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ORIGINAL RESEARCH

The asymptotics of price and strategy in the buyer's bid double auction

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Abstract
In a model with correlated and interdependent values/costs, we identify for the buyer's bid double auction the asymptotic distributions of the price and of two order statistics in the first order conditions for optimal bidding/asking, all of which are normal. Substitution of the asymptotic distributions into the first order conditions can permit the solution for approximately optimal bids/asks that provide insight into what is " first order" in a trader's strategic decision-making, which has been difficult to obtain through analysis of equilibrium.

Keywords Double auction · Rational expectations equilibrium · Interdependent value · Common value · Central limit theorem

JEL Classification C63 · D44 · D82

1 Introduction

In a double auction environment, this paper establishes a central limit theorem for the market price and for the order statistics among bids/asks that are critical in a trader's decision problem. The informational environment in which the result is established allows for both correlation among the informational signals of traders and interdependence of their values/costs. Interdependence of values/costs means that a trader draws an inference from the price and the event that he trades in determining his optimal bid/ask. This inference problem has impeded the study of double auction equilibrium in all but very large markets, wherein inference and strategic behavior are simplified.

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We apply the asymptotic distributions established in our central limit theorem to simplifying a trader's decision problem and thereby gain insight into what is "first order" in this problem.

The model. We consider a double auction in an informational environment from Satterthwaite, Williams, and Zachariadis (2020) with a linear structure that facilitates analysis. Fix $m, n \in \mathbb{N}$. For *market size* $\eta \in \mathbb{N}$, we consider ηm buyers, each of whom wishes to buy one item, and ηn sellers, each of whom has one item to sell. A state μ is drawn from the *uniform improper prior* on $\mathbb{R}$. We discuss our use of this improper distribution below. A *preference term* ε_i is independently drawn for each trader i from a distribution G_ε on $\mathbb{R}$ to determine his value/cost $\mu + \varepsilon_i$. Utility for each trader is quasilinear in his value/cost and money, with utility normalized to zero in the case of no trade and no monetary transfer. A *noise term* δ_i is independently drawn for each trader i from a distribution G_δ on $\mathbb{R}$, with trader i observing the signal $\sigma_i = \mu + \varepsilon_i + \delta_i$. Conditional on the observation of his signal, a trader's probabilistic beliefs about the signal of every other trader are well-defined. We thus study a *correlated interdependent values* model, with correlation of values/costs occurring through the state μ and interdependence referring to the fact that learning another trader's signal or value/cost may cause a trader to update his estimate of his own value/cost.

This informational environment is sufficiently restrictive to allow a deeper analysis of strategic behavior than in more general environments, while still retaining the correlation and interdependence of values/costs that are prominent features of actual markets. The uniform improper prior can be thought of informally as "the uniform distribution across the entire real line." DeGroot (1970, p. 190) motivates it as modeling a situation in which forming a proper prior ex ante is costly or complicated and the decision maker knows that he will receive valuable information at the interim stage on which to define his posterior beliefs, which are then well-defined. Its real value for our purposes is that it implies an *invariance property* for a trader's decision problem: a trader's beliefs about the values/costs and signals of others in relation to his own signal are the same for each possible value of his signal. A trader's decision problem is therefore simply translated linearly as his signal changes and he in this sense solves the same problem at every value of his signal. This greatly simplifies the analysis of his decision.

There are two cases in particular that we address in the paper: the case in which the noise distribution G_δ is degenerate and each trader directly observes his own value/cost is the *private values* special case, and the case in which preference distribution G_ε is degenerate and each trader observes a noisy signal of the state μ is the *common value* limiting case. Our distinction between the private values *special* case and the common value *limiting* case is meaningful: while all results from the general model define analogous results in the private values case, some results of the general model do not hold near the common value case. This is a theme that we explore in this paper.

Following Satterthwaite and Williams (1989, p. 480), the buyer's bid double auction (BBDA) is defined as follows. Knowing their signals, each buyer submits a bid and each seller submits an ask. The bids/asks are ordered in a list $s_{(1)} \leq s_{(2)} \leq \cdots \leq s_{(\eta(m+n))}$. The BBDA's price equals $s_{(\eta m+1)}$ with buyers whose bids are at or above this price buying from sellers whose asks are strictly below this price. The following argument shows that this price equates demand with supply when $s_{(\eta m+1)} > s_{(\eta m)}$. Letting t denote the number of bids from the ηm buyers that are at least $s_{(\eta m+1)}$, the remaining $\eta m - t$ bids must be below $s_{(\eta m+1)}$, which means that t of the ηm bids/asks that are below $s_{(\eta m+1)}$ must be asks. When $s_{(\eta m+1)} > s_{(\eta m)}$, the number of bids/asks below $s_{(\eta m+1)}$ is less than ηm and so the number of asks below $s_{(\eta m+1)}$ is less than t. In this case of excess demand, the available supply from the sellers whose asks are less than $s_{(\eta m+1)}$ is allocated among buyers whose bids are at least $s_{(\eta m+1)}$ in decreasing order starting at the highest bid, with any tie that remains resolved using a fair lottery.

The BBDA simplifies behavior on one side of the market in the following sense. A seller trades only if his ask is below the price $s_{(\eta m+1)}$; he cannot influence the price at which he trades. He therefore acts as a price-taker and chooses his ask to place himself on the correct side of the realized market price, taking into account both his signal and the information that he learns from the market price in the event that he trades. In the private values case in which he knows his cost, this reduces to submitting it as his ask, i.e., $S(c) = c$ is a seller's dominant strategy. A buyer, however, sets the price at which he trades in the event that his bid equals $s_{(\eta m+1)}$. He takes this possibility into account in choosing his bid, bidding below what his bid would otherwise be if he instead acted as a price-taker.

Asymptotics of price and critical order statistics. Theorem 1 is the central limit theorem for the distributions of the BBDA's price and of the order statistics among bids/asks that are crucial in a trader's decision problem. The theorem identifies a bound on strategies that is sufficient to enable a standard proof of a central limit theorem to proceed. This bound is weaker than a bound that has been shown to hold by equilibria in a variety of trading models (though not by equilibria in the general model of this paper). Notably, however, the theorem does not use the constraint of equilibrium in its proof, and it in this sense holds for a broader range of behavior.

Theorem 1 begins by characterizing the asymptotic distribution of the BBDA's price. The pioneering paper (Reny and Perry 2006) shows that a double auction's equilibrium price in a continuum market coincides with the *rational expectations equilibrium* (REE) price. It is also proven that a noncooperative equilibrium exists in a sufficiently large but finite market that approximately implements the REE price of the continuum limit. Satterthwaite, Williams, and Zachariadis (2022, Thm. 7) proves convergence of the BBDA's price to the REE price as the market size η increases within the private values special case of this paper and shows that its expected error is $\Theta\left(1/\sqrt{\eta}\right)$. Theorem 1 goes further and characterizes the asymptotic distribution of the BBDA's price as normal with mean equal to the limiting REE price and with an explicit formula for its variance. Within the informational environment of this paper, the BBDA's price is therefore a consistent, asymptotically unbiased and normal estimator of the REE price.

We next turn to a trader's decision problem. In the ordered list $s_{(1)} \le s_{(2)} \le \cdots \le s_{(\eta(m+n)-1)}$ of bids/asks from the other traders, a trader focuses on $s_{(\eta m)}$, for a buyer trades if and only if his bid is above this bid/ask while a seller sells if and only if his ask is strictly below it. A buyer additionally focuses on $s_{(\eta m+1)}$ because his bid sets the price when it lands between $s_{(\eta m)}$ and $s_{(\eta m+1)}$, in which case it is the $(\eta m+1)^{\text{st}}$-smallest bid/ask overall. Theorem 1 characterizes the asymptotic distributions of the order statistics $s_{(\eta m)}$ and $s_{(\eta m+1)}$, both of which are also normal.

The asymptotic distributions of $s_{(\eta m)}$ and $s_{(\eta m+1)}$ are the foundation for our effort to simplify a trader's decision problem, which is accomplished by substituting them into the first order conditions (FOCs) for optimal bidding/asking. The resulting equations are the *asymptotic first order conditions* (AFOCs). We focus first on the private values case and then on the general model in which G_ε and G_δ are normal. The AFOCs can be solved in both of these cases for a buyer's bid and a seller's ask, thereby providing intuition into these complex decision problems.

We then explore the effectiveness of this bid and ask obtained using the asymptotic distributions as approximations to computed examples of equilibria. Interestingly, the approximations are quite accurate in the private value case, but diminish in accuracy as the informational environment is changed so that interdependence plays a greater role in a trader's decision problem. This is explained using intuition concerning how equilibria change as the common value limit is approached.

Organization. Section 2 completes the model and presents material that supports the main results of the paper. Section 3 then addresses the asymptotic distribution of the BBDA's price and the order statistics among bids/asks that are critical in a trader's decision-making. We then turn in Sect. 4 to analyzing a trader's decision problem by substitution into the FOCs for optimal bidding/asking. All proofs are in the Appendix.

2 The model and preliminary results

We begin by completing the model. We then address (i) the first order conditions for an optimal bid and ask and (ii) the limit market and rational expectations equilibrium.

2.1 The model

We make the following assumption on the distributions:
A1: *G_ε and G_δ are C^1 with finite first moments and positive densities g_ε and g_δ on $\mathbb{R}$ that are symmetric about* 0.

The following notation is useful in our discussion of convergence as the market size η increases. Let

$$q \equiv \frac{m}{m+n}, \tag{1}$$

the relative size of demand in the market. Define

$$\xi_q^{\varepsilon+\delta} \equiv G_{\varepsilon+\delta}^{-1}(q), \tag{2}$$

the q^{th} quantile of the distribution $G_{\varepsilon+\delta}$ of the sum of a trader's preference term ε and his noise term δ. This specializes to

$$\xi_q^{\varepsilon} \equiv G_{\varepsilon}^{-1}(q) \tag{3}$$

in the private values case. Letting z denote the value/cost of a trader, define

$$V(\sigma) \equiv \mathbb{E}[z|\mu = 0, \sigma] \tag{4}$$

as the expectation of z when the state μ equals 0 and $\sigma \in \mathbb{R}$ is his signal. We in some instances add the following assumption on G_ε and G_δ:

A2: *$V(\sigma)$ is strictly increasing.*
This is a strict version of first order stochastic dominance. It is satisfied in the case of G_ε, G_δ normal and in the private values case, wherein $V(\sigma) = \sigma$.

2.2 First order condition for an optimal bid/ask

Drawing from Satterthwaite, Williams, and Zachariadis (2020, sec. 3 and app. B), a buyer's marginal expected utility when σ_B is his signal, he bids b, and all other traders use increasing, differentiable functions to choose their bids/asks is

$$\big(\mathbb{E}[v|\sigma_B, x^\eta = b] - b\big) \cdot f_{x|\sigma}^B(b|\sigma_B) - \Pr[x^\eta < b < y^\eta|\sigma_B]. \tag{5}$$

We use x^η and y^η throughout the paper to denote the ηm^{th} and $(\eta m + 1)^{\text{st}}$ order statistics of the other traders' bids/asks from the perspective of the trader of interest, with their distributions determined by the strategies of the other traders. The product in (5) is the marginal expected gain to the buyer from raising his bid, which increases his expected utility when he passes x^η; the first term is his expected value for an item given his signal σ_B and the event that his bid b equals x^η minus the price $x^\eta = b$ that he pays in this event, and the second term is the density of x^η at b. The subtracted term in (5) is the marginal expected cost from raising his bid. It equals the probability that he sets the price in the sample of bids/asks, i.e., his bid b lies between x^η and y^η.

The marginal expected utility of a seller with signal σ_S who asks a is

$$-\big(a - \mathbb{E}[c|\sigma_S, x^\eta = a]\big) \cdot f_{x|\sigma}^B(a|\sigma_S). \tag{6}$$

This is the seller's marginal expected loss from raising his ask and thereby passing x^η and losing a sale.

Equating (5) and (6) to zero and solving for a buyer's bid b and a seller's ask a defines the FOCs for optimal bidding/asking:

$$b = \underbrace{\mathbb{E}[v|\sigma_B, x^\eta = b]}_{\text{price-taking term}} - \underbrace{\frac{\Pr[x^\eta < b < y^\eta|\sigma_B]}{f_{x|\sigma}^B(b|\sigma_B)}}_{\text{strategic term}}. \tag{7}$$

$$a = \underbrace{\mathbb{E}\left[c|\sigma_S, x^\eta = a\right]}_{\text{price-taking term}}. \tag{8}$$

The labeling of the terms in (7) and (8) reflect their roles in the FOCs: the strategic term of a buyer originates in the possibility of moving the price in his favor, while the price-taking term would determine his optimal bid if he ignored this possibility; a seller's FOC has only a price-taking term because he has no ability to influence price in his favor. Each price-taking term is simply the trader's value/cost in the private values case.

2.2.1 Symmetric constant offset equilibrium

The examples of Bayesian-Nash equilibria that we compute in this paper are *symmetric constant offset* profiles of strategies that satisfy:

Symmetry: Each buyer uses the same function $B^\eta : \mathbb{R} \to \mathbb{R}$ to select his bid as a function of his signal and each seller uses the same function $S^\eta : \mathbb{R} \to \mathbb{R}$ to select his ask.

Constant Offset: Each strategy B^η and S^η is a *constant offset strategy* in the sense that

$$B^\eta(\sigma) = \sigma + \lambda_B^\eta, S^\eta(\sigma) = \sigma + \lambda_S^\eta$$

for all values of the signal $\sigma \in \mathbb{R}$, where the offsets λ_B^η and λ_S^η are constants.

These two properties restrict the form of equilibrium strategies and not what is required for equilibrium. As noted in the Introduction, a trader's decision problem is translated linearly as his signal changes and he in this sense solves the same problem in selecting his bid/ask at every value of his signal. An offset strategy specifies the same solution to this problem at each value of the signal. It thus extends to strategic behavior the invariance property mentioned in the Introduction.

2.3 The limit market and rational expectations equilibrium

The *limit market* in state μ consists of m times a unit mass of buyers and n times a unit mass of sellers, with values/costs and signals generated using the distributions G_ε and G_δ. The *REE function* $p^{\text{REE}} : \mathbb{R} \to \mathbb{R}$ determines a price $p^{\text{REE}}(\mu)$ in the limit market for state μ and is defined by two properties. First, it is invertible and thus reveals the state. Let $\Lambda(\cdot)$ denote the inverse function. Second, $p^{\text{REE}}(\mu)$ clears the limit market in the state μ when each trader learns his signal σ, observes $p^{\text{REE}}(\mu)$, and calculates his expected value/cost $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}(\mu)\right), \sigma\right]$. If he is a buyer, he buys an item if and only if $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}(\mu)\right), \sigma\right] \geq p^{\text{REE}}(\mu)$, and if he is a seller, he sells his item if and only if $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}(\mu)\right), \sigma\right] \leq p^{\text{REE}}(\mu)$. Satterthwaite, Williams, and Zachariadis (2020, Thm. 3) states that if G_ε, G_δ satisfy A1 and A2, then the unique REE price in state μ is

$$p^{\text{REE}}(\mu) \equiv \mu + V\left(\xi_q^{\varepsilon+\delta}\right). \tag{9}$$

The inverse mapping from the REE price to the state μ is therefore $\Lambda\left(p^{\mathrm{REE}}\right) \equiv p^{\mathrm{REE}} - V\left(\xi_q^{\varepsilon+\delta}\right)$. Additionally, when the BBDA operates in the limit market, each trader in equilibrium adds the constant offset

$$\lambda^{\infty} \equiv V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta} \tag{10}$$

to his signal to determine his bid/ask, with the equilibrium price equaling $\mu + V\left(\xi_q^{\varepsilon+\delta}\right) = p^{\mathrm{REE}}(\mu)$. The BBDA thus implements the REE price in the limit market.

