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Modifications of the Hurwicz's decision rule

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Abstract The Hurwicz's criterion is one of the classical decision rules applied in decision making under uncertainty as a tool enabling to find an optimal pure strategy both for interval and scenarios uncertainty. The interval uncertainty occurs when the decision maker knows the range of payoffs for each alternative and all values belonging to this interval are theoretically probable (the distribution of payoffs is continuous). The scenarios uncertainty takes place when the result of a decision depends on the state of nature that will finally occur and the number of possible states of nature is known and limited (the distribution of payoffs is discrete). In some specific cases the use of the Hurwicz's criterion in the scenarios uncertainty may lead to quite illogical and unexpected results. Therefore, the author presents two new procedures combining the Hurwicz's pessimism-optimism index with the Laplace's approach and using an additional parameter allowing to set an appropriate width for the ranges of relatively good and bad payoffs related to a given decision. The author demonstrates both methods on the basis of an example concerning the choice of an investment project. The methods described may be used in each decision making process within which each alternative (decision, strategy) is characterized by only one criterion (or one synthetic measure).

Keywords Decision making under uncertainty · States of nature · Hurwicz's criterion · Laplace's criterion · Optimal pure strategy

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1 Introduction

The Hurwicz's rule is a procedure applied within the decision making process under uncertainty (DMUU). This uncertainty is a consequence of the fact that we are not able to anticipate the future effectively. One may just forecast various phenomena and events, but in many cases it is extremely difficult to estimate the exact value of particular parameters (temperature, company profit, size of the mature crops, demand for a product, product prices, production costs etc.). If these data were known, it would be easy to indicate the best alternative (decision), e.g. the best investment strategy. But when many future factors are not deterministic at the time of the decision, the decision maker (DM) has to choose the appropriate alternative on the basis of some scenarios (states of nature, events) predicted by experts, him- or herself. Let us add that the probability of these scenarios may be known or not. The situation where parameters can be presented by means of random variables is characteristic for the decision making under risk. When it is impossible to calculate the likelihoods aforementioned the choice of an alternative is made under uncertainty (Groenewald and Pretorius 2011; Render et al. 2006; Chronopoulos et al. 2011; Sikora 2008; Trzaskalik 2008). Knight (1921) first introduced the idea to apply risk and uncertainty in economics, but these two categories were formally integrated in economic theory by von Neumann and Morgenstern (1944). In this contribution we will focus on the second case which seems to be more frequent in realistic decision problems.

The literature offers dozens of procedures applied in DMUU, such as the classical rules, which will be discussed in Sect. 3, and many diverse extensions or hybrids of these methods (see e.g. Basili 2006; Basili and Zappia 2010; Basili et al. 2008; Ghirardato et al. 2004; Ellsberg 2001; Marinacci 2002). One of them is the Hurwicz's criterion method (Hurwicz's Optimism-Pessimism Approach). This procedure usually leads to reasonable answers, but in some specific situations the Hurwicz's results may be astonishing. Therefore, the target of this paper is to present two modified Hurwicz's rules which yield more logical results. The first method is designed for passive decision makers who are only asked to declare their level of pessimism and optimism. The second one requires an active attitude in the decision making process, i.e. the ability to analyze the payoffs matrix and to determine some additional parameters.

The remainder of the paper is organized as follows. Section 2 contains a short description of the decision making under uncertainty and the assumptions adopted in the contribution. In Sect. 3, the most well-known methods for the DMUU are briefly discussed. A deeper analysis of the Hurwicz's criterion procedure is presented in Sect. 4. The proposed modified Hurwicz's techniques are demonstrated in Sect. 5. Finally, conclusions are gathered in Sect. 6.

2 Decision making under uncertainty: characteristics and assumptions

As it was mentioned in the previous section the result of the choice made by the decision maker under uncertainty depends on two factors: which alternative will be selected and which scenario (state of nature) will occur in the future. The consequence of any decision is determined not just by the decision itself but also by an external

Table 1 Profit matrix/decision table (general case)

Scenarios and alternatives	I_1	I_j	I_n
S_1	a_{11}	a_{1j}	a_{1n}
S_i	a_{i1}	a_{ij}	a_{in}
S_m	a_{m1}	a_{mj}	a_{mn}

factor which is beyond the control of the decision maker. The true state of nature is determined by prices, taxes, exchange rates, unemployment rates, by external micro- and macroeconomic decisions, by weather conditions and by many unpredictable events. If he or she knew the state of nature which will actually hold, he or she could predict the consequence of any alternative with certainty. The DMUU may be presented with the aid of a profits matrix (Table 1) where m is the number of mutually exclusive scenarios (let us denote them by S_1, S_2, ..., S_m). n signifies the number of alternatives (I_1, I_2, ..., I_n) and a_{ij} is the profit connected with the scenario S_i and alternative I_j. The goal of the DM consists in selecting this decision which maximizes the profit.

