Bipolar sorting and ranking of multistage alternatives

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Abstract

The present paper proposes an extension of the multicriteria Bipolar method, introduced by E. Konarzewska-Gubała, and its application to the control of multistage, multicriteria decision processes with a fixed number of stages. At each stage, two sets of reference points are determined, which constitute a reference system for the evaluation of stage decisions. At the end of the process, multistage alternatives—compositions of stage alternatives—are evaluated. The procedure proposed, which includes elements of the Electre methodology, allows to assign each multistage alternative to one of the six predefined, hierarchical classes, and then to perform ranking within each class. The purpose of the paper is to present and substantiate the dynamic Bipolar procedure. An essential part of the paper is a numerical example which illustrates the notions and relationships introduced.

Keywords Bipolar method · Multistage decision processes · MCDA · Reference sets · Multistage alternative

1 Introduction

The Bipolar method, proposed by Konarzewska-Gubała (1987, 1989), is an MCDA method; here we will call it the classic Bipolar method. The individual phases of this method use elements of the Electre methodology (Roy 1985; Roy and Bouyssou 1993), as well as algorithms of confrontation (Merighi 1980). This method has been described in many earlier papers (Konarzewska-Gubała 1996, 2002; Trzaskalik and Sitarz 2012; Trzaskalik et al. 2013), modified (Górecka 2014) and applied in practice (Dominiak 1997; Konarzewska-Gubała 1996, 2002; Górecka 2017; Tłuczak 2018). A detailed description of the classic Bipolar method, published recently in Trzaskalik et al. (2019), compares it with other multicriteria methods.

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A fundamental feature of the classic Bipolar method is that the decision alternatives are not compared directly with each other, but by means of two sets of reference points: objects with desired characteristics, called “good” objects, and objects with undesired characteristics, called “bad” objects. In the first phase of the classic Bipolar method decision alternatives are compared with good and bad objects. In the second phase of the Bipolar method the position of each alternative with respect to the Bipolar reference system is established. In the third stage, the alternatives are classified: first separately, with respect to the good and bad reference sets, then jointly. The alternatives are divided into indexed classes so that each alternative from a lower-indexed class is preferred over any alternative from a higher-indexed class. Within each class, a linear ordering is defined.

Almost simultaneously with the introduction of the classic Bipolar method the issue of a possible extension of this approach to the analysis of multistage, multicriteria decision processes arose (Trzaskalik 1987). That paper formulated the problem of searching for the best solution using an extended Bipolar approach. That attempt, however, had not been entirely successful and research in this direction was discontinued.

The present paper tackles this problem again. It aims at presenting possible applications of an essential fragment of the classic Bipolar approach—which is a single-stage procedure—to control multistage discrete decision processes. This requires that new notions be defined, directly related to the extension being constructed, such as stage alternative, multistage alternative, stage reference sets, or the importance of the criteria in the consecutive process stages. These notions, together with the notation used further in the paper [related to the description of the classic Bipolar method, presented in Trzaskalik et al. (2019)] are introduced in Sect. 2. Section 3 presents evaluation of stage alternatives, which corresponds to the first two stages of the classic Bipolar method. Section 4 contains definitions of indicators determining the position of the multistage alternatives with respect to the multistage reference sets. These indicators allow to divide the multistage alternatives into hierarchical, disjoint classes containing elements with similar characteristics. This section also proposes a method of ranking of the multistage alternatives within the individual classes. Section 5 consists of a numerical illustrative example, while Sect. 6 contains conclusions.

2 Assumptions and notation

The classic Bipolar method is a single-stage method. When considering multistage decision processes we deal with at least two stages. We will discuss multistage processes with a predefined number of T stages.

We will briefly describe the dynamics of the multistage process in question (Bellman 1957; Rapoport 1967; Trzaskalik 1998; Ahmad 2018). At the beginning of each stage t ($t = 1, \dots, T$), the process is in one of the feasible states. All the states of the process that can occur at the beginning of the consecutive stages form the sets of feasible states Y_t for the consecutive stages t . Furthermore, for each feasible state y_t at each stage t one can determine the possible decision x_t . The set of all such

decisions $X_t(y_t)$ is called the set of feasible decisions for the given state $y_t \in Y_t$. If at the beginning of a stage t the process is in a certain feasible state y_t and an feasible decision x_t is made, then at the end of this stage, the process will be in a final state y_{t+1} , determined by the transformation called the transition function Ω_t . The final state y_{t+1} of the process for any stage is at the same time an initial state for the next stage. The decision process and all the notions discussed in the next paragraphs are illustrated by the numerical example in Sect. 5.

In the classic Bipolar method, we deal with decision alternatives. In the multistage processes discussed in this paper, we distinguish two kinds of alternatives: stage and multistage ones. At the beginning of each stage the process is in an feasible state and we can make an feasible decision at this stage. The pair (y_t, x_t) consisting of a state and a decision feasible at this stage will be called a stage alternative $\mathbf{a}_t$. Using the transition function, we perform the transformation of the process to the initial state of the next stage. This action is repeated until the process terminates. Hence, we can talk about sequences of feasible states and decisions which start with a certain feasible state at the first stage and end in an feasible final state of the process. Such sequences $((y_1, x_1), \dots, (y_T, x_T))$ will be called multistage alternative $\mathbf{a}$. Obviously, each multistage alternative is at the same time a sequence of stage alternatives, that means $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_T)$.

Another problem to be solved is that of determining reference sets. We assume that at each stage of the process, two reference sets are defined: one containing “good” objects, the other one, “bad” objects, denoted as $\mathbf{G}_t$ and $\mathbf{B}_t$, respectively. We will call them stage reference sets. At each stage, the reference set of good objects is disjoint with the reference set of bad objects. Since the process consists of stages, we assume that the position of a multistage alternative with respect to the reference sets for the entire process will be evaluated on the basis of the positions of the stage alternatives which form this multistage alternative with respect to the stage reference sets.

For this purpose, we will use K stage criteria functions f_t^k ($k=1, \dots, K$) assuming, for simplicity (as in the original Bipolar method), that larger values are preferred over smaller values (“more is better”). Of course, it is possible to transform the remaining types of criteria to the form used here. Moreover, we assume that at each stage of the process each good reference object dominates each bad reference object. This assumption allows to avoid the situation in which a stage alternative is at the same time overgood (that is, it is evaluated higher than the set of good objects) and underbad (that is, it is evaluated lower than the set of bad objects) (Trzaskalik and Sitarz 2012).

An issue that arises here is that of the importance of the criteria at the consecutive stages. In the classic Bipolar method, all the criteria were obviously essential. In the multistage approach it is possible to have the case in which certain criteria are important in certain stages only, while in other stages other criteria are essential. Criteria which are taken into account at the given stage are called essential for this stage.

The decision maker determines the importance of each criterion by giving the values w_t^k of the weights. We assume that the weights have been normalized and that the sum of the values of the weights for all the criteria is equal to 1 for the given

stage. The weights of a given criterion can vary depending of the stage. The importance of the criteria and their weights are described in the weight matrix. Its rows correspond to the consecutive stages and its columns, to the criteria. If a matrix element is equal to 0, the corresponding criterion is not essential at that stage.

The sorting and ranking procedure proposed below consists of three phases. In the first phase, we analyze the consecutive stage alternatives from the stage sets of feasible alternatives, comparing them with the elements of the stage reference sets. As a result, we obtain outranking stage indicators. In the second phase, we determine the position of a stage alternative with respect to the Bipolar stage reference system, assuming that the concordance level is equal to 0.5. For each stage alternative, we calculate the stage success achievement and stage failure avoidance degrees. In the third phase, we determine the relationships in the set of multistage alternatives, using the values calculated in the second phase. The set of all alternatives is divided into six disjoint classes so that each multistage alternative from a lower-indexed class is evaluated higher than any multistage alternative from a higher-indexed class. Next, we determine a linear ordering in each class.

Let us summarize our notation. We have:

T	the number of process stages,
t	the index of the process stage considered ($t = 1, \dots, T$),
y_t	a feasible state of the process at the beginning of stage t ,
$\mathbf{Y}_t$	the set of feasible states at the beginning of stage t ,
x_t	a feasible decision for state y_t ,
$\mathbf{X}_t(y_t)$	the set of feasible decisions at the beginning of stage t for state y_t
Ω_t	the transition function for stage t . We have: $y_{t+1} = x_t$,
$\mathbf{a}_t$	a feasible realization for stage t ($\mathbf{a}_t = (y_t, x_t) = (y_t, y_{t+1})$)—a stage alternative,
$\mathbf{A}_t$	the set of stage alternatives for stage t ,
$\mathbf{a}$	a feasible realization of the entire process—a multistage alternative,

$$\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_T) = ((y_1, x_1), \dots, (y_T, x_T)) = (y_1, \dots, y_{T+1})$$

$\mathbf{A}$	the set of all multistage alternatives,
$\mathbf{G}_t$	the stage set of good objects,
$\mathbf{B}_t$	the stage set of bad objects,
$\mathbf{R}_t = (\mathbf{G}_t, \mathbf{B}_t)$	the stage reference system ($\mathbf{G}_t \cap \mathbf{B}_t = \emptyset$)
$\mathbf{R} = (\mathbf{R}_1, \dots, \mathbf{R}_T)$	the multistage reference system
K	the number of the all the criteria considered ($k = 1, \dots, K$)
C_t^k	k th criterion at stage t ,
C_t	the set of essential criteria at stage t
f_t^k	the stage criterion function for stage t ($f_t^k: \mathbf{A}_t \cup \mathbf{R}_t \rightarrow K^k$ for $k = 1, \dots, K$, and K_k is a cardinal, ordinal or binary scale). We have:

$$\forall t = 1, \dots, T \quad \forall k \in C_t \quad \forall \mathbf{g}_t \in \mathbf{G}_t \quad \forall \mathbf{b}_t \in \mathbf{B}_t \quad f_t^k(\mathbf{b}_t) < f_t^k(\mathbf{g}_t) \quad (1)$$

w_t^k the weight of the relative importance of criterion k in stage t . We have:

$$\sum_{k=1}^K w_t^k = 1 \quad (2)$$

and

$$\forall_{k=1, \dots, K} w_t^k \geq 0. \quad (3)$$

3 Stage alternatives

3.1 Comparison of stage alternatives with stage reference objects

We compare a selected stage alternative $\mathbf{a}_t$ with an arbitrary stage reference object $\mathbf{r}_t$, taking into account criterion C_t^k which is an essential stage criterion at stage t .

