\text{opt}_{\|\cdot\|}^{\text{sos}} = \|\text{id}_N\|^{n/d} v(b_{d,n})$ .

The first claim follows from convexity properties of the norm, the cone of positive semidefinite matrices and the volume function. Existence and uniqueness of an optimal solution is derived from the fact that the volume function is lower-semicontinuous and strictly convex and from the fact that the map (1.14) sending a real symmetric matrix to a real form is linear. The  $O(N)$ -invariance of the norm and of the volume function combined with the uniqueness of the optimal solution imply the last claim. A more detailed proof of Theorem 1.10 is given in Sect. 3.

**Remark 1.11** Theorem 1.10 implies that the Euclidean ball in  $\mathbf{R}^n$  of radius  $\|\text{id}_N\|^{1/d}$  has smallest volume among sublevel sets of sums of squares corresponding to Gram matrices from the unit ball  $\{G \in \mathcal{S}_N : \|G\| \leq 1\}$  in  $\mathcal{S}_N$  defined by an  $O(N)$ -invariant norm.

### 1.5 Schatten $p$ -norms

Given a real symmetric matrix  $G \in \mathcal{S}_N$  its Schatten  $p$ -norm,  $p \geq 1$ , is defined by

$$\|G\|_p = \left( \sum_{i=1}^N |\lambda_i(G)|^p \right)^{1/p}, \tag{1.16}$$

where  $\lambda_1(G), \dots, \lambda_N(G) \in \mathbf{R}$  are the eigenvalues of  $G$ . When  $p = 1$  this norm is also known as *nuclear norm* and if  $p = 2$  we recover *Frobenius norm* which is classically used in the context of low-rank approximation of matrices [5]. Since eigenvalues do not change under conjugation by orthogonal matrices, all Schatten  $p$ -norms are  $O(N)$ -invariant.

We next compute the optimal value of problem  $\mathbf{P}_{\|\cdot\|}^{\text{sos}}$  for Schatten  $p$ -norms. Again, as in the above case of general nonnegative forms, by Theorem 1.10 this task reduces to computing the norm of  $\text{id}_N \in \mathcal{S}_N$ .

**Corollary 1.12** (Schatten  $p$ -norms). *Let  $p \geq 1$ . Then for any sum of squares form  $f = m_{d/2}(x)^t G m_{d/2}(x) \in \mathcal{F}_{d,n}$ ,  $G \in \mathcal{PSD}_N$ , with  $\|G\|_p \leq 1$*

$$v\left(\frac{b_{d,n}}{\|\text{id}_N\|_p}\right) = N^{n/dp} \frac{\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2} + 1\right)} \leq v(f) \tag{1.17}$$

and equality holds if and only if  $f = b_{d,n}/N^{1/p}$ .

**Remark 1.13** In [11] the second author of the present paper considered an analogous problem to  $\mathbf{P}_{\|\cdot\|_2}^{\text{SOS}}$ , where  $m_{d/2}(x)$  is replaced by the vector of monomials  $x^\alpha$ ,  $|\alpha| = d/2$ , (without coefficients  $\sqrt{\binom{d/2}{\alpha}}$ ), and proved that  $b_{d,n}$  is (up to a multiple) a unique optimal solution when  $d = 2, 4$  and when  $d \in 4\mathbb{N}$  provided that  $n$  is large [11, Thm. 5.1].

Corollary 1.12 immediately follows from Theorem 1.10, definition of Schatten  $p$ -norms (1.16) and formula (1.4) for the volume of the sublevel set of  $b_{d,n}$ .

**1.6 Spectral norm**

The *Spectral norm* of  $G \in \mathcal{S}_N$  defined by

$$\|G\|_\sigma = \max_{i=1,\dots,N} |\lambda_i(G)| \tag{1.18}$$

can be considered as the limit of Schatten  $p$ -norms (1.16) as  $p \rightarrow +\infty$ .

**Corollary 1.14** (Spectral norm). *For a sum of squares  $f = m_{d/2}(x)^t G m_{d/2}(x) \in \mathcal{F}_{d,n}$ ,  $G \in \mathcal{PSD}_N$ , with  $\|G\|_\sigma \leq 1$*

$$v\left(\frac{b_{d,n}}{\|\text{id}_N\|_\sigma}\right) = \frac{\sqrt{\pi}^n}{\Gamma\left(\frac{n}{2} + 1\right)} \leq v(f) \tag{1.19}$$

and equality holds if and only if  $f = b_{d,n}$ .