Notice that the decision making under uncertainty may not necessarily signify the occurrence of a finite number of scenarios with a set of m profits for each decision. In the other concept of DMUU it is assumed that the exact profit connected with the alternative I_j is not known, but belongs to an interval $[w_j, m_j]$ and then we deal with the decision making under interval uncertainty (Huynh et al. 2007). In such a case each value from this interval is probable. In this contribution we will consider the scenarios' approach for DMMU which is characterized by a lower degree of uncertainty than the interval approach because only several values from this range are probable.

In the uncertainty case the decision maker may search an optimal pure strategy or an optimal mixed strategy. A pure strategy, in contradiction to a mixed strategy, is a solution assuming that the decision maker chooses and completely executes one and only one alternative. Meanwhile the mixed strategy allows the decision maker to select and perform a weighted combination of several accessible alternatives. The whole paper concerns techniques dedicated for optimal pure strategy's searching.

When talking about the selection of an optimal alternative we must be aware of the fact that each decision may be defined by a vector of values representing the performance of different criteria and then the DMUU is brought to a multicriteria decision making under uncertainty—MDMUU (Dominiak 2006, 2009). Here we will assume that each alternative may be characterized either by one essential criterion's value or by one synthetic aggregated value denoting the overall realization of all significant criteria. Therefore only one-criterion problems will be discussed in this article.

3 The most well-known methods for the decision making under uncertainty

This section contains a brief description of possible classical approaches applied in DMUU with scenarios when the decision maker is interested in finding an optimal pure strategy (Ignasiak 1996; Kaufmann and Faure 1974; Pazek and Rozman 2009; Sikora 2008; Trzaskalik 2008). Let us recall that in the scenarios' approach decision makers

have to choose one of a set number of alternatives with complete information about their outcomes but in the absence of any information or data about the probabilities of the various states of nature (Pazek and Rozman 2009). The decision rules presented below prescribe how an individual faced with a decision under uncertainty should go about choosing a course of action consistent with the individual's basic judgments and preferences (Pazek and Rozman 2009).

Wald's maximin criterion (Wald 1950a, b; Wen and Iwamura 2008) is a very pessimistic approach assuming that the worst will happen and that the smaller payoffs may have a higher probability of occurrence. Under the alternative I_j the worst consequence that can occur has a value of

$$w_j = \min_i \{a_{ij}\} \tag{1}$$

where w_j is the Wald's measure or the security level of I_j. Wald suggested that he or she choose this decision that has as large security level as possible:

$$w_j^* = \max_j \{w_j\} = \max_j \min_i \{a_{ij}\} \tag{2}$$

The *maximax criterion* is a very optimistic approach for gamblers attracted by high payoffs. Under I_j the best consequence that can occur has a value of

$$m_j = \max_i \{a_{ij}\} \tag{3}$$

where m_j is the optimism level of I_j. According to the maximax criterion it is recommended to choose this alternative that fulfills the condition (4):

$$m_j^* = \max_j \{m_j\} = \max_j \max_i \{a_{ij}\} \tag{4}$$

Hurwicz (1951; 1952) argued that the decision maker should rank alternatives according to the weighted average of the security and the optimism levels:

$$h_j = \alpha \cdot w_j + (1 - \alpha) \cdot m_j \tag{5}$$

where h_j is the Hurwicz's criterion and α is the coefficient of pessimism which fulfills the following condition: $\alpha \in [0, 1]$. The parameter α is close to 0 for extreme optimists, i.e. adventurous decision makers (risk-prone behavior), and it tends to 1 for radical pessimists, i.e. cautious decision makers (risk-averse behavior). The optimal alternative should satisfy the Eq. (6):

$$h_j^* = \max_j \{h_j\} = \max_j \{\alpha \cdot w_j + (1 - \alpha) \cdot m_j\} \tag{6}$$

Notice that one may find in the literature examples where the parameter α describes the user's degree of optimism (Groenewald and Pretorius 2011; Huynh et al. 2007; Pazek and Rozman 2009) and not the pessimism index.

The *Savage's minimax regret criterion* (Savage 1961) appeals to cautious people. Savage suggested to replace the payoffs matrix with a new regrets table computed according to the formula (7) and to assign an index to each decision on the basis of the Eq. (8) which represents the worst regret from the alternative I_j:

$$r_{ij} = \max_j\{a_{ij}\} - a_{ij} \tag{7}$$

$$s_j = \max_i\{r_{ij}\} \tag{8}$$

where r_{ij} denotes the non-negative opportunity loss. Within the Savage's criterion it is recommended to choose this decision that has as low index as possible:

$$s_j^* = \min_j\{s_j\} \tag{9}$$

Laplace argued that if one knows nothing at all about the true scenario, it means that all states have an equal probability (Render et al. 2006). That means that the decision maker faces an expected value of his choice and ought to select the alternative fulfilling the Eqs. (10) and (11):

$$l_j = \frac{1}{m}\sum_i a_{ij} \tag{10}$$

$$l_j^* = \max_j\{l_j\} \tag{11}$$

The use of expected values distinguishes this approach from the criteria using only extreme payoffs. This characteristic makes the procedure similar to the decision making under risk (Pazek and Rozman 2009).