The comparison of the values $f_t^k(\mathbf{a}_t)$ and $f_t^k(\mathbf{r}_t)$ can result in one of the following situations:

$$f_t^k(\mathbf{a}_t) > f_t^k(\mathbf{r}_t) \quad (4)$$

$$f_t^k(\mathbf{a}_t) = f_t^k(\mathbf{r}_t) \quad (5)$$

$$f_t^k(\mathbf{a}_t) < f_t^k(\mathbf{r}_t) \quad (6)$$

In the first case, the stage alternative $\mathbf{a}_t$ is evaluated higher with respect to criterion k than the stage reference object $\mathbf{r}_t$. In the second case, the evaluations are the same. In the third case, the stage alternative $\mathbf{a}_t$ is evaluated lower with respect to criterion k than the stage reference object $\mathbf{r}_t$.

We define 0–1 indicators $\varphi_t^{k+}(\mathbf{a}_t, \mathbf{r}_t)$, $\varphi_t^{k=}(\mathbf{a}_t, \mathbf{r}_t)$ i $\varphi_t^{k-}(\mathbf{a}_t, \mathbf{r}_t)$ as follows:

$$\varphi_t^{k+}(\mathbf{a}_t, \mathbf{r}_t) = \begin{cases} 1, & \text{if } f_t^k(\mathbf{a}_t) - f_t^k(\mathbf{r}_t) > 0 \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

$$\varphi_t^{k=}(\mathbf{a}_t, \mathbf{r}_t) = \begin{cases} 1, & \text{if } f_t^k(\mathbf{a}_t) = f_t^k(\mathbf{r}_t) \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

$$\varphi_t^{k-}(\mathbf{a}_t, \mathbf{r}_t) = \begin{cases} 1, & \text{if } f_t^k(\mathbf{a}_t) - f_t^k(\mathbf{r}_t) < 0 \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

For the pair $(\mathbf{a}_t, \mathbf{r}_t)$, $\mathbf{a}_t \in \mathbf{A}_t$, $\mathbf{r}_t \in \mathbf{R}_t$, we calculate the stage indicators $c_t^+(\mathbf{a}_t, \mathbf{r}_t)$, $c_t^-(\mathbf{a}_t, \mathbf{r}_t)$, $c_t^0(\mathbf{a}_t, \mathbf{r}_t)$ as follows:

$$c_t^+(\mathbf{a}_t, \mathbf{r}_t) = \sum_{k=1}^n w_t^k \phi_t^{k+}(\mathbf{a}_t, \mathbf{r}_t), \quad (10)$$

$$c_t^-(\mathbf{a}_t, \mathbf{r}_t) = \sum_{k=1}^n w_t^k \phi_t^{k-}(\mathbf{a}_t, \mathbf{r}_t), \quad (11)$$

$$c_t^-(\mathbf{a}_t, \mathbf{r}_t) = \sum_{k=1}^n w_t^k \phi_t^{k-}(\mathbf{a}_t, \mathbf{r}_t). \quad (12)$$

The value $c_t^+(\mathbf{a}_t, \mathbf{r}_t)$ is the sum of the weights of these stage criteria, for which $f_t^k(\mathbf{a}_t) > f_t^k(\mathbf{r}_t)$, that is, $\mathbf{a}_t$ is preferred over $\mathbf{r}_t$.

The value $c_t^-(\mathbf{a}_t, \mathbf{r}_t)$ is the sum of the weights of these stage criteria, for which $f_t^k(\mathbf{a}_t) = f_t^k(\mathbf{r}_t)$.

The value $c_t^-(\mathbf{a}_t, \mathbf{r}_t)$ is the sum of the weights of these stage criteria, for which $f_t^k(\mathbf{a}_t) < f_t^k(\mathbf{r}_t)$, that is, $\mathbf{r}_t$ is preferred over $\mathbf{a}_t$.

For the pair $(\mathbf{a}_t, \mathbf{r}_t)$, $\mathbf{a}_t \in \mathbf{A}_t$, $\mathbf{r}_t \in \mathbf{R}_t$, we calculate the stage outranking indicators $d_t^+(\mathbf{a}_t, \mathbf{r}_t)$ of the stage alternative $\mathbf{a}_t \in \mathbf{A}_t$ with respect to the stage reference object $\mathbf{r}_t \in \mathbf{R}_t$.

If

$$c_t^+(\mathbf{a}_t, \mathbf{r}_t) > c_t^-(\mathbf{a}_t, \mathbf{r}_t), \quad (13)$$

then the stage alternative $\mathbf{a}_t \in \mathbf{A}_t$ outranks the reference object $\mathbf{r}_t \in \mathbf{R}_t$. The outranking stage indicators are defined as follows:

$$d_t^+(\mathbf{a}_t, \mathbf{r}_t) = c_t^+(\mathbf{a}_t, \mathbf{r}_t) + c_t^-(\mathbf{a}_t, \mathbf{r}_t), \quad d_t^-(\mathbf{a}_t, \mathbf{r}_t) = 0. \quad (14)$$

If

$$c_t^+(\mathbf{a}_t, \mathbf{r}_t) < c_t^-(\mathbf{a}_t, \mathbf{r}_t), \quad (15)$$

the reference object $\mathbf{r}_t \in \mathbf{R}_t$ outranks the stage alternative $\mathbf{a}_t \in \mathbf{A}_t$. The outranking stage indicators are defined as follows:

$$d_t^+(\mathbf{a}_t, \mathbf{r}_t) = 0, \quad d_t^-(\mathbf{a}_t, \mathbf{r}_t) = c_t^-(\mathbf{a}_t, \mathbf{r}_t) + c_t^-(\mathbf{a}_t, \mathbf{r}_t). \quad (16)$$

If

$$c_t^+(\mathbf{a}_t, \mathbf{r}_t) = c_t^-(\mathbf{a}_t, \mathbf{r}_t), \quad (17)$$

the stage alternative $\mathbf{a}_t \in \mathbf{A}_t$ is evaluated as equally good as the reference object $\mathbf{r}_t \in \mathbf{R}_t$. The outranking stage indicators are then defined as follows:

- if $\mathbf{r}_t$ is a good object, then

$$d_t^+(\mathbf{a}_t, \mathbf{r}_t) = c_t^+(\mathbf{a}_t, \mathbf{r}_t) + c_t^-(\mathbf{a}_t, \mathbf{r}_t), \quad d_t^-(\mathbf{a}_t, \mathbf{r}_t) = 0, \quad (18)$$

- if $\mathbf{r}_t$ is a bad object, then

$$d_t^+(\mathbf{a}_t, \mathbf{r}_t) = 0, \quad d_t^-(\mathbf{a}_t, \mathbf{r}_t) = c_t^+(\mathbf{a}_t, \mathbf{r}_t) + c_t^-(\mathbf{a}_t, \mathbf{r}_t). \quad (19)$$

Using the outranking indicators, we define two stage relationships: stage preference L_t and stage indifference I_t :

$$\mathbf{a}_t L_t \mathbf{r}_t \text{ iff } c_t^+(\mathbf{a}_t, \mathbf{r}_t) \geq c_t^-(\mathbf{a}_t, \mathbf{r}_t), \quad (20)$$

$$\mathbf{r}_t L_t \mathbf{a}_t \text{ iff } c_t^+(\mathbf{a}_t, \mathbf{r}_t) \leq d_t^-(\mathbf{a}_t, \mathbf{r}_t), \quad (21)$$

$$\mathbf{a}_t I_t \mathbf{r}_t \text{ iff } c_t^+(\mathbf{a}_t, \mathbf{r}_t) = c_t^-(\mathbf{a}_t, \mathbf{r}_t). \quad (22)$$

3.2 Position of a stage alternative with respect to the Bipolar stage reference system

We will now determine the position of the stage alternative $\mathbf{a}_t$ with respect to the stage reference set of good objects. For a given stage alternative $\mathbf{a}_t \in \mathbf{A}_t$, the auxiliary sets of indices are defined as follows:

$$\mathcal{L}_t(\mathbf{a}_t, \mathbf{G}_t) = \left\{ h: \mathbf{a}_t L_t \mathbf{g}_t^{(h)}, \mathbf{g}_t^{(h)} \in \mathbf{G}_t \right\}, \quad (23)$$

$$\mathcal{L}_t(\mathbf{G}_t, \mathbf{a}_t) = \left\{ h: \mathbf{g}_t^{(h)} L_t \mathbf{a}_t, \mathbf{g}_t^{(h)} \in \mathbf{G}_t \right\}, \quad (24)$$

$$I_t(\mathbf{a}_t, \mathbf{G}_t) = \left\{ h: \mathbf{a}_t I_t \mathbf{g}_t^{(h)}, \mathbf{g}_t^{(h)} \in \mathbf{G}_t \right\}. \quad (25)$$

The set $\mathcal{L}_t(\mathbf{a}_t, \mathbf{G}_t)$ contains the indices of these good stage reference objects which are outranked by the stage alternative $\mathbf{a}_t$.

The set $\mathcal{L}_t(\mathbf{G}_t, \mathbf{a}_t)$ contains the indices of these good stage reference objects which outrank the stage alternative $\mathbf{a}_t$.

The set $I_t(\mathbf{a}_t, \mathbf{G}_t)$ contains the indices of these good stage reference objects which are regarded as equally good with respect to the stage alternative $\mathbf{a}_t$.

The position of the stage alternative $\mathbf{a}_t$ with respect to the stage reference system is described by two values: $d_G^+(\mathbf{a}_t)$ and $d_G^-(\mathbf{a}_t)$. The value $d_G^+(\mathbf{a}_t)$ characterizes the outranking of the given stage alternative with respect to the stage reference system, while $d_G^-(\mathbf{a}_t)$ characterizes the outranking of the stage reference system with respect to the given stage alternative $\mathbf{a}_t$. The values $d_G^+(\mathbf{a}_t)$ and $d_G^-(\mathbf{a}_t)$ are defined in such a way that at least one of them is equal to 0.