Example: the private values case. In the private values case in which G_δ is degenerate, we substitute into (4) to obtain

$$V(\sigma) \equiv \mathbb{E}[z|\mu = 0, \sigma] = \sigma,$$

i.e., the expectation of a trader's value/cost in the state $\mu = 0$ given his signal σ equals his signal, which is exactly his value/cost. Substituting into (9) produces

$$p^{\mathrm{REE}}(\mu) \equiv \mu + V\left(\xi_q^{\varepsilon+\delta}\right) = \mu + \xi_q^{\varepsilon},$$

where ξ_q^{ε} is defined in (3). As to bids/asks in the BBDA, substitution into (10) produces the constant offset

$$\lambda^{\infty} \equiv V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta} = \xi_q^{\varepsilon} - \xi_q^{\varepsilon} = 0.$$

With the ability of a buyer to influence price in his favor eliminated through the assumption of a continuum of traders, each trader submits his value/cost as his bid/ask and the distribution of bids/asks in each state μ is the same as the distribution of values/costs. The BBDA's equilibrium price in the state μ is $p^{\mathrm{REE}}(\mu) = \mu + \xi_q^{\varepsilon}$, which equates the mass $m(1-q)$ of buyers above ξ_q^{ε} with the mass nq of sellers below this quantile: applying $q = m/(m+n)$ from (1),

$$m \cdot (1-q) = \frac{mn}{m+n} = nq.$$

3 Asymptotic distributions

3.1 Statement of result

Let

$$\left(\boldsymbol{B}^{\eta}, \boldsymbol{S}^{\eta}\right)_{\eta\in\mathbb{N}} \equiv \left(\left(B_i^{\eta}, S_j^{\eta}\right)_{1\le i\le \eta m, 1\le j\le \eta n}\right)_{\eta\in\mathbb{N}}$$

denote a sequence of strategy profiles in the sequence of markets. We use the following assumption:

A3: The sequence $(\boldsymbol{B}^{\eta}, \boldsymbol{S}^{\eta})_{\eta\in\mathbb{N}}$ has the following two properties:

1. each strategy B_i^{η}, S_j^{η} in the sequence is an increasing, C^1 function;
2. there exists constants $K(G_\varepsilon, G_\delta, m, n), \epsilon > 0$ such that

$$\left|\left(B_i^{\eta}(\sigma) - \sigma\right) - \lambda^{\infty}\right|, \left|\left(S_j^{\eta}(\sigma) - \sigma\right) - \lambda^{\infty}\right| < \frac{K(G_\varepsilon, G_\delta, m, n)}{\eta^{1/2+\epsilon}} \tag{11}$$

for all $\eta \in \mathbb{N}$, $1 \le i \le \eta m$, $1 \le j \le \eta n$, and $\sigma \in \mathbb{R}$, where λ^{∞} is the equilibrium offset of the limit market defined in (10).

The assumption in 1. that strategies are C^1 ensures that densities for the critical order statistics in a trader's decision problem exist and are continuous, which is useful in the first order approach. Turning to (11) in 2., a "standard" central limit theorem characterizes the distribution of a statistic as a larger and larger sample is drawn from a fixed distribution. The problem is different here because the distribution of bids/asks from which the sample is drawn changes as the market size η increases and traders change their behavior. A central limit theorem for the BBDA's price and the critical order statistics therefore requires some restriction on the strategies of traders. The bound (11) is what we have found to be restrictive enough to allow the proof to go through while remaining relatively simple to state. Notice that (11) holds uniformly across the domain $\mathbb{R}$ of the signal σ, which is an important aspect of A3.

Though the bound (11) and its application in Theorem 1 below are not limited to equilibrium, (11) is motivated by a large literature on the $O(1/\eta)$ rate of convergence of equilibria to their values in the limit market, which is more restrictive than the $O(1/\eta^{1/2+\epsilon})$ rate in (11). This literature is summarized at the end of the subsection. While (11) has not been proven to hold for equilibria in our general model due to the complexity of addressing interdependence among values/costs, it has been proven to hold for symmetric offset equilibria in the private values special case (Satterthwaite, Williams, and Zachariadis 2022, Thm.4), and numerical evidence supports the claim that it holds generally (Satterthwaite, Williams, and Zachariadis 2020, sec. 5.3). We thus believe it is a plausible assumption for gaining insight into the problem of interdependent values/costs.

Theorem 1 *For fixed m and n, consider a sequence of markets indexed by the market size η, and $(\boldsymbol{B}^{\eta}, \boldsymbol{S}^{\eta})_{\eta\in\mathbb{N}}$ be a sequence of strategy profiles that satisfies A3.*

1. ***The market price.*** *Let $p^{\eta}(\mu)$ denote the random variable of the BBDA's price in the market of size η given the state μ, as determined by this sequence. For each state $\mu \in \mathbb{R}$ and for its corresponding REE price $p^{\mathrm{REE}}(\mu) = \mu + V(\xi_q^{\varepsilon+\delta})$, we have*

$$p^{\eta}(\mu) \sim \mathcal{AN}\left(p^{\mathrm{REE}}(\mu), \frac{mn}{\eta(m+n)^3}\frac{1}{g^2_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta})}\right). \tag{12}$$

2. ***The critical order statistics and their difference.*** *From the perspective of either a buyer or a seller faced with the strategies of others in the profile $(\boldsymbol{B}^{\eta}, \boldsymbol{S}^{\eta})_{\eta\in\mathbb{N}}$, the*

random variables $x^{\eta}(\mu)$ and $y^{\eta}(\mu)$ of the ηm^{th} and $(\eta m+1)^{st}$ order statistics of other traders' bids and asks given the state μ satisfy

$$x^{\eta}(\mu),\, y^{\eta}(\mu) \sim \mathcal{AN}\left(p^{\mathrm{REE}}(\mu)\,,\, \frac{mn}{(\eta(m+n)-1)\,(m+n)^2}\frac{1}{g^2_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta})}\right). \tag{13}$$

Additionally, the random variable of the difference $w^{\eta}(\mu)\equiv y^{\eta}(\mu)-x^{\eta}(\mu)$ given the state μ satisfies

$$w^{\eta}(\mu) \sim \mathcal{A}\exp\left((\eta(m+n)-1)\, g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)\right), \tag{14}$$

and its asymptotic expectation in the state μ therefore equals

$$\frac{1}{(\eta(m+n)-1)\, g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)}. \tag{15}$$

3. ***The private values special case.*** *Statements (12)–(15) holds in the private values case by replacing $g_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta})$ with $g_{\varepsilon}(\xi_q^{\varepsilon})$ and $V(\xi_q^{\varepsilon+\delta})$ with $V(\xi_q^{\varepsilon})$.*

As a consequence of (12) and (13), $p^{\eta}(\mu)$, $x^{\eta}(\mu)$, $y^{\eta}(\mu)$ are consistent, asymptotically unbiased and normal estimators of the REE price in each state μ. The difference between the asymptotic distribution of the BBDA's price $p^{\eta}(\mu)$ in (12) and the asymptotic distribution of the order statistics $x^{\eta}(\mu)$ and $y^{\eta}(\mu)$ in (13) lies in the variance, with the term $\eta(m+n)$ in (12) replaced with $\eta(m+n)-1$ in (13). This reflects the fact that $p^{\eta}(\mu)$ is determined by a sample of $\eta(m+n)$ bids/asks while $x^{\eta}(\mu)$ and $y^{\eta}(\mu)$ are determined by samples of $\eta(m+n)-1$ bids/asks.

The proofs of (12) and (13) generalize standard results (Serfling (1980, sec. 2.3.3)) on the asymptotic distribution of a sample quantile to a case where the sample is not identically distributed, which reflects the asymmetry here between buyer and seller behavior along with the fact that A3 allows asymmetry among the strategies on each side of the market. Notice that all results in the theorem are conditional on the state μ, in which case the finite samples of values/costs, signals and consequently bids/asks are independent. The correlation among signals and interdependence among values/costs that complicate a trader's decision problem are therefore inconsequential in generalizing the standard results to our setting. Finally, using the asymptotic distributions of $x^{\eta}(\mu)$ and $y^{\eta}(\mu)$, we then derive (14) by applying (Siddiqui 1960), who shows that a suitable rescaling of their difference $w^{\eta}(\mu)$ is asymptotically exponential.

It is straightfoward to show that the characterization (12) of the asymptotic distribution of the BBDA's price $p^{\eta}(\mu)$ also holds for other formulas for selecting a price from the interval $[s_{(\eta m)}, s_{(\eta m+1)}]$ of market-clearing prices for the market of size η, and not just the BBDA's rule of selecting $s_{(\eta m+1)}$ as the price. In particular, it holds for every k -double auction (which, for $k\in[0,1]$, sets $ks_{(\eta m+1)}+(1-k)s_{(\eta m)}$ as the price), along with randomized rules for price selection in this interval. The key is bound (11) of A3. We have chosen to state this theorem solely for the BBDA because it

Table 1 For δ standard normal, $\varepsilon \sim \mathcal{N}(0, \upsilon_\varepsilon)$, $m = 2$, $n = 1$, and market size $\eta = 2$, the equilibrium constant offsets of buyers and of sellers are calculated as the variance υ_ε of the preference term in the model grows small and the common value case is approached

υ_ε	Buyer's Offset λ_B^η	Seller's Offset λ_S^η	$\lambda_B^\eta - \lambda_S^\eta$
1	−1.2189	0.1332	−1.3521
1/2	−1.3379	0.3485	−1.6864
1/4	−1.6323	0.7760	−2.4083
1/8	−2.1556	1.5226	−3.6782
1/16	−2.9979	2.6090	−5.6069
1/32	−4.1762	3.9268	−8.1030
1/64	−5.7898	5.6143	−11.4041
1/128	−8.0939	7.9694	−16.0633
1/256	−11.3794	11.2912	−22.6706

is in this special case that we have made progress in understanding a trader's decision problem, which is where we apply the theorem in the remaining sections.

The $O(1/\eta)$ rate of convergence of equilibrium bids/asks. This rate has been proven for a variety of double auction models in the private values case, including Fudenberg, Mobius, and Szeidl (2007), Satterthwaite and Williams (1989), Williams (1991), and Rustichini, Satterthwaite, and Williams (1994). The $O(1/\eta)$ rate is also established by Vives (2011) for bid shading by firms with private information about their cost functions that compete in submitting supply schedules and by Kovalenkov and Vives (2014) for strategic and competitive equilibria in a Kyle (1989) model.

3.2 Numerical results: the relevance of Theorem 1 in a small market

We first summarize two numerical examples from Williams and Zachariadis (2021, secs. 3.2-3) that address the effectiveness of the asymptotic distribution (12) of the BBDA's price as an approximation. The first is a "central" case of our model, i.e., $m = n = 1$, G_ε, G_δ standard normal, and market sizes $\eta = 2, 4, 8, 16$. Using a computed sequence of symmetric offset equilibria, the example demonstrates the high accuracy of the approximating distribution even in the case of market $\eta = 2$. With this well-behaved case in hand, we then consider a standard approach to challenge convergence to normality by choosing G_ε, G_δ that determine a bimodal distribution of a trader's signal. Even with the bimodality of the sampled distribution, however, the asymptotic normal accurately approximates the distribution of the equilibrium price once the market size η reaches 16.

A more interesting challenge to convergence is made by fixing m, n, and the market size η and then approaching the common value case. This is investigated here with $m = 2$, $n = 1$, $\eta = 2$, $\varepsilon \sim \mathcal{N}(0, \upsilon_\varepsilon)$, and $\delta \sim \mathcal{N}(0, 1)$. Table 1 lists the equilibrium constant offsets for a sequence of values of υ_ε as it decreases to 0. Consistent with standard intuition concerning the common value case, the buyer's offset converges to

$-\infty$ and the seller's offset to ∞ as υ_ε decreases to zero, with the equilibrium thus converging to a no-trade outcome at the common value case.[1]

The implication of this for the accuracy of the asymptotic distribution (12) of price as an approximation is as follows. Assumption A3 posits a constant $K(G_\varepsilon, G_\delta, m, n)$ so that the bound (11) holds. If for fixed market size η the values

$$\left|\left(B_i^\eta(\sigma)-\sigma\right)-\lambda^\infty\right| = \left|\lambda_B^\eta-\lambda^\infty\right|, \left|\left(S_j^\eta(\sigma)-\sigma\right)-\lambda^\infty\right| = \left|\lambda_S^\eta-\lambda^\infty\right| \quad (16)$$

are large, then a large constant $K(G_\varepsilon, G_\delta, m, n)$ is required so that (11) holds, which means that a large market size η is needed to make these expressions small and the asymptotic distribution accurate as an approximation. Similarly, holding the market size η constant and allowing υ_ε to go to zero, the terms in (16) grow large and the approximation becomes less and less meaningful.

Figures 1 and 2 illustrate this problem. Figure 1 depicts the sample density of the BBDA's equilibrium normalized price together with its asymptotic limit in four of the cases in Table 1. With $m = 2$, $n = 1$, and $\eta = 1$ fixed, the asymptotic distribution grows worse and worse as an approximation as υ_ε decreases. Figure 2 illustrates that the issue is that the market size must increase as υ_ε becomes small in order for the approximation to remain meaningful. It depicts in the case of $\upsilon_\varepsilon = 1/16$, $m = 2$, $n = 1$, and $\eta = 2, 4, 8$, and 16 the sample density in comparison to its asymptotic limit. In contrast to the standard normal case mentioned at the beginning of this section, we see that the approximation does not become accurate until η increases to 16. The accuracy of the approximation is measured in each graph by the error of approximation $EA(F_{eq}^\eta) \equiv \sup_{t\in\mathbb{R}} \left|F_{eq}^\eta(t) - \Phi(t)\right|$, where $F_{eq}^\eta(\cdot)$ is the sample distribution of the normalized equilibrium price in market size η and Φ is its limit, the cdf of $\mathcal{N}(0, 1)$.

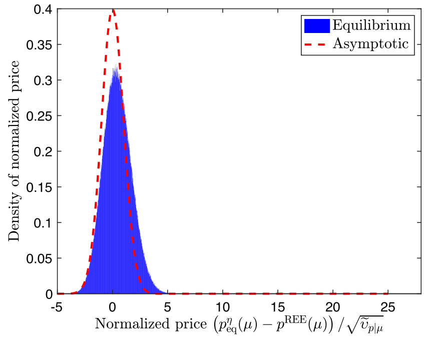
4 An asymptotic analysis of a trader's decision problem

The *asymptotic first order conditions* (AFOCs) are obtained by substituting the asymptotic distributions of $x^\eta(\mu)$ and $y^\eta(\mu)$ from Theorem 1 into the FOCs (7) and (8) for optimal bidding/asking. In (i) the private values case and (ii) the general model in which ε and δ are both normal, the AFOCs can be solved to determine the *asymptotic bid/ask.* These asymptotic expressions provide insight into both strategic behavior and the complicated inference from the market price that a trader addresses in the general model. We explore below their accuracy as approximations to computed examples of equilibria.