At the end of this section it is worth emphasizing that all criteria of choice may suggest a different optimal strategy for the same problem.

Notice that the first four rules find application when the decision maker intends to perform the selected alternative only once. When he or she contemplates to realize this decision many times, it is recommended to use the Laplace's criterion.

Let us also add that the rules aforementioned were described for problems with a target maximized (profits, effects, sales). When the objective function tends to the minimum (cost, time etc.), i.e. when payoffs are given as negative-flow rewards, one should apply one of the approaches suggested in Gaspars-Wieloch (2012).

4 Hurwicz's criterion method: a case study analysis

In this section we are going to apply the classical Hurwicz's rule in a case study concerning the choice of an optimal investment project. Assume that an investor wants to select the best investment project from three possible business plans. Table 2 presents profits (they are given in million dollars) connected with projects P1, P2 and P3, respectively. These payoffs (rewards) narrowly depend on the scenarios which will finally take place (S1, S2, S3, S4 or S5).

Let us have a look on these data. The profits related to P1 belong to the largest interval $[w_1, m_1] = [1, 10]$ and in one out of five scenarios this project gives the

Table 2 Payoffs matrix—case study

Investment projects	P1	P2	P3
Scenarios			
S1	10	1	4
S2	1	9.5	5
S3	1	9.5	2
S4	1	9.5	5
S5	1	9.5	4

Table 3 Hurwicz's measures for projects P1, P2 and P3 (optimist and pessimist type)

Decision maker	DM A (optimist type)	DM B (pessimist type)
Project		
P1	$h_{P1}^A = 0.3 \times 1 + 0.7 \times 10 = 7.3$	$h_{P1}^B = 0.7 \times 1 + 0.3 \times 10 = 3.7$
P2	$h_{P2}^A = 0.3 \times 1 + 0.7 \times 9.5 = 6.95$	$h_{P2}^B = 0.7 \times 1 + 0.3 \times 9.5 = 3.55$
P3	$h_{P3}^A = 0.3 \times 2 + 0.7 \times 5 = 4.1$	$h_{P3}^B = 0.7 \times 2 + 0.3 \times 5 = 2.9$

highest benefit. The project P2, with an interval $[w_2, m_2] = [1, 9.5]$, is the best in four out of five states of nature. The last business plan (P3) yields, independently on the situation, quite congenial and relatively low gains ($[w_3, m_3] = [2, 5]$).

Now, let us analyze the results obtained with the aid of the Hurwicz's criterion for two decision maker's types (A and B). The first one is rather an optimist ($\alpha^A = 0.3$). The second person is a moderate pessimist ($\alpha^B = 0.7$). The Hurwicz's measures for all potential business plans are gathered in Table 3.

According to the Eq. (6) both decision makers (the optimist and the pessimist one) ought to select the first project since in both cases its Hurwicz's measures attend the highest values. Have we expected such results? Are they rational and logical? Probably rather not. Why are we so surprised with the answer given by the Hurwicz's rule? There are at least three reasons:

1. The project P1 yields a very high profit only in the first scenario. In the four remaining states of nature it leads to the worst gains. For an optimist decision maker, prone to risk, this alternative seems to be the most appropriate. But let us focus on the pessimist one. If he or she assumes that the probability of the worst value is equal 0.7 (i.e. quite much), it would be more reasonable to choose P3 for which the worst value w_3 is twice as high as w_1. That is why we state that the optimal pure strategy set for the pessimist decision maker does not reflect his or her risk aversion. If the scenario S2, S3, S4 or S5 occurs, the loss of the pessimist DM choosing P1 will be in most cases higher than it would be if P2 or P3 were selected.
2. The project P2 gives the best results in four out of five states of nature. We intuitively conclude that this strategy should be safer for the pessimist decision maker than the project P1 which is the best merely in one out of five scenarios. Thus, it is fairly difficult to understand why P2 has obtained a lower Hurwicz's measure than P1, especially in the pessimist case.