We will consider the following cases:

Case S1

$$\mathcal{L}_t(\mathbf{a}_t, \mathbf{G}_t) \cup I_t(\mathbf{a}_t, \mathbf{G}_t) \neq \emptyset. \quad (26)$$

There exists at least one stage reference object outranked by $\mathbf{a}_t$ or equally good as $\mathbf{a}_t$. This situation is favorable for the decision maker. In this case, $d_{\mathbf{G}}^+(\mathbf{a}_t)$ is positive, while the second indicator $d_{\mathbf{G}}^-(\mathbf{a}_t)$ is then 0.

We calculate the value of the stage success achievement degree for the stage alternative $\mathbf{a}_t$ as follows:

$$d_{\mathbf{G}}^+(\mathbf{a}_t) = \max \left\{ d_t^+ \left(\mathbf{a}_t, \mathbf{g}_t^{(h)} \right) : h \in \mathcal{L}_t(\mathbf{a}_t, \mathbf{G}_t) \cup I_t(\mathbf{a}_t, \mathbf{G}_t) \right\}, \quad d_{\mathbf{G}}^-(\mathbf{a}_t) = 0. \quad (27)$$

Case S2

$$\mathcal{L}_t(\mathbf{a}_t, \mathbf{G}_t) \cup I_t(\mathbf{a}_t, \mathbf{G}_t) = \emptyset \wedge \mathcal{L}_t(\mathbf{G}_t, \mathbf{a}_t) \neq \emptyset. \quad (28)$$

No good stage reference object is outranked by the given stage alternative $\mathbf{a}_t$, nor is any such object evaluated as equally good as this alternative. There exists, however, at least one good stage reference object which outranks the stage alternative $\mathbf{a}_t$. This situation is not favorable for the decision maker—it can be described as “average”—at the same time, this is the most frequent occurrence. The indicator $d_{\mathbf{G}}^+(\mathbf{a}_t)$ is then equal to 0, while the indicator $d_{\mathbf{G}}^-(\mathbf{a}_t)$, describing the outranking of $\mathbf{a}_t$ by the reference system of good objects, is positive.

We calculate the value of the stage success achievement degree for the stage alternative $\mathbf{a}_t$ as follows:

$$d_{\mathbf{G}}^+(\mathbf{a}_t) = 0, \quad d_{\mathbf{G}}^-(\mathbf{a}_t) = \min \left\{ d_t^+ \left(\mathbf{a}_t, \mathbf{g}_t^{(h)} \right) : h \in \mathcal{L}_t(\mathbf{a}_t, \mathbf{G}_t) \cup I_t(\mathbf{a}_t, \mathbf{G}_t) \right\}. \quad (29)$$

We will now determine the position of the stage alternative $\mathbf{a}_t$ with respect to the stage reference set of bad objects. For a given stage alternative $\mathbf{a}_t \in \mathbf{A}_t$ the auxiliary sets of indices are defined as follows:

$$\mathcal{L}_t(\mathbf{a}_t, \mathbf{B}_t) = \left\{ h : \mathbf{a}_t L_t \mathbf{b}_t^{(h)}, \mathbf{b}_t^{(h)} \in \mathbf{B}_t \right\}, \quad (30)$$

$$\mathcal{L}_t(\mathbf{B}_t, \mathbf{a}_t) = \left\{ h : \mathbf{b}_t^{(h)} L_t \mathbf{a}_t, \mathbf{b}_t^{(h)} \in \mathbf{B}_t \right\}, \quad (31)$$

$$I_t(\mathbf{B}_t, \mathbf{a}_t) = \left\{ h : \mathbf{b}_t^{(h)} I_t \mathbf{a}_t, \mathbf{b}_t^{(h)} \in \mathbf{B}_t \right\}. \quad (32)$$

The set $\mathcal{L}_t(\mathbf{a}_t, \mathbf{B}_t)$ contains the indices of these bad stage reference objects that are outranked by the stage alternative $\mathbf{a}_t$.

The set $\mathcal{L}_t(\mathbf{B}_t, \mathbf{a}_t)$ contains the indices of these bad stage reference objects that outrank the stage alternative $\mathbf{a}_t$.

The set $I_t(\mathbf{B}_t, \mathbf{a}_t)$ contains the indices of these bad stage reference objects that are evaluated as equally good with respect to the stage alternative $\mathbf{a}_t$.

The position of the stage alternative $\mathbf{a}_t$ with respect to the stage reference system is described by two values: $d_{\mathbf{B}}^+(\mathbf{a}_t)$ and $d_{\mathbf{B}}^-(\mathbf{a}_t)$.

The value $d_{\mathbf{B}}^+(\mathbf{a}_t)$ describes how the given stage alternative outranks the bad stage reference set, while $d_{\mathbf{B}}^-(\mathbf{a}_t)$ describes how the bad stage reference set outranks the given stage alternative $\mathbf{a}_t$.

To determine the position of alternative $\mathbf{a}_t$ with respect to set $\mathbf{B}_t$, we consider:

Case F1

$$\mathcal{L}_t(\mathbf{B}_t, \mathbf{a}_t) \cup I_t(\mathbf{B}_t, \mathbf{a}_t) = \emptyset \wedge \mathcal{L}_t(\mathbf{a}_t, \mathbf{B}_t) \neq \emptyset. \quad (33)$$

No bad stage reference object outranks the given stage alternative $\mathbf{a}_t$, nor is equally good as the stage alternative $\mathbf{a}_t$. This situation is favorable for the decision maker. The indicator $d_{\mathbf{B}}^+(\mathbf{a}_t)$, describing the outranking of the stage alternative $\mathbf{a}_t$ over the bad stage reference set, is then positive. The value $d_{\mathbf{B}}^-(\mathbf{a}_t)$ is then 0.

We calculate the value of the stage failure avoidance degree for the stage alternative $\mathbf{a}_t$ as follows:

$$d_{\mathbf{B}}^+(\mathbf{a}_t) = \min \left\{ d_t^+ \left(\mathbf{a}_t, \mathbf{b}_t^{(h)} \right) : h \in \mathcal{L}_t(\mathbf{a}_t, \mathbf{B}_t) \right\}, \quad d_{\mathbf{B}}^-(\mathbf{a}_t) = 0. \quad (34)$$

Case F2

$$\mathcal{L}_t(\mathbf{B}_t, \mathbf{a}_t) \cup I_t(\mathbf{B}_t, \mathbf{a}_t) \neq \emptyset. \quad (35)$$

There exists at least one bad stage reference object which dominates the given multistage alternative $\mathbf{a}_t$ or is evaluated as equally good as this alternative. This situation is not favorable for the decision maker. The indicator $d_{\mathbf{B}}^+(\mathbf{a}_t)$ is then equal to 0, while the indicator $d_{\mathbf{B}}^-(\mathbf{a}_t)$, describing the domination of the set of bad reference objects over $\mathbf{a}_t$, is positive.

We calculate the value of the stage failure avoidance degree for the stage alternative $\mathbf{a}_t$ as follows:

$$d_{\mathbf{B}}^+(\mathbf{a}_t) = 0, \quad d_{\mathbf{B}}^-(\mathbf{a}_t) = \max \left\{ d_t^- \left(\mathbf{a}_t, \mathbf{b}_t^{(h)} \right) : h \in \mathcal{L}_t(\mathbf{B}_t, \mathbf{a}_t) \cup I_t(\mathbf{B}_t, \mathbf{a}_t) \right\}. \quad (36)$$

4 Relationships in the set of multistage alternatives

With the stage success achievement indicators determined, for each multistage alternative $\mathbf{a}$, we define the multistage success achievement degree:

$$d_{\mathbf{G}}^+(\mathbf{a}) = \frac{1}{T} \sum_{t=1}^T d_{\mathbf{G}}^+(\mathbf{a}_t), \quad (37)$$

$$d_{\mathbf{G}}^-(\mathbf{a}) = \frac{1}{T} \sum_{t=1}^T d_{\mathbf{G}}^-(\mathbf{a}_t). \quad (38)$$

We define these indicators as arithmetic means of the stage indicators. They admit values from the interval $[0, 1]$.

With the stage failure avoidance indicators determined, for each multistage alternative $\mathbf{a}$, we define the multistage failure avoidance degree:

$$d_{\mathbf{B}}^{+}(\mathbf{a}) = \frac{1}{T} \sum_{t=1}^T d_{\mathbf{B}}^{+}(\mathbf{a}_t), \quad (39)$$

$$d_{\mathbf{B}}^{-}(\mathbf{a}) = \frac{1}{T} \sum_{t=1}^T d_{\mathbf{B}}^{-}(\mathbf{a}_t). \quad (40)$$

Analogously to the previous indicators, we define these indicators as arithmetic means of the stage indicators. They too admit values from the interval $[0, 1]$.

4.1 Sorting multistage alternatives

The values of the multistage indicators allow to classify the multistage alternatives. For a given multistage alternative $\mathbf{a}$, we consider the components of the vector:

$$d(\mathbf{a}) = [d_{\mathbf{G}}^{+}(\mathbf{a}), d_{\mathbf{G}}^{-}(\mathbf{a}), d_{\mathbf{B}}^{+}(\mathbf{a}), d_{\mathbf{B}}^{-}(\mathbf{a})]. \quad (41)$$

First, we consider the class $\mathbf{A}^1$ of multistage alternatives, defined as follows:

$$\mathbf{A}^1 = \{\mathbf{a} \in \mathbf{A}: d_{\mathbf{G}}^{+}(\mathbf{a}) > 0, d_{\mathbf{G}}^{-}(\mathbf{a}) = 0, d_{\mathbf{B}}^{+}(\mathbf{a}) > 0, d_{\mathbf{B}}^{-}(\mathbf{a}) = 0\}. \quad (42)$$

This class includes multistage alternatives with the following property: at each stage, their components—stage alternatives—outrank the sets of good stage objects. In other words, at each stage, for each stage alternative from the given multistage alternative, there exists at least one good reference object, which is outranked by this particular multistage alternative. As a result, we have $d_{\mathbf{G}}^{-}(\mathbf{a}) = 0$. At the same time, at each stage, the stage alternatives from the given multistage alternative $\mathbf{a}_t$ outrank the stage bad reference sets, hence we have $d_{\mathbf{B}}^{-}(\mathbf{a}) = 0$. Therefore, the multistage alternatives satisfying condition (42) constitute the class of the best alternatives.