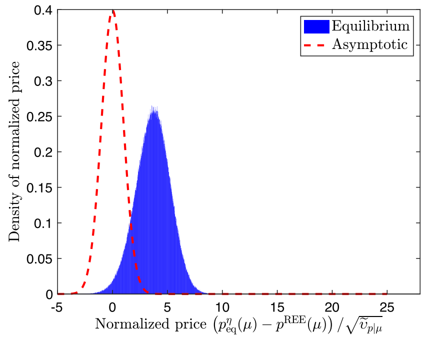
4.1 Asymptotic analysis in the private values case

We begin with a formal statement of our approximation result.

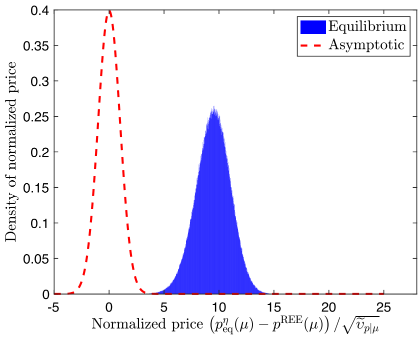
[1] A related example is computed in Gresik (1991, ex. 2, p. 15), where the constrained efficient trading mechanism of Myerson and Satterthwaite (1983) is evaluated for different sizes of markets as the common value case is approached.



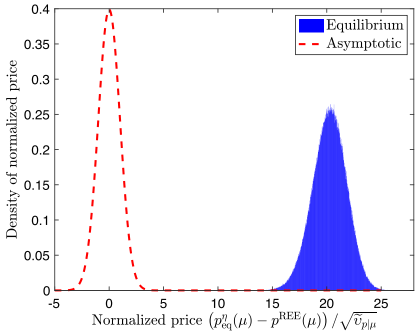
(a) $\upsilon_{\varepsilon} = \frac{1}{4}$, $EA(F^2_{eq}) = 0.1640$



(b) $\upsilon_{\varepsilon} = \frac{1}{16}$, $EA(F^2_{eq}) = 0.8504$



(c) $\upsilon_{\varepsilon} = \frac{1}{64}$, $EA(F^2_{eq}) = 0.9996$



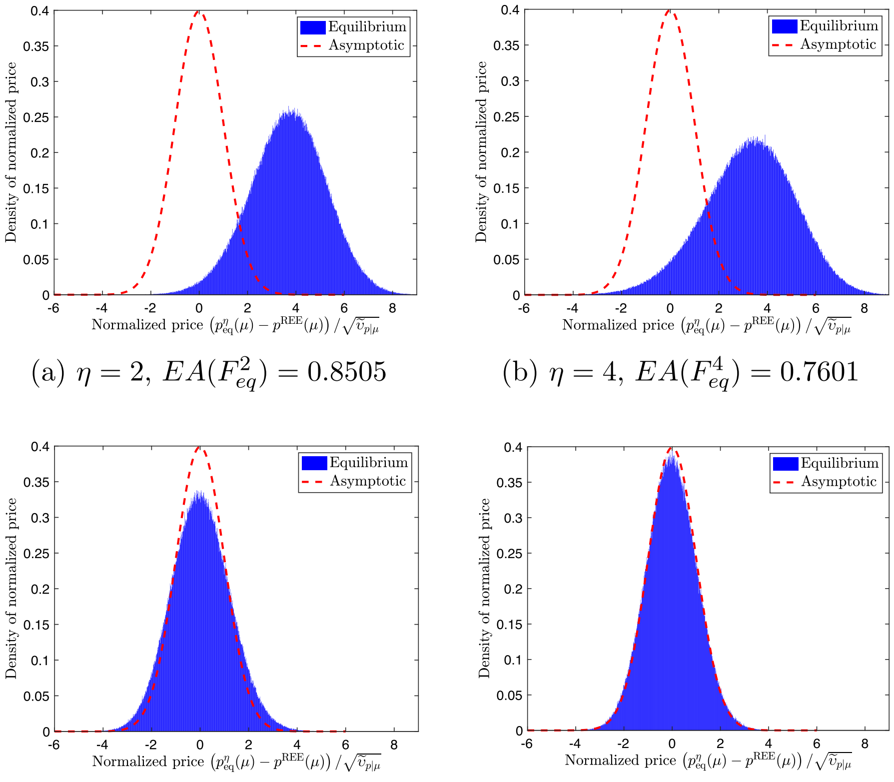
(d) $\upsilon_{\varepsilon} = \frac{1}{256}$, $EA(F^2_{eq}) = 1.0000$

Fig. 1 The figure depicts the sample density of the BBDA's normalized equilibrium price $\left(p^{\eta}_{\mathrm{eq}}(\mu) - p^{\mathrm{REE}}(\mu)\right)/\sqrt{\widetilde{\upsilon}_{p|\mu}}$ calculated using Monte Carlo simulations in the case of δ standard normal, $\varepsilon \sim \mathcal{N}(0, \upsilon_{\varepsilon})$, $m = 2$, $n = 1$, market size $\eta = 2$, and variance of the preference term $\upsilon_{\varepsilon} = 1/4, 1/16, 1/64, 1/256$. It is graphed against its asymptotic limit (i.e., the pdf of $\mathcal{N}(0, 1)$) from Theorem 1. The error of approximation $EA(F^{\eta}_{eq})$ is reported below each graph, where F^{η}_{eq} is the cumulative distribution of the equilibrium price for market size η. The error of approximation $EA(F^{\eta}_{eq})$ is reported below each graph

Theorem 2 *For fixed m and n, consider a sequence of markets indexed by the market size η. Let $(\boldsymbol{B}^{\eta}, \boldsymbol{S}^{\eta})_{\eta\in\mathbb{N}}$ be a sequence of strategy profiles that satisfies A3. Faced with the strategies in this sequence, suppose a buyer considers a sequence of strategies $\left(\widehat{B}(\cdot;\eta)\right)_{\eta\in\mathbb{N}}$ in which his underbidding $\left|\widehat{B}(v;\eta) - v\right|$ is $O\left(1/\eta^{\epsilon}\right)$ for some $\epsilon > 0$ and all $v \in \mathbb{R}$. Then the unique strategy that solves the buyer's AFOC is given by the offset*

$$\widetilde{\lambda}^{\eta}_{B} \equiv -\frac{1}{\eta(m+n)-1}\frac{1}{g_{\varepsilon}(\xi^{\varepsilon}_{q})} + O\left(\frac{1}{\eta^2}\right) \tag{17}$$

that he adds to his value to determine his bid.



(a) $\eta = 2,\ EA(F^2_{eq}) = 0.8505$ (b) $\eta = 4,\ EA(F^4_{eq}) = 0.7601$

(c) $\eta = 8,\ EA(F^8_{eq}) = 0.0509$ (d) $\eta = 16,\ EA(F^{16}_{eq}) = 0.0154$

Fig. 2 The figure depicts the sample density of the BBDA's normalized equilibrium price $\left(p^{\eta}_{\rm eq}(\mu) - p^{\rm REE}(\mu)\right) / \sqrt{\tilde{v}_{p|\mu}}$ calculated using Monte Carlo simulations in the case of δ standard normal, $\varepsilon \sim \mathcal{N}(0, 1/16)$, $m = 2,\ n = 1$, and market sizes $\eta = 2, 4, 8, 16$. It is graphed against its asymptotic limit (i.e., the pdf of $\mathcal{N}(0, 1)$) from Theorem 1. The error of approximation $EA(F^{\eta}_{eq})$ is reported below each graph

The odd but rather weak assumption that the buyer considers strategies whose underbidding is $O\left(1/\eta^{\epsilon}\right)$ for some $\epsilon > 0$ is used in the proof to start the derivation of (17) by allowing a Taylor polynomial expansion of $\widehat{B}(\cdot; \eta)$.

Formula (17) defines the first order *approximate offset* of a buyer,

$$\lambda^{\eta}_{B,\rm approx} = -\frac{1}{\eta(m+n) - 1} \frac{1}{g_{\varepsilon}(\xi^{\varepsilon}_{q})}. \tag{18}$$

This approximation is the negative of the asymptotic expectation of the difference $y^{\eta}(\mu) - x^{\eta}(\mu)$ of the $(\eta m + 1)^{\rm st}$ and $\eta m^{\rm th}$ order statistics among the bids/asks of the other traders in each state μ, as reported in (15) of Theorem 1. The order statistic $y^{\eta}(\mu)$ is the upper endpoint of the interval in which the buyer's bid sets the price; the order statistic $x^{\eta}(\mu)$ is the lower endpoint of this interval and the bid/ask at which he passes from trading to not trading as he lowers his bid. The approximation $\lambda^{\eta}_{B,\rm approx}$

therefore suggests that the buyer underbids so as to push the price across this interval to its lower limit.

The uniqueness of the solution $\widetilde{\lambda}_B^\eta$ together with the fact that its first-order approximation $\lambda_{B,\text{approx}}^\eta$ is a constant are consistent with the support provided in Satterthwaite, Williams, and Zachariadis (2020, sec. 4.1.2) for the conjecture that a symmetric equilibrium in the BBDA is uniquely determined in the private values case and with the property that the difference between each trader's bid/ask and his value/cost is constant. We also note that $\widetilde{\lambda}_B^\eta$ and its approximation $\lambda_{B,\text{approx}}^\eta$ are $O(1/\eta)$, which is consistent with the rate of convergence of a trader's bid/ask to its limit that has been derived a variety of models in the private values case (as noted in Sect. 3.1).

Finally, we note that $\lambda_{B,\text{approx}}^\eta$ identifies what is first order in a buyer's decision problem in this case, namely, (i) the total number $\eta(m+n)-1$ of traders that he faces and (ii) the value of the density $g_\varepsilon(\xi_q^\varepsilon)$ at the single point

$$\xi_q^\varepsilon = G_\varepsilon^{-1}\left(\frac{m}{m+n}\right),$$

which is the quantile of interest in the private values case. Part (i) reflects the fact that buyers and sellers increasingly bid/ask in the same way as the market size η increases. Part (ii) reflects the fact that the focus of the buyer's decision problem in every state μ is near its limiting equilibrium price, namely, $p^{\text{REE}}(\mu) = \mu + \xi_q^\varepsilon$, and the uncertainty that he faces is thus summarized from a first order perspective with the single term $g_\varepsilon(\xi_q^\varepsilon)$.

Numerical investigation of $\lambda_{B,\text{approx}}^\eta$. Williams and Zachariadis (2021, app. F) investigates the accuracy of $\lambda_{B,\text{ approx}}^\eta$ as an approximation to the computed value of the buyer's equilibrium constant offset. Four different choices of the distribution G_ε of a trader's preference are considered in the case of $m = n = 1$. The accuracy of the approximation depends upon the market size η and the distribution G_ε, with a bimodal case again presenting the greatest difficulty. Absolute error as a fraction of the magnitude of the equilibrium offset is less than 6% in all four cases, however, once $\eta = 8$.

4.2 Asymptotic analysis in the case of G_ε, G_δ normal

Let $\widetilde{\upsilon}_{x|\mu}$ denote the asymptotic variance of x^η conditional on μ as given in (13) and let $\widetilde{e}_{w|\mu}$ denote the asymptotic expectation of $w^\eta(\mu) \equiv y^\eta(\mu) - x^\eta(\mu)$ conditional on μ as given in (15). Our approximation result in the normal case of the general model is as follows.

Theorem 3 *Assume that $\varepsilon \sim \mathcal{N}(0, \upsilon_\varepsilon)$ and $\delta \sim \mathcal{N}(0, \upsilon_\delta)$. For fixed m and n, consider a sequence of markets indexed by the market size η. Let $(\boldsymbol{B}^\eta, \boldsymbol{S}^\eta)_{\eta\in\mathbb{N}}$ be a sequence of strategy profiles that satisfies A3. Faced with the strategies of the others in this sequence, suppose a buyer considers strategies $\widehat{B}(\cdot;\eta)$ in which the difference $\left|\widehat{B}(v;\sigma) - \sigma - \lambda^\infty\right|$ between his offset $\widehat{B}(v;\sigma) - \sigma$ and the equilibrium offset λ^∞*

of the limit market is $O\left(1/\eta^{\epsilon}\right)$ *for some* $\epsilon > 0$ *and all* $\sigma \in \mathbb{R}$. *The AFOCs imply the following.*

1. *A buyer with signal* σ_B *bids*

$$b = \widetilde{\mathbb{E}}[v|\sigma_B, x = b] - \left(\widetilde{e}_{w|\mu} + O\left(\frac{1}{\eta^2}\right)\right), \tag{19}$$

where his asymptotic price-taking term is

$$\widetilde{\mathbb{E}}[v|\sigma_B, x = b] = \sigma_B \frac{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon}{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon + \upsilon_\delta} + \left(b - V\left(\xi_q^{\varepsilon+\delta}\right)\right) \frac{\upsilon_\delta}{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon + \upsilon_\delta}, \tag{20}$$

and his asymptotic strategic term is

$$\widetilde{e}_{w|\mu} + O\left(\frac{1}{\eta^2}\right) = -\frac{1}{(\eta(m+n)-1)\, g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)} + O\left(\frac{1}{\eta^2}\right). \tag{21}$$

2. *A seller with signal* σ_S *asks*

$$a = \widetilde{\mathbb{E}}[c|\sigma_S, x = a] = \sigma_S \frac{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon}{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon + \upsilon_\delta} + \left(a - V\left(\xi_q^{\varepsilon+\delta}\right)\right) \frac{\upsilon_\delta}{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon + \upsilon_\delta}, \tag{22}$$

which is the same as formula (20), with the ask a *replacing the bid* b *and* σ_S *replacing* σ_B.