Table 4 Hurwicz's measures for project P1, P2, P3 (for different values of α)

α	P1	P2	P3
0.00	10.00	9.50	5.00
0.10	9.10	8.65	4.70
0.20	8.20	7.80	4.40
0.30	7.30	6.95	4.10
0.40	6.40	6.10	3.80
0.50	5.50	5.25	3.50
0.60	4.60	4.40	3.20
0.70	3.70	3.55	2.90
0.80	2.80	2.70	2.60
0.83	2.50	2.42	2.50
0.90	1.90	1.85	2.30
1.00	1.00	1.00	2.00

3. We are surprised not only with the optimal pure strategy selected by Hurwicz's rule, but also with the ranking of projects. The order is totally the same for both pessimism indices (P1 dominates P2 and P2 is better than P3). This conclusion is astonishing as well since the parameters α are significantly different.

Table 4 presents the Hurwicz's measures for different values of α and enables to make some additional conclusions:

1. The order of projects changes only when α is higher than 0.8! P3 is the best only for decision makers whose coefficient of pessimism exceeds 0.8.
2. Despite the fact that P1 and P2 are Pareto-optimal and that P2 offers the highest payoffs in four out of five scenarios, the project P2 according to the Hurwicz's rule does never take the first place in the ranking and does never obtain a higher index than the P1 measure.
3. When the parameter α equals 83.3 %, the Hurwicz's rule treats the project P1 on equal terms with the project P3, though the first one is much more risky than the third one.

Usually the Hurwicz's criterion leads to sensible results in the DMUU process, but in this particular case the answer seems to be contradictory with the logic and does not reflect decision maker's preferences. What are the reasons of such a defect? The factors are quite obvious:

1. The Hurwicz's rule takes only extreme payoffs into consideration. Transitional values, i.e. $a_{ij} \in (w_j, m_j)$, are ignored. This state entails the following consequence. The position of an alternative in the ranking is merely determined by parameters w_j and m_j. This factor explains why, according to the Hurwicz's rule, the project P2 is always dominated by P1 in the problem discussed:

$$(w_1 = w_2) \wedge (m_1 > m_2) \Rightarrow (h_1 > h_2) \tag{12}$$

$$(w_1 > w_2) \wedge (m_1 = m_2) \Rightarrow (h_1 > h_2) \tag{13}$$

2. The Hurwicz's criterion does not take into account the frequency of relatively high and small payoffs belonging to the set of all profits assigned to a given alternative. Therefore, two decisions with the same minimal and maximal profits always obtain an identical Hurwicz's index, even if one of them contains many small payoffs and the second one—many high payoffs:

$$(w_1 = w_2) \wedge (m_1 = m_2) \Rightarrow (h_1 = h_2) \tag{14}$$

For example, if the decision maker may choose one out of two alternatives: A1 with rewards 5,1,1,1,1 and A2 with rewards 5,5,5,5,1, the Hurwicz's rule gives the same value for both: $h_{A1} = h_{A2} = \alpha \cdot 1 + (1 - \alpha) \cdot 5 = 5 - 4\alpha$, which is rather unfair for the second one.

As within the classical Hurwicz's procedure the DM's utility function only uses extreme profits, one may put forward a hypothesis that this rule ought to be applied exclusively in two cases:

1. in problems with an even distribution of payoffs for each alternative, i.e. when the number of rather optimistic scenarios is similar to the number of scenarios with rather bad results or
2. in the process of decision making under interval uncertainty, i.e. when merely the parameters w_j and m_j are known and when each value between them may theoretically occur.

Two modified Hurwicz's criteria for scenarios DMMU are proposed in Sect. 5. In contradiction to the original method, they take into consideration both the level of pessimism and how the payoffs related to a given decision are distributed.

5 Two modified Hurwicz's criteria

The observations aforementioned encourage us to create a new method (or new methods) which takes into consideration both the minimal and the maximal value for each alternative and the frequency of the worst and the best payoffs. The Sects. 5.1 and 5.2 contain a description of both proposed modified Hurwicz's methods. The first technique (the APO method) is designed for a passive decision maker who is not interested in carrying out a meticulous analysis of the payoffs matrix. The second procedure (the SAPO method) takes better into account the DM's preferences, but requires a more active and conscious attitude in the decision making process.

5.1 Method I (the averages of good and bad results weighted by the pessimism and optimism index: APO)

The first approach (the APO method) suggested combines elements of the Hurwicz's and the Laplace's criterion. The procedure consists of five steps:

1. For each alternative I_j present the payoffs as a non-increasing sequence $Sq_j = (a_{1j}, \ldots, a_{sj}, \ldots, a_{mj})$ containing m terms (where m still denotes the number of scenarios and s is the number of the term).