**1.7 Probabilistic interpretation of results**

If for  $f \in \mathcal{V}_{d,n}$  the sublevel set  $\{f \leq 1\}$  has finite Lebesgue volume, then by [10, Theorem 1]

$$v(f) = \frac{1}{\Gamma(1 + n/d)} \int_{\mathbf{R}^n} \exp(-f(x)) dx, \tag{1.20}$$

see also [13], (3.1). When  $\int_{\mathbf{R}^n} \exp(-f(x)) dx = 1$  the function  $x \mapsto \exp(-f(x))$  is the density of a probability measure  $\mu_f$  on  $\mathbf{R}^n$ . In particular, if  $f^*(x) = \kappa |x|^d$  with

$$\kappa = \left(\frac{\Gamma(1 + n/d)}{\Gamma(1 + n/2)}\right)^{d/n} \pi^{d/2}, \tag{1.21}$$

then  $\mu_{f^*}$  is a Gaussian-like probability measure in the sense that all of its moments are easily obtained from those of a Gaussian measure, that is,

$$\begin{aligned} & \int_{\mathbf{R}^n} x^\alpha \exp(-f^*(x)) dx \\ &= \frac{\Gamma(1 + (n + |\alpha|)/d)}{\Gamma(1 + (n + |\alpha|)/2)} \int_{\mathbf{R}^n} x^\alpha \exp(-\kappa^{2/d} |x|^2) dx, \quad \forall \alpha \in \mathbf{N}^n. \end{aligned} \tag{1.22}$$



By (1.20) and homogeneity,  $f$  is the unique optimal solution of  $\mathbf{P}_{\|\cdot\|}$  if and only if  $(\text{opt}_{\|\cdot\|} \Gamma(1 + n/d))^{d/n} f$  is the unique optimal solution of the convex optimization problem

$$\mathbf{P}_{\|\cdot\|}^* : \text{opt}_{\|\cdot\|}^* = \inf \left\{ \|f\| : \int_{\mathbf{R}^n} \exp(-f(x)) dx \leq 1, f \in \mathcal{F}_{d,n} \right\}. \tag{1.23}$$

In light of this fact Theorem 1.2 can be equivalently stated as follows.

**Theorem 1.15** *Let  $d$  be even and  $\|\cdot\| : \mathcal{F}_{d,n} \rightarrow \mathbf{R}$  be a norm. Then*

- *the convex optimization problem  $\mathbf{P}_{\|\cdot\|}^*$  has a unique optimal solution  $f^* \in \mathcal{V}_{d,n}$ .*
- *If the norm  $\|\cdot\|$  is  $O(n)$ -invariant, then  $f^* = \kappa b_{d,n}$ , where  $b_{d,n}(x) = |x|^d$ ,  $\kappa$  is as in (1.21), and  $\text{opt}_{\|\cdot\|}^* = \kappa \|b_{d,n}\|$ .*

**Remark 1.16** If  $\|\cdot\|$  is  $O(n)$ -invariant, then Theorem 1.15 implies that the Gaussian-like probability density  $x \mapsto \exp(-\kappa |x|^d)$  minimizes  $\|f\|$  over all probability measures with density  $x \mapsto \exp(-f(x))$ , where  $f \in \mathcal{F}_{d,n}$  is a nonnegative form of degree  $d$ .

## 2 Preliminaries and auxiliary results

In this section we give necessary definitions and prove some auxiliary results that are needed in Sect. 3.