As in the private values case, the assumption that a buyer considers strategies in which $\left|\widehat{B}(v;\sigma) - \sigma - \lambda^\infty\right|$ is $O\left(1/\eta^{\epsilon}\right)$ for some $\epsilon > 0$ is used in the proof to start the derivation by allowing a Taylor polynomial expansion of $\widehat{B}(\cdot;\eta)$. After substituting (20), (21) into (19) and dropping the second order terms, (19) and (22) become linear equations in the bid b and the ask a that are easily solved for the first order *approximate offsets*

$$\lambda^{\eta}_{B,\text{approx}} = -\left(\frac{\upsilon_\delta}{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon}\right) V\left(\xi_q^{\varepsilon+\delta}\right) - \left(\frac{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon + \upsilon_\delta}{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon}\right) \frac{1}{(\eta(m+n)-1)\, g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)} \tag{23}$$

and

$$\lambda^{\eta}_{S,\text{approx}} = -\left(\frac{\upsilon_\delta}{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon}\right) V\left(\xi_q^{\varepsilon+\delta}\right). \tag{24}$$

We begin with two observations concerning these approximate offsets and the buyer's asymptotic strategic term (21). First, we note continuity at the limit market of the approximate offsets $\lambda^{\eta}_{B,\,\text{approx}}, \lambda^{\eta}_{S,\text{approx}}$ and their effectiveness as approximations to symmetric equilibrium constant offsets. Applying formula (10) for λ^{∞} and $V\left(\xi_q^{\varepsilon+\delta}\right) = \upsilon_{\varepsilon}\xi_q^{\varepsilon+\delta}/(\upsilon_{\varepsilon}+\upsilon_{\delta})$ (which holds in the normal case considered here), it is straightforward to show that $\left|\lambda^{\infty} - \lambda^{\eta}_{B,\text{approx}}\right|$ and $\left|\lambda^{\infty} - \lambda^{\eta}_{S,\text{approx}}\right|$ are both $O(1/\eta)$. Consequently, $(\lambda^{\eta}_{B,\text{approx}}, \lambda^{\eta}_{S,\text{approx}})_{\eta\in\mathbb{N}}$ is an $O(1/\eta)$-appproximation of any sequence of symmetric equilibrium offsets $(\lambda^{\eta}_{B}, \lambda^{\eta}_{S})_{\eta\in\mathbb{N}}$ for which $\left|\lambda^{\infty} - \lambda^{\eta}_{B}\right|$ and $\left|\lambda^{\infty} - \lambda^{\eta}_{S}\right|$ are $O(1/\eta)$, which Satterthwaite, Williams, and Zachariadis (2020, sec. 5.3) argues is true of all such sequences.

Second, the buyer's asymptotic strategic term (21) extends formula (17) from the private values case, with $g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)$ replacing $g_{\varepsilon}\left(\xi_q^{\varepsilon}\right)$ in the denominator. As in the private values case, its first term is the negative of the asymptotic expectation of the difference $y^{\eta}(\mu)-x^{\eta}(\mu)$ given μ, as stated in Theorem 1. It has the same intepretation as before. In the approximate offset $\lambda^{\eta}_{B,\text{approx}}$, this term is weighted with the function

$$\frac{\widetilde{\upsilon}_{x|\mu} + \upsilon_{\varepsilon} + \upsilon_{\delta}}{\widetilde{\upsilon}_{x|\mu} + \upsilon_{\varepsilon}} \tag{25}$$

of the variances υ_{ε}, υ_{δ} of preference and noise parameters along with the asymptotic variance $\widetilde{\upsilon}_{x|\mu}$ of the order statistic $x^{\eta}(\mu)$. This reflects the fact that while a buyer's concerns are purely strategic in the private values case, a buyer in the general model weighs his strategic incentive against his necessity of protecting himself from a winner's curse in the selection of his bid. While the weight (25) converges to one (its value in the private values case) as $\upsilon_{\delta} \to 0$, it does not converge to zero as $\upsilon_{\varepsilon} \to 0$ and the common value case is approached, for a buyer retains the ability to influence price in his favor in this limit.

The asymptotic price-taking term. We now turn to a buyer's asymptotic price-taking term (20), with the seller's term (22) interpreted similarly. It is a convex combination of his signal σ_B and $b - V\left(\xi_q^{\varepsilon+\delta}\right)$, which, from the left side of (20), is calculated given $b = x^{\eta}(\mu)$. This estimates his value, which is the sum of the state μ and a preference term ε. As the rest of the market can provide no information about his preference term, the best that a buyer can do using market data to estimate his value is to estimate μ. The term $b - V\left(\xi_q^{\varepsilon+\delta}\right)$ in (20) serves exactly this purpose: Theorem 1 implies that $x^{\eta}(\mu)$ is an unbiased estimate of $p^{\text{REE}}(\mu) = \mu + V(\xi_q^{\varepsilon+\delta})$, and so $b - V\left(\xi_q^{\varepsilon+\delta}\right) = x^{\eta}(\mu) - V\left(\xi_q^{\varepsilon+\delta}\right)$ is an unbiased estimate of μ. A buyer's asymptotic price-taking term (20) is thus a weighted average of his own signal, which contains information about both the state and his value, together with the information he infers from the market, which is an unbiased estimate of the state μ.

Approaching the private values and common value cases. We next turn to the dependence of the asymptotic price-taking term (20) and the approximate offsets (23) and (24) on the variances υ_{ε} and υ_{δ} of the preference and noise distributions. In the normal case considered here, the values of the quantile $\xi_q^{\varepsilon+\delta}$ and the density

$g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)$ are determined by the sum $\upsilon_\varepsilon + \upsilon_\delta$ of these variances; consequently, the value of $\widetilde{\upsilon}_{x|\mu}$ in the above formulas remains fixed as the variances change so long as their sum remains fixed. Keeping $\upsilon_\varepsilon + \upsilon_\delta = K$ for some constant $K > 0$, we thus explore how the weights in the asymptotic price-taking term change as υ_δ varies across $[0, K]$, where υ_δ measures how much noise there is in a trader's signal concerning his value/cost. Recall that (i) as $\upsilon_\delta \to 0$ and $\upsilon_\varepsilon \to K$, the general model converges to the private values case as noise is eliminated from the signals, and (ii) as $\upsilon_\delta \to K$ and $\upsilon_\varepsilon \to 0$, the general model approaches the common value case in which all values/costs are the same and equal to the state μ.

Consider first approaching the private values case, in which the asymptotic price-taking term and the approximate offsets converge continuously to their values in this case. As $\upsilon_\varepsilon \to K$, $\upsilon_\delta \to 0$, and noise is eliminated from the signals, the weight on a trader's signal converges to one and the weight on the inference from the market converges to zero; a trader's asymptotic price-taking term thus converges to his signal, which in the limit is his value/cost. The seller's approximate offset (24) converges to zero, which corresponds to his dominant strategy of asking his cost in the private values case. The buyer's approximate offset (23) converges to its value (18) in the private values case, which, as discussed in Sect. 4.1, can be an accurate approximation of his equilibrium offset.

Approaching the common value case is more interesting because while the asymptotic price-taking term and the approximate offsets converge, their limits do not correspond to the no-trade outcome of equilibrium in this case. As $\upsilon_\delta \to K$ and $\upsilon_\varepsilon \to 0$, the weight on a trader's signal in his asymptotic price-taking term coverges to $\widetilde{\upsilon}_{x|\mu}/\left(\widetilde{\upsilon}_{x|\mu} + K\right) \neq 0$; unlike convergence to the private values case in which the weight placed upon the market inference goes to zero, the weight a trader places upon his private signal remains positive as the common value case is approached because his signal remains informative about the state even in the common value limit. The seller's approximate offset $\lambda^{\eta}_{S,\text{approx}}$ converges to 0 while the buyer's approximate offset $\lambda^{\eta}_{B,\text{approx}}$ converges to the finite value

$$-\left(\frac{\widetilde{\upsilon}_{x|\mu} + K}{\widetilde{\upsilon}_{x|\mu}}\right)\frac{1}{((m+n)\eta - 1)\, g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)}.$$

This disagrees with the intuition that the offset of buyers should go to $-\infty$ while the offset of sellers goes to $+\infty$ as the common value case is approached, consistent with the equilibria reported in Table 1.

Why do the approximations $\lambda^{\eta}_{B,\text{approx}}$ and $\lambda^{\eta}_{S,\text{approx}}$ perform so poorly as the common value case is approached? The approximations rest upon the accuracy of the asymptotic distribution (13) in Theorem 1 as an approximation of the equilibrium distributions of the order statistics $x^{\eta}(\mu)$ and $y^{\eta}(\mu)$. As suggested by the results presented in Fig. 1 for the BBDA's equilibrium price $p^{\eta}_{\text{eq}}(\mu)$, the accuracy of this approximation diminishes as the common value case is approached with the market size η held constant. The exercise we carry out here is for fixed market size η; because we fail to increase η as $\upsilon_\varepsilon \to 0$ as needed to maintain the accuracy of the approximation, the asymptotic distribution (13) becomes worse as an approximation as the common value case is approached, which explains why the approximations $\lambda^{\eta}_{B,\text{approx}}$ and $\lambda^{\eta}_{S,\text{approx}}$ similarly become less meaningful.

Numerical investigation of $\lambda^{\eta}_{B,\text{approx}}$ and $\lambda^{\eta}_{S,\text{approx}}$. Williams and Zachariadis (2021, App. F) investigates the accuracy of the approximations $\lambda^{\eta}_{B,\text{approx}}$ and $\lambda^{\eta}_{S,\text{approx}}$ to equilibrium offsets in numerical examples. Their effectiveness is explored first for G_ε, G_δ standard normal, then for fixed market size η as the common value case is approached, and finally for an instance near the common value case in which the market size η is increased. Starting with the case of G_ε, G_δ standard normal, while the approximations become more accurate as the market size increases, they are notably less accurate than the approximations computed in the private values case with G_ε standard normal. We attribute this to the noise in a trader's signal, which increases the error through a trader's computation of his price-taking term. For η fixed and as the common value case is approached, the errors in the approximations grow larger and larger. This is as expected, for the asymptotic distributions on which the approximations are based become less and less meaningful as the common value case is approached. Finally, near the commmon value case, the accuracy of the approximations is recovered as the market size η is increased sufficiently. This is consistent with a theme of this paper, namely, while convergence in the market size η to the asymptotic limit is at the same rate for all instances of the general model, it takes a larger and larger η to make the asymptotic limit accurate as an approximation as one nears the common value case.

5 Conclusion

We analyze the price and a trader's decision problem in the buyer's bid double auction from an asymptotic perspective. The asymptotic distribution of the price is characterized. It reveals that the price is a consistent, asymptotically unbiased and normal estimator of the rational expectations price. The rational expectations price is thus approximately implemented in a finite market by a market-clearing price, and numerical examples suggests that this can be true even in small markets.

The asymptotic first order conditions are determined by identifying the asymptotic values of probabilities in a trader's first order condition for his optimal bid/ask. They are solvable for an approximate bid and ask in the private values case and in the normal case of the general model. The approximations provide insight into what is first order in a buyer's strategic effort to influence the price in his favor, namely, the total number of traders (and not their roles as buyers and sellers), along with the value of the prior density at a quantile of interest. In the normal case of the general model, we also resolve a trader's effort to protect himself from a winner's curse in his price-taking term into a weighted average of his own private information and an inference that he draws from the market, with the weights dependent in an intuitive way upon the relative variances of preference and noise terms in the model.

All of our approximations perform well in the general model in a "sufficiently large" market; numerical examples suggest, however, that all approximations become less and less accurate as the common value case is approached with the market size fixed. Expressed in another way, a larger and larger market is needed as one approaches the common value case for our approximations to be accurate. In particular, as one nears the common value case for a fixed market size, the expected error of the market price as an estimate of the rational expectations price goes to infinity.

Acknowledgements This paper originates in our work with Mark Satterthwaite, who we thank for his contributions and his support. We also thank Moritz Meyer-ter-Vehn for his advice as editor of the paper.

Appendix

A Proof of Theorem 1

Our proof follows Serfling (1980, secs. 2.3–2.5). The state μ is fixed throughout and so we suppress it from most of the notation. We first prove in Lemmas 1 and 2 the asymptotic normality of the q^{th} quantile in the relevant sample and then apply these results to the order statistics of Theorem 1.

In the market of size $\eta \in \mathbb{N}$, write the vector of signals as $(\sigma_1, \dots, \sigma_{\eta(m+n)})$ with the first ηn signals belonging to sellers and the last ηm signals to buyers. This vector determines the vector of bid/asks $(a_1, \dots, a_{\eta n}, b_1, \dots, b_{\eta m})$ through the strategies

$$S_j^\eta(\sigma_j) = a_j,\ 1 \le j \le \eta n, \text{and } B_i^\eta(\sigma_{i+\eta n}) = b_i,\ 1 \le i \le \eta m.$$

Select a trader and let $\widetilde{F}_{\eta(m+n)-1}$ denote the sample distribution of bids/asks of the $\eta(m+n)-1$ other traders: in sampling $\eta(m+n)-1$ bids/asks, $\widetilde{F}_{\eta(m+n)-1}(t)$ states the proportion of the sample that is smaller than $t \in \mathbb{R}$. We define the q^{th} quantile for this distribution as

$$\tilde{\xi}_{q[\eta(m+n)-1]} \equiv \inf\{t : \widetilde{F}_{\eta(m+n)-1}(t) \ge q\}.$$

In the case in which the selected trader is a buyer with signal $\sigma_{\eta(m+n)}$, $\widetilde{F}_{\eta(m+n)-1}(t)$ is given by

$$\widetilde{F}_{\eta(m+n)-1}(t) \equiv \frac{1}{\eta(m+n)-1} \sum_{i=1}^{\eta m-1} \mathbb{I}\{b_i \le t\} + \frac{1}{\eta(m+n)-1} \sum_{j=1}^{\eta n} \mathbb{I}\{a_j \le t\}, \tag{26}$$

where $\mathbb{I}\{\cdot\}$ denotes the indicator function. As the proof of Lemma 1 is essentially the same regardless of whether the selected trader is a buyer or a seller, we use (26) in its proof. Finally, recall from (9) that $p^{\mathrm{REE}}(\mu) = \mu + V\left(\xi_q^{\varepsilon+\delta}\right)$.

Lemma 1 establishes the asymptotic relationship between $\tilde{\xi}_{q[\eta(m+n)-1]}$ and $p^{\mathrm{REE}}(\mu)$ conditional on μ.

Lemma 1 *Let $(\boldsymbol{B}^\eta, \boldsymbol{S}^\eta)_{\eta\in\mathbb{N}}$ be a sequence of strategy profiles that satisfies A3 and select a trader. Then for all $t \in \mathbb{R}$ and $0 < q < 1$,*

$$\lim_{\eta\to\infty} \Pr\left(\frac{\sqrt{\eta(m+n)-1}\left(\tilde{\xi}_{q[\eta(m+n)-1]} - p^{\mathrm{REE}}(\mu)\right)}{\sqrt{q(1-q)}/g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)} \le t\right) = \Phi(t),$$

where $\Phi(\cdot)$ is the standard normal distribution function. Therefore,

$$\tilde{\xi}_{q[\eta(m+n)-1]} \sim \mathcal{AN}\left(p^{\mathrm{REE}}(\mu), \frac{q(1-q)}{[\eta(m+n)-1]\, g^2_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)}\right).$$

This is proven below, after the proof of Theorem 1. Similarly, we can define $\widetilde{F}_{\eta(m+n)}$ for the sample distribution of the entire sample of $\eta(m+n)$ bids/asks by replacing $\eta(m+n)-1$ with $\eta(m+n)$ in (26). Let $\tilde{\xi}_{q[\eta(m+n)]}$ denote the corresponding q^{th} quantile. Lemma 2 establishes the asymptotic relationship between $\tilde{\xi}_{q[\eta(m+n)]}$ and $p^{\mathrm{REE}}(\mu)$ conditional on μ. The proof is essentially the same as that of Lemma 1, with $\eta m - 1$ and $\eta(m+n)-1$ replaced by ηm and $\eta(m+n$.It is therefore omitted.