2. Fix the value of the parameter C which signifies the number of good and bad terms in the sequences [Eq. (15)]:

$$C = \max\{1, \lceil m \cdot \min\{\alpha, 1 - \alpha\}\rceil\} \tag{15}$$

3. For each alternative calculate the average of good results and the average of bad results according to the Eqs. (16)–(17).

$$A_j^{I,\max} = \frac{1}{C} \sum_{s=1}^{C} a_{sj} \quad j = 1, \ldots, n \tag{16}$$

$$A_j^{I,\min} = \frac{1}{C} \sum_{s=m-C+1}^{m} a_{sj} \quad j = 1, \ldots, n \tag{17}$$

4. For each decision calculate the modified Hurwicz's measure (H^I) using the following expression:

$$H_j^I = \alpha \cdot A_j^{I,\min} + (1 - \alpha) \cdot A_j^{I,\max} \tag{18}$$

5. Select the strategy fulfilling the condition (19):

$$H_j^{I*} = \max_j \{H_j^I\} \tag{19}$$

As mentioned before the technique presented above is similar both to the Hurwicz's rule and to the Laplace's rule. On one side, the APO method takes advantage of the pessimism index. On the other side, it consists in calculating the mean value on the basis of all (when α equals 0.5) or almost all payoffs (when the decision maker is a moderate optimist or pessimist) connected with a given alternative.

In the formulas (16) and (17) the averages of good and bad results are reduced to the values m_j and w_j, respectively (i.e. the maximal and minimal payoff) when $C = 1$. Such a situation occurs for $\min\{\alpha, 1 - \alpha\} \leq \frac{1}{m}$.

The condition (15) allows to fit the cardinality (i.e. the number of elements) of the subsequence of good and bad results to the level of optimism and pessimism. Thus, a radical optimist or pessimist calculates the averages only on the basis of the extreme values of particular alternatives, whereas the quantity of terms considered in both averages is pointedly larger for moderate decision makers, which means that in the second case the transitional values $a_{ij} \in (w_j, m_j)$ have an influence on the modified Hurwicz's measure.

5.2 Method II (the shortened averages of good and bad results weighted by the pessimism and optimism index: SAPO)

In the first suggested method the subsequences of good and bad payoffs are determined by the level of the parameter α. It is a quite comfortable technique, nevertheless such

an approach entails the risk of inserting rather bad results in the "good" subsequence or rather good results in the "bad" subsequence. For example, if $\alpha = 0.4$ and the rewards of the alternative A1 are 5,1,1,1,1, the average of good results will include two payoffs: 5 and 1, though the second value is not high at all.

In the second technique (the SAPO method) the cardinality of both subsequences is determined by the level of the pessimism index (α) and by an additional parameter allowing the decision maker to specify which range of values is really attractive for him or her ("good" range) and which range of values is treated as a bad one ("bad" range). The initial range may be determined by means of diverse procedures:

Method II.A. Separately for each alternative in a relative way (using deviation degrees).
Method II.B. Separately for each alternative in an absolute way (using bounds).
Method II.C. Together for all alternatives in a relative way (using deviation degrees).
Method II.D. Together for all alternatives in an absolute way (using bounds).

When the initial ranges are set in a relative way one can use parameters $d_j^{\max}$ and $d_j^{\min}$ which signify the allowable degree of deviation from m_j and w_j, respectively (where m_j and w_j still denote the maximal and the minimal payoff for the alternative I_j). When the initial ranges are set together, then for all alternatives the deviation degrees are $d^{\max}$ and $d^{\min}$.

When the initial ranges are calculated in an absolute way one may use parameters $b_j^{\max}$ and $b_j^{\min}$ which signify the lower bound of the "good" range and the upper bound of the "bad" range, respectively. If the initial ranges are defined together for all alternatives, then the bounds are the same and equal to $b^{\max}$ and $b^{\min}$.

The parameters $d_j^{\max}$ and $d_j^{\min}$ or $b_j^{\max}$ and $b_j^{\min}$ are set arbitrarily by the decision maker. The connection between parameters $d_j^{\max}$, $d_j^{\min}$ and $b_j^{\max}$, $b_j^{\min}$ is shown by the Eqs. (20)–(21).

$$d_j^{\max} = \frac{m_j - b_j^{\max}}{m_j - w_j} \tag{20}$$

$$d_j^{\min} = \frac{b_j^{\min} - w_j}{m_j - w_j} \tag{21}$$

Remember that the parameters $b_j^{\max}$, $b_j^{\min}$, $d_j^{\max}$ and $d_j^{\min}$ ought to satisfy the following conditions:

$$w_j \leq b_j^{\min} < b_j^{\max} \leq m_j \tag{22}$$

$$d_j^{\max} + d_j^{\min} < 1 \tag{23}$$

$$d_j^{\max}, d_j^{\min} \geq 0 \tag{24}$$

Of course, the methods II.A and II.D are these approaches which enable, in some specific cases, to obtain a similar order of magnitude for each "good" range and for each "bad" range. For example, if $n = 2$, $w_1 = 4$, $w_2 = 3$, $m_1 = 20$, $m_2 =$

13, $b^{\max} = 12$, $b^{\min} = 8$, $d_1^{\max} = 0.5$, $d_1^{\min} = 0.25$, $d_2^{\max} = 0.1$, $d_2^{\min} = 0.5$, then the initial ranges are the same for the relative (II.A) and absolute (II.D) approach: [4;8] and [12;20] for the first alternative and [3;8] and [12;13] for the second one. Analogical remarks may be formulated for the pair of methods II.B and II.C.

