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Characterizations of convex functions by level sets

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Abstract
It is shown that a lsc function is convex if and only if the minimal set of any linear perturbation of the function is convex. That fact also yields that the convexity of a function is equivalent to the quasiconvexity of its all linear perturbations.

Keywords Minimal set · Linear perturbation · Convexity of functions

Mathematics Subject Classification 26B25 · 46N10

1 Introduction

The examination of convex functions through their level sets is arrestive thing in itself. But it has some relevance in theory of concave games, as well. In 1986, Joó in an article (Joó 1986) argued the necessity of concavity of the response funtions at their relevant variable, in the Nikaido-Isoda theorem (Theorem 3.1 in Nikaidô and Isoda 1955). Certainly Joó cannot mean the necessity in the direct sense since the concavity can be obviously replaced by quasiconcavity in the previously mentioned theorem. Instead, he meant the necessity in a "perturbative" sense, i.e., he supposed the existence of the equilibrium for all additive perturbations of the response functions taken by concave functions, and hence concluded that the original response functions are concave in the relavant variable. To achieve his result, Joó used the following: a continuous function $f : [a, b] \to \mathbb{R}$ is concave if and only if the set of maximum points of $f + c \cdot id$ form a closed interval for any $c \in \mathbb{R}$ (Lemma 1 in Joó (1986)). But the author left it unproven. Recently Forgó and me used this lemma and also proved it in Forgó and Kánnai (2020). However, the idea of the proof was typically one-dimensional, while the relevance of convexity/concavity appears principally for multivariate functions. Hereinafter we prove the convex analogue of Joó's lemma for multivariate lower semicontinuous functions. It makes possible also to characterize the convexity by the quasiconvexity of all linear perturbations, and to

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clarify the mathematical nature of the necessity in Joó's result. The equivalence of points 1. and 2. of the characterization (Corollary 2 below) also appears in Proposition 2.1. of Aussel et al. (1994).

2 The result

Theorem 1 *Let* $(X, \langle \cdot \mid \cdot \rangle)$ *be a Euclidean space*, $K \subseteq X$ *be a nonempty convex compact subset*, *and* $f : K \to \mathbb{R}$ *be a lower semicontinuous function such that the minimal set* (*in other words*, argmin *set*)

$$\left\{ x \in K \ : \ f(x) + \langle c \mid x \rangle = \min_{y \in K} (f(y) + \langle c \mid y \rangle) \right\}$$

of the perturbed function $f + \langle c \mid \cdot \rangle$, *is convex for all* $c \in X$. *Then* f *is a convex function*.

Proof Lower semicontinuity obviously implies that f is bounded from below on the compact set K. Denote by

$$\mathrm{epi}^{\mathrm{o}} f := \{(x, \alpha) : f(x) < \alpha\} \subseteq X \times \mathbb{R}$$

the strict epigraph of f. By elementary calculations we obtain that the convex hull of $\mathrm{epi}^{\mathrm{o}} f$ coincides with the strict epigraph of a suitable convex function $f_* : K \to \mathbb{R}$ bounded from below $\left(f \geq f_*\right)$:

$$\mathrm{conv}\,(\,\mathrm{epi}^{\mathrm{o}} f) = \ \mathrm{epi}^{\mathrm{o}} f_*.$$

Let K_0 be the relative interior of K. We can assume (without loss of generality) that $\mathbf{0}_X \in K_0$. We can also assume (otherwise restrict the Euclidean space X to the linear hull of K) that $K_0 = \ \mathrm{int}\, K$. Let $x_0 \in K_0$ be arbitrary. On account of the weak separation form of the Hahn-Banach theorem, we get a nonzero element (y, r) of the Euclidean space $X \times \mathbb{R}$ which separates the convex set $\mathrm{epi}^{\mathrm{o}} f_*$ from the point $\left(x_0, f_*\left(x_0\right)\right)$, namely

$$\inf_{(x,\alpha) \in \ \mathrm{epi}^{\mathrm{o}} f_*} (\langle y \mid x \rangle + r\alpha) \geq \langle y \mid x_0 \rangle + r f_*\left(x_0\right).$$