Recall that  $\mathcal{F}_{d,n}$  denotes the space of  $n$ -ary real forms (or homogeneous polynomials) of degree  $d$  endowed with a norm  $\|\cdot\| : \mathcal{F}_{d,n} \rightarrow \mathbf{R}$ . Furthermore, recall that the group  $O(n) = \{\rho \in \mathbf{R}^{n \times n} : \rho \rho^t = \text{id}\}$  of orthogonal transformations acts on  $\mathcal{F}_{d,n}$  as follows

$$\rho \in O(n), f \in \mathcal{F}_{d,n} \mapsto \rho^* f \in \mathcal{F}_{d,n}, \rho^* f(x) = f(\rho^{-1}x). \tag{2.1}$$

For even  $d$  the form

$$b_{d,n}(x) = (x_1^2 + \dots + x_n^2)^{d/2} = \sum_{|\beta|=d/2} \binom{d/2}{\beta} x_1^{2\beta_1} \dots x_n^{2\beta_n}, x \in \mathbf{R}^n, \tag{2.2}$$

is obviously invariant with respect to (2.1). The following easy lemma asserts that  $b_{d,n}$  is essentially the only invariant form.

**Lemma 2.1** *Let  $f \in \mathcal{F}_{d,n}$  be a non-zero form invariant under  $O(n)$ -action (1.3). Then  $d$  is even and  $f$  is proportional to  $b_{d,n}$ .*

**Proof**  $O(n)$ -invariance of  $f$  implies  $f(x) = c$  whenever  $|x| = 1$ , for some constant  $c$ . If the degree  $d$  is odd, we have  $f(-x) = -f(x)$  for any  $x \in \mathbf{R}^n$  and hence  $f = 0$ . Thus  $d$  must be even and by homogeneity of  $f$

$$f(x) = f\left(|x| \frac{x}{|x|}\right) = c|x|^d = c b_{d,n}(x) \quad \text{for any } x \neq 0. \tag{2.3}$$

□

For two real forms  $f, g \in \mathcal{F}_{d,n}$  define their *Bombieri product* as

$$\langle f, g \rangle_B = \sum_{|\alpha|=d} f_\alpha g_\alpha, \tag{2.4}$$

where  $\{f_\alpha\}_{|\alpha|=d}$  and  $\{g_\alpha\}_{|\alpha|=d}$  are the coefficients of  $f$  and  $g$  in the basis of rescaled monomials (1.5). Equivalently, denoting by  $f(\partial)$  the differential operator obtained from  $f$  by replacing variable  $x_i, i = 1, \dots, n$ , with partial derivative  $\partial/\partial x_i$ , one can show that

$$\langle f, g \rangle_B = \frac{1}{d!} f(\partial)g(x). \tag{2.5}$$

From this, taking Bombieri product with a power of a linear form  $x \mapsto f(x) = (y \cdot x)^d, y \in \mathbf{R}^n$ , amounts to evaluation at  $y$ , that is,

$$\langle f, g \rangle_B = g(y). \tag{2.6}$$

Recall that a form  $f \in \mathcal{F}_{d,n}$  of even degree  $d$  is called a *sum of squares* if  $f = s_1^2 + \dots + s_r^2$  for some  $s_1, \dots, s_r \in \mathcal{F}_{d/2,n}$ . The following characterization of sums of squares is well-known; we state it here as our version concerns rescaled monomials, cf. [4, §2].

**Lemma 2.2** *A form  $f \in \mathcal{F}_{d,n}$  is a sum of squares if and only if there exists a positive semidefinite real symmetric matrix  $G \in \mathcal{PSD}_N$  such that*

$$f(x) = m_{d/2}(x)^t G m_{d/2}(x), \quad x \in \mathbf{R}^n, \tag{2.7}$$

where  $N = \dim \mathcal{F}_{d/2,n} = \binom{d/2+n-1}{n-1}$  and  $m_{d/2}(x)$  is the column-vector of rescaled monomials  $\sqrt{\binom{d/2}{\alpha}} x^\alpha, |\alpha| = d/2$ , ordered with respect to a fixed order  $\leq$ .

**Proof** If  $f = s_1^2 + \dots + s_r^2$ , then  $f = m_{d/2}(x)^t G m_{d/2}(x)$ , where  $G = \sum_{i=1}^r \mathbf{s}_i \mathbf{s}_i^t \in \mathcal{PSD}_N$  and  $\mathbf{s}_i$  denotes the column-vector of coefficients of form  $s_i, i = 1, \dots, r$ , in the basis of rescaled monomials ordered with respect to  $\leq$ . Conversely, if (2.7) holds for some positive semidefinite matrix  $G = G^{1/2} G^{1/2} \in \mathcal{PSD}_N$ , then  $f = s_1^2 + \dots + s_N^2$ , where  $s_1, \dots, s_N \in \mathcal{F}_{d/2,n}$  are the entries of the  $N$ -dimensional vector of forms  $G^{1/2} m_{d/2}(x)$ . □

### 3 Proof of main results

In this section we prove our main results, Theorem 1.2 and Theorem 1.10.