Lemma 2 *Let $(\boldsymbol{B}^\eta, \boldsymbol{S}^\eta)_{\eta\in\mathbb{N}}$ be a sequence of strategy profiles that satisfies A3. For all $t \in \mathbb{R}$, and $0 < q < 1$,*

$$\lim_{\eta\to\infty} \Pr\left(\frac{\sqrt{\eta(m+n)}\left(\tilde{\xi}_{q[\eta(m+n)]} - p^{\mathrm{REE}}(\mu)\right)}{\sqrt{q(1-q)}/g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)} \le t\right) = \Phi(t)$$

and therefore

$$\tilde{\xi}_{q[\eta(m+n)]} \sim \mathcal{AN}\left(p^{\mathrm{REE}}(\mu), \frac{q(1-q)}{\eta(m+n)\, g^2_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)}\right).$$

Proof of Theorem 1. Recall that $p^\eta(\mu)$ is the $(\eta m+1)^{\mathrm{st}}$ order statistic in a sample of ηm bids and ηn asks, and $x^\eta(\mu)$ and $y(\eta)$ are the ηm^{th} and $(\eta m+1)^{\mathrm{st}}$ order statistics in a sample of bids/asks from the other traders. For small $\epsilon > 0$, the ratio between the order of each statistic and the cardinality of its sample size satisfies the following as $\eta \to \infty$:

$$\frac{\eta m+1}{\eta(m+n)} = \frac{m}{m+n} + \frac{1}{\eta(m+n)} = \frac{m}{m+n} + o\left(\frac{1}{\eta^{1-\epsilon}}\right); \tag{27}$$

$$\frac{\eta m+1}{\eta(m+n)-1} = \frac{m}{m+n} + \frac{2m+n}{(m+n)(\eta(m+n)-1)} = \frac{m}{m+n} + o\left(\frac{1}{\eta^{1-\epsilon}}\right); \tag{28}$$

$$\frac{\eta m}{\eta(m+n)-1} = \frac{m}{m+n} + \frac{m}{(m+n)(\eta(m+n)-1)} = \frac{m}{m+n} + o\left(\frac{1}{\eta^{1-\epsilon}}\right). \tag{29}$$

Lemmas 1 and 2 establish the asymptotic distribution of the q^{th} quantile in particular samples. Equations (27)–(29) link the order statistics $p^\eta(\mu)$, $x^\eta(\mu)$, and $y^\eta(\mu)$ with the q^{th} quantile of the corresponding sample. An application of Serfling (1980, Thm. and Cor. 2.5.2) then implies the asymptotic relationships between $p^\eta(\mu)$, $x^\eta(\mu)$, and $y^\eta(\mu)$ with $p^{\text{REE}}(\mu) = \mu + V(\xi_q^{\varepsilon+\delta})$ conditional on μ, as stated in the theorem.

Relying on the asymptotic distributions of $x^\eta(\mu)$ and $y^\eta(\mu)$ in the case of sampling from two populations (i.e., bids and asks), it is straightforward to adapt the result in Siddiqui (1960) on the asymptotic distribution of their difference $w^\eta(\mu) = y^\eta(\mu) - x^\eta(\mu)$. In particular, Siddiqui (1960) shows that a suitable rescaling of $w^\eta(\mu)$ (i.e., by the number $\eta(m+n)-1$ of the other traders and twice the density at the quantile of interest $g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)$) is asymptotically exponential with rate $1/2$ and independent of $x^\eta(\mu)$. This implies (14). Finally, the entire argument goes through in the private values case in which G_δ is degenerate, with $g_\varepsilon(\xi_q^\varepsilon)$ and $V(\xi_q^\varepsilon)$ replacing $g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)$ and $V\left(\xi_q^{\varepsilon+\delta}\right)$, respectively, in the results. □

Proof of Lemma 1. The following notation is used in this section. For $1 \le i \le \eta m - 1$, let $\mathcal{S}_{i,B}^\eta(\cdot)$ be the inverse bid function of buyer i, and for $1 \le j \le \eta n$, let $\mathcal{S}_{j,S}(\cdot)$ be the inverse ask function for seller j. For $t \in \mathbb{R}$, define

$$\lambda_i^\eta(t) \equiv \begin{cases} t - \mathcal{S}_{i,S}^\eta(t) & 1 \le i \le \eta n, \\ t - \mathcal{S}_{i-\eta n,B}^\eta(t) & \eta n + 1 \le i \le \eta(m+n) - 1. \end{cases} \tag{30}$$

Using this notation, rewrite (11) of A3 as

$$\left|\lambda_i^\eta(t) - \lambda^\infty\right| < \frac{K(G_\varepsilon, G_\delta, m, n)}{\eta^{1/2+\epsilon}}, \tag{31}$$

which holds for all $\eta \in \mathbb{N}$, $1 \le i \le \eta\,(m+n) - 1$, $t \in \mathbb{R}$, and some $\epsilon > 0$.

As noted above, we prove Lemma 1 assuming the selected trader is a buyer with signal $\sigma_{\eta(m+n)}$. Starting from (26), we: (i) apply $\mathcal{S}_{i,B}^\eta(\cdot)$ to both sides of the inequality inside the first indicator function and $\mathcal{S}_{j,S}^\eta(\cdot)$ to both sides of the inequality inside the second indicator function; (ii) change the index of summation in the first indicator function to $i' = i + \eta n$; (iii) combine the two summations under the same index; (iv) apply definition (30) of $\lambda_i^\eta(t)$. For $t \in \mathbb{R}$, we then have

$$\widetilde{F}_{\eta(m+n)-1}(t) = \frac{1}{\eta(m+n)-1} \sum_{i=1}^{\eta(m+n)-1} \mathbb{I}\{\sigma_i \leq t - \lambda_i^\eta(t)\}.$$

In order to align our notation with Serfling (1980) and thereby clarify the relationship between our proofs and the proofs that inspire them, we now depart from the notation used in the text of the paper and denote the distribution and density of signals conditional on μ as F_μ and f_μ, respectively. Conditional on μ, signals are i.i.d. with $F_\mu(t) = G_{\varepsilon+\delta}(t-\mu)$ and $f_\mu(t) = g_{\varepsilon+\delta}(t-\mu)$ for all $(t, \mu) \in \mathbb{R}^2$, and let $\overline{F}_\mu \equiv 1 - F_\mu$. By definition, for all $t \in \mathbb{R}$ we have

$$F_\mu(t) = \lim_{\eta \to \infty} \frac{1}{\eta(m+n)-1} \sum_{i=1}^{\eta(m+n)-1} \mathbb{I}\{\sigma_i \leq t\}. \tag{32}$$

The proof extends Serfling (1980, Thm. 2.3.3 A) to our model in which bids/asks conditional on μ are independent but not identically distributed, for we allow here each trader i to use his own non-constant function $\lambda_i^\eta(t)$ in (30).

Let $A > 0$ be a normalizing constant to be specified later. Define

$$\begin{aligned} G_{\eta(m+n)-1}(t) &\equiv \Pr\left(\frac{\sqrt{\eta(m+n)-1}\left(\tilde{\xi}_{q[\eta(m+n)-1]} - p^{\mathrm{REE}}(\mu)\right)}{A} \leq t\right) \\ &= \Pr\left(\widetilde{F}_{\eta(m+n)-1}\left(p^{\mathrm{REE}}(\mu) + t\, A\sqrt{\eta(m+n)-1}^{\,-1}\right) \geq q\right), \end{aligned} \tag{33}$$

where the last line follows from Serfling (1980, Lem. 1.1.4 (iii)). Setting

$$\Delta^\eta \equiv p^{\mathrm{REE}}(\mu) + t\,\frac{A}{\sqrt{\eta(m+n)-1}}, \tag{34}$$

$\widetilde{F}_{\eta(m+n)-1}(\Delta^\eta)$ is a random variable with mean and variance

$$\mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta^\eta)] = \frac{1}{\eta(m+n)-1} \sum_{i=1}^{\eta(m+n)-1} F_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)), \tag{35}$$

$$\begin{aligned} \mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta^\eta)] = \frac{1}{(\eta(m+n)-1)^2} \sum_{i=1}^{\eta(m+n)-1} & F_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)) \\ & \times \overline{F}_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)). \end{aligned} \tag{36}$$

After some algebra (33) reduces to

$$G_{\eta(m+n)-1}(t) = \Pr\left(\widetilde{F}^{*}_{\eta(m+n)-1}(\Delta^{\eta}) \geq c(\Delta^{\eta})\right), \tag{37}$$

where (37) corresponds to ($\star$) in Serfling (1980, Thm. 2.3.3 A). Define

$$\widetilde{F}^{*}_{\eta(m+n)-1}(\Delta^{\eta}) \equiv \frac{\widetilde{F}_{\eta(m+n)-1}(\Delta^{\eta}) - \mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta^{\eta})]}{\sqrt{\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta^{\eta})]}} \text{ and}$$
$$c(\Delta^{\eta}) \equiv \frac{q - \mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta^{\eta})]}{\sqrt{\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta^{\eta})]}}. \tag{38}$$

For $t = 0$ the Lindeberg–Feller Central Limit Theorem (Serfling (1980, Thm. 1.9.2 A)) leads to

$$\lim_{\eta\to\infty} \Pr\left[\sqrt{\eta(m+n)-1}\left(\tilde{\xi}_{q[\eta(m+n)-1]} - p^{\mathrm{REE}}(\mu)\right) \geq 0\right] = \Phi(0) = \frac{1}{2}.$$

Using the Berry–Esseen Theorem (Serfling (Serfling 1980, Thm. 1.9.5 and p. 33)), we have

$$\sup_{t\in\mathbb{R}} \left|\Pr\left(\widetilde{F}^{*}_{\eta(m+n)-1}(\Delta^{\eta}) \leq t\right) - \Phi(t)\right| \leq K \frac{\beta(\Delta^{\eta})}{\left[\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta^{\eta})]\right]^{3/2}}, \tag{39}$$

where K is a constant and (39) corresponds to ($\star\star$) in Serfling (1980, Thm. 2.3.3 A). For $\tau \in \mathbb{R}$, define

$$\beta(\tau) \equiv \frac{1}{(\eta(m+n)-1)^3} \sum_{i=1}^{\eta(m+n)-1} \mathbb{E}\left[\left|\mathbb{I}\{\sigma_i \leq \tau - \lambda_i^{\eta}(\tau)\} - F_{\mu}\left(\tau - \lambda_i^{\eta}(\tau)\right)\right|^3\right]. \tag{40}$$

Combining (37) and (39) we have:

$$\begin{aligned}\left|\Pr\left(\widetilde{F}^{*}_{\eta(m+n)-1}(\Delta^{\eta}) \geq c(\Delta^{\eta})\right) - \Phi(t)\right| \ &\leq K \frac{\beta(\Delta^{\eta})}{\left[\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta^{\eta})]\right]^{3/2}} \\ &\quad + \left|\Phi(t) - \Phi\left(-c(\Delta^{\eta})\right)\right|. \end{aligned} \tag{41}$$

We need to show that

$$\lim_{\eta\to\infty} \frac{\beta(\Delta^{\eta})}{\left[\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta^{\eta})]\right]^{3/2}} = 0. \tag{42}$$

After some algebra for the terms inside the summation in (40), we arrive at

$$\mathbb{E}\left[\left|\mathbb{I}\{\sigma_i \le \tau - \lambda_i^\eta(\tau)\} - F_\mu\left(\tau - \lambda_i^\eta(\tau)\right)\right|^3\right]$$
$$= F_\mu\left(\tau - \lambda_i^\eta(\tau)\right)\overline{F}_\mu\left(\tau - \lambda_i^\eta(\tau)\right)\left[F_\mu^2\left(\tau - \lambda_i^\eta(\tau)\right) + \overline{F}_\mu^2\left(\tau - \lambda_i^\eta(\tau)\right)\right].$$

Substituting the above in the definition (40) for $\beta(\tau)$, the fraction in the left hand side of (42) becomes

$$\begin{aligned}
&\frac{\beta(\Delta^\eta)}{\left[\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta^\eta)]\right]^{3/2}}\\
&= \frac{\sum_{i=1}^{\eta(m+n)-1} F_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right)\overline{F}_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right)\left[F_\mu^2\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right) + \overline{F}_\mu^2\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right)\right]}{\left[\sum_{i=1}^{\eta(m+n)-1} F_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right)\overline{F}_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right)\right]^{3/2}}\\
&= \frac{1}{\sqrt{\eta(m+n)-1}}\\
&\times \frac{\frac{1}{\eta(m+n)-1}\sum_{i=1}^{\eta(m+n)-1} F_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right)\overline{F}_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right)\left[F_\mu^2\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right) + \overline{F}_\mu^2\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right)\right]}{\left[\frac{1}{\eta(m+n)-1}\sum_{i=1}^{\eta(m+n)-1} F_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right)\overline{F}_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right)\right]^{3/2}}.
\end{aligned} \tag{43}$$

In the second line, we substitute for $\beta(\Delta^\eta)$ from (40) and $\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta^\eta)]$ from (36), and in the final two lines we multiply and divide by $1/(\eta(m+n)-1)^{3/2}$. In Lemma 3 below we establish that the limit as $\eta \to \infty$ of the numerator of the fraction immediately below (43) equals $q(1-q)(q^2+(1-q)^2)$, while the limit as $\eta \to \infty$ of the term in brackets in its denominator is $q(1-q)$. With the term in (43) remaining, the limit (42) follows immediately.

To complete the proof, we need to find a constant A such that

$$\lim_{\eta\to\infty} c(\Delta^\eta) = -t, \tag{44}$$

where the dependence of Δ^η on A can be seen in (34) and $c(\Delta^\eta)$ is defined in (38).

We rewrite $c(\Delta^\eta)$ in (38) by (i) substituting $q = F_\mu(\mu + \xi_q^{\varepsilon+\delta})$ and for $\mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta^\eta)]$, $\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta^\eta)]$ from (35) and (36), and (ii) multiplying and dividing by $\sqrt{\eta(m+n)-1}\left(\Delta^\eta - p^{\mathrm{REE}}(\mu)\right)$:

$$\begin{aligned}
c(\Delta^\eta) &= -\frac{\sqrt{\eta(m+n)-1}\left(\Delta^\eta - p^{\mathrm{REE}}(\mu)\right)}{\sqrt{\frac{1}{\eta(m+n)-1}\sum_{i=1}^{\eta(m+n)-1} F_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta))\overline{F}_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta))}}\\
&\quad\times \frac{1}{\eta(m+n)-1}\,\frac{\sum_{i=1}^{\eta(m+n)-1} F_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)) - F_\mu(\mu+\xi_q^{\varepsilon+\delta})}{\Delta^\eta - p^{\mathrm{REE}}(\mu)}\\
&= -\frac{\sqrt{\eta(m+n)-1}\left(\Delta^\eta - p^{\mathrm{REE}}(\mu)\right)}{\sqrt{\frac{1}{\eta(m+n)-1}\sum_{i=1}^{\eta(m+n)-1} F_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta))\overline{F}_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta))}}\\
&\quad\times \frac{1}{\eta(m+n)-1}\sum_{i=1}^{\eta(m+n)-1}\left\{\frac{\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu+\xi_q^{\varepsilon+\delta})}{\Delta^\eta - p^{\mathrm{REE}}(\mu)}\right.
\end{aligned} \tag{45}$$

$$\left.\times \frac{F_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta}) + (\mu + \xi_q^{\varepsilon+\delta})\right) - F_\mu(\mu + \xi_q^{\varepsilon+\delta})}{\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta})}\right\}. \tag{46}$$

In the last two lines we (i) add and subtract $\mu + \xi_q^{\varepsilon+\delta}$ in the argument of $F_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta))$ and (ii) multiply and divide each term in the sum by $\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta})$.