The class $\mathbf{A}^2$ is defined as follows:

$$\mathbf{A}^2 = \{\mathbf{a} \in \mathbf{A}: d_{\mathbf{G}}^{+}(\mathbf{a}) > 0, d_{\mathbf{G}}^{-}(\mathbf{a}) > 0, d_{\mathbf{B}}^{+}(\mathbf{a}) > 0, d_{\mathbf{B}}^{-}(\mathbf{a}) = 0\}. \quad (43)$$

At some stages, the stage alternatives from each alternative from $\mathbf{A}^2$ outrank the stage set of good reference objects, but at one stage at least, it is the good reference set that outranks a stage alternative from the given multistage alternative. As a result, the multistage alternatives from $\mathbf{A}^2$ are evaluated lower than the ones from $\mathbf{A}^1$. At the same time, all the stage alternatives from the multistage alternatives from $\mathbf{A}^2$ outrank all the bad stage reference sets, hence we have $d_{\mathbf{B}}^{-}(\mathbf{a}) = 0$.

The next class, $\mathbf{A}^3$, includes the following multistage alternatives:

$$\mathbf{A}^3 = \{\mathbf{a} \in \mathbf{A} : d_{\mathbf{G}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) > 0\}. \quad (44)$$

As in the case of $\mathbf{A}^2$, some stage alternatives from the multistage alternatives belonging to $\mathbf{A}^3$ outrank the good stage reference sets, while at other stages, the good stage reference sets outrank the stage alternatives. A similar situation occurs when we compare the stage alternatives which constitute the multistage alternatives from $\mathbf{A}^3$ with the bad reference sets: some of them outrank these sets, some are outranked. For this reason, the multistage alternatives from $\mathbf{A}^3$ are evaluated lower than those from $\mathbf{A}^2$.

Class $\mathbf{A}^4$ of multistage alternatives is defined as follows:

$$\mathbf{A}^4 = \{\mathbf{a} \in \mathbf{A} : d_{\mathbf{G}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) = 0\}. \quad (45)$$

At all stages, all the stage alternatives from the multistage alternatives from class $\mathbf{A}^4$ are outranked by the sets of good objects, hence $d_{\mathbf{G}}^+(\mathbf{a})=0$. This also means that each multistage alternative from $\mathbf{A}^4$ is evaluated lower than any multistage alternative from $\mathbf{A}^3$. At the same time, at all stages, the stage alternatives of all the multistage alternatives from $\mathbf{A}^4$ outrank the sets of bad objects, hence $d_{\mathbf{B}}^-(\mathbf{a})=0$.

The fifth class $\mathbf{A}^5$ consists of these multistage alternative, for which:

$$\mathbf{A}^5 = \{\mathbf{a} \in \mathbf{A} : d_{\mathbf{G}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) > 0\} \quad (46)$$

As in the case of $\mathbf{A}^4$, all the stage alternatives of the multistage alternatives from $\mathbf{A}^5$ are outranked by the good reference sets. At the same time, among the stage alternatives of the multistage alternatives from $\mathbf{A}^4$ there are some that outrank the bad stage reference sets, as well as some that are outranked by these sets. As a result, the multistage alternatives from $\mathbf{A}^5$ are evaluated as worse than those from $\mathbf{A}^4$.

The last class considered is $\mathbf{A}^6$, defined as follows:

$$\mathbf{A}^6 = \{\mathbf{a} \in \mathbf{A} : d_{\mathbf{G}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) > 0\}. \quad (47)$$

All the stage alternatives of the multistage alternatives from $\mathbf{A}^6$ are outranked by the good reference sets, hence $d_{\mathbf{G}}^+(\mathbf{a})=0$. At the same time, these stage alternatives are outranked, in all stages, by the bad reference sets, hence $d_{\mathbf{B}}^+(\mathbf{a})=0$. This means that all the multistage alternatives from $\mathbf{A}^6$ are evaluated lower than those from $\mathbf{A}^5$.

Using $d_{\mathbf{G}}^+(\mathbf{a})$, $d_{\mathbf{G}}^-(\mathbf{a})$, $d_{\mathbf{B}}^+(\mathbf{a})$ and $d_{\mathbf{B}}^-(\mathbf{a})$, we could distinguish another ten classes as follows:

$$\mathbf{A}^7 = \{\mathbf{a} \in \mathbf{A} : d_{\mathbf{G}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) > 0\} \quad (48)$$

$$\mathbf{A}^8 = \{\mathbf{a} \in \mathbf{A} : d_{\mathbf{G}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) > 0\} \quad (49)$$

$$\mathbf{A}^9 = \{\mathbf{a} \in \mathbf{A} : d_{\mathbf{G}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) > 0\} \quad (50)$$

$$\mathbf{A}^{10} = \{\mathbf{a} \in \mathbf{A} : d_{\mathbf{G}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) = 0\} \quad (51)$$

$$\mathbf{A}^{11} = \{\mathbf{a} \in \mathbf{A}: d_{\mathbf{G}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) = 0\} \quad (52)$$

$$\mathbf{A}^{12} = \{\mathbf{a} \in \mathbf{A}: d_{\mathbf{G}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) = 0\} \quad (53)$$

$$\mathbf{A}^{13} = \{\mathbf{a} \in \mathbf{A}: d_{\mathbf{G}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) = 0\} \quad (54)$$

$$\mathbf{A}^{14} = \{\mathbf{a} \in \mathbf{A}: d_{\mathbf{G}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) > 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) > 0\} \quad (55)$$

$$\mathbf{A}^{15} = \{\mathbf{a} \in \mathbf{A}: d_{\mathbf{G}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) > 0\} \quad (56)$$

$$\mathbf{A}^{16} = \{\mathbf{a} \in \mathbf{A}: d_{\mathbf{G}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{G}}^-(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^+(\mathbf{a}) = 0, \quad d_{\mathbf{B}}^-(\mathbf{a}) = 0\} \quad (57)$$

In this way, all the possibilities of creating such classes have been exhausted, hence:

$$\mathbf{A}^1 \cup \mathbf{A}^2 \cup \dots \cup \mathbf{A}^{16} = \mathbf{A}. \quad (58)$$

Because of the construction of these classes, we have:

$$\mathbf{A}^1 \cap \mathbf{A}^2 \cap \dots \cap \mathbf{A}^{16} = \emptyset. \quad (59)$$

At the same time, our assumptions easily lead to the conclusion that:

$$\mathbf{A}^i = \emptyset \quad (60)$$

for $i=7, \dots, 16$.

Class $\mathbf{A}^7$, defined by formula (48), is empty. Multistage alternatives from this class have to satisfy the conditions: $d_{\mathbf{G}}^+(\mathbf{a}) > 0$ and $d_{\mathbf{G}}^-(\mathbf{a}) = 0$. It follows that, at all stages, the stage alternatives (their components) are outranked by the good stage reference objects. Since at the same time $d_{\mathbf{B}}^+(\mathbf{a}) > 0$ and $d_{\mathbf{B}}^-(\mathbf{a}) > 0$, this means that at least in one stage, bad reference objects outrank stage alternatives. Furthermore, it follows that at least in one stage, stage alternatives are at the same time overgood and underbad. But this, because of the assumption in formula (1), is not possible, which is proven in Trzaskalik and Sitarz (2012).

For a similar reason, class $\mathbf{A}^8$, defined by formula (49), is also empty. In this case, the multistage alternatives from this class have to satisfy the following conditions: $d_{\mathbf{B}}^+(\mathbf{a}) = 0$ and $d_{\mathbf{B}}^-(\mathbf{a}) > 0$. It follows that, at all stages, the stage alternatives (their components) are outranked by bad reference objects. Since at the same time, $d_{\mathbf{G}}^+(\mathbf{a}) > 0$ and $d_{\mathbf{G}}^-(\mathbf{a}) > 0$, this means that at least in one step, the stage alternatives outrank the reference objects. Again, it follows that at least in one stage, stage alternatives would be both overgood and underbad, which is not possible.

The class $\mathbf{A}^9$ defined by (50) is empty, because the multistage alternatives from this set would have to satisfy the conditions: $d_{\mathbf{G}}^+(\mathbf{a}) > 0$ and $d_{\mathbf{G}}^-(\mathbf{a}) = 0$ as well as $d_{\mathbf{B}}^+(\mathbf{a}) = 0$ and $d_{\mathbf{B}}^-(\mathbf{a}) > 0$. From this it follows that at all stages we would have also $d_{\mathbf{G}}^+(\mathbf{a}_i) > 0$ and $d_{\mathbf{G}}^-(\mathbf{a}_i) = 0$ as well as $d_{\mathbf{B}}^+(\mathbf{a}_i) = 0$ and $d_{\mathbf{B}}^-(\mathbf{a}_i) > 0$. This would mean,

however, that all the stage alternatives constituting the given multistage alternative are at the same time overgood and underbad, which is not possible.

The classes $\mathbf{A}^{10}$, $\mathbf{A}^{11}$ and $\mathbf{A}^{12}$ defined by (51), (52) and (53) are empty because the multistage alternatives which would belong to them would have to satisfy the following conditions: $d_{\mathbf{B}}^+(\mathbf{a})=0$ and $d_{\mathbf{B}}^-(\mathbf{a})=0$. This is not possible, since at each stage one of the outranking indicators $d_{\mathbf{B}}^+(\mathbf{a}_t, \mathbf{r}_t)$ or $d_{\mathbf{B}}^-(\mathbf{a}_t, \mathbf{r}_t)$ (formulas 34 and 36) is positive, therefore—from (39) and (40)—at least one of the numbers $d_{\mathbf{B}}^+(\mathbf{a})$ and $d_{\mathbf{B}}^-(\mathbf{a})$ has to be positive.