Here, for simplicity's sake, we will discuss only the case when the "good" and the "bad" range is defined by means of deviation degrees and these relative ranges have the same allowable deviations for all alternatives. In such a situation we need only two parameters: $d^{\max}$ and $d^{\min}$, since $d_1^{\max} = d_2^{\max} = \ldots = d_j^{\max} = d^{\max}$ and $d_1^{\min} = d_2^{\min} = \ldots = d_j^{\min} = d^{\min}$.

Remark that the final ranges of "good" and "bad" results are usually shorter than the initial ones due to the parameter α which represents the DM's risk aversion and has an impact on C (see the second step of the following method).

In this case the procedure consists of five steps:

1. For each alternative I_j present the payoffs as a non-increasing sequence $Sq_j = (a_{1j}, \ldots, a_{sj}, \ldots, a_{mj})$ containing m terms (where m still denotes the number of scenarios and s is the number of the term).
2. For each alternative generate the subsequence of good results ($SSq_j^{\max}$) and the subsequence of bad results ($SSq_j^{\min}$) using Eqs. (25) and (26):

$$SSq_j^{\max} = \Big\{ a_{sj} \in Sq_j : (m_j - d^{\max}(m_j - w_j) \leq a_{sj} \leq m_j) \wedge \big(\big| SSq_j^{\max} \big| \leq C \big) \\ \wedge (a_{sj} \rightarrow \max) \Big\} \quad j = 1, \ldots, n \tag{25}$$

$$SSq_j^{\min} = \Big\{ a_{sj} \in Sq_j : (w_j \leq a_{sj} \leq w_j + d^{\min}(m_j - w_j)) \wedge \big(\big| SSq_j^{\min} \big| \leq C \big) \\ \wedge (a_{sj} \rightarrow \min) \Big\} \quad j = 1, \ldots, n \tag{26}$$

where $|SSq_j^{\max}|$ and $|SSq_j^{\min}|$ signify the final cardinalities of both subsequences and the parameter C is computed according to the constraint (15).

The Eq. (25) allows the decision maker to include in the subsequence $SSq_j^{\max}$ only these elements of the whole sequence which belong to the range determined by the deviation degree $d^{\max}$. For example, if $d^{\max} = 0.2$, $m_j = 20$, $w_j = 5$, then the elements of $SSq_j^{\max}$ should satisfy the following constraint $a_{sj} \in [20 - 0.2(20 - 5); 20] = [17; 20]$. Notice that the final cardinality of $SSq_j^{\max}$ is additionally limited by C which depends on the pessimism and optimism indices. Closer to 0 and 1 they are, less elements the subsequence $SSq_j^{\max}$ may contain. Such a relation may be explained by the fact that more radical the decision maker is, more likely, in his or her opinion, one of the extreme values is. If $m = 10$ and $\alpha = 0.2$, $SSq_j^{\max}$ may constist of at least two elements which, due to the last part of the Eq. (25), must be the highest. Thanks to the parameters $d^{\max}$ and α the decision maker is able to set a subsequence $SSq_j^{\max}$ which, from his or her point of view, is composed of appropriate payoffs, because the formula (25) takes into consideration both the subjective evaluation of "good" values and the DM's risk aversion. The Eq. (26) has an analogical interpretation.

3. For each alternative calculate the average of good results and the average of bad results, see Eqs. (27)–(28).

$$A_j^{II,\max} = \frac{1}{\left|SSq_j^{\max}\right|} \sum_{a_{sj} \in SSq_j^{\max}} a_{sj} \quad j = 1, \ldots, n \tag{27}$$

$$A_j^{II,\min} = \frac{1}{\left|SSq_j^{\min}\right|} \sum_{a_{sj} \in SSq_j^{\min}} a_{sj} \quad j = 1, \ldots, n \tag{28}$$

4. For each decision compute the modified Hurwicz's measure (H^{II}) using the following expression:

$$H_j^{II} = \alpha \cdot \frac{m+1-\left|SSq_j^{\min}\right|}{m} A_j^{II,\min} + (1 - \alpha) \cdot \frac{m-1+\left|SSq_j^{\max}\right|}{m} A_j^{II,\max} \tag{29}$$