We next reduce (45) and (46). For (45), we have from (34) that

$$\sqrt{\eta(m+n)-1}\left(\Delta^\eta - p^{\mathrm{REE}}(\mu)\right) = \sqrt{\eta(m+n)-1}\, t\, A \sqrt{\eta(m+n)-1}^{\,-1} = t\, A, \tag{47}$$

and from Lemma 3,

$$\lim_{\eta\to\infty} \sqrt{\frac{1}{\eta(m+n)-1} \sum_{i=1}^{\eta(m+n)-1} F_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)) \overline{F}_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta))} = \sqrt{q(1-q)}. \tag{48}$$

For (46), we first note from (9)–(10) that $\mu + \xi_q^{\varepsilon+\delta} = p^{\mathrm{REE}}(\mu) - \lambda^\infty$. From (34) we therefore have for each $1 \le i \le \eta\,(m+n) - 1$ that

$$\begin{aligned} \lim_{\eta\to\infty} \frac{\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta})}{\Delta^\eta - p^{\mathrm{REE}}(\mu)} &= \lim_{\eta\to\infty} \frac{\Delta^\eta - p^{\mathrm{REE}}(\mu) - (\lambda_i^\eta(\Delta^\eta) - \lambda^\infty)}{\Delta^\eta - p^{\mathrm{REE}}(\mu)} \\ = \lim_{\eta\to\infty} \frac{t\, A \sqrt{\eta(m+n)-1}^{\,-1} - (\lambda_i^\eta(\Delta^\eta) - \lambda^\infty)}{t\, A \sqrt{\eta(m+n)-1}^{\,-1}} &= \lim_{\eta\to\infty} \frac{t\, A - O\left(1/\eta^\epsilon\right)}{t\, A} = 1, \end{aligned} \tag{49}$$

where in the last equality we use (31). This is the point at which the "$+\epsilon$" in (11) and (31) is consequential. Applying (34) and (31) produces

$$\lim_{\eta\to\infty} \Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta}) = \lim_{\eta\to\infty} \Delta^\eta - p^{\mathrm{REE}}(\mu) - (\lambda_i^\eta(\Delta^\eta) - \lambda^\infty) = 0. \tag{50}$$

Because $F_\mu(t) = G_{\varepsilon+\delta}(t-\mu)$ for $t, \mu \in \mathbb{R}$ and $G_{\varepsilon+\delta}$ is differentiable with $G'_{\varepsilon+\delta} = g_{\varepsilon+\delta}$, we have

$$\begin{aligned} &\lim_{\eta\to\infty} \frac{F_\mu(\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta}) + (\mu + \xi_q^{\varepsilon+\delta})) - F_\mu(\mu + \xi_q^{\varepsilon+\delta})}{\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta})} \\ &\quad = G'_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta}) = g_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta}), \end{aligned}$$

for each $1 \le i \le \eta\,(m+n) - 1$. Combining (49)–(50) and applying Lemma 3 produces

$$\lim_{\eta\to\infty} \frac{1}{\eta(m+n)-1} \sum_{i=1}^{\eta(m+n)-1} \left\{ \frac{\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta})}{\Delta^\eta - p^{\text{REE}}(\mu)} \right.$$
$$\left. \times \frac{F_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta}) + (\mu + \xi_q^{\varepsilon+\delta})\right) - F_\mu(\mu + \xi_q^{\varepsilon+\delta})}{\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta})} \right\}$$
$$= g_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta}). \tag{51}$$

We now take the limit of $c(\Delta^\eta)$, given by the product of (45) and (46), as $\eta \to \infty$. Applying (47), (48), and (51) we obtain

$$\lim_{\eta\to\infty} c(\Delta^\eta) = -\frac{t\,A}{\sqrt{q(1-q)}} g_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta}).$$

To produce (44) we choose $A = \sqrt{q(1-q)}/g_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta})$. Using (42) and (44), we obtain from (41) that $\lim_{\eta\to\infty} \left|\Pr\left(\widetilde{F}^*_{\eta(m+n)-1}(\Delta^\eta) \geq c(\Delta^\eta)\right) - \Phi(t)\right| = 0$, which establishes the result. □

Lemma 3 *Let $(\boldsymbol{B}^\eta, \boldsymbol{S}^\eta)_{\eta\in\mathbb{N}}$ be a sequence of strategy profiles that satisfies A3. The limit as $\eta \to \infty$ of the numerator of the fraction immediately below (43) equals $q(1-q)(q^2+(1-q)^2)$, the limit as $\eta \to \infty$ of the term in brackets in the denominator of this fraction is $q(1-q)$, and the limits (48) and (51) hold.*

Proof. The essential issue in proving the four limits is incorporating the uniform convergence across the traders' offsets (indexed by i) in (31) to their limit as $\eta \to \infty$. We prove here the second limit in the lemma, i.e., as $\eta \to \infty$, the term in brackets in the denominator below (43) converges to $q(1-q)$. This argument clarifies how to address the uniform convergence and makes the proof of the other three limits straightforward.

The first step is to prove that there exists for each $t \in \mathbb{R}$ a constant $\Lambda(t, A, G_\varepsilon, G_\delta, m, n)$ such that

$$\left|\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta})\right| < \frac{\Lambda(t, A, G_\varepsilon, G_\delta, m, n)}{\sqrt{\eta(m+n)-1}} \tag{52}$$

for all $\eta \in \mathbb{N}$ and $1 \leq i \leq \eta(m+n)-1$. We have

$$\begin{aligned}
&\left|\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (\mu + \xi_q^{\varepsilon+\delta})\right| \\
&= \left|\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - \left(\mu + V\left(\xi_q^{\varepsilon+\delta}\right) - V\left(\xi_q^{\varepsilon+\delta}\right) + \xi_q^{\varepsilon+\delta}\right)\right| \\
&= \left|\Delta^\eta - \lambda_i^\eta(\Delta^\eta) - (p^{\text{REE}}(\mu) - \lambda^\infty)\right| = \left|\Delta^\eta - p^{\text{REE}}(\mu) - (\lambda_i^\eta(\Delta^\eta) - \lambda^\infty)\right| \\
&\leq \left|\Delta^\eta - p^{\text{REE}}(\mu)\right| + \left|\lambda_i^\eta(\Delta^\eta) - \lambda^\infty\right| < \left|\Delta^\eta - p^{\text{REE}}(\mu)\right| + \frac{K(G_\varepsilon, G_\delta, m, n)}{\sqrt{\eta}}
\end{aligned}$$

$$= \left| t \frac{A}{\sqrt{\eta(m+n)-1}} \right| + \frac{K(G_\varepsilon, G_\delta, m, n)}{\sqrt{\eta}} < \frac{\Lambda(t, A, G_\varepsilon, G_\delta, m, n)}{\sqrt{\eta(m+n)-1}}.$$

The second line applies formulas (9) and (10) for $p^{\text{REE}}(\mu)$ and λ^∞ and the third line applies the bound (31). Definition (34) of Δ^η is substituted in line four, and the last inequality results by choosing an appropriate constant Λ independent of i. Recall that

$$F_\mu(\mu + \xi_q^{\varepsilon+\delta}) = G_{\varepsilon+\delta}(\mu + \xi_q^{\varepsilon+\delta} - \mu) = G_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta}) = q).$$

Turning to the second limit in the statement of the lemma, we wish to show that

$$\lim_{\eta\to\infty} \frac{1}{\eta(m+n)-1} \sum_{i=1}^{\eta(m+n)-1} F_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right) \overline{F}_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right) = q(1-q). \tag{53}$$

Recall from definition (34) of Δ^η that it is a function of $t \in \mathbb{R}$. For each t,

$$\begin{aligned} &\left| \frac{1}{\eta(m+n)-1} \cdot \left(\sum_{i=1}^{\eta(m+n)-1} F_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right) \overline{F}_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right) \right) - q(1-q) \right| \\ &\le \frac{1}{\eta(m+n)-1} \sum_{i=1}^{\eta(m+n)-1} \left| F_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right) \overline{F}_\mu\left(\Delta^\eta - \lambda_i^\eta(\Delta^\eta)\right) - q(1-q) \right| \\ &\le \frac{1}{\eta(m+n)-1} \cdot (\eta(m+n)-1) \cdot \sup_{z\in B^\eta} \left| F_\mu(z)\overline{F}_\mu(z) - F_\mu(\mu + \xi_q^{\varepsilon+\delta})\overline{F}_\mu(\mu + \xi_q^{\varepsilon+\delta}) \right| \\ &= \sup_{z\in B^\eta} \left| F_\mu(z)\overline{F}_\mu(z) - F_\mu(\mu + \xi_q^{\varepsilon+\delta})\overline{F}_\mu(\mu + \xi_q^{\varepsilon+\delta}) \right|. \end{aligned} \tag{54}$$

Here, B^η is the closed ball of radius $\Lambda(t, A, G_\varepsilon, G_\delta, m, n)/\sqrt{\eta(m+n)-1}$, i.e., the bound in (52), centered at $\mu + \xi_q^{\varepsilon+\delta}$. Continuity of F_μ implies that the limit of (54) is zero, which implies (53). ◻

B Proof of Theorem 2

We prove Theorem 2 by deriving the solution to the AFOC for a buyer's strategy in the private values case, where we need only consider the strategic term in (7) as the price-taking term reduces to $\sigma_B = v$. Comparing (21) in Theorem 3 to (17), the asymptotic strategic term in the general model has the same form as in the private values case, with $g_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta})$ replacing $g_\varepsilon(\xi_q^\varepsilon)$. Therefore, in Lemma 4 below we state the form of the asymptotic strategic term in the general model with Theorem 2 then following as a corollary.

Lemma 4 *For fixed m and n, consider a sequence of markets indexed by the market size η. Let $(\boldsymbol{B}^\eta, \boldsymbol{S}^\eta)_{\eta\in\mathbb{N}}$ be a sequence of strategy profiles that satisfies A3. Faced with the strategies of the others in this sequence, suppose a buyer considers strategies $\widehat{B}(\cdot;\eta)$ in which the difference $\left|\widehat{B}(v;\sigma) - \sigma - \lambda^\infty\right|$ between his offset $\widehat{B}(v;\sigma) - \sigma$*

and the equilibrium offset λ^{∞} *of the limit market is* $O\left(1/\eta^{\epsilon}\right)$ *for some* $\epsilon > 0$ *and all* $\sigma \in \mathbb{R}$. *Then the buyer's asymptotic strategic term is*

$$\begin{aligned}\frac{\widetilde{\Pr}[x^{\eta} < b < y^{\eta}|\sigma_B]}{\widetilde{f}_x^B(b|\sigma_B)} &\equiv \widetilde{e}_{w|\mu} + O\left(\frac{1}{\eta^2}\right) \\ &= -\frac{1}{(\eta(m+n)-1)\, g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)} + O\left(\frac{1}{\eta^2}\right) \end{aligned} \tag{55}$$

for any $(b, \sigma_B) \in \mathbb{R}^2$ *such that* $b = \widehat{B}(\sigma_B; \eta)$.

Lemma 4 is proven after the proof of Theorem 2.

Proof of Theorem 2. The FOC for a buyer in the private values case is

$$(v-b)\cdot f_{x|v}^{B}(b|v) - \Pr[x^{\eta} < b < y^{\eta}|v] = 0, \tag{56}$$

where we have substituted $b = \sigma_B + \lambda_B^{\eta}$ and $\sigma_B = v$ in (5). Substituting the asymptotic distributions of $x^{\eta}(\mu)$, $y^{\eta}(\mu)$, and $w^{\eta}(\mu)$ from Theorem 1 into (56) results in

$$(v-b)\cdot \widetilde{f}_x^B(b|v) - \widetilde{\Pr}[x^{\eta} < b < y^{\eta}|v] = 0 \Leftrightarrow v - b - \frac{\widetilde{\Pr}[x^{\eta} < b < y^{\eta}|v]}{\widetilde{f}_x^B(b|v)} = 0. \tag{57}$$

Here, "$\sim$" denotes the use of the asymptotic distributions to calculate the density and probability, and the equivalence follows from $\widetilde{f}_x^B(b|v) > 0$. We substitute for the strategic term in (57) from (55) in Lemma 4, as applied to the private values case. The unique solution is the asymptotic offset $\widetilde{\lambda}_B^{\eta} = v - b$ given by (17). □

Proof of Lemma 4. For a market of size η, we want to express the probability that a buyer sets the price conditional on his signal σ_B in terms of the distributions of $w^{\eta}(\mu) = y^{\eta}(\mu) - x^{\eta}(\mu)$ and $x^{\eta}(\mu)$ (henceforth simply w, y and x). Our notational convention for density functions is illustrated by $f_{xw|\sigma}(x, w|\sigma_B)$, which denotes the joint density of x and w conditional on $\sigma = \sigma_B$, where the market size η is implicit in w and x. We have

$$\begin{aligned} &\Pr[x < b < y|\sigma_B] \\ &\quad = \Pr[0 < b - x < w|\sigma_B] = \int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{xw|\sigma}\left(x, w|\sigma_B\right) dw dx \\ &\quad = \int_{\mu=-\infty}^{\infty}\int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{xw|\mu,v}\left(x, w|\mu, \sigma_B\right) f_{\mu|\sigma}\left(\mu|\sigma_B\right) dw dx d\mu, \end{aligned} \tag{58}$$

where the first equality applies the definition of w and the third equality introduces μ into the marginal by integrating over $\mu \in \mathbb{R}$. Signals are independent conditional on μ, and so the order statistics of bids/asks from the other traders are also independent of the signal σ_B of the buyer. We can therefore write $f_{xw|\mu,\sigma}(x, w|\mu, \sigma_B) = f_{xw|\mu}(x, w|\mu) = f_{w|x,\mu}(w|x, \mu; \eta)\, f_{x|\mu}(x|\mu)$ in (58) so that

$$\begin{aligned}&\Pr[x < b < y|\sigma_B]\\&\quad = \int_{\mu=-\infty}^{\infty} \underbrace{\int_{x=-\infty}^{b} \overline{F}_{w|x,\mu}(b - x|x, \mu)\, f_{x|\mu}(x|\mu)\, dx}_{\Pr[x<b<y|\mu]}\, f_{\mu|\sigma}(\mu|\sigma_B)\, d\mu,\end{aligned} \tag{59}$$

where the equality follows using the right-hand distribution function of w, $\overline{F}_{w|x,\mu}(\cdot|x, \mu) \equiv 1 - F_{w|x,\mu}(\cdot|x, \mu)$.