The classes $\mathbf{A}^{13}$, $\mathbf{A}^{14}$, $\mathbf{A}^{15}$ and $\mathbf{A}^{16}$ defined by (54)–(57) are empty because the multistage alternatives from them would have to satisfy the conditions: $d_{\mathbf{G}}^+(\mathbf{a})=0$ and $d_{\mathbf{G}}^-(\mathbf{a})=0$. This is not possible, since at each stage, one of the outranking indicators $d_{\mathbf{G}}^+(\mathbf{a}_t)=0$ or $d_{\mathbf{G}}^-(\mathbf{a}_t)=0$ (formulas (27) and (29)) is positive, hence—from (37) and (38)—at least one of the numbers $d_{\mathbf{G}}^+(\mathbf{a})$ and $d_{\mathbf{G}}^-(\mathbf{a})$ must be positive.

Therefore, the multistage alternatives can be sorted into the six classes $\mathbf{A}^1$, ..., $\mathbf{A}^6$ and

$$\mathbf{A}^1 \cup \mathbf{A}^2 \cup \mathbf{A}^3 \cup \mathbf{A}^4 \cup \mathbf{A}^5 \cup \mathbf{A}^6 = \mathbf{A}. \quad (61)$$

We have shown above that our construction of the above classes implies that if $k < l$, then each multistage alternative from class $\mathbf{A}^k$ should be preferred over any multistage alternative from class $\mathbf{A}^l$.

4.2 Ranking the multistage alternatives

We will now discuss the ranking of the multistage alternatives within each class. Note that two multistage indicators, $d_{\mathbf{G}}^+(\mathbf{a})$ and $d_{\mathbf{B}}^+(\mathbf{a})$, describe the outranking of the stage reference sets $\mathbf{G}_t$ and $\mathbf{B}_t$ by the multistage alternative $\mathbf{a}$. Positive values of these indicators are favorable for the decision maker. Multistage indicators $d_{\mathbf{G}}^-(\mathbf{a})$ and $d_{\mathbf{B}}^-(\mathbf{a})$, on the other hand, describe the outranking of the multistage alternative $\mathbf{a}$ by the reference sets $\mathbf{G}_t$ and $\mathbf{B}_t$. Positive values of these indicators are unfavorable for the decision maker.

Let:

$$d(\mathbf{a}^{(i)}) = d_{\mathbf{G}}^+(\mathbf{a}^{(i)}) - d_{\mathbf{G}}^-(\mathbf{a}^{(i)}) + d_{\mathbf{B}}^+(\mathbf{a}^{(i)}) - d_{\mathbf{B}}^-(\mathbf{a}^{(i)}). \quad (62)$$

Within the classes the ordering of the alternatives is defined as follows:

$$\begin{aligned} \mathbf{a}^{(i)} \text{ is preferred to } \mathbf{a}^{(j)}, \text{ iff} \\ d(\mathbf{a}^{(i)}) > d(\mathbf{a}^{(j)}) \end{aligned} \quad (63)$$

$$\begin{aligned} \mathbf{a}^{(i)} \text{ is equivalent to } \mathbf{a}^{(j)}, \text{ iff} \\ d(\mathbf{a}^{(i)}) = d(\mathbf{a}^{(j)}) \end{aligned} \quad (64)$$

The best multistage alternative $\mathbf{a}^{**}$ is defined as a multistage alternative which belongs to the non-empty class with the lowest index m and satisfies the relationship

$$\forall_{\mathbf{a}' \in \mathbf{A}} {}^m d(\mathbf{a}^{**}) \geq d(\mathbf{a}'). \quad (65)$$

5 Numerical illustration

We consider a three-stage decision process. The sets of feasible states and decisions are as follows:

$$\mathbf{Y}_t = \{0, 1\} \text{ for } t = 1, \dots, 4, \quad \mathbf{X}_t(0) = \{0, 1\}, \quad \mathbf{X}_t(1) = \{0, 1\} \quad \text{for } t = 1, 2, 3.$$

The structure of the process discussed is shown in Fig. 1.

We have four stage alternatives at each stage $t = 1, 2, 3$ of the process:

$$\mathbf{A}_t = \left\{ \mathbf{a}_t^{(0)}, \mathbf{a}_t^{(1)}, \mathbf{a}_t^{(2)}, \mathbf{a}_t^{(3)} \right\} \\ \mathbf{a}_t^{(0)} = (0, 0), \quad \mathbf{a}_t^{(1)} = (0, 1) \quad \mathbf{a}_t^{(2)} = (1, 0) \quad \mathbf{a}_t^{(3)} = (1, 1).$$

At each stage we have two reference sets:

$$\mathbf{G}_t = \left\{ \mathbf{g}_t^{(0)}, \mathbf{g}_t^{(1)} \right\} \quad \text{and} \quad \mathbf{B}_t = \left\{ \mathbf{b}_t^{(0)}, \mathbf{b}_t^{(1)} \right\}.$$

In the first stage, the essential criteria are: $C_1^1, C_2^1, C_3^1, C_4^1, C_5^1$, in the second stage, $C_1^2, C_2^2, C_3^2, C_4^2, C_5^2$, and in the third stage, $C_1^3, C_2^3, C_3^3, C_4^3, C_5^3$. The matrix of stage criteria weights is given in Table 1.

The results of the comparisons of stage alternatives $\mathbf{a}_t \in \mathbf{A}_t$ with the elements of the reference sets are given in Table 2.

Using our numerical data, we compare the stage alternative $\mathbf{a}_1^{(0)}$ with the good object $\mathbf{g}_1^{(0)}$. In stage 1, the essential criteria are: $C_1^1, C_2^1, C_3^1, C_4^1, C_5^1$, and their weights:

$$w_1^1 = 0.17, \quad w_2^1 = 0.23, \quad w_3^1 = 0.23, \quad w_4^1 = 0.12, \quad w_5^1 = 0.25.$$

From the comparisons in Table 2 we obtain, according to formulas (7)–(9):

Fig. 1 The structure of the process discussed

$$\begin{aligned}
f_1^1(\mathbf{a}_1^{(0)}) &< f_1^1(\mathbf{g}^{(0)}) \\
\Rightarrow \varphi_1^{1+}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) &= 0, \quad \varphi_1^{1-}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) = 1, \quad \varphi_1^{1=}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) = 0 \\
f_1^2(\mathbf{a}_1^{(0)}) &= f_1^2(\mathbf{g}^{(0)}) \\
\Rightarrow \varphi_1^{2+}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) &= 0, \quad \varphi_1^{2-}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) = 0, \quad \varphi_1^{2=}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) = 1 \\
f_1^3(\mathbf{a}_1^{(0)}) &< f_1^3(\mathbf{g}^{(0)}) \\
\Rightarrow \varphi_1^{3+}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) &= 0, \quad \varphi_1^{3-}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) = 1, \quad \varphi_1^{3=}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) = 0 \\
f_1^4(\mathbf{a}_1^{(0)}) &> f_1^4(\mathbf{g}^{(0)}) \\
\Rightarrow \varphi_1^{4+}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) &= 1, \quad \varphi_1^{4-}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) = 0, \quad \varphi_1^{4=}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) = 0 \\
f_1^5(\mathbf{a}_1^{(0)}) &= f_1^5(\mathbf{g}^{(0)}) \\
\Rightarrow \varphi_1^{5+}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) &= 0, \quad \varphi_1^{5-}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) = 0, \quad \varphi_1^{5=}(\mathbf{a}_1^{(0)}, \mathbf{g}^{(0)}) = 1
\end{aligned}$$

From (10)–(12) we calculate:

$$\begin{aligned}
c_1^+(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)}) &= 0 \cdot 0.17 + 0 \cdot 0.23 + 0 \cdot 0.23 + 1 \cdot 0.12 + 0 \cdot 0.25 = 0.12, \\
c_1^-(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)}) &= 0 \cdot 0.17 + 1 \cdot 0.23 + 0 \cdot 0.23 + 0 \cdot 0.12 + 1 \cdot 0.25 = 0.48, \\
c_1^-(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)}) &= 1 \cdot 0.17 + 0 \cdot 0.23 + 1 \cdot 0.23 + 0 \cdot 0.12 + 0 \cdot 0.25 = 0.40.
\end{aligned}$$

Since $c_1^+(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)}) < c_1^-(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)})$, that is, condition (15) is satisfied, from (16) we obtain:

$$d_1^+(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)}) = 0, \quad d_1^-(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)}) = c_1^-(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)}) + c_1^-(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)}) = 0.88.$$

From the comparisons in Table 2 according to formulas (7)–(9) we obtain:

Table 1 Values of the stage criteria weights

	w_t^1	w_t^2	w_t^3	w_t^4	w_t^5	w_t^6	w_t^7	w_t^8
$t=1$	0.17	0.23	0.23	0.12	0.25	0	0	0
$t=2$	0.17	0	0.23	0.12	0	0.28	0	0.2
$t=3$	0	0	0.23	0	0.25	0.27	0.11	0.14