**Proof of Theorem 1.2** The proof of the fact that  $\mathbf{P}_{\|\cdot\|}$  has a unique optimal solution is analogous to the one of [11, Thm. 3.2]; we give it here for the sake of completeness.

Let  $\{f_k\}_{k \in \mathbb{N}}$  be a minimizing sequence of the optimization problem  $\mathbf{P}_{\|\cdot\|}$ , that is  $\|f_k\| \leq 1, k \in \mathbb{N}$ , and  $\lim_{k \rightarrow +\infty} v(f_k) = \text{opt}_{\|\cdot\|}$ . By compactness of the unit

ball of norm  $\|\cdot\|$  there exists a subsequence  $\{f_{k_m}\}_{m \in \mathbb{N}}$  and  $f^* \in \mathcal{F}_{d,n}$  such that  $\lim_{m \rightarrow +\infty} \|f_{k_m} - f^*\| = 0$  and  $\|f^*\| \leq 1$ , meaning that  $f^*$  is feasible. By [11, Lemma 2.3], the function  $v : \mathcal{F}_{d,n} \rightarrow \mathbf{R}_{\geq 0} \cup \{+\infty\}$  is lower-semicontinuous. This implies

$$\text{opt}_{\|\cdot\|} = \liminf_{m \rightarrow +\infty} v(f_{k_m}) \geq v(f^*), \tag{3.1}$$

that is,  $f^*$  is an optimal solution of  $\mathbf{P}_{\|\cdot\|}$ .

Now, by [11, Thm. 2.2], the function  $v$  is strictly convex. Thus, if  $\mathbf{P}_{\|\cdot\|}$  had two different optimal solutions  $f_1^*$  and  $f_2^*$ , then for  $\alpha \in (0, 1)$  we would have

$$\begin{aligned} \|\alpha f_1^* + (1 - \alpha)f_2^*\| &\leq \alpha\|f_1^*\| + (1 - \alpha)\|f_2^*\| \leq 1, \\ v(\alpha f_1^* + (1 - \alpha)f_2^*) &< \alpha v(f_1^*) + (1 - \alpha)v(f_2^*) = \text{opt}_{\|\cdot\|}, \end{aligned} \tag{3.2}$$

a contradiction. Thus an optimal solution of  $\mathbf{P}_{\|\cdot\|}$  is unique. Moreover, homogeneity of  $\|\cdot\|$  and  $v$  implies that the unique optimal solution  $f^*$  of  $\mathbf{P}_{\|\cdot\|}$  must satisfy  $\|f^*\| = 1$ .

Let us now consider the case of an  $O(n)$ -invariant norm. Observe first that the volume function  $v$  is  $O(n)$ -invariant, that is,  $v(\rho^* f) = v(f)$  for any  $f \in \mathcal{F}_{d,n}$  and  $\rho \in O(n)$ . Indeed, this follows directly from the definition of  $v(f)$  and invariance of Lebesgue measure on  $\mathbf{R}^n$ . We claim that the optimal solution  $f^*$  of  $\mathbf{P}_{\|\cdot\|}$  is  $O(n)$ -invariant. If not there exists  $\rho \in O(n)$  such that  $\rho^* f^* \neq f^*$ . Then in view of  $O(n)$ -invariance of  $v$  and  $\|\cdot\|$ ,  $f^*$  and  $\rho^* f^*$  are two different optimal solutions of  $\mathbf{P}_{\|\cdot\|}$ , which is impossible by the above. Lemma 2.1 implies that  $f^*$  is proportional to  $b_{d,n}$  and since  $\|f^*\| = 1$  we must have  $f^* = b_{d,n}/\|b_{d,n}\|$ . As  $v$  is homogeneous of degree  $-n/d$ , we obtain  $\text{opt}_{\|\cdot\|} = v(b_{d,n}/\|b_{d,n}\|) = \|b_{d,n}\|^{n/d} v(b_{d,n})$ .  $\square$

If  $\|\cdot\|$  is a particular norm then by Theorem 1.2, computing the optimal value of  $\mathbf{P}_{\|\cdot\|}$  reduces to computing  $\|b_{d,n}\|$ . We next evaluate  $\|b_{d,n}\|$  for Bombieri norm,  $L^p(\mathbb{S}^{n-1})$ -norm, uniform norm on  $\mathbb{S}^{n-1}$ , nuclear norm, and thus prove Corollaries 1.5, 1.6, 1.7 and 1.8.