Since α can be arbitrarily large in the term of the left-hand side, the number r cannot be negative. If r were 0, then the linear functional $\langle y \mid \cdot \rangle$ would separate the set K from the point x_0. But it is impossible because x_0 is an interior point of K. So $r > 0$. Divided by r the latest inequality, and by choosing $c := -\frac{y}{r}$, we obtain that

$$\inf_{(x,\alpha) \in \ \mathrm{epi}^{\mathrm{o}} f_*} (\langle -c \mid x \rangle + \alpha) \geq \langle -c \mid x_0 \rangle + f_*\left(x_0\right),$$

hence immediately we have that

$$\alpha \geq \langle c \mid x - x_0 \rangle + f_*\left(x_0\right)$$

for every $(x, \alpha) \in \operatorname{epi}^{\mathrm{o}} f_*$, whence we easily obtain that

$$f_*(x) \geq \langle c \mid x - x_0 \rangle + f_*\left(x_0\right)$$

for every $x \in K$, consequently by $f(x) \geq f_*(x)$,

$$f(x) \geq \langle c \mid x - x_0 \rangle + f_*\left(x_0\right).$$

According to the premise, the minimal set

$$M := \left\{ x \in K \; : \; f(x) - \langle c \mid x \rangle = \min_{y \in K} (f(y) - \langle c \mid y \rangle) \right\}$$

is convex, on the other hand, by lower semicontinuity of f, it is also nonempty and compact. The set M is also the minimal set of the correspondence $x \mapsto f(x) - \langle c \mid x - x_0 \rangle - f_*\left(x_0\right)$. In consequence of the latest inequality, this term is nonnegative on the set K, hence its minimum (taken on the set M) is also nonnegative. Therefore this term is positive out of M. Now suppose, by contradiction, that $x_0 \notin M$. Then on account of the strict separation form of the Hahn-Banach theorem, we get a vector $d \in X$ of unit length, such that

$$\max_{x \in M} \langle d \mid x \rangle < \langle d \mid x_0 \rangle.$$

Furthermore, by M being compact and $\langle d \mid \cdot \rangle$ being continuous, there is a positive number ε such that the relation

$$\langle d \mid x \rangle < \langle d \mid x_0 \rangle$$

remains valid on the set

$$K \cap \mathcal{B}^{\mathrm{o}}(M, \varepsilon) = K \cap \bigcup_{x \in M} \mathcal{B}^{\mathrm{o}}(x, \varepsilon)$$

(where the symbol $\mathcal{B}^{\mathrm{o}}$ denotes open balls and neighborhoods). Since the term $f(x) - \langle c \mid x - x_0 \rangle - f_*\left(x_0\right)$ is lower semicontinuous and positive at the points of the compact set $K \setminus \mathcal{B}^{\mathrm{o}}(M, \varepsilon)$, it takes its positive minimum on this set. That is, there exists a positive number A such that

$$f(x) - \langle c \mid x - x_0 \rangle - f_*\left(x_0\right) \geq A$$

for every $x \in K \setminus \mathcal{B}^{\mathrm{o}}(M, \varepsilon)$. Denote by h the diameter of K. If $z \in X$ with $\|z - c\| \leq \frac{A}{2h}$, then

$$\begin{aligned} & f(x) - \langle z \mid x - x_0 \rangle - f_*\left(x_0\right) \\ & \quad \geq f(x) - \langle c \mid x - x_0 \rangle - f_*\left(x_0\right) - \|z - c\| \left\| x - x_0 \right\| \geq A - \frac{A}{2h} h = \frac{A}{2} > 0 \end{aligned}$$

for every $x \in K \setminus \mathcal{B}^{\circ}(M,\varepsilon)$. Take $0 < t \leq \frac{A}{2h}$, then $\|t \cdot d\| = t \cdot \|d\| = t \leq \frac{A}{2h}$, hence by the latter we obtain that

$$f(x) - \langle c + td \mid x - x_0 \rangle - f_*(x_0) > 0$$

for every $x \in K \setminus \mathcal{B}^{\circ}(M,\varepsilon)$. At the same time, for every $x \in K \cap \mathcal{B}^{\circ}(M,\varepsilon)$ by $\langle d \mid x \rangle < \langle d \mid x_0 \rangle$ we have $\langle d \mid x - x_0 \rangle < 0$, so