Table 2 Results of the comparisons of the stage alternatives with the elements of the reference sets

t	A_t	G_t	f_t^1	f_t^2	f_t^3	f_t^4	f_t^5	f_t^6	f_t^7	f_t^8	B_t	f_t^1	f_t^2	f_t^3	f_t^4	f_t^5	f_t^6	f_t^7	f_t^8
1	$a_1^{(0)}$	$g_1^{(0)}$	-	=	-	+	=	=			$b_1^{(0)}$	+	+	=	-	-	-	-	-
		$g_1^{(1)}$	-	-	=	-	=	=			$b_1^{(1)}$	=	=	-	-	-	-	-	-
		$g_1^{(0)}$	=	+	=	-	+	=			$b_1^{(0)}$	+	=	+	=	-	-	-	-
	$a_1^{(1)}$	$g_1^{(1)}$	+	-	=	-	=	=			$b_1^{(1)}$	-	=	-	-	-	-	-	-
		$g_1^{(0)}$	-	-	=	-	+	=			$b_1^{(0)}$	+	=	+	-	-	-	-	-
		$g_1^{(1)}$	-	-	=	-	+	=			$b_1^{(1)}$	-	=	-	-	-	-	-	-
	$a_1^{(2)}$	$g_1^{(0)}$	=	-	=	-	=	=			$b_1^{(0)}$	+	=	+	-	-	-	-	-
		$g_1^{(1)}$	-	-	=	-	+	=			$b_1^{(1)}$	-	=	-	-	-	-	-	-
		$g_1^{(0)}$	-	+	=	-	=	+			$b_1^{(0)}$	+	=	+	-	-	-	-	-
	$a_1^{(3)}$	$g_1^{(1)}$	-	-	=	-	+	=			$b_1^{(1)}$	-	=	-	-	-	-	-	-
2	$a_2^{(0)}$	$g_2^{(0)}$	-	+	-	-	+	=			$b_2^{(0)}$	+	+	+	+	+	+	+	+
		$g_2^{(1)}$	-	-	-	-	+	=			$b_2^{(1)}$	+	+	+	+	+	+	+	+
		$g_2^{(0)}$	-	-	-	-	+	=			$b_2^{(0)}$	+	+	+	+	+	+	+	+
	$a_2^{(1)}$	$g_2^{(1)}$	-	-	-	-	+	=			$b_2^{(1)}$	-	-	-	-	-	-	-	-
		$g_2^{(0)}$	-	-	-	-	+	=			$b_2^{(0)}$	-	-	-	-	-	-	-	-
		$g_2^{(1)}$	-	-	-	-	+	=			$b_2^{(1)}$	-	-	-	-	-	-	-	-
	$a_2^{(2)}$	$g_2^{(0)}$	-	-	-	-	+	=			$b_2^{(0)}$	-	-	-	-	-	-	-	-
		$g_2^{(1)}$	-	-	-	-	+	=			$b_2^{(1)}$	-	-	-	-	-	-	-	-
		$g_2^{(0)}$	-	-	-	-	+	=			$b_2^{(0)}$	-	-	-	-	-	-	-	-
	$a_2^{(3)}$	$g_2^{(1)}$	-	-	-	-	+	=			$b_2^{(1)}$	-	-	-	-	-	-	-	-
3	$a_3^{(0)}$	$g_3^{(0)}$	-	-	-	-	+	=			$b_3^{(0)}$	+	+	+	+	+	+	+	+
		$g_3^{(1)}$	-	-	-	-	+	=			$b_3^{(1)}$	+	+	+	+	+	+	+	+
		$g_3^{(0)}$	-	-	-	-	+	=			$b_3^{(0)}$	+	+	+	+	+	+	+	+
	$a_3^{(1)}$	$g_3^{(1)}$	-	-	-	-	+	=			$b_3^{(1)}$	+	+	+	+	+	+	+	+
		$g_3^{(0)}$	-	-	-	-	+	=			$b_3^{(0)}$	+	+	+	+	+	+	+	+
		$g_3^{(1)}$	-	-	-	-	+	=			$b_3^{(1)}$	+	+	+	+	+	+	+	+
	$a_3^{(2)}$	$g_3^{(0)}$	-	-	-	-	+	=			$b_3^{(0)}$	+	+	+	+	+	+	+	+
		$g_3^{(1)}$	-	-	-	-	+	=			$b_3^{(1)}$	+	+	+	+	+	+	+	+
		$g_3^{(0)}$	-	-	-	-	+	=			$b_3^{(0)}$	+	+	+	+	+	+	+	+
	$a_3^{(3)}$	$g_3^{(1)}$	-	-	-	-	+	=			$b_3^{(1)}$	+	+	+	+	+	+	+	+
		$g_3^{(0)}$	-	-	-	-	+	=			$b_3^{(0)}$	+	+	+	+	+	+	+	+
		$g_3^{(1)}$	-	-	-	-	+	=			$b_3^{(1)}$	+	+	+	+	+	+	+	+

$$\begin{aligned}
f_1^1(\mathbf{a}_1^{(0)}) &> f_1^1(\mathbf{b}^{(0)}) \\
\Rightarrow \varphi_1^{1+}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) &= 1, \quad \varphi_1^{1-}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) = 0, \quad \varphi_1^{1=}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) = 0. \\
f_1^2(\mathbf{a}_1^{(0)}) &> f_1^2(\mathbf{b}^{(0)}) \\
\Rightarrow \varphi_1^{2+}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) &= 1, \quad \varphi_1^{2-}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) = 0, \quad \varphi_1^{2=}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) = 0. \\
f_1^3(\mathbf{a}_1^{(0)}) &= f_1^3(\mathbf{b}^{(0)}) \\
\Rightarrow \varphi_1^{3+}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) &= 0, \quad \varphi_1^{3-}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) = 0, \quad \varphi_1^{3=}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) = 1. \\
f_1^4(\mathbf{a}_1^{(0)}) &< f_1^4(\mathbf{b}^{(0)}) \\
\Rightarrow \varphi_1^{4+}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) &= 0, \quad \varphi_1^{4-}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) = 1, \quad \varphi_1^{4=}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) = 0. \\
f_1^5(\mathbf{a}_1^{(0)}) &< f_1^5(\mathbf{b}^{(0)}) \\
\Rightarrow \varphi_1^{5+}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) &= 0, \quad \varphi_1^{5-}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) = 1, \quad \varphi_1^{5=}(\mathbf{a}_1^{(0)}, \mathbf{b}^{(0)}) = 0.
\end{aligned}$$

From (10)–(12) we calculate:

$$\begin{aligned}
c_1^+(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)}) &= 1 \cdot 0.17 + 1 \cdot 0.23 + 0 \cdot 0.23 + 0 \cdot 0.12 + 0 \cdot 0.25 = 0.40, \\
c_1^-(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)}) &= 0 \cdot 0.17 + 0 \cdot 0.23 + 1 \cdot 0.23 + 0 \cdot 0.12 + 0 \cdot 0.25 = 0.23, \\
c_1^-(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)}) &= 0 \cdot 0.17 + 0 \cdot 0.23 + 0 \cdot 0.23 + 1 \cdot 0.12 + 1 \cdot 0.25 = 0.37.
\end{aligned}$$

Since $c_1^+(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)}) > c_1^-(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)})$, that is, condition (13) is satisfied, from (14) we obtain:

$$d_1^+(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)}) = c_1^+(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)}) + c_1^-(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)}) = 0.63, \quad d_1^-(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)}) = 0.$$

Performing further calculations analogously, we obtain the remaining indicators describing the comparisons of the stage alternatives for stage 1 with the elements of the reference system. Below is the complete set of these values:

$$\begin{aligned}
d_1^+(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)}) &= 0 & d_1^-(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(0)}) &= 0.88 & d_1^+(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)}) &= 0.63 & d_1^-(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(0)}) &= 0 \\
d_1^+(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(1)}) &= 0 & d_1^-(\mathbf{a}_1^{(0)}, \mathbf{g}_1^{(1)}) &= 1 & d_1^+(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(1)}) &= 0.88 & d_1^-(\mathbf{a}_1^{(0)}, \mathbf{b}_1^{(1)}) &= 0 \\
d_1^+(\mathbf{a}_1^{(1)}, \mathbf{g}_1^{(0)}) &= 0 & d_1^-(\mathbf{a}_1^{(1)}, \mathbf{g}_1^{(0)}) &= 0.77 & d_1^+(\mathbf{a}_1^{(1)}, \mathbf{b}_1^{(0)}) &= 0.77 & d_1^-(\mathbf{a}_1^{(1)}, \mathbf{b}_1^{(0)}) &= 0 \\
d_1^+(\mathbf{a}_1^{(1)}, \mathbf{g}_1^{(1)}) &= 0.65 & d_1^-(\mathbf{a}_1^{(1)}, \mathbf{g}_1^{(1)}) &= 0 & d_1^+(\mathbf{a}_1^{(1)}, \mathbf{b}_1^{(1)}) &= 1 & d_1^-(\mathbf{a}_1^{(1)}, \mathbf{b}_1^{(1)}) &= 0 \\
d_1^+(\mathbf{a}_1^{(2)}, \mathbf{g}_1^{(0)}) &= 0 & d_1^-(\mathbf{a}_1^{(2)}, \mathbf{g}_1^{(0)}) &= 1 & d_1^+(\mathbf{a}_1^{(2)}, \mathbf{b}_1^{(0)}) &= 0 & d_1^-(\mathbf{a}_1^{(2)}, \mathbf{b}_1^{(0)}) &= 0.61 \\
d_1^+(\mathbf{a}_1^{(2)}, \mathbf{g}_1^{(1)}) &= 0 & d_1^-(\mathbf{a}_1^{(2)}, \mathbf{g}_1^{(1)}) &= 0.75 & d_1^+(\mathbf{a}_1^{(2)}, \mathbf{b}_1^{(1)}) &= 0 & d_1^-(\mathbf{a}_1^{(2)}, \mathbf{b}_1^{(1)}) &= 0.64 \\
d_1^+(\mathbf{a}_1^{(3)}, \mathbf{g}_1^{(0)}) &= 0 & d_1^-(\mathbf{a}_1^{(3)}, \mathbf{g}_1^{(0)}) &= 0.88 & d_1^+(\mathbf{a}_1^{(3)}, \mathbf{b}_1^{(0)}) &= 0.75 & d_1^-(\mathbf{a}_1^{(3)}, \mathbf{b}_1^{(0)}) &= 0 \\
d_1^+(\mathbf{a}_1^{(3)}, \mathbf{g}_1^{(1)}) &= 0 & d_1^-(\mathbf{a}_1^{(3)}, \mathbf{g}_1^{(1)}) &= 0.52 & d_1^+(\mathbf{a}_1^{(3)}, \mathbf{b}_1^{(1)}) &= 0.52 & d_1^-(\mathbf{a}_1^{(3)}, \mathbf{b}_1^{(1)}) &= 0
\end{aligned}$$