**Proof of Corollary 1.5** By [15, (8.19)] we have

$$\|b_{d,n}\|_B = \sqrt{\prod_{i=0}^{d/2-1} \frac{2i+n}{2i+1}}. \tag{3.3}$$

Combining this formula with (1.4) yields (1.7).  $\square$

**Proof of Corollary 1.6** One has

$$\|b_{d,n}\|_{L^p(\mathbb{S}^{n-1})} = \left( \int_{\mathbb{S}^{n-1}} |b_{d,n}(x)|^p d\mathbb{S}^{n-1}(x) \right)^{1/p} = \text{vol}(\mathbb{S}^{n-1})^{1/p} = \left( \frac{2\sqrt{\pi}^n}{\Gamma(\frac{n}{2})} \right)^{1/p}, \tag{3.4}$$

which together with Theorem 1.2 and (1.4) yields (1.9).  $\square$

Corollary 1.7 follows from Theorem 1.2, (1.4) and the formula  $\|b_{d,n}\|_{L^\infty(\mathbb{S}^{n-1})} = \max_{x \in \mathbb{S}^{n-1}} |x|^d = 1$ .

**Proof of Corollary 1.8** From a result of Hilbert [9] it follows that there exist  $r \in \mathbb{N}$ ,  $\lambda_1, \dots, \lambda_r > 0$  and  $y^1, \dots, y^r \in \mathbb{S}^{n-1}$  such that

$$b_{d,n}(x) = \sum_{k=1}^r \lambda_k (y^k \cdot x)^d \tag{3.5}$$

and thus, invoking [14, Example 1.1], we have

$$\|b_{d,n}\|_* = \sum_{k=1}^r \lambda_k. \tag{3.6}$$

On the other hand, by (2.6) and (3.5),

$$\|b_{d,n}\|_* = \sum_{k=1}^r \lambda_k = \sum_{k=1}^r \lambda_k \langle (y^k \cdot \bullet)^d, b_{d,n} \rangle_B = \langle b_{d,n}, b_{d,n} \rangle_B = \|b_{d,n}\|_B^2, \tag{3.7}$$

and (1.8) follows from (3.3) and (1.4). □

We now prove Theorem 1.10.

**Proof of Theorem 1.10** Let us observe first that convexity of the feasible set

$$\{f \in \mathcal{F}_{d,n} : f = m_{d/2}(x)^t G m_{d/2}(x), G \in \mathcal{PSD}_N, \|G\| \leq 1\} \tag{3.8}$$

of optimization problem  $\mathbf{P}_{\|\cdot\|}^{\text{SOS}}$  follows directly from convexity of the cone  $\mathcal{PSD}_N$  of positive semidefinite matrices and convexity of norm  $\|\cdot\|$ . This fact combined with convexity of the function  $v$  (see [11, Thm. 2.2]) implies that  $\mathbf{P}_{\|\cdot\|}^{\text{SOS}}$  is a convex sequence of the optimization problem.

Let  $\{f_k = m_{d/2}(x)^t G_k m_{d/2}(x)\}_{k \in \mathbb{N}}$  be a minimizing sequence of the optimization problem  $\mathbf{P}_{\|\cdot\|}^{\text{SOS}}$ , i.e.,  $G_k \in \mathcal{PSD}_N$ ,  $\|G_k\| \leq 1$ ,  $k \in \mathbb{N}$ , and  $\lim_{k \rightarrow +\infty} v(f_k) = \text{opt}_{\|\cdot\|}^{\text{SOS}}$ . Since  $\{G \in \mathcal{PSD}_N : \|G\| \leq 1\}$  is compact, there is a subsequence  $\{G_{k_m}\}_{m \in \mathbb{N}}$  and a matrix  $G^* \in \mathcal{PSD}_N$ ,  $\|G^*\| \leq 1$ , such that  $\lim_{m \rightarrow +\infty} \|G_{k_m} - G^*\| = 0$ . In particular, the sum of squares form  $f^* = m_{d/2}(x)^t G^* m_{d/2}(x) \in \mathcal{F}_{d,n}$  is feasible for  $\mathbf{P}_{\|\cdot\|}^{\text{SOS}}$  and coefficients of  $f_{k_m}$  converge to coefficients of  $f^*$ , as  $m \rightarrow +\infty$ . Next, since the function  $v : \mathcal{F}_{d,n} \rightarrow \mathbf{R}_{\geq 0} \cup \{+\infty\}$  is lower-semicontinuous [11, Lemma 2.3] we have