We now substitute into the inner integral of (59)—which as noted above is equal to the probability that the buyer sets the price conditional on the state μ—the asymptotic distributions of w and x as given by Theorem (1). In this sense we compute the asymptotic probability of being pivotal for the buyer conditional on μ. We again let "$\sim$" denote an asymptotic distribution. From (14) we have

$$\widetilde{\overline{F}}_{w|x,\mu}(t|x, \mu) = \widetilde{\overline{F}}_{w|\mu}(t|\mu) = e^{-t/\widetilde{e}_{w|\mu}}, \tag{60}$$

for $t \in \mathbb{R}^+$, where

$$\widetilde{e}_{w|\mu} = \frac{1}{(\eta(m+n) - 1)\, g_{\varepsilon+\delta}\left(\xi_q^{\varepsilon+\delta}\right)}. \tag{61}$$

The first equality in (60) follows because conditional on μ, w is asymptotically independent from x. From (13) we have

$$\widetilde{f}_{x|\mu}(t|\mu) = \frac{1}{\sqrt{2\pi \widetilde{\upsilon}_{x|\mu}}} e^{-\left(t - p^{\mathrm{REE}}(\mu)\right)^2/(2\widetilde{\upsilon}_{x|\mu})} \tag{62}$$

for $t \in \mathbb{R}$, where $p^{\mathrm{REE}}(\mu) = \mu + V\left(\xi_q^{\varepsilon+\delta}\right)$ as in (9), and

$$\widetilde{\upsilon}_{x|\mu} = \frac{mn}{(\eta(m+n) - 1)\,(m+n)^2} \frac{1}{g_{\varepsilon+\delta}^2(\xi_q^{\varepsilon+\delta})}. \tag{63}$$

Substituting (60) and (62) in the inner integral in (59), we write the asymptotic probability that the buyer sets the price conditional on μ as

$$\begin{aligned}\widetilde{\Pr}[x < b < y|\mu] &= \int_{-\infty}^{b} \widetilde{\overline{F}}_{w|x,\mu}(b - x|x, \mu)\, \widetilde{f}_{x|\mu}(x|\mu)\, dx\\&= \int_{-\infty}^{b} e^{-(b-x)/\widetilde{e}_{w|\mu}} \frac{1}{\sqrt{2\pi \widetilde{\upsilon}_{x|\mu}}} e^{-\left(x - p^{\mathrm{REE}}(\mu)\right)^2/(2\widetilde{\upsilon}_{x|\mu})} dx.\end{aligned} \tag{64}$$

Although the integral (64) is available in closed form, we take a different route to get a simpler expression through approximation. We first change the variable of integration in (64) to $z = b - x$:

$$\begin{aligned}\widetilde{\Pr}[x < b < y|\mu] &= \int_0^{+\infty} \frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} e^{-z/\widetilde{e}_{w|\mu} - \left(b - z - p^{\mathrm{REE}}(\mu)\right)^2/(2\widetilde{\upsilon}_{x|\mu})} dz \\ &= \frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} \int_0^{+\infty} \Big(\underbrace{e^{-z/(\eta\widetilde{e}_{w|\mu}) - \left(b - z - p^{\mathrm{REE}}(\mu)\right)^2/(2\eta\widetilde{\upsilon}_{x|\mu})}}_{R(z)}\Big)^{\eta} dz.\end{aligned} \tag{65}$$

In the second line we multiply and divide the exponent by η. We now apply (Fibich and Gavious 2010, Lem. 2) to approximate the integral in (65): for $R(z)$ equal to the indicated term in (65), we have

$$\int_0^{\infty} R^{\eta}(z) dz = -\frac{1}{\eta} \frac{R^{\eta+1}(0)}{R'(0)} \left[1 + O\left(\frac{1}{\eta}\right)\right].$$

After some algebra, (65) becomes

$$\begin{aligned}\widetilde{\Pr}[x < b < y|\mu] = \widetilde{e}_{w|\mu} &\frac{1}{1 - (b - p^{\mathrm{REE}}(\mu))\,\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}} \\ &\times \underbrace{\frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} e^{-\left(b - p^{\mathrm{REE}}(\mu)\right)^2/(2\widetilde{\upsilon}_{x|\mu})}}_{\widetilde{f}_{x|\mu}(b|\mu)} \left[1 + O\left(\frac{1}{\eta}\right)\right],\end{aligned} \tag{66}$$

where we annotate a term as $\widetilde{f}_{x|\mu}(b|\mu)$ that corresponds to the asymptotic density of x conditional on μ (see (69)). Substituting the expression for $\widetilde{\Pr}[x < b < y|\mu]$ given by (66) in (59) produces the asymptotic probability that the buyer sets the price conditional on σ_B:

$$\begin{aligned}&\widetilde{\Pr}[x < b < y|\sigma_B] \\ &= \left[1 + O\left(\frac{1}{\eta}\right)\right] \widetilde{e}_{w|\mu} \int_{-\infty}^{\infty} \frac{\frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} e^{-\left(b - p^{\mathrm{REE}}(\mu)\right)^2/(2\widetilde{\upsilon}_{x|\mu})}}{1 - (b - p^{\mathrm{REE}}(\mu))\,\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}} f_{\mu|\sigma}(\mu|\sigma_B) d\mu. \\ &= \left[1 + O\left(\frac{1}{\eta}\right)\right] \widetilde{e}_{w|\mu} \int_{-\infty}^{\infty} \frac{\frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} e^{-\left(b - p^{\mathrm{REE}}(\mu)\right)^2/(2\widetilde{\upsilon}_{x|\mu})}}{1 - (b - p^{\mathrm{REE}}(\mu))\,\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}} g_{\varepsilon+\delta}(\sigma_B - \mu) d\mu. \\ &= \left[1 + O\left(\frac{1}{\eta}\right)\right] \widetilde{e}_{w|\mu} \int_{-\infty}^{\infty} \frac{\frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})}}{1 - \alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}} g_{\varepsilon+\delta}\left(\widehat{\lambda} + \xi_q^{\varepsilon+\delta} + \alpha\right) d\alpha.\end{aligned} \tag{67}$$

The second line follows because $f_{\mu|\sigma}(\mu|\sigma_B) = g_{\varepsilon+\delta}(\sigma_B - \mu)$ due to the uniform improper prior assumption on μ. The last line follows by: (i) changing the variable of integration to $\alpha = b - p^{\mathrm{REE}}(\mu) = b - \mu - V\left(\xi_q^{\varepsilon+\delta}\right)$ using (9); (ii) defining $\widehat{\lambda} \equiv \lambda^{\infty} - (b - \sigma_B)$, with $\lambda^{\infty} \equiv V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta}$ (i.e., the limit market offset (10)). Note that we do not assume that the difference $b - \sigma_B$ is a constant offset. Moreover, observe from (61) that $\widetilde{e}_{w|\mu}$ is $O(1/\eta)$ and so $[1 + O(1/\eta)]\,\widetilde{e}_{w|\mu} = \widetilde{e}_{w|\mu} + O\left(1/\eta^2\right)$.

Even for a particular choice of the density $g_{\varepsilon+\delta}$, the integral in (67) is still not computable in closed form due to the term $\left(1 - \alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^{-1}$. In order to proceed we take a Taylor's series expansion of $\left(1 - \alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^{-1}$ around zero,

$$\frac{1}{1 - \alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}} = 1 + \sum_{i=1}^{\infty} \left(\alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^{i}.$$

Substituting in the integral of (67), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{1 - \alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}} \frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})} g_{\varepsilon+\delta}\left(\widehat{\lambda} + \xi_q^{\varepsilon+\delta} + \alpha\right) d\alpha \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})} g_{\varepsilon+\delta}\left(\widehat{\lambda} + \xi_q^{\varepsilon+\delta} + \alpha\right) d\alpha \\
&\quad + \int_{-\infty}^{\infty} \sum_{i=1}^{\infty} \left(\alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^{i} \frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})} g_{\varepsilon+\delta}\left(\widehat{\lambda} + \xi_q^{\varepsilon+\delta} + \alpha\right) d\alpha.
\end{aligned} \tag{68}$$

Using the asymptotic density of x given μ, we write the asymptotic density of x given the signal σ of the buyer as

$$\begin{aligned}
\widetilde{f}_{x|\sigma}(t|\sigma_B) &= \int_{-\infty}^{\infty} \widetilde{f}_{x|\mu}(t|\mu)\, f_{\mu|\sigma}(\mu|\sigma_B) d\mu \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} e^{-\left(t - p^{\mathrm{REE}}(\mu)\right)^2/(2\widetilde{\upsilon}_{x|\mu})} g_{\varepsilon+\delta}(\sigma_B - \mu) d\mu,
\end{aligned} \tag{69}$$

where the second equality applies (60). This is the term in the integral in the second line of (68) (which follows by changing the variable of integration to $\alpha = t - p^{\mathrm{REE}}(\mu)$). Substituting (68) back to (67) using (69) produces the following expression for the asymptotic probability that the buyer sets the price conditional on his signal:

$$\begin{aligned}
\widetilde{\Pr}[x < b < y|\sigma_B] &= \left[\widetilde{e}_{w|\mu} + O\left(\frac{1}{\eta^2}\right)\right] \widetilde{f}_{x|\sigma}(b|\sigma_B) \\
&+ \left[\widetilde{e}_{w|\mu} + O\left(\frac{1}{\eta^2}\right)\right] \int_{-\infty}^{\infty} \sum_{i=1}^{\infty} \left(\alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^{i} \frac{e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})}}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} g_{\varepsilon+\delta}\left(\widehat{\lambda} + \xi_q^{\varepsilon+\delta} + \alpha\right) d\alpha.
\end{aligned} \tag{70}$$

Our goal now is to show that the integral in (70) is $O(1/\eta)\widetilde{f}_{x|\sigma}(b|\sigma_B)$. This will lead to a simple expression for the asymptotic probability that the buyer sets the price. If

$$\int_{-\infty}^{\infty}\sum_{i=1}^{\infty}\left(\alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^i \frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}}e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})}g_{\varepsilon+\delta}\left(\widehat{\lambda}+\xi_q^{\varepsilon+\delta}+\alpha\right)d\alpha$$
$$= O\left(\frac{1}{\eta}\right)\widetilde{f}_{x|\sigma}(b|\sigma_B), \tag{71}$$

then the asymptotic probability (70) that the buyer sets the price becomes

$$\begin{aligned}\widetilde{\Pr}[x < b < y|\sigma_B] &= \left[\widetilde{e}_{w|\mu} + O\left(\frac{1}{\eta^2}\right)\right]\widetilde{f}_{x|\sigma}(b|\sigma_B)\\ &\quad + \left[\widetilde{e}_{w|\mu} + O\left(\frac{1}{\eta^2}\right)\right]O\left(\frac{1}{\eta}\right)\widetilde{f}_{x|\sigma}(b|\sigma_B)\\ &= \left[\widetilde{e}_{w|\mu} + O\left(\frac{1}{\eta^2}\right)\right]\widetilde{f}_{x|\sigma}(b|\sigma_B),\end{aligned}$$

where in the second line we again use that $\widetilde{e}_{w|\mu}$ is $O(1/\eta)$ (see (61)). Dividing both sides above by the density $\widetilde{f}_{x|\sigma}(b|\sigma_B)$, we get (55).

In the remainder, we first prove that (71) holds for $g_{\varepsilon+\delta}$ normal and then extend it to the case of mixture of normals. The normal and mixtures of normals allow us to compute the integral in (70) in closed form and thus makes it possible to establish (71). In particular, consider the density $g_{\varepsilon+\delta}(t) = \sum_{k=1}^K w_k\phi(t; m_k, \upsilon_k)$, $t \in \mathbb{R}$, with $w_k > 0$, $\sum_{k=1}^K w_k = 1$. Let $\phi_k(t) \equiv \phi(t; m_k, \upsilon_k)$ for $t \in \mathbb{R}$, the density of a $\mathcal{N}(m_k, \upsilon_k)$ random variable. Mixtures of normals approximate arbitrarily closely any continuous density in a variety of different norms, including L^1 (McLachlan and Peel 2000); as the integral in (71) is continuous in the density $g_{\varepsilon+\delta}$ in the L^1 norm, the expression that we derive in the mixtures of normals case thus holds generally for all choices of the continuous density function $g_{\varepsilon+\delta}$ (whose continuity follows from A1).

Finally, recall that we assume that the buyer considers a sequence of strategies $\left(\widehat{B}(\cdot;\eta)\right)_{\eta\in\mathbb{N}}$ such that $\left|\left(\widehat{B}(\sigma;\eta)-\sigma\right)-\lambda^{\infty}\right|$ is $O\left(1/\eta^{\epsilon}\right)$ for some $\epsilon > 0$ and all $\sigma \in \mathbb{R}$. Let $\widehat{\lambda}(\sigma;\eta) \equiv \left(\widehat{B}(\sigma;\eta)-\sigma\right) - \lambda^{\infty}$. In what follows, we suppress the dependence of $\widehat{\lambda}(\sigma;\eta)$ on σ and η and write it simply as $\widehat{\lambda}$.

Normal distribution case ($K = 1$). We assume here that $g_{\varepsilon+\delta} = \phi_k$. Substituting in the integral in (70) yields

$$\int_{-\infty}^{\infty}\sum_{i=1}^{\infty}\left(\alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^i \frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}}e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})}$$
$$\frac{1}{\sqrt{2\pi\upsilon_k}}\exp\left(-\frac{(\widehat{\lambda}+\xi_q^{\varepsilon+\delta}+\alpha-m_k)^2}{2\upsilon_k}\right)d\alpha. \tag{72}$$

Focusing on the first three terms of the infinite sum, we have

$$\int_{-\infty}^{\infty} \alpha \widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu} \frac{1}{\sqrt{2\pi \widetilde{\upsilon}_{x|\mu}}} e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})} \frac{1}{\sqrt{2\pi \upsilon_k}} \exp\left(-\frac{(\widehat{\lambda}+\xi_q^{\varepsilon+\delta}+\alpha-m_k)^2}{2\upsilon_k}\right) d\alpha \tag{73}$$

$$+\int_{-\infty}^{\infty} \left[\alpha \widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right]^2 \frac{1}{\sqrt{2\pi \widetilde{\upsilon}_{x|\mu}}} e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})} \frac{1}{\sqrt{2\pi \upsilon_k}} \exp\left(-\frac{(\widehat{\lambda}+\xi_q^{\varepsilon+\delta}+\alpha-m_k)^2}{2\upsilon_k}\right) d\alpha \tag{74}$$

$$+\int_{-\infty}^{\infty} \left[\alpha \widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right]^3 \frac{1}{\sqrt{2\pi \widetilde{\upsilon}_{x|\mu}}} e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})} \frac{1}{\sqrt{2\pi \upsilon_k}} \exp\left(-\frac{(\widehat{\lambda}+\xi_q^{\varepsilon+\delta}+\alpha-m_k)^2}{2\upsilon_k}\right) d\alpha. \tag{75}$$

Observe that when $g_{\varepsilon+\delta} = \phi_k$ equation (69) implies

$$\begin{aligned}\widetilde{f}_{x|\sigma}(b|\sigma_B) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \widetilde{\upsilon}_{x|\mu}}} \exp\left(-\frac{\left(b-p^{\mathrm{REE}}(\mu)\right)^2}{2\widetilde{\upsilon}_{x|\mu}}\right) \frac{1}{\sqrt{2\pi \upsilon_k}} \exp\left(-\frac{(\mu-\sigma_B-m_k)^2}{2\upsilon_k}\right) d\mu \\ &= \frac{1}{\sqrt{2\pi\left(\upsilon_k+\widetilde{\upsilon}_{x|\mu}\right)}} \exp\left(-\frac{(\widehat{\lambda}+\xi_q^{\varepsilon+\delta}-m_k)^2}{2\left(\upsilon_k+\widetilde{\upsilon}_{x|\mu}\right)}\right),\end{aligned} \tag{76}$$

that is, the asymptotic density of x given the signal $\sigma = \sigma_B$ is also normal with mean $m_k - \sigma_B - V\left(\xi_q^{\varepsilon+\delta}\right)$ and variance $\upsilon_k + \widetilde{\upsilon}_{x|\mu}$. In the last equality above we substitute for $\widehat{\lambda} = \lambda^\infty - (b-\sigma_B) = V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta} - (b-\sigma_B)$.