The parameters m, $\left|SSq_j^{\min}\right|$, $\left|SSq_j^{\max}\right|$ inserted in the condition (29) enable to take into consideration the size of both subsequences, i.e. the frequency of particular payoffs. As one can see the index H_j^{II} is proportional to the number of good payoffs, i.e. the final cardinality of $SSq_j^{\max}$, and inversely proportional to the number of bad values, i.e. the final cardinality of $SSq_j^{\min}$, because a given alternative is more attractive when it contains many high profits and few low results. The fractions $\frac{m+1-\left|SSq_j^{\min}\right|}{m}$ and $\frac{m-1+\left|SSq_j^{\max}\right|}{m}$ are equal to 1 when particular subsequences consist of one term. If $|SSq_j^{\min}|$ increases, then the first fraction is smaller than 1, but bigger than 0. The weight $\frac{m+1-\left|SSq_j^{\min}\right|}{m}$ is a kind of punishment for the alternative which number of bad results is high because such a distribution of payoffs is not desirable for the decision maker. On the other hand, if $|SSq_j^{\max}|$ increases, then the second fraction is bigger than 1, but smaller than 2. The weight $\frac{m-1+\left|SSq_j^{\max}\right|}{m}$ is a kind of bonus for the alternative which number of good results is high because such a distribution of payoffs is much-desired.

5. Select the strategy fulfilling the condition (30):

$$H_j^{II*} = \max_j\{H_j^{II}\} \tag{30}$$

5.3 Demonstration and results

Let us analyze the results generated by both modified Hurwicz's criteria (APO and SAPO) for the case presented in Sect. 4 (Table 2). The first decision maker (A) was an optimist ($\alpha^A = 0.3$). The second one (B) was a moderate pessimist ($\alpha^B = 0.7$).

Table 5 Modified Hurwicz's measures (H^I) for projects P1, P2, P3 (optimist and pessimist type)—method I (APO)

Decision maker	DM A (optimist type)	DM B (pessimist type)
Project		
P1	$h_{P1}^{I,A} = 0.3 \times 1 + 0.7 \times 5.5 = 4.15$	$h_{P1}^{I,B} = 0.7 \times 1 + 0.3 \times 5.5 = 2.35$
P2	$h_{P2}^{I,A} = 0.3 \times 5.25 + 0.7 \times 9.5 = 8.23$	$h_{P2}^{I,B} = 0.7 \times 5.25 + 0.3 \times 9.5 = 6.53$
P3	$h_{P3}^{I,A} = 0.3 \times 3 + 0.7 \times 5 = 4.4$	$h_{P3}^{I,B} = 0.7 \times 3 + 0.3 \times 5 = 3.6$

Method I.

1. The non-increasing sequences of payoffs:

$$Sq_{P1} = (10, 1, 1, 1, 1) \quad Sq_{P2} = (9.5, 9.5, 9.5, 9.5, 1) \quad Sq_{P3} = (5, 5, 4, 4, 2)$$

2. The parameters C are the same for both decision makers:

$$C^A = \max\{1, \lceil 5 \cdot \min\{0.3, 0.7\}\rceil\} = 2, C^B = \max\{1, \lceil 5 \cdot \min\{0.7, 0.3\}\rceil\} = 2,$$

3. The average of good and bad results (see Eqs. 16–17):

$$A_{P1}^{I,\max} = \frac{1}{2}(10 + 1) = 5.5 \quad A_{P1}^{I,\min} = \frac{1}{2}(1 + 1) = 1$$
$$A_{P2}^{I,\max} = 9.5 \quad A_{P2}^{I,\min} = 5.25$$
$$A_{P3}^{I,\max} = 5 \quad A_{P3}^{I,\min} = 3$$

4. Table 5 presents the modified Hurwicz's measures H_j^I for all business plans (see Eq. 18).
5. In both cases (optimist and pessimist type) the alternative fulfilling the condition (19) is the project P2. The ranking of projects changes depending on the value of the pessimism index because in this example: $A_{P2}^{I,\max} > A_{P1}^{I,\max} > A_{P3}^{I,\max}$ and $A_{P2}^{I,\min} > A_{P3}^{I,\min} > A_{P1}^{I,\min}$.

Notice that according to the method I the project P1 could have been the best for the optimist type of decision maker if at least one following condition had been satisfied:

- its highest value had significantly exceeded the highest payoffs of other projects.
- its highest value had occurred in more than one state of nature.
- the parameter α had been close to 0 (radical optimist).

As one can see the first modified Hurwicz's rule is a little bit similar to the Laplace's approach, since it is based on an average of almost all payoffs treated as equally likely.