$$\text{opt}_{\|\cdot\|}^{\text{SOS}} = \liminf_{m \rightarrow +\infty} v(f_{k_m}) \geq v(f^*), \tag{3.9}$$

that is,  $f^*$  is an optimal solution of  $\mathbf{P}_{\|\cdot\|}^{\text{SOS}}$ . Exactly in the same way as in the proof of Theorem 1.2, strict convexity of  $v$  implies that  $f^*$  is a unique optimal solution of  $\mathbf{P}_{\|\cdot\|}^{\text{SOS}}$ . Also, from homogeneity of  $v$  and  $\|\cdot\|$  we obtain  $\|G^*\| = 1$ .

Let now  $\|\cdot\| : \mathcal{S}_N \rightarrow \mathbf{R}$  be an  $O(N)$ -invariant norm. If  $f = m_{d/2}(x)^t G m_{d/2}(x)$  is feasible for  $\mathbf{P}_{\|\cdot\|}^{\text{sos}}$ , i.e.,  $G \in \mathcal{PSD}_N$  and  $\|G\| \leq 1$ , then so is  $\rho^* f$  for any  $\rho \in O(n)$ . Indeed, since Bombieri product (2.4) is invariant under  $O(n)$ -action (1.3) and since the rescaled monomials  $\sqrt{\binom{d/2}{\alpha}} x^\alpha$ ,  $|\alpha| = d/2$ , form an orthonormal basis of  $\mathcal{F}_{d/2,n}$  with respect to Bombieri product, for any  $\rho \in O(n)$  there exists  $R = R(\rho) \in O(N)$  such that

$$\rho^* f = m_{d/2}(x)^t R^t G R m_{d/2}(x). \tag{3.10}$$

Hence, by  $O(N)$ -invariance of  $\mathcal{PSD}_N$  and  $\|\cdot\|$ , we have  $R^t G R \in \mathcal{PSD}_N$  and  $\|R^t G R\| = \|G\| \leq 1$  or, in other words,  $\rho^* f$  is feasible for  $\mathbf{P}_{\|\cdot\|}^{\text{sos}}$ . Therefore the unique optimal solution  $f^*$  of  $\mathbf{P}_{\|\cdot\|}^{\text{sos}}$  must be  $O(n)$ -invariant, that is,  $\rho^* f^* = f^*$  for all  $\rho \in O(n)$ .

Next, by Lemma 2.1,  $f^*$  is proportional to  $b_{d,n}$ . From (2.2) we have that  $b_{d,n} = m_{d/2}(x)^t m_{d/2}(x)$ , namely the identity matrix  $\text{id}_N \in \mathcal{PSD}_N$  is a Gram matrix of  $b_{d,n}$ . Therefore  $f^* = b_{d,n}/\|\text{id}_N\|$  as its Gram matrix satisfies  $\|\text{id}_N/\|\text{id}_N\|\| = 1$ . Also,  $\text{opt}_{\|\cdot\|}^{\text{sos}} = v(b_{d,n}/\|\text{id}_N\|) = \|\text{id}_N\|^{n/d} v(b_{d,n})$  by homogeneity of the volume function. □

## 4 Conclusion

We have provided new volume-minimizing properties of the Euclidean unit ball. In contrast to its intrinsic geometric properties, they are attached to its representation as the sublevel set of a form of fixed even degree. The minimum is over all nonnegative forms of same degree with bounded norm or over sum of squares forms of same degree whose Gram matrix has bounded norm, for certain families of norms.

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