Let I_1, I_2, I_3 denote the integrals in (73), (74), (75), respectively. These integrals are easily computed in closed form. Using formula (76) for $\widetilde{f}_{x|\sigma}(b|\sigma_B)$ in the normal case we have

$$I_1 = -\widetilde{f}_{x|\sigma}(b|\sigma_B)\left(\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right) \frac{\widetilde{\upsilon}_{x|\mu}}{\upsilon_k+\widetilde{\upsilon}_{x|\mu}} (\widehat{\lambda}+\xi_q^{\varepsilon+\delta}-m_k), \tag{77}$$

$$I_2 = \widetilde{f}_{x|\sigma}(b|\sigma_B)\left(\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^2 \frac{\widetilde{\upsilon}_{x|\mu}}{\left(\upsilon_k+\widetilde{\upsilon}_{x|\mu}\right)^2} \left(1+\widetilde{\upsilon}_{x|\mu}\left(1+(\widehat{\lambda}+\xi_q^{\varepsilon+\delta}-m_k)^2\right)\right), \tag{78}$$

$$I_3 = -\widetilde{f}_{x|\sigma}(b|\sigma_B)\left(\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^3 \frac{\left(\widetilde{\upsilon}_{x|\mu}\right)^2}{\left(\upsilon_k+\widetilde{\upsilon}_{x|\mu}\right)^3} \cdot \left(3+\widetilde{\upsilon}_{x|\mu}\left(3+(\widehat{\lambda}+\xi_q^{\varepsilon+\delta}-m_k)^2\right)\right)$$

$$(\widehat{\lambda} + \xi_q^{\varepsilon+\delta} - m_k). \tag{79}$$

Notice that $\widetilde{f}_{x|\sigma}(b|\sigma_B)$ is a common factor in I_1, I_2, and I_3. These terms differ in their exponents on the factors $\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}$, $\widetilde{\upsilon}_{x|\mu}$ and $\widehat{\lambda} + \xi_q^{\varepsilon+\delta} - m_k$. To derive a simple formula for the buyer's strategy, we focus on the dependence of these terms on η. Using (61) and (63) we have

$$\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu} = \frac{g_{\varepsilon+\delta}(\xi_q^{\varepsilon+\delta})(m+n)^2}{mn},$$

which is a constant that does not depend on η. Further, again from (63) we see that $\widetilde{\upsilon}_{x|\mu}$ is $O(1/\eta)$. We also have that $\widetilde{\upsilon}_{x|\mu}/\left(1+\widetilde{\upsilon}_{x|\mu}\right)^2$ is $O(1/\eta)$ and $\left(\widetilde{\upsilon}_{x|\mu}\right)^2/\left(1+\widetilde{\upsilon}_{x|\mu}\right)^3$ is $O(1/\eta^2)$.

We assumed that the buyer restricts attention to $\widehat{\lambda}$ that is $O(1/\eta^{\epsilon})$ for some $\epsilon > 0$. It is then easy to see from (77)–(79) for I_1, I_2, I_3, that

$$I_1 = -\widetilde{f}_{x|\sigma}(b|\sigma_B) O\left(\frac{1}{\eta}\right),\ I_2 = \widetilde{f}_{x|\sigma}(b|\sigma_B) O\left(\frac{1}{\eta}\right),\ I_3 = -\widetilde{f}_{x|\sigma}(b|\sigma_B) O\left(\frac{1}{\eta^2}\right).$$

Similarly, if we compute the integrals corresponding to the higher terms (i.e., I_n for $n > 3$) in the series expansion of $\left(1 - \alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^{-1}$ we obtain expressions of the form $\widetilde{f}_{x|\sigma}(b|\sigma_B) O\left(1/\eta^{\kappa}\right)$ for $\kappa \geq 2$. Summing the I_n's, the integral in (72) is therefore $I_1 + I_2 + I_3 + \cdots = O\left(1/\eta\right) \widetilde{f}_{x|\sigma}(b|\sigma_B)$. This means that (71) is satisfied in the normal distribution case.

Mixture of normals ($K > 1$). Consider a nondegenerate mixture of normals $g_{\varepsilon+\delta} = \sum_{k=1}^{K} w_k\phi_k$. We substitute in the integral in (70) to obtain $g_{\varepsilon+\delta}\left(\widehat{\lambda} + \xi_q^{\varepsilon+\delta} + \alpha\right) = \sum_{k=1}^{K} w_k\phi_k(\widehat{\lambda} + \xi_q^{\varepsilon+\delta} + \alpha)$.

We proved above that (71) holds for each ϕ_k, i.e.,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \sum_{i=1}^{\infty} \left(\alpha\widetilde{e}_{w|\mu}/\widetilde{\upsilon}_{x|\mu}\right)^i \frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} e^{-\alpha^2/(2\widetilde{\upsilon}_{x|\mu})} \\
&\quad \frac{1}{\sqrt{2\pi\upsilon_k}} \exp\left(-\frac{(\widehat{\lambda} + \xi_q^{\varepsilon+\delta} + \alpha - m_k)^2}{2\upsilon_k}\right) d\alpha \\
&= O\left(\frac{1}{\eta}\right) \frac{1}{\sqrt{2\pi\left(\upsilon_k + \widetilde{\upsilon}_{x|\mu}\right)}} \exp\left(-\frac{(\widehat{\lambda} + \xi_q^{\varepsilon+\delta} - m_k)^2}{2\left(\upsilon_k + \widetilde{\upsilon}_{x|\mu}\right)}\right).
\end{aligned} \tag{80}$$

Furthermore, for the mixture of normals case from (69),

$$\widetilde{f}_{x|\sigma}(b|\sigma_B) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\widetilde{\upsilon}_{x|\mu}}} \exp\left(-\frac{\left(b - p^{\mathrm{REE}}(\mu)\right)^2}{2\widetilde{\upsilon}_{x|\mu}}\right)$$

$$
\begin{aligned}
&\sum_{k=1}^{K} w_k \frac{1}{\sqrt{2\pi \upsilon_k}} \exp\left(-\frac{(\mu-\sigma_B-m_k)^2}{2\upsilon_k}\right) d\mu \\
&= \sum_{k=1}^{K} w_k \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \widetilde{\upsilon}_{x|\mu}}} \exp\left(-\frac{\left(b-p^{\mathrm{REE}}(\mu)\right)^2}{2\widetilde{\upsilon}_{x|\mu}}\right) \\
&\quad \frac{1}{\sqrt{2\pi \upsilon_k}} \exp\left(-\frac{(\mu-\sigma_B-m_k)^2}{2\upsilon_k}\right) d\mu \\
&= \sum_{k=1}^{K} w_k \frac{1}{\sqrt{2\pi\left(\upsilon_k+\widetilde{\upsilon}_{x|\mu}\right)}} \exp\left(-\frac{(\widehat{\lambda}+\xi_q^{\varepsilon+\delta}-m_k)^2}{2\left(\upsilon_k+\widetilde{\upsilon}_{x|\mu}\right)}\right). \qquad (81)
\end{aligned}
$$

Multiplying both sides of (80) by w_k and summing across $k=\{1,\ldots,K\}$ using (81) shows that (71) is satisfied also for a mixture of normals. □

C Proof of Theorem 3

We begin with the asymptotic problem of a buyer and then proceed to that of a seller. **Buyer.** From 5), the FOC for a buyer is

$$
(\mathbb{E}[v|\sigma_B, x=b]-b)\cdot f^B_{x|\sigma}(b|\sigma_B) - \Pr[x<b<y|\sigma_B] = 0, \qquad (82)
$$

where we substitute b for $\sigma_B+\lambda^\eta_B$ and omit the dependence of x and y on η. As shown in (Satterthwaite, Williams, and Zachariadis 2020, sec. B.1), the price-taking term of a buyer can be expressed as

$$
\begin{aligned}
\mathbb{E}[v|\sigma_B, x=b] &= \frac{\int \mathbb{E}[v|\mu,\sigma_B] f^B_{x|\mu}(b|\mu) g_{\varepsilon+\delta}(\sigma_B-\mu)d\mu}{f^B_{x|\sigma}(b|\sigma_B)} \\
&= \frac{\int \mathbb{E}[v|\mu,\sigma_B] f^B_{x|\mu}(b|\mu) g_{\varepsilon+\delta}(\sigma_B-\mu)d\mu}{\int f^B_{x|\mu}(b|\mu) g_{\varepsilon+\delta}(\sigma_B-\mu)d\mu}, \qquad (83)
\end{aligned}
$$

where the second equality follows because σ and x are independent conditional on μ. Substituting in (83) for the density of x conditional on μ with its asymptotic counterpart given by Theorem 1 produces the asymptotic price-taking term, denoted by a "$\sim$":

$$
\widetilde{\mathbb{E}}[v|\sigma_B, x=b] = \frac{\int \mathbb{E}[v|\mu,\sigma_B] \widetilde{f}_{x|\mu}(b|\mu) g_{\varepsilon+\delta}(\sigma_B-\mu)d\mu}{\widetilde{f}_x(b|\sigma_B)}. \qquad (84)
$$

Here, $\widetilde{f}_x(b|\sigma_B) = \int \widetilde{f}_{x|\mu}(b|\mu) g_{\varepsilon+\delta}(\sigma_B-\mu)d\mu$ is the asymptotic density of x conditional on σ, which is the same function for a buyer as for a seller (since $\widetilde{f}_{x|\mu}$ is the same). We thus drop the superscript B.

To proceed with the calculation of (84), we consider $\varepsilon \sim \mathcal{N}(0, \upsilon_\varepsilon)$ and $\delta \sim \mathcal{N}(0, \upsilon_\delta)$ so that $\varepsilon + \delta \sim \mathcal{N}(0, \upsilon_\varepsilon + \upsilon_\delta)$, that is

$$g_{\varepsilon+\delta}(t) = \frac{1}{\sqrt{2\pi\,(\upsilon_\varepsilon + \upsilon_\delta)}} \exp\left(\frac{-t^2}{2\,(\upsilon_\varepsilon + \upsilon_\delta)}\right) \tag{85}$$

for all $t \in \mathbb{R}$. From standard results with normal random variables we have

$$\mathbb{E}[v|\mu, \sigma] = \mathbb{E}[c|\mu, \sigma] = \frac{\upsilon_\varepsilon \sigma + \upsilon_\delta \mu}{\upsilon_\varepsilon + \upsilon_\delta} \tag{86}$$

for all $\mu, \sigma \in \mathbb{R}$. Moreover, from Theorem 1 we have

$$\widetilde{f}_{x|\mu}(t|\mu) = \frac{1}{\sqrt{2\pi \widetilde{\upsilon}_{x|\mu}}} \exp\left(\frac{-\left(t - p^{\mathrm{REE}}(\mu)\right)^2}{2\widetilde{\upsilon}_{x|\mu}}\right) \tag{87}$$

for all $t, \mu \in \mathbb{R}$, where $p^{\mathrm{REE}}(\mu) = \mu + V\left(\xi_q^{\varepsilon+\delta}\right)$ as in (9) and in this normal-normal case $V(t) = t\upsilon_\varepsilon/(\upsilon_\varepsilon + \upsilon_\delta)$ for all $t \in \mathbb{R}$.

Substituting the expressions for $\mathbb{E}[v|\mu, \sigma_B]$, $\widetilde{f}_{x|\mu}(b|\mu)$ and $g_{\varepsilon+\delta}(\sigma_B - \mu)$ from (86), (87) and (85), we calculate the numerator of (84) in closed form,

$$\begin{aligned} &\int \mathbb{E}[v|\mu, \sigma_B]\widetilde{f}_{x|\mu}(b|\mu) g_{\epsilon+\delta}(\sigma_B - \mu) d\mu \\ &\quad = \frac{\sigma_B(\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon) + \left(b - V\left(\xi_q^{\varepsilon+\delta}\right)\right)\upsilon_\delta}{\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon + \upsilon_\delta} \widetilde{f}_x^B\,(b|\sigma_B)\,, \end{aligned}$$

where we used the fact that in this case, by direct calculation using the expressions for $\widetilde{f}_{x|\mu}(b|\mu)$ and $g_{\varepsilon+\delta}(\sigma_B - \mu)$ from (87) and (85),

$$\widetilde{f}_{x|\sigma}^B\,(b|\sigma_B) = \frac{1}{\sqrt{2\pi\,\left(\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon + \upsilon_\delta\right)}} \exp\left(-\frac{\left(b - \sigma_B - V\left(\xi_q^{\varepsilon+\delta}\right)\right)^2}{2\left(\widetilde{\upsilon}_{x|\mu} + \upsilon_\varepsilon + \upsilon_\delta\right)}\right).$$

Using (88), we get from (84) that the asumptotic price-taking term is given by (20). Moreover, the asymptotic strategic term is reported in (55) of Lemma 4 in Section B of the Appendix and appears as (21) in the main text.

Hence, by using the asymptotic price-taking and strategic terms, the AFOC corresponding to (82) is

$$\begin{aligned} &\left(\widetilde{\mathbb{E}}[v|\sigma_B, x = b] - b\right) \cdot \widetilde{f}_x^B(b|\sigma_B) - \widetilde{\Pr}[x < b < y|\sigma_B] = 0 \Leftrightarrow b \\ &\quad = \widetilde{\mathbb{E}}[v|\sigma_B, x = b] - \frac{\widetilde{\Pr}[x < b < y|\sigma_B]}{\widetilde{f}_x^B(b|\sigma_B)}, \end{aligned}$$

which, by substituting for (55), yields (19).

Seller. From (6), the FOC for a seller is $-(\mathbb{E}[c|\sigma_S, x = a] - a) \cdot f^S_{x|\sigma}(a|\sigma_S) = 0$. We substitute a for $\sigma_S + \lambda^\eta_S$ in (6) and drop the dependence on x of η for brevity. By Theorem 1, the asymptotic distribution of x is the same for a buyer and seller. The price-taking term of a seller is therefore the same as that of a buyer with the only changes of σ_B to σ_S and b to a. This is reported in (22). □

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