From comparisons performed for stage 2, we obtain:

$$\begin{aligned}
d_2^+(\mathbf{a}_2^{(0)}, \mathbf{g}_2^{(0)}) &= 0 & d_2^-(\mathbf{a}_2^{(0)}, \mathbf{g}_2^{(0)}) &= 0.88 & d_2^+(\mathbf{a}_2^{(0)}, \mathbf{b}_2^{(0)}) &= 0.72 & d_2^-(\mathbf{a}_2^{(0)}, \mathbf{b}_2^{(0)}) &= 0 \\
d_2^+(\mathbf{a}_2^{(0)}, \mathbf{g}_2^{(1)}) &= 0 & d_2^-(\mathbf{a}_2^{(0)}, \mathbf{g}_2^{(1)}) &= 1 & d_2^+(\mathbf{a}_2^{(0)}, \mathbf{b}_2^{(1)}) &= 0.63 & d_2^-(\mathbf{a}_2^{(0)}, \mathbf{b}_2^{(1)}) &= 0 \\
d_2^+(\mathbf{a}_2^{(1)}, \mathbf{g}_2^{(0)}) &= 0 & d_2^-(\mathbf{a}_2^{(1)}, \mathbf{g}_2^{(0)}) &= 1 & d_2^+(\mathbf{a}_2^{(1)}, \mathbf{b}_2^{(0)}) &= 0 & d_2^-(\mathbf{a}_2^{(1)}, \mathbf{b}_2^{(0)}) &= 1 \\
d_2^+(\mathbf{a}_2^{(1)}, \mathbf{g}_2^{(1)}) &= 0 & d_2^-(\mathbf{a}_2^{(1)}, \mathbf{g}_2^{(1)}) &= 1 & d_2^+(\mathbf{a}_2^{(1)}, \mathbf{b}_2^{(1)}) &= 0.83 & d_2^-(\mathbf{a}_2^{(1)}, \mathbf{b}_2^{(1)}) &= 0 \\
d_2^+(\mathbf{a}_2^{(2)}, \mathbf{g}_2^{(0)}) &= 0 & d_2^-(\mathbf{a}_2^{(2)}, \mathbf{g}_2^{(0)}) &= 0.88 & d_2^+(\mathbf{a}_2^{(2)}, \mathbf{b}_2^{(0)}) &= 0.72 & d_2^-(\mathbf{a}_2^{(2)}, \mathbf{b}_2^{(0)}) &= 0 \\
d_2^+(\mathbf{a}_2^{(2)}, \mathbf{g}_2^{(1)}) &= 0.51 & d_2^-(\mathbf{a}_2^{(2)}, \mathbf{g}_2^{(1)}) &= 0 & d_2^+(\mathbf{a}_2^{(2)}, \mathbf{b}_2^{(1)}) &= 0.77 & d_2^-(\mathbf{a}_2^{(2)}, \mathbf{b}_2^{(1)}) &= 0 \\
d_2^+(\mathbf{a}_2^{(3)}, \mathbf{g}_2^{(0)}) &= 0 & d_2^-(\mathbf{a}_2^{(3)}, \mathbf{g}_2^{(0)}) &= 0.72 & d_2^+(\mathbf{a}_2^{(3)}, \mathbf{b}_2^{(0)}) &= 0.80 & d_2^-(\mathbf{a}_2^{(3)}, \mathbf{b}_2^{(0)}) &= 0 \\
d_2^+(\mathbf{a}_2^{(3)}, \mathbf{g}_2^{(1)}) &= 0 & d_2^-(\mathbf{a}_2^{(3)}, \mathbf{g}_2^{(1)}) &= 0.72 & d_2^+(\mathbf{a}_2^{(3)}, \mathbf{b}_2^{(1)}) &= 0 & d_2^-(\mathbf{a}_2^{(3)}, \mathbf{b}_2^{(1)}) &= 0.63
\end{aligned}$$

From comparisons performed for stage 3, we obtain:

$$\begin{aligned}
d_3^+(\mathbf{a}_3^{(0)}, \mathbf{g}_3^{(0)}) &= 0 & d_3^-(\mathbf{a}_3^{(0)}, \mathbf{g}_3^{(0)}) &= 0.73 & d_3^+(\mathbf{a}_3^{(0)}, \mathbf{b}_3^{(0)}) &= 0.59 & d_3^-(\mathbf{a}_3^{(0)}, \mathbf{b}_3^{(0)}) &= 0 \\
d_3^+(\mathbf{a}_3^{(0)}, \mathbf{g}_3^{(1)}) &= 0.77 & d_3^-(\mathbf{a}_3^{(0)}, \mathbf{g}_3^{(1)}) &= 0 & d_3^+(\mathbf{a}_3^{(0)}, \mathbf{b}_3^{(1)}) &= 0.73 & d_3^-(\mathbf{a}_3^{(0)}, \mathbf{b}_3^{(1)}) &= 0 \\
d_3^+(\mathbf{a}_3^{(1)}, \mathbf{g}_3^{(0)}) &= 0 & d_3^-(\mathbf{a}_3^{(1)}, \mathbf{g}_3^{(0)}) &= 1 & d_3^+(\mathbf{a}_3^{(1)}, \mathbf{b}_3^{(0)}) &= 0 & d_3^-(\mathbf{a}_3^{(1)}, \mathbf{b}_3^{(0)}) &= 0.73 \\
d_3^+(\mathbf{a}_3^{(1)}, \mathbf{g}_3^{(1)}) &= 0 & d_3^-(\mathbf{a}_3^{(1)}, \mathbf{g}_3^{(1)}) &= 0.61 & d_3^+(\mathbf{a}_3^{(1)}, \mathbf{b}_3^{(1)}) &= 0.77 & d_3^-(\mathbf{a}_3^{(1)}, \mathbf{b}_3^{(1)}) &= 0 \\
d_3^+(\mathbf{a}_3^{(2)}, \mathbf{g}_3^{(0)}) &= 0 & d_3^-(\mathbf{a}_3^{(2)}, \mathbf{g}_3^{(0)}) &= 0.75 & d_3^+(\mathbf{a}_3^{(2)}, \mathbf{b}_3^{(0)}) &= 0 & d_3^-(\mathbf{a}_3^{(2)}, \mathbf{b}_3^{(0)}) &= 0.77 \\
d_3^+(\mathbf{a}_3^{(2)}, \mathbf{g}_3^{(1)}) &= 0 & d_3^-(\mathbf{a}_3^{(2)}, \mathbf{g}_3^{(1)}) &= 0.75 & d_3^+(\mathbf{a}_3^{(2)}, \mathbf{b}_3^{(1)}) &= 0 & d_3^-(\mathbf{a}_3^{(2)}, \mathbf{b}_3^{(1)}) &= 0.64 \\
d_3^+(\mathbf{a}_3^{(3)}, \mathbf{g}_3^{(0)}) &= 0 & d_3^-(\mathbf{a}_3^{(3)}, \mathbf{g}_3^{(0)}) &= 0.77 & d_3^+(\mathbf{a}_3^{(3)}, \mathbf{b}_3^{(0)}) &= 0.75 & d_3^-(\mathbf{a}_3^{(3)}, \mathbf{b}_3^{(0)}) &= 0 \\
d_3^+(\mathbf{a}_3^{(3)}, \mathbf{g}_3^{(1)}) &= 0 & d_3^-(\mathbf{a}_3^{(3)}, \mathbf{g}_3^{(1)}) &= 0.75 & d_3^+(\mathbf{a}_3^{(3)}, \mathbf{b}_3^{(1)}) &= 1 & d_3^-(\mathbf{a}_3^{(3)}, \mathbf{b}_3^{(1)}) &= 0
\end{aligned}$$

A comparison of the stage alternative $\mathbf{a}_1^{(0)}$ with the set $\mathbf{G}_1$ shows that we deal here with Case S2. Using formula (29), we calculate:

$$d_{\mathbf{G}}^+(\mathbf{a}_1^{(0)}) = 0, \quad d_{\mathbf{G}}^-(\mathbf{a}_1^{(0)}) = 0.88.$$

A comparison of the stage alternative $\mathbf{a}_1^{(0)}$ with the set $\mathbf{B}_1$ shows that we deal here with Case F1. Using formula (34), we obtain:

$$d_{\mathbf{B}}^+(\mathbf{a}_1^{(0)}) = 0.63, \quad d_{\mathbf{B}}^-(\mathbf{a}_1^{(0)}) = 0.$$

Further stage indicators are calculated in the same way. The results of the calculations are shown in Table 3.

For each multistage alternative $\mathbf{a}^{(0)}, \dots, \mathbf{a}^{(15)}$, the multistage values $d_{\mathbf{G}}^+(\mathbf{a})$, $d_{\mathbf{G}}^-(\mathbf{a})$, $d_{\mathbf{B}}^+(\mathbf{a})$, $d_{\mathbf{B}}^-(\mathbf{a})$, the index of the class to which alternative $\mathbf{a}$ is assigned, and the position of alternative $\mathbf{a}$ in the Bipolar ranking are calculated. The results are given in Table 4.

Multistage alternative $\mathbf{a}^{(4)}$ is assigned to class $\mathbf{A}^1$; two alternatives, $\mathbf{a}^{(12)}$ and $\mathbf{a}^{(0)}$, are assigned to class $\mathbf{A}^2$; five: $\mathbf{a}^{(5)}$, $\mathbf{a}^{(6)}$, $\mathbf{a}^{(7)}$, $\mathbf{a}^{(8)}$, $\mathbf{a}^{(13)}$, to class $\mathbf{A}^3$; seven: $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$, $\mathbf{a}^{(3)}$, $\mathbf{a}^{(9)}$, $\mathbf{a}^{(11)}$, $\mathbf{a}^{(14)}$, $\mathbf{a}^{(15)}$, to class $\mathbf{A}^5$; and one, $\mathbf{a}^{(10)}$, to class $\mathbf{A}^6$. The multistage alternatives are ranked as follows: $\mathbf{a}^{(4)}$, $\mathbf{a}^{(12)}$, $\mathbf{a}^{(0)}$, $\mathbf{a}^{(5)}$, $\mathbf{a}^{(7)}$, $\mathbf{a}^{(13)}$, $\mathbf{a}^{(8)}$, $\mathbf{a}^{(6)}$, $\mathbf{a}^{(15)}$, $\mathbf{a}^{(1)}$, $\mathbf{a}^{(3)}$, $\mathbf{a}^{(14)}$, $\mathbf{a}^{(9)}$, $\mathbf{a}^{(11)}$, $\mathbf{a}^{(2)}$, $\mathbf{a}^{(10)}$.