$$\begin{aligned} f(x) - \langle c + td \mid x - x_0 \rangle - f_*(x_0) &= f(x) - \langle c \mid x - x_0 \rangle - f_*(x_0) - t\langle d \mid x - x_0 \rangle \\ &> f(x) - \langle c \mid x - x_0 \rangle - f_*(x_0) \geq 0, \end{aligned}$$

because the term $f(x) - \langle c \mid x - x_0 \rangle - f_*(x_0)$ is nonnegative on the whole K. Thus, overall, the term $f(x) - \langle c + td \mid x - x_0 \rangle - f_*(x_0)$ is positive on the whole K. This term being lower semicontinuous in x, takes its positive minimum on the compact set K. That is, there is a suitable constant $B > 0$ such that

$$f(x) - \langle c + td \mid x - x_0 \rangle - f_*(x_0) \geq B$$

for every $x \in K$. Therefore the halfspace consisting of the vectors (x, α) satisfying the inequality

$$\alpha \geq \langle c + td \mid x - x_0 \rangle + f_*(x_0) + B,$$

contains $\mathrm{epi}^{\mathrm{o}} f$ and its convex hull $\mathrm{epi}^{\mathrm{o}} f_*$ as well. Hence we easily obtain that

$$f_*(x) \geq \langle c + td \mid x - x_0 \rangle + f_*(x_0) + B$$

for every $x \in K$, in particular $f_*(x_0) \geq f_*(x_0) + B$, which is a contradiction. Thus, $x_0 \in M$. Hence the term $f(x) - \langle c \mid x - x_0 \rangle$ defined on K, takes its minimum at x_0, namely with value $f(x_0)$; what means that the value of the latter term is at least $f(x_0)$ for every $x \in K$, that is,

$$f(x) \geq \langle c \mid x - x_0 \rangle + f(x_0)$$

whenever $x \in K$. It just means that the term $\langle c \mid x - x_0 \rangle + f(x_0)$ defines an affine support to the function f at the point x_0. Since it is true for every $x_0 \in \operatorname{int} K$, hence the upper envelope $\widehat{f}$ of all these affine support functions is a lower semicontinuous convex function such that $\widehat{f} \leq f$, moreover $\widehat{f}$ coincides with f in the interior of K. Fix a point $x_* \in \operatorname{int} K$. By restricting the functions f and $\widehat{f}$ to the line segments with endpoint x_*, and by applying the facts that f is lower semicontinuous, $\widehat{f}$ is convex lower semicontinuous, $\widehat{f} \leq f$, and $\widehat{f}$ coincides with f in the interior of K, we easily obtain that $\widehat{f}$ coincides with f on the whole K. Consequently f is a convex function. □

Recall that a function $f : X \rightarrowtail \mathbb{R}$ defined on a convex set, is *quasiconvex* if the level set $(f \leq \alpha)$ is convex for every $\alpha \in \mathbb{R}$.

Corollary 2 *Let X be a Euclidean space and $K \subseteq X$ be a nonempty convex compact subset. For a lower semicontinuous function $f : K \to \mathbb{R}$ the following three condition are equivalent*:

1. *f is convex*;
2. *the correspondence $K \to \mathbb{R}$, $x \mapsto f(x) + \langle c \mid x \rangle$ is quasiconvex for every $c \in X$*;
3. *the minimal set*

$$\left\{ x \in K \ : \ f(x) + \langle c \mid x \rangle = \min_{y \in K} (f(y) + \langle c \mid y \rangle) \right\}$$

is convex for every $c \in X$.

Proof 1→2: If f is convex then also the correspondences $x \mapsto f(x) + \langle c \mid x \rangle$ are convex, hence quasiconvex.

2→3: If the minimum of a term $f(x) + \langle c \mid x \rangle$ is α, then the minimal set coincides the convex level set $(f(x) + \langle c \mid x \rangle \leq \alpha)$.

3→1: It follows from the previous theorem. □

According to Corollary 2, the partial concavity result in Joó 's necessity theorem (Theorem 2 in Joó 1986) is due to the excessively strong perturbation condition. Namely if all additive perturbations of a continuous function f taken by concave functions are quasiconcave then f is concave. This pure logical fact is independent from game theory.

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Declarations

conflict of interest I declare that I have no conflict of interest to disclose.

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