Table 6 Modified Hurwicz's measures (H^{II}) for projects P1, P2, P3 (optimist and pessimist type)—method II (SAPO)

Decision maker	DM A (optimist type)	DM B (pessimist type)
Project		
P1	$H_{P1}^{II,A} = 0.3 \times \frac{4}{5} \times 1 + 0.7 \times \frac{5}{5} \times 10 = 7.24$	$H_{P1}^{II,B} = 0.7 \times \frac{4}{5} \times 1 + 0.3 \times \frac{5}{5} \times 10 = 3.56$
P2	$H_{P2}^{II,A} = 0.3 \times \frac{5}{5} \times 1 + 0.7 \times \frac{6}{5} \times 9.5 = 8.28$	$H_{P2}^{II,B} = 0.7 \times \frac{5}{5} \times 1 + 0.3 \times \frac{6}{5} \times 9.5 = 4.12$
P3	$H_{P3}^{II,A} = 0.3 \times \frac{5}{5} \times 2 + 0.7 \times \frac{6}{5} \times 5 = 4.8$	$H_{P3}^{II,A} = 0.7 \times \frac{5}{5} \times 2 + 0.3 \times \frac{6}{5} \times 5 = 3.2$

<u>Method II.</u>

We will assume that $d^{\max} = d^{\min} = 0.35$.

1. The non-increasing sequences of payoffs:

$$Sq_{P1} = (10, 1, 1, 1, 1) \quad Sq_{P2} = (9.5, 9.5, 9.5, 9.5, 1) \quad Sq_{P3} = (5, 5, 4, 4, 2)$$

2. The parameter C is still equal to 2:

$$C^A = \max\{1, \lceil 5 \cdot \min\{0.3, 0.7\}\rceil\} = 2, \quad C^B = \max\{1, \lceil 5 \cdot \min\{0.7, 0.3\}\rceil\} = 2$$

The subsequences of good and bad results (see Eqs. 25–26):

$$SSq_{P1}^{\max} = \left\{a_{s,P1} \in Sq_{P1} : (10 - 0.35(10 - 1) \le a_{s,P1} \le 10) \land \left|SSq_{P1}^{\max}\right| \right.$$
$$\left. \le 2 \land (a_{s,P1} \to \max)\right\} = (10)$$
$$SSq_{P1}^{\min} = \left\{a_{s,P1} \in Sq_{P1} : (1 \le a_{s,P1} \le 1 + 0.35(10 - 1)) \land \left|SSq_{P1}^{\min}\right| \right.$$
$$\left. \le 2 \land (a_{s,P1} \to \min)\right\} = (1, 1)$$
$$SSq_{P2}^{\max} = (9.5, 9.5) \quad SSq_{P2}^{\min} = (1)$$
$$SSq_{P3}^{\max} = (5, 5) \quad SSq_{P3}^{\min} = (2)$$

3. The averages of good and bad results (see Eqs. 27–28):

$$A_{P1}^{II,\max} = 10 \quad A_{P1}^{II,\min} = 1 \quad A_{P2}^{II,\max} = 9.5 \quad A_{P2}^{II,\min} = 1 \quad A_{P3}^{II,\max} = 5$$
$$A_{P3}^{II,\min} = 2$$

4. Table 6 presents the modified Hurwicz's measures H_j^{II} for all business plans (see Eq. 29).
5. Here again the project P2 wins in both cases, but it is worth underlying that the ranking changes depending on the parameters α and d.

Therefore for a radical pessimist ($\alpha = 1$) the ranking is P3, P1 and P2. Meanwhile for an extreme optimist ($\alpha = 0$) the order is P1, P2, P3. For these two particular situations

$C = 1$ and the frequency of the good and bad payoffs is always equal to $1/m$. When the parameter α is close to 0 or 1, or the parameters $d^{\max}$ and $d^{\min}$ are close to 0, then the second modified Hurwicz's criterion is reduced to the classical Hurwicz's decision rule.

6 Conclusions

The contribution concerns the Hurwicz's criterion and its limited application for the scenarios' approach in decision making under uncertainty. As it was stated this rule leads to more logical and rational results when instead of several possible scenarios' payoffs each value from a given interval is theoretically probable or when the discrete distribution of payoffs is rather uniform. Therefore, two new procedures designed for DMMU with states of nature are proposed in the article. Both procedures enable to take into consideration not only the extreme rewards connected with each decision, but also these payoffs which are close to the minimal and maximal values. The first technique proposed (the APO method) is designed for passive decision makers, i.e. for people interested in getting a quick answer about the best decision just on the basis of their level of risk aversion. The second procedure suggested (the SAPO method) allows to better control the width and the contents of the range of good and bad results considered in the modified Hurwicz's measure. Nevertheless, this time the decision maker must present a more active and conscious attitude in the decision making process. He or she has to carefully analyze the payoffs assigned to each alternative and to determine the width of the "good" and "bad" intervals by means of bounds or deviation degrees. Here, we analyzed a case concerning the choice of an investment project, but actually the methods described may be used in each decision making process within which each alternative is characterized by only one criterion (or one synthetic measure) and the goal of the decision maker is to find the optimal pure strategy.

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