The enumerative method of finding the best multistage alternative, presented in this paper, may be compared to pure random sampling of the decision space to illustrate the benefits of the method against intuitive approaches.

Let us assume that in the example considered, we will be choosing a multistage alternative at random, and we will repeat that choice many times. We will assume (which is obvious in the case considered) that the choice of each of the sixteen multistage alternatives is now equally probable. Due to the small size of our numerical example and our knowledge of the numerical data, we will use the average number of class $\mathbf{A}^{avr}$ and the average values of the indices $d_{\mathbf{G}}^+(\mathbf{a}_t)$, $d_{\mathbf{G}}^-(\mathbf{a}_t)$, $d_{\mathbf{B}}^+(\mathbf{a}_t)$, $d_{\mathbf{B}}^-(\mathbf{a}_t)$. These average values will be denoted, respectively, as

$$d_{\mathbf{G}}^{+avr}, d_{\mathbf{G}}^{-avr}, d_{\mathbf{B}}^{+avr}, d_{\mathbf{B}}^{-avr}.$$

Using numerical data from Table 4, we obtain:

Table 3 Stage indicators

Stage	$\mathbf{A}_t$	$d_{\mathbf{G}}^+(\mathbf{a}_t)$	$d_{\mathbf{G}}^-(\mathbf{a}_t)$	$d_{\mathbf{B}}^+(\mathbf{a}_t)$	$d_{\mathbf{B}}^-(\mathbf{a}_t)$
$t = 1$	$\mathbf{a}_1^{(0)}$	0	0.88	0.63	0
	$\mathbf{a}_1^{(1)}$	0.65	0	0.77	0
	$\mathbf{a}_1^{(2)}$	0	0.75	0	0.64
	$\mathbf{a}_1^{(3)}$	0	0.52	0.52	0
$t = 2$	$\mathbf{a}_2^{(0)}$	0	0.88	0.63	0
	$\mathbf{a}_2^{(1)}$	0	1	0	1
	$\mathbf{a}_2^{(2)}$	0.51	0	0.77	0
	$\mathbf{a}_2^{(3)}$	0	0.72	0	0.63
$t = 3$	$\mathbf{a}_3^{(0)}$	0.77	0	0.59	0
	$\mathbf{a}_3^{(1)}$	0	0.61	0	0.73
	$\mathbf{a}_3^{(2)}$	0	0.75	0	0.77
	$\mathbf{a}_3^{(3)}$	0	0.75	0.75	0

Table 4 Bipolar sorting and Bipolar ranking

A	A_1, A_2, A_3	$d_G^+(a)$	$d_G^-(a)$	$d_B^+(a)$	$d_B^-(a)$	$d(a)$	A^i	Ranking
$a^{(0)}$	$a_1^{(0)}, a_2^{(0)}, a_3^{(0)}$	0.257	0.587	0.617	0	0.287	2	3
$a^{(1)}$	$a_1^{(0)}, a_2^{(0)}, a_3^{(1)}$	0	0.79	0.42	0.243	-0.61	5	10
$a^{(2)}$	$a_1^{(0)}, a_2^{(1)}, a_3^{(2)}$	0	0.877	0.21	0.59	-1.26	5	15
$a^{(3)}$	$a_1^{(0)}, a_2^{(1)}, a_3^{(3)}$	0	0.877	0.46	0.333	-0.75	5	11
$a^{(4)}$	$a_1^{(1)}, a_2^{(2)}, a_3^{(0)}$	0.643	0	0.71	0	1.353	1	1
$a^{(5)}$	$a_1^{(1)}, a_2^{(2)}, a_3^{(1)}$	0.387	0.203	0.513	0.243	0.453	3	4
$a^{(6)}$	$a_1^{(1)}, a_2^{(3)}, a_3^{(2)}$	0.217	0.49	0.257	0.467	-0.48	3	9
$a^{(7)}$	$a_1^{(1)}, a_2^{(3)}, a_3^{(3)}$	0.217	0.49	0.507	0.21	0.023	3	5
$a^{(8)}$	$a_1^{(2)}, a_2^{(0)}, a_3^{(0)}$	0.257	0.543	0.407	0.213	-0.09	3	7
$a^{(9)}$	$a_1^{(2)}, a_2^{(0)}, a_3^{(1)}$	0	0.747	0.21	0.457	-0.99	5	13
$a^{(10)}$	$a_1^{(2)}, a_2^{(1)}, a_3^{(2)}$	0	0.833	0	0.803	-1.64	6	16
$a^{(11)}$	$a_1^{(2)}, a_2^{(1)}, a_3^{(3)}$	0	0.833	0.25	0.547	-1.13	5	14
$a^{(12)}$	$a_1^{(3)}, a_2^{(2)}, a_3^{(0)}$	0.427	0.173	0.627	0	0.88	2	2
$a^{(13)}$	$a_1^{(3)}, a_2^{(2)}, a_3^{(1)}$	0.17	0.377	0.43	0.243	-0.02	3	6
$a^{(14)}$	$a_1^{(3)}, a_2^{(3)}, a_3^{(2)}$	0	0.663	0.173	0.467	-0.96	5	12
$a^{(15)}$	$a_1^{(3)}, a_2^{(3)}, a_3^{(3)}$	0	0.663	0.423	0.21	-0.45	5	8

$$A^{avr} = 4.0.$$

Interpreting the result obtained, we can note, first, that by performing many random selection attempts of a multistage alternative, we will receive, in average, multistage alternatives from class 4. This is a value near the arithmetic mean of class numbers from 1 through 6, which is $3\frac{1}{2}$. At the same time, it should be noted that in this numerical example none of the multistage alternatives belongs to this class. This situation is not in line with our initial expectations, since it would seem that there should be many multistage alternatives for which good stage reference objects outrank stage alternatives and for which stage alternatives outrank bad stage objects. Our example indicates that this assumption is not justified and that the number of multistage alternatives from each class depends on the structure of the reference system. At the same time, the value of A^{avr} constitutes a certain overall characteristic feature of the reference system. There is no doubt, however, that the use of the enumerative method of finding the best alternative, which allows to generate the best existing multistage alternative (in our case, one from class A^1) gives much better results than a single random selection.

Let us now look at the average values d_G^{+avr} , d_G^{-avr} , d_B^{+avr} and d_B^{-avr} . In our example, we obtain the following values:

$$\begin{aligned} d_G^{+avr} &= 0.161, & d_G^{-avr} &= 0.572, \\ d_B^{+avr} &= 0.388, & d_B^{-avr} &= 0.314. \end{aligned}$$

We have:

$$d_G^{+avr} < d_G^{-avr} \quad (66)$$

$$d_B^{+avr} > d_B^{-avr} \quad (67)$$

Inequality (66) indicates that good stage reference objects outrank stage alternatives and the difference

$$d_G^{avr} = d_G^{+avr} - d_G^{-avr} = -0.411$$

characterizes the degree of this outranking. Inequality (67) indicates that stage alternatives outrank bad stage reference objects, and the difference

$$d_B^{avr} = d_B^{+avr} - d_B^{-avr} = 0.074$$

characterizes the degree of this outranking.

Note that $d_G^{avr} < 0$ and $d_B^{avr} > 0$, which confirms our expectations as regards the signs of these values.

Using d_G^{avr} and d_B^{avr} we can calculate the value

$$d_R^{avr} = d_G^{avr} + d_B^{avr} = -0.411 + 0.074 = -0.337$$

which is another characteristic feature of the entire reference system. A negative value of this index indicates that multistage alternatives outrank bad stage reference objects more than they outrank good reference objects, and the value d_R^{avr} is a measure of that. This value is also an information for the decision maker that characterizes the reference system he/she has created.

6 Concluding remarks

In our example we have found exactly one multistage alternative belonging to class A^1 . This means that at each stage at least one of the stage alternatives of the multistage alternative $a^{(4)}$ outranks at least one reference object. Moreover, each of them outranks all the bad stage objects. The next multistage alternatives belong to class A^2 , evaluated as a much worse one. It is worth noting that a ranking taking into account the indicator $d(a)$ alone—that is, disregarding the assignment of the stage alternatives to the classes—would be different. In that case, the first two positions in both rankings would be taken by $a^{(4)}$ and $a^{(12)}$, while the third position would belong to $a^{(5)}$, an alternative from class A^3 , which would outrank $a^{(0)}$, an alternative from class A^2 . This shows that the division into classes has an essential influence on the ranking.

Worth noting are also Eqs. (18) and (19). According to the original version of the description of the Bipolar method, when $c_t^+(a, r_t) = c_t^-(a, r_t)$, the following equations should be used:

$$d_t^+(a, r_t) = c_t^+(a, r_t) + c_t^-(a, r_t), \quad d_t^-(a, r_t) = c_t^-(a, r_t) + c_t^+(a, r_t).$$

In the dynamic approach, such a definition would make it difficult to construct classes A^1 through A^6 , which would not be disjoint. In such a situation the arguments for the outranking of one of the objects by another are equally convincing for one as for the other one. We propose then formula (18), if the stage alternative is compared with a good reference object, and formula (19), if it is compared with a bad one.

The present paper originates a new research direction. Hence, at the beginning we have made certain simplifying assumptions, which should be eliminated in the future. They include: setting the equivalence threshold at 0 (cf. formulas (6)–(8)) and the concordance threshold at 0.5, as well as not using the veto coefficients. These are elements of the Electre methodology, used in the classic Bipolar method. Further research will aim at eliminating these limitations. In the general case it may happen that certain stage alternatives are not comparable with the stage reference sets. In such situations, certain multistage alternatives will also be non-comparable. One of the future directions of the extension of the method is the preparation of a general case description. Another direction is to design software for numerical simulations. A further research direction would be to replace the enumerative method presented in this paper (which may be difficult to apply to larger-size problems) by methods based on multicriteria dynamic programming and genetic algorithms.

As regards the possible applications of the dynamic Bipolar procedure, it seems that this procedure can be applied to create a long-term development strategy, in particular for Foresight processes and when using the sustainable development approach.

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