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PREFACE

Lifts of convex sets in optimization

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This special issue is dedicated to the geometry and complexity of *lifts* or *extended formulations* of convex sets which has been an active area of research in recent years. This developing field lies at the intersection of several areas such as convex geometry, polyhedral theory, real algebraic geometry, combinatorics, optimization, and computer science, among others.

The idea of finding efficient representations of convex sets (especially polytopes) by expressing them as projections of simple convex sets in higher dimensions is not new and is well-known in areas such as integer programming and real algebraic geometry. The current view of the topic is inspired by the 1991 paper of Mihalis Yannakakis who showed that the matching polytope does not admit a small "symmetric" polyhedral lift, in striking contrast to the classical result of Jack Edmonds that linear optimization over the matching polytope is possible in polynomial time. Yannakakis also showed that there is a precise connection between the nonnegative rank of the *slack matrix* of a polytope and the size of the smallest polyhedral lift that is possible for this polytope. Both of these results have been extended in multiple directions in recent papers. The restrictions imposed by symmetry on the size of lifts has been explored in multiple papers by now. Also, the connection between lifts and nonnegative ranks have been extended to general convex sets and closed cones. In particular, positive semidefinite lifts of polytopes have received a fair bit of attention lately, while the older notion of

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polyhedral lifts of polytopes have had some impressive breakthroughs, the culmination of which was the result that the matching polytope does not admit a small polyhedral lift even without symmetry restrictions, completing the Yannakakis story. It is also known by now that for specific polytopes from NP-hard problems such as max cut and traveling salesman, neither polyhedral nor psd lifts of small size are possible.

This issue contains research papers on both polyhedral and psd lifts of polytopes and other convex sets. The polyhedral part comes first, followed by the psd section.

Our volume starts with the paper *Smallest Compact Formulation of the Permutahedron* by Michel Goemans, in which he provides an asymptotically smallest possible extended formulation for the permutahedron starting from a sorting network. Our second constructive paper *Stable Sets and Graphs with no Even Holes* is by Michele Conforti, Bert Gerards, and Kanstantsin Pashkovich. They develop a decomposition theory for certain graphs that allows them to come up with polynomial size extended formulations for the respective stable set polytopes. Due to the above mentioned seminal work of Yannakakis, lower bounds on the nonnegative ranks of slack matrices are of great importance when bounding sizes of extended formulations from below. In *Lower bounds on nonnegative rank via nonnegative nuclear norms* Hamza Fawzi and Pablo Parrilo describe a new lower bound on the nonnegative rank in terms of a copositive program. Yaroslav Shitov generalizes Yannakakis' result to general real closed fields in his contribution *Tropical lower bounds for extended formulations* and shows how to use this generalization in order to construct polytopes with high extension complexity. Extending the relation to communication complexity already realized and exploited by Yannakakis, Yuri Faenza, Samuel Fiorini, Roland Grappe, and Hans Raj Tiwary show in their paper *Extended formulations, nonnegative factorizations, and randomized communication protocols* how to express the nonnegative rank of a matrix as the minimum complexity of a randomized protocol computing the matrix entries in expectation. Known exponential lower bounds on the extension complexity of the cut polytope of the complete graph are used to establish corresponding results for other polytopes associated with hard optimization problems in the contribution *On the extension complexity of combinatorial polytopes* by David Avis and Hans Raj Tiwary. Samuel Fiorini and Kanstantsin Pashkovich show in *Uncapacitated Flow-based Extended Formulations* that for several polytopes – among them the perfect matching polytopes of bipartite graphs – extended formulations based on network flows without capacities necessarily need to have exponential size.

The second set of papers in this issue are concerned with psd lifts of convex sets from various angles. This section begins with the article *Positive Semidefinite Rank* by Hamza Fawzi, João Gouveia, Pablo Parrilo, Richard Robinson and Rekha Thomas which studies psd rank for general nonnegative matrices. The paper *On the existence of* $0/1$ *polytopes with high extension complexity* by Jop Briët, Daniel Dadush, and Sebastian Pokutta show that not all $0/1$-polytopes of a given dimension can have psd lifts whose size is small as a function of the dimension. Methods for finding lower bounds to the psd rank of a polytope or matrix are still far and few between. In *Worst-case results for positive semidefinite rank*, João Gouveia, Richard Robinson, and Rekha Thomas exhibit several situations of high psd rank, in and outside the context of lifts. The geometry underlying psd lifts is that of spectrahedral shadows, namely projections of affine slices of the psd cone. In *Smooth hyperbolicity cones are* *spectrahedral shadows*, Tim Netzer and Raman Sanyal prove the result of their title settling part of the open question as to whether all hyperbolic cones have this property. Hyperbolic cones resurface in the paper *Hyperbolic polynomials, interlacers and sums of squares* by Daniel Plaumann and Cynthia Vinzant who show that the hyperbolic cone is a slice of the cone of nonnegative polynomials, and that it can be approximated by a spectrahedral shadow. The last paper *Linear Optimization with Cones of Moments and Nonnegative Polynomials* by Jiawang Nie steps out into the world of polynomial optimization and studies the general technique of moments and sums of squares in the situation of restricted supports.

We thank everyone who contributed to this volume in particular the reviewers and hope that these papers both showcase the current activity on the subject of extended formulations, and will inspire more research and results on this exciting interdisciplinary topic.